ON SOME PROBLEMS IN GENERAL TOPOLOGY

SH-E3

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§0. Introduction

This work was done in 1977 and was widely quoted but not submited.

In section 3 it is proved that Arhangelskii's problem has a consistent positive answer: if $V \models CH$, then for some $\aleph_1$-complete $\aleph_2$-c.c. forcing notion P of cardinality $\aleph_2$ we have $\Vdash_P$ "CH and there is a Lindelöf regular topological space of size $\aleph_2$ with clopen basis with every point of pseudo-character $\aleph_0$ (i.e. each singleton is the intersection of countably many open sets)".

Meanwhile this was continued in Hajnal and Juhasz [HJ], Stanley and Shelah [ShSt:167], I.Gorelic [Go] and Ch.Morgan.

In section 4 we prove the consistency of: $CH + 2^{\aleph_1} > \aleph_2 +$ there is no space as above with $\aleph_2$ points" (starting with a weakly compact cardinal).

Section 2 deals with $\beta(\mathbb{N})$, it is proved that the following is consistent with ZFC: $MA + 2^{\aleph_0} = \aleph_2 + (*)$ where

(∗) if $A^0_i, A^1_i \subseteq \omega$ (for $i < \omega_1$) and $A^0_i \cap A^1_j$ is finite for $i,j < \omega_1$; and $\mathfrak{D}^\ell_i$ is a non-principal ultrafilter over ω such that $A^\ell_i \in \mathfrak{D}^\ell_i$ (for $i < \omega_1$ and $\ell \in \{0,1\}$), *then* there is a $B \subseteq \omega$ such that $B \in \mathfrak{D}^\ell_i \Leftrightarrow \ell = 0$.

The scheme is as in Baumgartner [B].

Another problem on $\beta(\mathbb{N})$ which I remember was asked and published by E. van Douwen and G. Woods, is answered in §1: is there a discrete $D \subseteq \beta(\mathbb{N})$, of cardinality $\aleph_1$ and $A \subseteq D$ such that $cl(A) \cap cl(D \setminus A) \neq \emptyset$.

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§1. A problem on $\beta(\mathbb{N})$

1.1 Question. Does there exist a discrete $D \subseteq \beta(\mathbb{N})$ of cardinality $\aleph_1$, and an $A \subseteq D$ such that $cl(A) \cap cl(D \setminus A) \neq \emptyset$?

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1.2 Answer. Yes, moreover we can let D have any cardinality λ such that $\aleph_1 \leq \lambda = cf(\lambda) \leq 2^{\aleph_0}$ and have $\bigcap\{cl(D') : D' \subseteq D$ and $|D'| = |D|\}$ not empty.

1.3 Definition. Let B_i (for $i \leq \omega_1$) be the BA (Boolean algebra) freely generated by $\{x_\alpha : \alpha < i\}, B^c_i$ is the completion of B_i, $B = B^c_{\omega_1}$.

1.4 Claim. *In the space of ultrafilters of $B^c_{\omega_1}$ we can find such a D.*

PROOF: We define by induction on $i \leq \omega_1$, an ultrafilter $\mathfrak{D}_i$ of $B^c_{\omega_1}$ such that

(i) $x_\alpha \in \mathfrak{D}_i \Leftrightarrow \alpha = i$,

(ii) if $a \in B^c_i$, $a \in \mathfrak{D}_i$ then $a \in \mathfrak{D}_j$ for every $j > i$.

This is easy. Let $\mathfrak{D}_{\omega_1} = \bigcup_{i<\omega_1}(\mathfrak{D}_i \cap B^c_i)$. Now, $\{\mathfrak{D}_i : i < \omega_1\}$ is discrete by (i) and $\mathfrak{D}_{\omega_1} \in cl\{\mathfrak{D}_i : i \in S\}$ for any $S \subseteq \omega_1, |S| = \aleph_1$, because $B^c_{\omega_1} = \bigcup_{i<\omega_1} B^c_i$, as B_{ω_1} satisfies the countable chain condition.

1.5 Solution of the problem. Let $X_i \subseteq \omega$ ($i < \omega_1$) be independent, i.e. any non trivial Boolean combination of the X_i is not empty. Let $f : \mathcal{P}(\omega) \to B^c_{\omega_1}$ be any homomorphism such that $f(X_i) = x_i$ (exists as $B^c_{\omega_1}$ is complete and the X_i are independent). It is not hard to prove f is onto, and $\{f^{-1}(\mathfrak{D}_i) : i < \omega_1\}$ is as required ($\mathfrak{D}_i$ from the claim), and

$$f^{-1}(\mathfrak{D}_{\omega_1}) \in cl(\{f^{-1}(\mathfrak{D}_{2i}) : i < \omega_1\}) \cap cl(\{f^{-1}(\mathfrak{D}_{2i+1}) : i < \omega_1\}).$$

$\square_{1.5}$

§2 A question on $\beta(\mathbb{N}) \setminus \mathbb{N}$

2.1 Claim. *Assuming the consistency of ZFC we prove the consistency of the following assertion with*

$ZFC + 2^{\aleph_0} = \aleph_2 + MA$:

$\otimes$ *if $A^0_i, A^1_i \subseteq \omega$ (for $i < \omega_1$) and $A^0_i \cap A^1_j$ is finite for $i, j < \omega_1$; $\mathfrak{D}^\ell_i$ is a non-principal ultrafilter over ω such that $A^\ell_i \in \mathfrak{D}^\ell_i$ (for $i < \omega_1$ and $\ell \in \{0, 1\}$), then there is a $B \subseteq \omega$ such that $B \in \mathfrak{D}^\ell_i \Leftrightarrow \ell = 0$.*

PROOF: We assume $V \vDash 2^{\aleph_0} = \aleph_1 \ \wedge \ 2^{\aleph_1} = \aleph_2$. We repeat the proof of Solovay-Tennenbaum of $Con(ZFC+2^{\aleph_0} = \aleph_2+MA)$, similarly to Baumgartner [B]. That is, we define by induction on $\alpha < \omega_2$ a set of forcing conditions P_α, increasing (under $\subseteq$ and even $\lessdot$) and continuous with $\alpha, |P_\alpha| \leq \aleph_1$, and each P_α satisfies the countable chain condition. We start with $V \vDash 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2 + \diamondsuit_{\{\delta<\omega_2: cf(\delta)=\aleph_1\}}$.

Now, in addition, at some $\alpha < \omega_2$ we consider a system $\langle A^\ell_i : i < \omega_1, \ell = 0, 1\rangle$, $\langle \mathfrak{D}^\ell_i : i < \omega_1, \ell = 0, 1\rangle$, which belongs to $V^{P_\alpha}, A^\ell_i \subseteq \omega$, $A^\ell_i \in \mathfrak{D}^\ell_i$, $\mathfrak{D}^\ell_i$ a family of subsets of ω, such that the filter it generates in V^{P_α} (which we denote by the same letter) is $\aleph_1$-saturated (i.e. there are no $C_\alpha \neq \emptyset \ mod \ \mathfrak{D}^\ell_i, C_\alpha \subseteq \omega$ for $\alpha < \omega_1$ such that $C_\alpha \cap C_\beta = \emptyset \ mod \ \mathfrak{D}^\ell_i$ for $\alpha < \beta < \omega_1$). By the usual bookkeeping, every such system appears, and this is possible as $V \vDash 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$. Clearly $V^{P_\alpha} \vDash 2^{\aleph_0} = \aleph_1$. We define in V^{P_α} a set of forcing conditions Q satisfying the c.c.c., whose generic set gives a B such that $B \in \mathfrak{D}^0_i$ and $\omega \setminus B \in \mathfrak{D}^1_i$ (for

$i < \omega_1$), and define $P_{\alpha+1} = P_\alpha * Q$. This is sufficient, as if $p \in P_{\omega_2}$ forces that $\langle \underset{\sim}{A}^\ell_i : i < \omega_1\rangle, \langle \underset{\sim}{\mathfrak{D}}^\ell_i : i < \omega_1, \ell = 0,1\rangle$ contradict $\otimes$, then for a club of E of ω_2, for every $\alpha \in E$ of cofinality $\aleph_1$, $\langle \underset{\sim}{A}^\ell_i : \ell < \omega_1\rangle$ is a P_α-name, and $\langle \mathcal{P}(\omega)^{V^{P_\alpha}} \cap \underset{\sim}{\mathfrak{D}}^\ell_i : \ell, i\rangle$ is a P_α-name of an ultrafilter. Then, clearly for stationarily many $\alpha < \omega_2$ of cofinality $\aleph_1$, in V^{P_α} the above holds and as $\diamondsuit_{\{\delta<\omega_2 : cf(\delta)=\aleph_1\}}$ holds we can assume that we have considered the system in question at some α (of course in the bookkeeping we take care of MA too).

So it suffices to prove:

2.2 Claim. *If $V \models 2^{\aleph_0} = \aleph_1$, $A^\ell_i \subseteq \omega$ (for $\ell = 0,1$ and $i < \omega_1$), $A^0_i \cap A^1_j$ are finite for $i, j < \omega_1$, $\mathfrak{D}^\ell_i$ an $\aleph_1$-saturated filter over ω, $A^\ell_i \in \mathfrak{D}^\ell_i$, then we can find a partial order Q of size $\aleph_1$, satisfying c.c.c., and in V^Q there is an $X \subseteq \omega$ such that $X \in \mathfrak{D}^0_i$ and $\omega \setminus X \in \mathfrak{D}^1_i$ (for $i < \omega_1$; note: $\mathfrak{D}^\ell_i$ stands for the filter it generates).*

2.2A Remark. The sequence $\langle (B^0_i, B^1_i) : i < \omega_1\rangle$ constructed below should be just generic enough (we do not use this presentation because some people do not like it[1]). More specifically letting $f_{2i+\ell} \in {}^\omega 2$ be defined by

$$f^\ell_i(n) = 0 \Leftrightarrow [\text{the } n\text{th element of } A^\ell_i \text{ belongs to } B^\ell_i]$$

we demand: for every $n < \omega$ and open dense $J \subseteq {}^n({}^\omega 2)$, for some $\alpha_J < \omega_1$, for every $\alpha_0 < \ldots < \alpha_{n-1}$ from (α_J, ω_1) we have $\langle f_{\alpha_0}, \ldots, f_{\alpha_{n-1}}\rangle \in J$.

PROOF: We shall choose sets $B^\ell_i \subseteq A^\ell_i$, $B^\ell_i \notin \mathfrak{D}^\ell_i$, and let $Q = \{(f,g) : f, g$ are finite functions from ω_1 to ω, and $(A^0_i \setminus B^0_i \setminus f(i)) \cap (A^1_j \setminus B^1_j \setminus g(j)) = \emptyset$ when $i \in Dom(f)$ and $j \in Dom(g)\}$.

(let $Q_i = \{(f,g) \in Q : Dom(f) \cup Dom(g) \subseteq i\}$).

Q is ordered naturally: $(f_1, g_1) \le (f_2, g_2)$ <u>iff</u> $f_1 \subseteq f_2$ & $g_1 \subseteq g_2$

For a generic $G \subseteq Q$ the set $X = \bigcup\{B^0_i \setminus f(i) : \exists g\,((f,g) \in G)\}$ is as required, and $|Q| = \aleph_1$. We should show only that we can choose B^ℓ_i such that Q satisfies the c.c.c. (the density condition is easy: for every $i, j < \omega_1$ by the almost disjoint condition, $A^0_i \cap A^1_j$ is finite, hence for every $(f,g) \in Q$ and $i < \omega_1$ there is $n^* < \omega$ such that $A^0_i \subseteq n^*$ is disjoint to A^1_j for $j \in Dom(g) \cup \{i\}$ and $A^1_i \setminus n^*$ is disjoint to A^0_j for $j \in Dom(f) \cup \{i\}$. Let $f' = f \cup \{\langle i, n^*\rangle\}$, $g' = g \cup \{\langle i, n^*\rangle\}$. So, there is a $(f', g') \in Q$ such that $(f,g) \le (f', g')$ and $i \in Dom(f') \cap Dom(g')$).

Suppose (f_i, g_i) (for $i < \omega_1$) exemplify a contradiction to c.c.c. Then, by the well known techniques, we can assume that there is a $(f,g) \le (f_i, g_i)$, $Dom(f_i) = Dom(g_i) = w_i$, $Dom(f) = Dom(g) = w$, $w \subseteq w_i$ and the sets $w_i \setminus w$ are pairwise disjoint, $w_i \setminus w = \{\eta_i(0), \ldots, \eta_i(t)\}$, $i < j \Rightarrow \eta_i(t) < \eta_j(0)$, $\eta_i(0) < \eta_i(1) < \ldots < \eta_i(t)$, and for $m \le t$ we have $f_i(\eta_i(m)) = k_m$ and $g_i(\eta_i(m)) = j_m$. For some δ we can get an "elementary submodel" of the whole system (so $\eta_i(t) < \delta$ for $i < \delta$). Let $t_\zeta, \delta_\zeta, f^\zeta_i, g^\zeta_i$ (for $i < \delta$), $\eta^\zeta_i(m), k^\zeta_m, j^\zeta_m$ (for $\zeta < \omega_1$ and $m \le t$) enumerate all possible such systems. Now we shall choose B^ℓ_α by induction on α and then on ℓ with some restrictions: (say $\ell = 0$ for notational simplicity).

Let us try to explain the idea of the proof. If $\max\{\zeta, \delta_\zeta\} < \alpha < \omega_1$, then we think of $(\langle \alpha, k^\zeta_0\rangle, \emptyset) \in Q$ as a candidate to be a part of some (f_i, g_i). Now,

[1] at least they did not back in the seventies

either it is compatible with infinitely many (f^ζ_i, g^ζ_i) (for $i < \delta_\zeta$), or the condition on being an elementary submodel eliminates this possibility, and similarly if $Dom(f') = Dom(g') = \{\eta(0), \dots, \eta(m-1)\} \subseteq \alpha$ and (f, g) is compatible with infinitely many (f^ζ_i, g^ζ_i), then either so is $(f' \cup \{\langle \alpha, k^\zeta_m \rangle\}, g')$, or the condition on elementary submodels is violated.

There are countably many such conditions and we can find $2^{\aleph_0}$ pairwise almost disjoint infinite $Y_\xi \subseteq A^\ell_i$ (for $\xi < 2^{\aleph_0}$), such that $A^\ell_\alpha \setminus Y_\xi$ are as required and all but countably many of the Y_ξ's are $= \emptyset \ mod\ \mathfrak{D}^\ell_\alpha$ (as it is $\aleph_1$-saturated), so we have many candidates for B^ℓ_α.

Now we present the construction itself. Assume B^0_i, B^1_0 have been choosen for $i < \alpha$ and we shall define B^0_α, B^1_α. As $\langle B^0_i, B^1_i : i < \alpha \rangle$ is defined, so is Q_α.

Let

$$\begin{aligned} K_\alpha = \big\{(\zeta, m, n, i, f', g', \eta) \colon & \zeta < \alpha \wedge m \le t_\zeta \wedge n < \omega \wedge i < \delta_\zeta \wedge (f', g') \in Q_\alpha \\ & \wedge |Dom(f')| = m \wedge \eta \in {}^m\alpha \wedge \eta \text{ strictly increasing} \\ & \wedge Dom(f') = Dom(g') = \{\eta(0), \dots \eta(m-1)\} \\ & \wedge f'(\eta(s)) = k^\zeta_s (\text{ for } s < m) \\ & \wedge g'(\eta(s)) = j^\zeta_s (\text{ for } s < m)\big\}. \end{aligned}$$

Clearly, K_α is countable. Let $L_\alpha = \{(Y^0, Y^1) : Y^\ell \subseteq A^\ell_\alpha$ for $\ell = 0, 1\}$, and for $(Y^0, Y^1) \in L_\alpha$, let $Q_\alpha[Y^0, Y^1]$ be defined as $Q_{\alpha+1}$, had we chosen (B^0_α, B^1_α) to be $(A^0_\alpha \setminus Y^0, A^1_\alpha \setminus Y^1)$. We say that $(Y^0, Y^1) \in L_\alpha$ satisfies $(\zeta, m, n, i, f', g', \eta) \in K_\alpha$ if:

<u>either</u>

(α) for some $\beta < \delta_\zeta$, $(f^\zeta_\beta, g^\zeta_\beta)$ and $(f' \cup \{< \alpha, k^\zeta_m >\}, g' \cup \{< \alpha, j^\zeta_m >\}, h)$ are compatible conditions in $Q_\alpha[Y^0, Y^1]$ and $\beta e_{\zeta,n} i$ where $j e_{\zeta,n} i$ means:

$$\begin{aligned} (\forall \ell \le t^\zeta)[A^0_{\eta^\zeta_j(\ell)} \cap n = A^0_{\eta^\zeta_i(\ell)} \cap n \ \wedge \ A^1_{\eta^\zeta_j(\ell)} \cap n = A^1_{\eta^\zeta_i(\ell)} \cap n \\ \wedge B^0_{\eta^\zeta_j(\ell)} \cap n = B^0_{\eta^\zeta_i(\ell)} \cap n \ \wedge \ B^1_{\eta^\zeta_j(\ell)} \cap n = B^1_{\eta^\zeta_i(\ell)}]. \end{aligned}$$

<u>or</u>

(β) for some natural number $u < \omega$, for every $(Z^0, Z^1) \in L_\alpha$ satisfying $Z^0 \cap u = Y^0 \cap u \,\&\, Z^1 \cap u = Y^1 \cap u$, clause ($\alpha$) fails even for $Q_\alpha[Z^0, Z^1]$.

Now, L_α is a complete separable metric space (by the metric d defined as $d((Y^0, Y^1), (Z^0, Z^1)) = \min((Y^0 \triangle Z^0) \cup (Y^1 \triangle Z^1))$ where $\triangle$ is the symmetric difference i.e. $Y \triangle Z = (Y \setminus Z) \cup (Z \setminus Y)$.)

Clearly:

$(*)$ for each $(\zeta, m, n, i, f', g', \eta) \in K_\alpha$ the set

$$L^\alpha_{(\zeta,m,n,i,f',g',\eta)} = \big\{(Y^0, Y^1) \in L_\alpha : (Y^0, Y^1)$$

satisfies $(\zeta, m, n, i, f', g', \eta)\big\}$ is an open dense set.

As K_α is countable, we can find a $\langle (Y^0_\xi, Y^1_\xi) : \xi < 2^{\aleph_0} \rangle$ such that:

(a) $(Y^0_\xi, Y^1_\xi) \in \cap \{L^\alpha_{(\zeta,m,n,i,f',g',\eta)} : (\zeta, m, n, i, f', g', \eta) \in K_\alpha\}$

(b) for $\zeta < \xi$ the set $Y^0_\zeta \cap Y^0_\xi$ is finite.

(this is like building a perfect set of Cohen generic reals.)

So (by the $\aleph_1$-saturation), for some $\xi < 2^{\aleph_0}$ we have $Y^0_\xi = \emptyset \, mod \, \mathfrak{D}^0_\alpha$ and $Y^1_\xi = \emptyset \, mod \, \mathfrak{D}^1_\alpha$.

We let $(B^0_\alpha, B^1_\alpha) = (A^0 \setminus Y^0, A^1_\alpha \setminus Y^1)$.

So we have finished the inductive definition of the (B^0_α, B^1_α)'s.

Now we show that the construction quarantees that Q satisfies the c.c.c. Suppose $\{(f_i, g_i) : i < \omega_1\}$ is an uncountable antichain. As we explained above, we may assume there are t_ζ, $\delta_\zeta, \ldots$, and $\{(f_i, g_i) : i < \delta_\zeta\}$ is an "elementary subsystem". So in particular, if for some $u < \omega$ and $F^\ell : t_\zeta + 1 \to \mathcal{P}(u) \times \mathcal{P}(u)$ (where $\ell \in \{0,1\}$) we have an $i < \omega_1$ such that $(A^\ell_i \cap u, B^\ell_{\eta^\zeta_i(m)} \cap u) = F^\ell(m)$ for each $\ell < 2$ and $m \le t_\zeta$, then such i exists already below δ_ζ. So for each $u < \omega$ and $\gamma < \omega_1$ for some $i_{\gamma,u} < \delta_\zeta$ we have $\gamma e_{\zeta,u} i_{\gamma,u}$. But now consider $\gamma > \delta_\zeta$ such that $\eta^\zeta_\gamma(0) > \delta_\zeta$ and $\eta^\zeta_\gamma(0) > \zeta$, and let $m \le t_\zeta$ be the least ordinal $n^\zeta_\gamma(m)$ for which condition (α) fails for some finite $u < \omega$ and $i = i_{\gamma,u}$ which is $< \delta_\zeta$ (note that if $\alpha = \eta^\zeta_\gamma(t_\zeta)$ and $u = 0$ then condition (α) cannot possibly hold). Let it fail for $u = u_m$. Then condition (β) holds. But if u witnesses (β) then for $\max\{u_m, u\}$ we have a failure for $m-1$, contradiction.

The construction guarantees that Q satisfies the c.c.c, hence we have finished. $\square_{2.2}$

$\square_{2.1}$

2.3 Remark. 1) We can act as in [Wi:C 2], and then use really $\aleph_1$-saturation and all filters are in V.

2) Can we ignore CH and make $2^{\aleph_0}$ larger?

Assume $\lambda = \lambda^{\aleph_1} = cf(\lambda)$ and $\diamondsuit_S$ where $S = \{\delta < \lambda : cf(\delta) = \aleph_1\}$ and $(\forall \alpha < \lambda)[|\alpha|^{\aleph_0} < \lambda]$. We can find a forcing notion P, which is c.c.c. of cardinality λ, and $\Vdash_P$ "$2^{\aleph_0} = \lambda$ and MA and $(*)$ of 2.1 holds."

Why? we use finite support iteration, if $\alpha \notin S, Q_\alpha$ is adding a Cohen real; if $\delta \in S$ and $\diamondsuit_S$ guesses $P, \langle A^\ell_i : i < \omega_1, \ell < 2\rangle$, and $\langle \mathfrak{D}^\ell_\alpha \cap \mathcal{P}(\omega)^{V^{P_\alpha}} : \alpha < \omega_1, \ell < 2\rangle$ as in the proof, we imitate the proof, but for $\langle (B^0_\alpha, B^1_\alpha) : \alpha < \omega_1\rangle$ we use a sequence $\langle T_\alpha : \alpha < \omega_1\rangle$, T_α a perfect set of members of L_α such that for every large enough $\alpha < \omega_1$, all branches of T_α are Cohen generic over V^{P_δ}.

To answer the question of the referee:

2.4 Claim. *The statement $\otimes$ of 2.1 follows from PFA.*

PROOF Consider the forcing $P = Levy(\aleph_1, \aleph_2) * \underset{\sim}{Q}$ where $\underset{\sim}{Q}$ is constructed as in the proof of 2.1 in the universe $V^{Levy(\aleph_1,\aleph_2)}$ (note that forcing with $Levy(\aleph_1, \aleph_2)$ adds no reals, so $\mathfrak{D}_i$ is still an ultrafilter and CH holds) and let $\underset{\sim}{B}$ be the name of the desired set. So $\Vdash_P$ "for $i < \omega_1$, $\ell < 2$, for some $A \in \mathfrak{D}^\ell_i$ we have $[\ell = 0 \Rightarrow \underset{\sim}{B}^* \supseteq A]$ and $[\ell = 1 \Rightarrow \underset{\sim}{B} \cap A =^* \emptyset]$". Apply PFA.

§3. Concerning Arhangelskii's Problem

3.1 Theorem. *The following is consistent with $ZFC + GCH$:*

$(*)$ *There is a regular space of cardinality $\aleph_2$ which is Lindelöf and has pseudo-character $\aleph_0$.*

Remark. We had first said "a Hausdorff space..." but Kunen noted the proof actually yields a regular space.

PROOF: Assume V satisfies GCH. Let P be the set of tuples of the form $p = \langle A, f, E, T\rangle = \langle A^p, f^p, E^p, T^p\rangle$, where:

(1) A is a countable subset of ω_2,

(2) f is a two place function from A to $\omega+1$, where we write $f_x(y)$ instead of $f(x,y)$, and $f_x(y) = \omega \Leftrightarrow y = x$ holds.

(3) E is a three place relation on A, but we write $xE_\gamma y$, and demand that for each fixed γ, E_γ is an equivalence relation, and $\gamma \in A \wedge \gamma < \beta \wedge xE_\beta y \Rightarrow xE_\gamma y$, while $x < \gamma \wedge xE_\gamma y \Rightarrow x = y$. We stipulate E_{ω_2} as the equality on A.

(4) T is a countable set. Each member B of T will be called a formal cover, $|B| = \aleph_0$, and B is of the form $\{\tau_n^B : n < \omega\}$, where each τ_n^B is the formal intersection of finitely many $U_x^n (x \in A, n \in \mathbb{Z})$ $\big[$the intended meaning is: $U_x^n = \{y : f_x(y) \geq n\}$ for $n \geq 0$, and $U_x^{-n} = \omega_2 \setminus U_x^n$ for $n > 0$; we say $p \vDash$ "$y \in \tau$" in the natural case (i.e. $y \in A$, $\tau = \bigcap_{i<k} U_{x(i)}^{\ell(i)}$, where $k < \omega, \ell(i) \in \mathbb{Z}, x(i) \in A$ and for each $i < k$ we have: $f_{x(i)}(y) \geq |\ell(i)|$ iff $\ell(i) \geq 0)\big]$. We let, for a formal term τ, $dom(\tau) = \{x \in A : x$ is mentioned in $\tau\}, Dom(B) = \bigcup\{dom(\tau) : \tau \in B\}$.

The real restrictions are

(5) if $zE_\gamma y$, and $x < \gamma$ then $f_x(y) = f_x(z)$.

(6) (a) if $B \in T, x \in A$, then $p \vDash$ "$x \in \tau_n^B$" for some $\tau_n^B \in B$.

(b) moreover, for each finite $A^1 \subseteq A$ and $h : A^1 \to \omega$, there is a $\tau \in B$ such that

$$[x \in A^1 \wedge U_x^n \text{ appears in } \tau] \Rightarrow [[0 \leq n \leq h(x)] \vee [n < -h(x)]]$$

(c) moreover, for each $B \in T, x \in A$ and $\gamma \in \{\omega_2\} \cup Dom(B)$ satisfying $\gamma \leq x$, and a finite $A^1 \subseteq A$ and function $h : A^1 \to \omega$, there is a $\tau = \bigcap_{i<k} U_{x(i)}^{\ell(i)} \in B$ such that: $[x(i) < \gamma] \Rightarrow [x \in U_{x(i)}^{\ell(i)}]$ and $[x(i) \geq \gamma \wedge x(i) \in A^1] \Rightarrow [[0 \leq \ell(i) \leq h(y)] \vee [\ell(i) < -h(y)]]$. Note that for $\gamma = \min\ Dom(B)$, clause (c) reduces to clause (b) and for $\gamma = \omega_2$, clause (c) reduces to clause (a).

[<u>Explanation:</u> The set A approximates the set of points, the function f describes the U_x^n's which will generate the topology as clopen sets, T is a set of "countable covers", i.e. we think of a possible covering which is a counterexample to Lindelöfness and "promise" that a countable subfamily of such cover, will cover the entire space. In demand (6), clause (a) just says that each $B \in T$ really covers, clause (b) is necessary when we prove e.g. density of $\{p : x \in A^p\}$ for $x \in \omega_2$, (see the proof of Fact B). This is done by an increasing ω-sequence of descriptions of the important new values of f, so this clause tells us a finite information, so does not prevent us from preserving "B is a cover".

Still, why do we need clause (c) of demand (6)? We want that our forcing notion satisfies the $\aleph_2$-c.c., so we use the Δ-system lemma, and during the construction (i.e. the proof of Fact E i.e. the construction of a common upper bound of p_1,

p_2) we have finitely many commitements on new values of f, we want to make $x \in \bigcup_{n<\omega} \tau_n^B$ for $B \in T^{p_j}$, $x \in A^{p_i} \setminus A^{p_j}$, so $f_y(x)$ is determined for all $y \in A^{p_i} \cap A^{p_j}$ and for finitely many $y \in A^{p_j} \setminus A^{p_i}$, and clause (6)(c) guarantees we can deal with this. We have above avoided "justifying" the use of the equivalence relation E^p, it is needed when in the proof of Fact D (Lindelöfness holds), to the union of a generic enough sequence $\langle p_n : n < \omega\rangle$ for the union q we add to the T^q a covering thus defeating a possible counterexample, we need E to verify condition (6)(c).]

If there are several p's in consideration, we shall write $A^p, f^p, \dots$ or $p^\ell = \langle A^\ell, f^\ell, E^\ell, T^\ell\rangle$. Now we define the order on $P : p \le q$ iff $A^p \subseteq A^q, f^p = f^q \restriction A^p, E^p = E^q \restriction A^p, T^p \subseteq T^q$. In V^P we define the following topology on ω_2. For $x \in \omega_2, n < \omega$, let $U_x^n = \{y : f_x^p(y) \ge n \text{ for some } p \text{ in the generic set }\}$, and if $n > 0$, we let $U_x^{-n} = \omega_2 \setminus U_x^n$. Now, $\{U_x^n : x \in \omega_2, n \in \mathbb{Z}\}$ will be closed and open, and the topology X is the minimal one which satisfies this. So the set of finite intersections of U_x^n's forms a basis. By clause (2) in the definition of P, and the Fact B below, we know $\bigcap_{n<\omega} U_x^n = \{x\}$, so as each U_x^n is clopen, the space is Hausdorff and even regular, and has pseudo-character $\aleph_0$.

Fact A. P is $\aleph_1$-complete; in fact, any ascending ω-sequence has a naturally defined union.

In fact, we already use

Fact B. For every $p \in P$ and $z \in \omega_2$ there is a $q \in P$ such that $q \ge p$ and $z \in A^q$.

Moreover, if $z \notin A^p$, for any finite subset A^* of A^p and function h^* from A^* to $\mathbb{Z}$ we can demand $f_x^q(z) = h^*(x)$ for $x \in A^*$.

PROOF OF B: The non-trivial part is to satisfy clause (6). We first define $f_x(z)$ for $x \in A^p$ to satisfy (6)(c) when z here stands for x there. So let $\{(B_k, A_k, h_k, \gamma_k) : k < \omega\}$ be a list of all tuples (B, A, h, γ) such that $\gamma \in \{\omega_2\} \cup Dom(B)$, $\gamma \le z$, $B \in T^p$, $A \subseteq A^p$ is finite and $h : A \to \omega$. Now we define by induction on k a finite set $D_k \subseteq A^p$ and $f_x(z)$ for $x \in D_k$.

For $k = 0$, $D_0 = A^*$, $\bigwedge_{x\in D_0} f_x(z) = h^*(x)$. If we have defined D_k, let us define h'_k, A'_k as follows: $A'_k = A_k \cup D_k$ and $h'_k(x)$ is $h_k(x)$ if $x \in A_k$ and $f_x(z)$ if $x \in D_k \setminus A_k$; choose $\tau \in B_k$ as exists by (6)(c) (with A'_k in place of A^1 and h'_k in place of h), let $D_{k+1} = D_k \cup Dom(\tau)$, and define $f_x(z)$ for $k \in Dom(\tau) \setminus D_k$ as ℓ if U_z^ℓ appears in τ with $\ell \ge 0$ and as 0 otherwise.

We can at last complete the definition of $h_x(z)$ for $x \in A \setminus \bigcup_{k<\omega} D_k$. Lastly define $f_z(y)$ for $y \in A^p \cup \{z\}$ as ω if $y = z$ and 1 if $y \ne z$. If we let $q = \langle A \cup \{z\}, f^q$ is f expanded as described above, E^p (i.e. x is E_γ-equivalent only to itself), $T^p\rangle$ then q is O.K. $\square_B$

Similarly, we can prove

Fact C.

(1) For every $p \in P$, $z \in \omega_2 \setminus A^p$ and $\gamma \le y$ in $A^p, \gamma \le z$, there is a $q \in P$ such that $p \le q$ and $q \vDash$ "$zE_\gamma y$". Moreover, for a given finite $A^1 \subseteq A^p \setminus \gamma$, and a function $h^1 : A^1 \to \omega$, we can demand $q \vDash$ "$f_x(z) = h^1(x)$ for $x \in A$".

(2) The following is a dense subset of P, closed under unions:

$$\mathcal{I} = \Big\{ q \in P : \underline{\text{for every}}\ \gamma \in A^q \cup \{\omega_2\} \text{ and finite } A \subseteq A^q \text{ and function } h : A \to \omega \text{ and } x \in A^q \text{ satisfying } \gamma \leq x\ \underline{\text{there is}}\ x' \in A^q \text{ such that } \gamma \leq x', x' E_\gamma x \text{ and } (\forall y \in A \setminus \gamma)[f_x(y) = h(y)] \Big\}.$$

PROOF OF C: 1) like the proof of Fact B.

2) For any $p \in P$, choose $p_n \in P$, $p_0 = p$, $p_n \leq p_{n+1}$, each time use part (1) with a suitable bookkeeping and take the union by Fact A. $\square_B$

Fact D. The space is Lindelöf.

PROOF OF D: Let $\underset{\sim}{\sigma}$ be a name of a cover and $\underset{\sim}{\sigma}_x$ a member of it to which x belongs, and *w.l.o.g.* $\underset{\sim}{\sigma}_x$ is a member of the basis (i.e. is a τ). Now, for each $p \in P$ and $x \in A$, there is a $q \geq p$, such that $q \Vdash$ "$\underset{\sim}{\sigma}_x = \tau$" for some specific τ. By Fact A for every $p \in P$ there is $q \in P$ such that

$$p \leq q \in \mathcal{I} \ \wedge \bigwedge_{x \in A^p} \bigvee \{ q \Vdash \text{"}\underset{\sim}{\sigma}_x = \tau\text{"} : \tau \text{ as in demand (4)}\}.$$

So for every $p \in P$ we can find $\langle p_n : n < \omega \rangle$ such that $p \leq p_0$, $p_n \leq p_{n+1}$, $p_n \in \mathcal{I}$ and $x \in A^{p_n} \Rightarrow p_{n+1} \Vdash \ \underset{\sim}{\sigma}_x = \tau^x$". Let $q = \bigcup_{n<\omega} p_n$, now $q \in P$ is an upper bound of $\{p_n : n < \omega\}$ and $q \in \mathcal{I}$ by Fact C(2). Now

$$B^* = \{\tau : q \Vdash \text{"}\underset{\sim}{\sigma}_x = \tau\text{"} \text{ for some } x \in A^q \text{ and } \tau \text{ as in (4) (for } q)\},$$

satisfies the requirements on B in clause (4). Now B^* also satisfies the requirements of part (6) in the definition: clause (a) holds as

$$x \in A^q \Rightarrow \bigvee_n x \in A^{p_n} \Rightarrow \bigvee_n (p_{n+1} \text{ forces a value to } \underset{\sim}{\sigma}_x).$$

Clause (c) holds by the above as $q \in \mathcal{I}$. So $q^* \overset{\text{df}}{=} \langle A^q, f^q, E^q, T^q \cup \{B^*\}\rangle \in P$ [why? check; the main point is (6)(c) which holds as $q \in \mathcal{I}$]. Also $q \leq q^*$, so q^* forces that B^* is (essentially) a countable subcover, as required. $\square_D$

Fact E. P satisfies the $\aleph_2$-chain condition. (Hence in V^P, G has power $> 2^{\aleph_0}$).

PROOF: Let $p_i \in P(i < \omega_2)$. It is well known that we can assume that for some p and α_i (for $i < \omega_2$): α_i is increasing, $A^{p_i} \cap \alpha_i = A^p, A^{p_i} \subseteq \alpha_{i+1}, p \leq p_i$. Now, like in the proof of Fact B, we can prove p_0, p_1 can be extended to a condition $\langle A^{p_0} \cup A^{p_1}, f^{p_0} \cup f^{p_1}, E^{p_0} \cup E^{p_1}, T^{p_0} \cup T^{p_1}\rangle$. $\square_E$

$\square_{3.1}$

Remarks.

(1) The proof works with $\aleph_0$ replaced by any λ such that $\lambda^{<\lambda} = \lambda, 2^\lambda = \lambda^+$; countable is replaced by "of cardinality $\leq \lambda$" and τ are still finite formal intersections, $Rang(f) \leq \omega = 1$. We get λ-Lindelöf but still pseudo-character $\aleph_0$.

(2) It is well known that there is no Lindelöf space of pseudo-character $\aleph_0$ of power $\geq$ " first measurable".

§4. More on Arhangelskii's problem

We prove:

4.1 Theorem. *If $(ZFC + GCH + \exists$ a weakly compact) is consistent, then so is the following: $ZFC + CH + not(*)$ where*

$(*)$ *There is a regular space of cardinality $\aleph_2$ which is Lindelöf and has pseudo-character $\aleph_0$.*

Question. (CH) Is there a Lindelöf space with pseudo-character $\aleph_0$ and with $2^{\aleph_1}$ points? More than $\aleph_2$ points?

PROOF: Let $V \models$ "$GCH \wedge \kappa$ weakly compact."

Let P_i $(i < \kappa^+)$ be the forcing for adding a Cohen subset to ω_1 (so P_i is $\aleph_1$-complete).

Q_2^κ is the Levy collapse of κ to $\aleph_2$ (i.e. every $\theta \in (\aleph_1, \kappa)$ is collapsed to $\aleph_1$, and each condition is countable).

$Q_1 = \Pi_{i<\kappa^+} P_i = \{p \in \Pi_{i<\kappa^+} P_i : p$ has countable support (i.e. $p(i) = \emptyset$ for all except countably many i)$\}$.

$Q = Q_2^\kappa \times Q_1$.

In V^Q we know κ is $\aleph_2$, CH holds, $2^{\aleph_1} = \kappa^+$ (and only the cardinals $\theta \in (\aleph_1, \kappa)$ were collapsed.)

We prove that in V^Q there is no Lindelöf space X with countable pseudo-character, such that $|X| = \kappa$.

If there is such an X, we can assume its set of points is ω_2, and $x \in U_x^n, U_x^n$ open, $\bigcap_n U_x^n = \{x\}$ for all $x \in \omega_2$ (by the countable pseudo-character).

So the topology is the minimal one to which all U_x^n belong; this is O.K. as if we decrease family of open sets, the Lindelöfness is preserved.

We $w.l.o.g.$ identify X with $\langle (x, n, U_x^n) : x < \omega_2, n < \omega \rangle$. So the topology is the minimal one to which all U_x^n belong; this is O.K. as if we decrease the family of open sets, the Lindelöfness is preserved. Note

$\oplus$ if $X \in V_1 \subseteq V_2$ and $U \subseteq X$ with $U \in V_1$, then $V_1 \models$ "U is open in X" iff $V_2 \models$ "U is open in X".

(because U is open iff $\forall x \in U \exists x_1, \dots, x_k \exists n_1, \dots, n_k (x \in \bigcap_{\ell=1}^{k} U_{k_\ell}^{n_\ell} \subseteq U)$)

Claim A. *In V^Q (but CH is used), for some closed unbounded $C \subseteq \kappa$, we have $\alpha \in C \wedge cf(\alpha) > \aleph_0 \Rightarrow (X \restriction \alpha$ is not Lindelöf, moreover, there is a $g : \alpha \to \omega$, so that $\{U_x^{g(x)} : x \in \alpha\}$ has no countable subcover).*

PROOF OF A: C will be the family of $\alpha < \omega_2$ such that:

if $Dom(h)$ is a countable bounded subset of α, $Range(h) \subseteq \omega, \omega_2 \neq \bigcup\{U_x^{h(x)} : x \in Dom(h)\}$ *then* there is a $\beta < \alpha$, $\beta \notin \bigcup\{U_x^{h(x)} : x \in Dom(h)\}$.

Clearly, C is closed and by CH it is unbounded.

If $cf(\alpha) > \aleph_0$, we can omit the "bounded", as every countable subset of α is bounded.

For $\alpha \in C$ such that $cf(\alpha) > \aleph_0$, define $g : \alpha \to \omega$ as follows : $g(x)$ is the first $n < \omega$ such that $\alpha \notin U_x^n$ (exists as $\bigcap_{n<\omega} U_x^n = \{x\}$).

Clearly $\{U_x^{g(x)} : x < \alpha\}$ cover α. Suppose $\{U_x^{g(x)} : x < \alpha\}$ has a countable subcover $\{U_x^{g(x)} : x \in Y\}, |Y| \leq \aleph_0$. Let $h = g \restriction Y$ and we get a contradiction to the definition of C (because α witnesses that the union is not ω_2). $\square_A$

Claim B. *Suppose V satisfies CH and X and $U_x^n (x \in X)$ are as above. Suppose X is Lindelöf, P is an $\aleph_1$-complete forcing, but in V^P, the space X is not Lindelöf.*

Then also in V^{P_0} (remember that P_0 is adding one Cohen subset to ω_1), the space is not Lindelöf.

PROOF OF B: Suppose $\underset{\sim}{\tau}$ is a P-name for a cover contradicting Lindelöfness, *wlog* the cover consists of old sets. Let $p \in P$.

We define $p_\eta \in P$ for $\eta \in {}^{\omega_1>}\omega$ by induction on the length $\ell(\eta)$ of η, an old open set U_η (where $\ell(\eta)$ is a successor) such that:

(1) $p_{\eta\restriction\alpha} \leq p_\eta$ ($\leq$ means "weaker than")
(2) $p_\eta \Vdash_P$ "$U_\eta \in \underset{\sim}{\tau}$" (for $\ell(\eta)$ a successor)
(3) $X = \bigcup_{n<\omega} U_{\eta\hat{}\langle n\rangle}$.

For $\ell(\eta) = 0, \eta = \langle\,\rangle, p_\eta = p$, for limit only (1) applies and we use $\aleph_1$-completeness. If $\eta \in ({}^\alpha\omega)$ and $p_{\eta\restriction(\beta+1)}$ is defined for $\beta < \alpha$, let

$$F_\eta = \{U : U \text{ an open set of } X \text{ (in the universe } V) \text{ and for some } q \geq p_\eta, q \Vdash \text{"}U \in \underset{\sim}{\tau}\text{"}\}.$$

Clearly, F_η is a cover.

F_η is a cover, but X is Lindelöf, so for some countable $F' \subseteq F_\eta$ we have $X = \bigcup F'$. Let $F' = \{U_n^\eta : n < \omega\}$ (maybe with repetitions). Let $U_{\eta\hat{}\langle n\rangle} \overset{\text{def}}{=} U_n^\eta$, and let $p_{\eta\hat{}\langle n\rangle} \geq p_\eta$ be chosen so that $p_{\eta\hat{}\langle n\rangle} \Vdash_P$ "$U_{\eta\hat{}\langle n\rangle} \in \underset{\sim}{\tau}$."

Now we show that in V^{P_0} the space X is not Lindelöf. For a generic $g \in ({}^{\omega_1}2)$ let σ be the family $\{U_{g\restriction(\alpha+1)} : \alpha < \omega_1\}$. It is easily seen that σ is a cover of X. Suppose $X \subseteq \bigcup\{U_{g\restriction(\alpha+1)} : \alpha < \beta\}$ for some $\beta < \omega_1$. Without loss of generality, β is a limit ordinal. Then $p_{g\restriction\beta} \Vdash_P$ "$\sigma \subseteq \underset{\sim}{\tau}$", in contradiction with the choice of $\underset{\sim}{\tau}$. $\square_B$

CONTINUATION OF THE PROOF OF 4.1: W.l.o.g. $y \in U_x^n$ is determined in $Q_3 = Q_2^\kappa \times \Pi_{i<\kappa} P_i$. [Why? As Q satisfies the κ^+-c.c. there are κ maximal antichains $\mathcal{I}_{y,n,x}$ of elements forcing a truth value to "$y \in U_x^n$". So $|\bigcup_{y,n,x} \mathcal{I}_{y,n,x}| \leq \kappa$, so for some α we have $\bigcup \mathcal{I}_{y,n,x} \subseteq Q_2^\alpha \times \prod_{i<\kappa} P_i$.] Also in V^{Q_3}, the space X (i.e. the space defined by letting a subset be open iff it is forced to be open, see $\oplus$ above) is Lindelöf, noting that a cover in V^{Q_3} is also a cover in V^Q and those two universes have the same ω-sequences of members of V^{Q_3} as no ω-sequences are added by $\Pi_{i\in[\kappa,\kappa^+)}P_i$). So, forcing by $P_0 = P_\kappa$ over V^{Q_3} does not contradict Lindelöfness, by Claim B.

As κ is weakly compact, for some stationary set S we have

$$S \subseteq \{\alpha < \kappa : \alpha \text{ strongly inaccessible}\},$$

and in addition, for each $\alpha \in S$, we can split the forcing Q to $Q'_\alpha \times Q''_\alpha$, both Q'_α and Q''_α are $\aleph_1$-complete forcings, and $Q'_\alpha = Q_2^\alpha \times \Pi_{i<\alpha}P_i$ so that:

$$Q'_\alpha \text{ determines "} y \in U^n_x \text{" for } \; y, x < \alpha, n < \omega,$$

and in $V^{Q'_\alpha}$, α becomes $\aleph_2$. Also, the part of the space X that we get after forcing with $Q^{'}_\alpha$, X_α that is $X \restriction \alpha$, is Lindelöf of pseudo-character $\aleph_0$, as exemplified by the U^n_x, and adding a P_0-generic does not contradict Lindelöfness. (Here we use the weak compactness of κ i.e. Π^1_1-indescribability of κ.)

Now, by claim A, for some such α, in $V^{Q'_\alpha \times Q''_\alpha}$ the space X_α is no longer Lindelöf. Therefore, forcing by Q''_α abolishes Lindelöfness. Also Q'_α, Q''_α are $\aleph_1$-complete, so Q''_α is $\aleph_1$-complete in $V^{G'_\alpha}$, hence (Claim B) P_0 forcing abolishes Lindelöfness, a contradiction. $\square_{4.1}$

4.2 Remark. Note that during the proof we did not use the regularity of X.

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