Graphs with no unfriendly partitions

Saharon Shelah* and E. C. Milner†

Abstract

An unfriendly n-partition of a graph $G = (V, E)$ is a map $c: V \to \{0, 1, \ldots, n-1\}$ such that, for every vertex x, there holds

$$|\{y \in E(x) : c(x) = c(y)\}| \leqslant |\{y \in E(x) : c(x) \neq c(y)\}|,$$

where $E(x)$ is the set of vertices joined to x by an edge of G. We disprove a conjecture of Cowen & Emerson by showing that there is a graph which has no unfriendly 2-partition. However, we also show that every graph has an unfriendly 3-partition.

1 Introduction

Let $G = (V, E)$ be a simple graph. A map $c: V \to \{0, 1, \ldots, n-1\}$ is called an unfriendly n-partition of G (see [1]) if, for every vertex x, there holds

$$|\{y \in E(x) : c(x) = c(y)\}| \leqslant |\{y \in E(x) : c(x) \neq c(y)\}|,$$

where $E(x)$ is the set of vertices joined to x by an edge of G.

It is easily seen that any finite graph has an unfriendly 2-partition and hence, by compactness, so does every locally finite graph. Cowan & Emerson [2] conjectured that every graph has an unfriendly 2-partition and Aharoni, Milner & Prikry [1] proved this for graphs satisfying either (1) *there are only finitely many vertices with infinite degrees*, or (2) *there are a finite number of infinite cardinals* $m_0 < m_1 < \cdots < m_k$ *such that*

* No. 356. The first author wishes to express his thanks to the Niedersächsches Ministerium für Wissenshaft und Kunst for partially supporting this research.

† Research supported by NSERC grant #A5198.

m_i is regular for $0 < i \leqslant k$, every vertex of infinite degree has degree m_i for some $i \leqslant k$ and the number of vertices of finite degree is less than m_0.

The following result disproves the conjecture of [2]. For a cardinal $\lambda = \omega_\alpha$ and an ordinal β, we use the notation $\lambda^{(+\beta)}$ to denote the cardinal $\omega_{\alpha+\beta}$.

Theorem 1 *There is a graph $G = (V, E)$, of size $|V| = (2^\omega)^{(+\omega)}$, which has no unfriendly 2-partition and in which every vertex has infinite degree.*

A similar argument also proves the following more general version of Theorem 1.

Theorem 2 *For any infinite cardinal λ, there is a graph $G = (V, E)$, of size $|V| = \kappa = (2^\lambda)^{(+\omega)}$, which has no unfriendly 2-partition and in which every vertex has infinite degree.*

Before giving proofs of these results, we shall prove the following consistency result which, although weaker, illustrates the main idea in a simpler setting.

Theorem 3 *It is consistent that there is a graph $G = (V, E)$ of size $|V| = \omega_\omega$ which has no unfriendly 2-partition and the degree of each vertex is either ω, or ω_1, or ω_ω.*

We conclude the paper with a proof of the following positive result.

Theorem 4 *Every graph has an unfriendly 3-partition.*

2 Proof of Theorem 3

For subsets A and B of ω, we write $A > B$ if $|A \setminus B| = \omega$ and $|B \setminus A| < \omega$. It is well known that the following statement (*) is independent of the axioms of set theory. (For example, CH $\Rightarrow$ (*) and $(\mathrm{MA} + 2^\omega > \omega_1) \Rightarrow \neg(*)$.)

(*) There is a uniform, non-principal ultrafilter $\mathfrak{U}$ on ω which is generated by ω_1 sets A_ξ $(\xi < \omega_1)$ such that $A_\xi > A_\zeta$ for $\xi < \zeta < \omega_1$ so that, for any set $A \in \mathfrak{U}$, there is some $\xi < \omega_1$ such that $|A_\zeta \setminus A| < \omega$ for $\xi \leqslant \zeta < \omega_1$.

We show that (*) implies there is a graph with the properties stated in Theorem 3.

We construct the desired graph $G = (V, E)$ as follows. Let $V = X \cup Y \cup Z$, where $X = \{x_n : n < \omega\}$, $Y = \{y_{\alpha,\xi} : \alpha < \omega_\omega, \xi < \omega_1\}$ and

$Z = \{z_\alpha : \alpha < \omega_\omega\}$ and let $E = E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{(x_n, y_{\alpha,\xi}) : \alpha \leqslant \omega, \xi < \omega_1, n \in A_\xi\},$$

$$E_2 = \{(y_{\alpha,\xi}, z_\alpha) : \alpha < \omega_\omega, \xi < \omega_1\},$$

$$E_3 = \{(x_n, z_\alpha) : \alpha < \omega_\omega, n < \omega\}.$$

Note that each vertex of X has degree ω_ω, each vetex of Y has degree ω and each vertex of Z has degree ω_1.

We want to show that G has no unfriendly 2-partition. Suppose for a contradiction that $c: V \to \{0,1\}$ is an unfriendly partition of G. Since $\mathfrak{U}$ is an ultrafilter on ω, there are $\epsilon < 2$ and $A \in \mathfrak{U}$ such that $c(x_n) = \epsilon$ if and only if $n \in A$. There is $\xi < \omega_1$ such that $|A_\zeta \backslash A| < \omega$ for $\xi \leqslant \zeta < \omega_1$. Since, by assumption, c is an unfriendly partition, since

$$E(y_{\alpha,\zeta}) = \{z_\alpha\} \cup \{x_n : n \in A_\zeta, \alpha \leqslant \omega_n\} \qquad (\alpha < \omega_\omega, \zeta < \omega_1)$$

and since $c(x_n) = \epsilon$ for $n \in A$, it follows that $c(y_{\alpha,\zeta}) = 1 - \epsilon$ for $\alpha < \omega_\omega$ and $\xi \leqslant \zeta < \omega_1$. Further, since $E(z_\alpha) = X \cup \{y_{\alpha,\zeta} : \zeta < \omega_1\}$ for $\alpha < \omega_\omega$, we must also have $c(z_\alpha) = \epsilon$. But, for $n \in A$,

$$E(x_n) = \{y_{\alpha,\zeta} : n \in A_\zeta, \alpha \leqslant \omega_n\} \cup Z,$$

and this contradicts the assumption that c is an unfriendly partition since $c(x_n) = c(z)$ $(z \in Z)$ and $|E(x_n) \backslash Z| < |Z|$. $\square$

3 Proof of Theorem 1

We will use the following notation. For an ordinal we α define $\|\alpha\|$ to be $|\alpha|$ if α is infinite and 0 if α is finite. If $\mathfrak{u} = (u_0, u_1, \ldots, u_{l-1})$ is a sequence of ordinals, the length of $\mathfrak{u}$ is $l(\mathfrak{u}) = l$, and the *last term* of $\mathfrak{u}$ is

$$\mathrm{lt}(\mathfrak{u}) = \begin{cases} u_{l-1} & \text{if } l \geqslant 1, \\ 2^\omega & \text{if } l = 0. \end{cases}$$

If $\mathfrak{v} = (v_0, v_1, \ldots, v_l)$ has length $l+1$ and $v_i = u_i$ $(i < l)$, then we write $\mathfrak{v} = \mathfrak{u}^\wedge v_l$ and we also write $\mathfrak{u} = \mathfrak{v}^*$ to indicate that $\mathfrak{u}$ is obtained from $\mathfrak{v}$ by omitting the last term v_l. Put

$$\mathcal{I} = \{(u_0, u_1, \ldots, u_{l-1}) : 2^\omega > \|u_0\| > \|u_1\| > \ldots > \|u_{l-1}\|\},$$

$$\mathcal{J} = \{(v_0, v_1, \ldots, v_{l-1}) : v_i < \omega_1 \ (i < l)\}.$$

Let $\mathfrak{U}$ be a uniform, non-principal ultrafilter on ω. We shall define sets $A_{\mathfrak{i},\rho} \in \mathfrak{U}$ for $\mathfrak{i} \in \mathcal{I}$ and $\rho < |\mathrm{lt}(\mathfrak{i})|$ by induction on $l(\mathfrak{i})$ as follows. Let $A_{\square,\rho}$ $(\rho < 2^\omega)$ be any enumeration of the members of $\mathfrak{U}$, where $\square$

denotes the empty sequence. Now suppose that $A_{\mathfrak{i},\rho}$ has been defined for $\mathfrak{i} \in \mathcal{J}$, $l(\mathfrak{i}) \leq l$ and $\rho < |\mathrm{lt}(\mathfrak{i})|$. For $\mathfrak{i} \in \mathcal{J}$, $l(\mathfrak{i}) = l+1$ and $\rho < |\mathrm{lt}(\mathfrak{i})|$, put

$$A_{\mathfrak{i},\rho} = A_{\mathfrak{i}^*,h(\theta,\rho)},$$

where $\theta = \mathrm{lt}(\mathfrak{i})$ and $h(\theta,\cdot)$ is any one–one map from $|\theta|$ onto θ.

Put $\kappa_n = (2^\omega)^{(+n)}$ and $\kappa = \sum\{\kappa_n : n < \omega\}$. We define the graph $G = (V, E)$ of size κ as follows. Put $V = X \cup Y \cup Z$, where

$$X = \{x_n : n < \omega\}, \qquad Y = \{y^\alpha_{\mathfrak{i},\mathfrak{j}} : \alpha < \kappa,\ \mathfrak{i} \in \mathcal{J},\ \mathfrak{j} \in \mathcal{J},\ l(\mathfrak{j}) = l(\mathfrak{i})+1\},$$

$$Z = \{z^\alpha_{\mathfrak{i},\mathfrak{j}} : \alpha < \kappa,\ \mathfrak{i} \in \mathcal{J},\ \mathfrak{j} \in \mathcal{J},\ l(\mathfrak{j}) = l(\mathfrak{i})\}.$$

The edge set of G is $E = E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{\{x_n, y^\alpha_{\mathfrak{i},\mathfrak{j}}\} : y^\alpha_{\mathfrak{i},\mathfrak{j}} \in Y,\ k = \mathrm{lt}(\mathfrak{i}) < \omega,\ n \in \bigcap\{A_{\mathfrak{i},\rho} : \rho < k\} \text{ and } \alpha \leq \kappa_n\},$$

$$E_2 = \{\{y^\alpha_{\mathfrak{i},\mathfrak{j}}, z^\alpha_{\mathfrak{i}_1,\mathfrak{j}_1}\} : y^\alpha_{\mathfrak{i},\mathfrak{j}} \in Y,\ z^\alpha_{\mathfrak{i}_1,\mathfrak{j}_1} \in Z,\ \alpha < \kappa \text{ and either } \mathfrak{i} = \mathfrak{i}_1^*,\ \mathfrak{j}_1 = \mathfrak{j} \text{ or } \mathfrak{i} = \mathfrak{i}_1,\ \mathfrak{j}_1 = \mathfrak{j}^*\},$$

$$E_3 = \{\{x_n, z^\alpha_{\square,\square}\} : n < \omega,\ \alpha < \kappa\}.$$

Note that every vertex has infinite degree.

We will assume that there is an unfriendly partition $c : V \to \{0,1\}$ of G and derive a contradiction.

Since $\mathfrak{U}$ is an utrafilter, there are $A \in \mathfrak{U}$ and $\epsilon \in \{0,1\}$ such that $c(x_n) = \epsilon$ if and only if $n \in A$. We will prove that, whenever

$$\alpha < \kappa, \qquad \mathfrak{i} \in \mathcal{J}, \qquad \mathfrak{j} \in \mathcal{J}, \qquad l(\mathfrak{j}) = l(\mathfrak{i})+1, \qquad \gamma = \mathrm{lt}(\mathfrak{i}), \tag{1}$$

and there is a $\rho < \gamma$ such that $A_{\mathfrak{i},\rho} = A$ holds, then

$$c(y^\alpha_{\mathfrak{i},\mathfrak{j}}) = 1-\epsilon \tag{2}$$

and

$$c(z^\alpha_{\mathfrak{i},\mathfrak{j}^*}) = \epsilon. \tag{3}$$

Note first that (3) follows from (2). For $E(z^\alpha_{\mathfrak{i},\mathfrak{j}^*}) = C_1 \cup C_2$, where

$$C_1 = \{y^\alpha_{\mathfrak{i},\mathfrak{j}^*\frown\zeta} : \zeta < \omega_1\} \qquad \text{and} \qquad C_2 = \begin{cases} \{y^\alpha_{\mathfrak{i}^*,\mathfrak{j}^*}\} & \text{if } \mathfrak{i} \neq \square, \\ X & \text{if } \mathfrak{i} = \mathfrak{j}^* = \square. \end{cases}$$

Since $|C_1| = \omega_1 > |C_2|$ and since, by (2), $c(y_{\mathfrak{i},\mathfrak{j}^*\frown\zeta}) = 1-\epsilon$, (3) follows.

We will prove (2) by induction on $\gamma = \mathrm{lt}(\mathfrak{i})$.

Consider first the case when $\gamma < \omega$. In this case there is no $\mathfrak{i}_1 \in \mathcal{J}$ such that $\mathfrak{i} = \mathfrak{i}_1^*$. Therefore,

$$E(y^{\alpha}_{i,j}) = \{x_n : \kappa_n > \alpha \text{ and } n \in \bigcap\{A_{i,\sigma} : \sigma < \gamma\} \cup \{z^{\alpha}_{i,j^*}\}.$$

But $\bigcap\{A_{i,\sigma} : \sigma < \gamma\}$ is an infinite subset of $A_{i,\rho} = A$ and only finitely many $n < \omega$ fail to satisfy the condition $\kappa_n > \alpha$. Therefore, since c is an unfriendly partition of G, it follows that $c(y^{\alpha}_{i,j}) = 1-\epsilon$.

Now suppose that $\gamma \geqslant \omega$. In this case,

$$E(y^{\alpha}_{i,j}) = \{z^{\alpha}_{i^\frown\tau,j} : \tau < |\gamma|\} \cup \{z^{\alpha}_{i,j^*}\}.$$

By the hypothesis (1), there is some $\rho < |\gamma|$ such that $A = A_{i,\rho}$. Also, for any τ such that $\rho < \tau < |\gamma|$, there is some $\sigma < |\tau|$ such that $h(\tau,\sigma) = \rho$, and so

$$A_{i^\frown\tau,\sigma} = A_{i,\rho} = A.$$

Thus, by the inductive hypothesis, $c(z^{\alpha}_{i^\frown\tau,j}) = \epsilon$. It follows that $c(y^{\alpha}_{i,j}) = 1-\epsilon$, and this completes the proof of (2) and (3) under the hypothesis (1).

In particular, by (3), $c(z^{\alpha}_{\square,\square}) = \epsilon$ for every $\alpha < \kappa$.

For $n \in A$, we have that

$$E(x_n) = D_1 \cup D_2,$$

where

$$D_1 = \{y^{\alpha}_{i,j} \in Y : \gamma = \text{lt}(i) < \omega,\ n \in \bigcap\{A_{i,\rho} : \rho < \gamma\} \text{ and } \alpha \leqslant \kappa_n\},$$

$$D_2 = \{z^{\alpha}_{\square,\square} : \alpha < \kappa\}.$$

Since $|D_1| \leqslant |\mathcal{J}|\kappa_n < \kappa = |D_2|$ and $c(x_n) = c(z)$ for all $z \in D_2$, this contradicts the assumption that c is an unfriendly partition. □

4 Sketch of the proof of Theorem 2

The proof is similar to the proof of Theorem 1. First we choose an ultrafilter $\mathfrak{U}$ on λ such that

$$B_n = \{\omega\alpha + n : \alpha < \lambda\} \notin \mathfrak{U} \qquad (n < \omega).$$

Now continue as in the proof of Theorem 1 using this ultrafilter and replacing 2^ω by 2^λ, ω_1 by λ^+, the cardinal successor of λ, X by $\{x_\xi : \xi < \lambda\}$ and replacing E_1 by

$$\{\{x_\xi, y^{\alpha}_{i,j}\} : \gamma = \text{lt}(i) < \omega,\ \xi \in \bigcap\{A_{i,\rho} : \rho < \gamma\}$$
$$\text{and } (\exists\, n)(\xi \in B_n \text{ and } \alpha \leqslant \kappa_n)\}. \quad \square$$

5 Unfriendly 3-partitions

The following Bernstein-type lemma is probably known.

Lemma 1 *Let* $\mathcal{A} = \langle A_i : i \in I \rangle$ *be a family of sets such that* $|A_i| \geqslant |I| \geqslant \omega$. *Then there are pairwise disjoint sets* $B_i \subseteq A_i$ $(i \in I)$ *such that* $|B_i| = |A_i|$.

Proof Let

$$D = \{|A_i| : i \in I\}, \qquad R = \{\kappa \in D : \kappa > \sum\{\mu : \mu < \kappa, \mu \in D\}\}$$

and, for $\kappa \in R$, let $I(\kappa) = \{i \in I : |A_i| \geqslant \kappa\}$. We can inductively choose subsets $A_i(\kappa) \subseteq A_i$ for $\kappa \in R$ and $i \in I(\kappa)$ so that $|A_i(\kappa)| = \kappa$ and so that $A_i(\kappa) \cap A_j(\mu) = \varnothing$ if $(\kappa, i) \neq (\mu, j)$. The sets

$$B_i = \bigcup \{A_i(\kappa) : i \in I(\kappa), \kappa \in R\} \qquad (i \in I)$$

satisfy the conditions of the lemma. □

Let $G = (V, E)$ be a graph. For a subset $A \subseteq V$, we define

$$\mathrm{nbly}(A) = \{x \in V : |E(x) \cap A| = |E(x)|\}.$$

The set A is *closed* if $\mathrm{nbly}(A) \subseteq A$, and the *closure* of A is $\bar{A}$, the smallest closed set containing A. Note that, if we write $A^* = A \cup \mathrm{nbly}(A)$, then $\bar{A} = A_\alpha$, where $\langle A_\xi : \xi \leqslant \alpha \rangle$ is a continuous increasing sequence of sets such that $A_0 = A$, $A_{\xi+1} = A_\xi^*$ and $A_\alpha^* = A_\alpha$. Thus we may write $\bar{A} \backslash A = \{a_i : i < \lambda\}$, where

$$|E(a_i)| = |E(a_i) \cap (A \cup \{a_j : j < i\}| \qquad (i < \lambda).$$

If h is a function defined on a subset $A \subseteq V$, then we say that h is *satisfactory* for the element $a \in A$ if

$$|\{y \in A \cap E(a) : h(y) = h(a)\}| \leqslant |\{y \in A \cap E(a) : h(y) \neq h(a)\}|,$$

and h is *completely satisfactory* for a if

$$|\{y \in E(a) : y \notin A \text{ or } h(y) = h(a)\}| \leqslant |\{y \in A \cap E(a) : h(y) \neq h(a)\}|.$$

Of course, if h is satisfactory on the set $B \subseteq A$, then it is completely satisfactory on $B \cap \mathrm{nbly}(A)$. It is also clear that, if h is completely satisfactory on $B \subseteq A$, then so also is any extension of h. In particular, if the domain of h is V, the terms satisfactory and completely satisfactory coincide. An *unfriendly 3-partition* of the graph G is a function $h: V \to \{0, 1, 2\}$ which is satisfactory for every vertex.

Lemma 2 *Let A, $B \subseteq V$, B infinite and $A \cap B = \emptyset$, and suppose that, for $z \in B$,*

$$|E(z)\backslash A| \leqslant |B| \Rightarrow E(z)\backslash A \subseteq B,$$

$$|E(z)\backslash A| > |B| \Rightarrow |[E(z)\backslash A] \cap B| = |B|.$$

If $h\colon A \cup B \to \{0,1,2\}$, then there is $g\colon \overline{A \cup B} \to \{0,1,2\}$ extending h which is satisfactory for every element of $[\overline{A \cup B}\backslash(A \cup B)] \cup B'$, where $B' = \{b \in B : |E(b) \cap [\overline{A \cup B}\backslash(A \cup B)]| > |E(b) \cap (A \cup B)|\}$.

Proof Let $b \in B'$. If $|E(b)\backslash A| \leqslant |B|$, then $E(b)\backslash A \subseteq B$ and so

$$E(b) \cap \overline{A \cup B} = E(b) \cap (A \cup B),$$

which is a contradiction. Therefore, $|E(b)\backslash A| > |B|$ and hence

$$|[E(b)\backslash A] \cap B| = |B|.$$

It follows that

$$|E(b) \cap [\overline{A \cup B}\backslash(A \cup B)]| > |B|$$

for $b \in B'$ and, hence, by Lemma 1, there are pairwise disjoint sets

$$F(b) \subseteq E(b) \cap [\overline{A \cup B}\backslash(A \cup B)] \qquad (b \in B')$$

such that $|F(b)| = |E(b) \cap \overline{A \cup B}|$.

Let $\{z_i : i < \lambda\}$ be an enumeration of the elements of $\overline{A \cup B}\backslash(A \cup B)$ such that

$$|E(z_i)| = |E(z_i) \cap (A \cup B \cup \{z_j : j < i\})| \qquad (i < \lambda).$$

We extend h to the function $g\colon \overline{A \cup B} \to \{0,1,2\}$ by choosing $g(z_i) \in \{0,1,2\}$ inductively for $i < \lambda$. At the i-th step there are two possible choices for $g(z_i)$ that will ensure that g is satisfactory for z_i; consequently, if $z_i \in F(b)$ for some $b \in B'$, then we may also choose $g(z_i)$ different from $g(b)$. The function g so constructed satisfies the requirements of the lemma. □

We now prove Theorem 4 that *every graph has an unfriendly 3-partition.*

Proof We will prove by induction on the infinite cardinal μ that the following assertion holds.

$\mathcal{P}_\mu$: *Let $G = (V, E)$ be a graph and let $A, B \subseteq V$ be subsets such that*

$$A = \bar{A}, \qquad A \cup B = V, \qquad A \cap B = \emptyset, \qquad |B| = \mu.$$

If $x \in B$, $c < 3$ *and* $h\colon A \to \{0,1,2\}$, *then there is* $g\colon V \to \{0,1,2\}$ *extending* h *such that* $g(x) \neq c$ *and* g *is satisfactory for every element of* B.

The theorem follows from this since every finite graph has an unfriendly 2-partition and $\mathcal{P}_\mu$ (with $A = \varnothing$ and $B = V$) implies that every graph of cardinality μ has an unfriendly 3-partition.

Case $\mu = \omega$.

Since A is closed and B is denumerable, it follows that $0 < |E(y)| \leqslant \omega$ for $y \in B$. We define an ordinal $\alpha < \omega_1$ and subsets B_β ($\beta \leqslant \alpha$) of B so that

$$B_0 = \{y \in B : |E(y)| < \omega\},$$

$$B_\beta = \left\{ y \in B \Big\backslash \bigcup_{\gamma<\beta} B_\gamma : \left| E(y) \cap \bigcup_{\gamma<\beta} B \right| = \omega \right\} \qquad (0 < \beta < \alpha)$$

and $E(y) \cap \bigcup_{\beta<\alpha} B_\beta$ is finite for all $y \in B_\alpha = B \Big\backslash \bigcup_{\beta<\alpha} B_\beta$. Let $\{\epsilon_1, \epsilon_2, \epsilon_3\} = \{0,1,2\}$ be such that

$$c \notin \begin{cases} \{\epsilon_0, \epsilon_2\} & \text{if } x \in B_0, \\ \{\epsilon_0, \epsilon_1\} & \text{if } x \notin B_0. \end{cases}$$

We will construct the extension g of h so that $\operatorname{range}(g \restriction B_0) \subseteq \{\epsilon_0, \epsilon_2\}$ and $\operatorname{range}(g \restriction B \backslash B_0) \subseteq \{\epsilon_0, \epsilon_1\}$. This will ensure that $g(x) \neq c$.

First define $g_1 = \{(y, \epsilon_1) : y \in B_1\}$. Now inductively define $g_\beta\colon B_\beta \to \{\epsilon_0, \epsilon_1\}$ for $1 < \beta < \alpha$ in such a way that, for each $y \in B_\beta$,

$$|\{z : z \in B_\gamma \ (1 \leqslant \gamma < \beta),\ g_\gamma(z) \neq g_\beta(y)\}| = \omega.$$

The set B_α is either empty or denumerable and every vertex of $G \restriction B_\alpha$ has infinite degree; so there is a map $g_\alpha\colon B_\alpha \to \{\epsilon_0, \epsilon_1\}$ that is satisfactory for every element of B_α. The function $g' = h \cup \bigcup_{1 \leqslant \beta \leqslant \alpha} g_\beta$, defined on $V \backslash B_0$, is completely satisfactory for the elements of $\bigcup_{1<\beta\leqslant\alpha} B_\beta$.

We now imitate the proof that any locally finite graph has an unfriendly 2-partition to define $g''\colon B_0 \to \{\epsilon_0, \epsilon_1\}$. For each finite set $K \subseteq B_0$, we can choose a map $g_K\colon K \to \{\epsilon_0, \epsilon_2\}$ so that $g_K \cup g'$ is satisfactory for elements of K. Since every vertex of B_0 has finite degree, it follows by compactness that there is $g\colon V \to \{0,1,2\}$ extending g' which is satisfactory for elements of B_0 and satisfies $\operatorname{range}(g \restriction B_0) \subseteq \{\epsilon_0, \epsilon_2\}$. Since g is constantly ϵ_1 on B_1, it follows that g is satisfactory for all elements of B.

Case $\mu > \omega$.

We may assume without loss of generality that

(a) $\overline{A \cup B'} \neq V$ for $B' \subseteq B$ with $|B'| < |B|$.

For, suppose that $\overline{A \cup B'} = V$, where $\omega \leqslant \kappa = |B'| < \mu$. We can assume that $x \in B'$ and also that, for all $y \in B'$,

$$|E(y) \backslash A| \leqslant \kappa \Rightarrow E(y) \backslash A \subseteq B',$$

$$|E(y) \backslash A| > \kappa \Rightarrow |[E(y) \backslash A] \cap B'| = \kappa.$$

By the inductive hypothesis, $\mathcal{P}_\kappa$ holds and so there is an extension h': $A \cup B' \to \{0,1,2\}$ of h which is satisfactory for elements of B'. Now it follows from Lemma 2 that there is g: $V = \overline{A \cup B'} \to \{0,1,2\}$ extending h' which is satisfactory for all elements of $[V \backslash (A \cup B')] \cup B''$, where

$$B'' = \{y \in B' : |E(y) \cap [V \backslash (A \cup B')]| > |E(y) \cap (A \cup B')|\}.$$

But if $y \in B' \backslash B''$, then $|E(y)| = |E(y) \cap (A \cup B')|$ and so h' is completely satisfactory for y and, hence, so also is g. Thus g is satisfactory for all the elements of $B = V \backslash A$.

By the assumption (a) it follows that there are subsets A_α $(\alpha \leqslant \mu)$ and B_α $(\alpha < \mu)$ of V such that

(b) $A_0 = A$, $A_{\alpha+1} = \overline{A_\alpha \cup B_\alpha}$, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ (α a limit) and $A_\mu = V$;

(c) $x \in B_0$ and $B_\alpha \subseteq B \backslash A_\alpha$ $(\alpha < \mu)$;

(d) $B_\alpha = \varnothing$ if α is a limit;

(e) if α is a non-limit then

$$|B_\alpha| = |\alpha| + \omega$$

and, for every $y \in \bigcup_{\beta \leqslant \alpha} B_\beta$,

$$|E(y) \backslash A_\alpha| \leqslant |\alpha| + \omega \Rightarrow E(y) \backslash A_\alpha \subseteq B_\alpha,$$

$$|E(y) \backslash A_\alpha| > |\alpha| + \omega \Rightarrow |[E(y) \backslash A_\alpha] \cap B_\alpha| = |B_\alpha|.$$

If $\alpha < \mu$ and B_β has been defined for $\beta < \alpha$, then A_α is defined by (b); and it follows by (a) and the fact that (e) holds for $\beta < \alpha$ that $|B \backslash A_\alpha| = |B|$, and so we can choose B_α satisfying (c), (d) and (e). At the same time, at non-limit stages, we can also choose the set B_α so that it contains the first element of $B \backslash A_\alpha$ in some well ordering of B (in type μ); this will ensure that the construction stops with $A_\mu = V$.

For an infinite cardinal $\kappa < \mu$, denote by Y_κ the set of all elements $y \in \bigcup_{\beta<\kappa^+} B_\beta$ such that $|E(y) \cap \bigcup_{\beta<\kappa^+} B_\beta| = \kappa^+$. Since $|Y_\kappa| \leqslant \kappa^+$, it follows that there are pairwise disjoint sets $I_\kappa(y) \subseteq \{\alpha : \kappa \leqslant \alpha < \kappa^+\}$

($y \in Y_\kappa$) each of cardinality κ^+ such that $E(y) \cap B_\alpha \neq \varnothing$ for $\alpha \in I(y)$. Now choose elements $x_\alpha \in B_\alpha$ for non-limit $\alpha < \mu$ so that $x_0 = x$ and $x_\alpha \in E_\kappa(y)$ if $\alpha \in I_\kappa(y)$ for some $\kappa < \mu$ and $y \in Y_\kappa \cap \bigcup_{\beta<\alpha} B_\beta$ (and x_α is chosen arbitrarily in B_α if there is no such y).

We shall define inductively a continuously increasing sequence of functions $g_\alpha: A_\alpha \to \{0,1,2\}$ for $\alpha < \mu$ so that, at non-limit stages, the following conditions hold:

(f) $g_{\alpha+1}(x_{\alpha+1}) \neq g(y)$ if there are $\kappa < \mu$ and $y \in Y_\kappa$ such that $\alpha+1 \in I_\kappa(y)$;

(g) $g_{\alpha+1}$ is satisfactory for every element of

$$[A_{\alpha+1} \backslash (A_\alpha \cup B_\alpha)] \cup B'_\alpha,$$

where

$$B'_\alpha = \left\{ y \in \bigcup_{\beta \leq \alpha} B_\beta : |E(y) \cap [A_{\alpha+1} \backslash (A_\alpha \cup B_\alpha)]| > |E(y) \cap (A_\alpha \cup B_\alpha)| \right\}.$$

Put $g_0 = h$. At limit stages we define $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$. Suppose that $\alpha < \mu$ and that $g_\alpha: A_\alpha \to \{0,1,2\}$ has already been defined. We want to define $g_{\alpha+1}$ so that (f) and (g) hold. If α is a non-limit, then A_α is closed and so, by the inductive hypothesis $\mathcal{P}_{|\alpha|+\omega}$ applied to the subgraph $G_\alpha = G \restriction A_\alpha \cup B_\alpha$, there is $g'_\alpha: A_\alpha \cup B_\alpha \to \{0,1,2\}$ which extends g and which is satisfactory for every element of B_α. Further, we may assume that $g'_\alpha(x_\alpha) \neq c_\alpha$, where $c_\alpha = g_\alpha(y)$ if $\alpha \in I_\kappa(y)$ for some $\kappa < \mu$ and $y \in Y_\kappa$, and $c_\alpha = c$ otherwise. If α is a limit ordinal, we simply put $g'_\alpha = g_\alpha$.

We want to apply Lemma 2, with

$$A = A_\alpha \Big\backslash \bigcup_{\beta \leq \alpha} B_\beta \quad \text{and} \quad B = \bigcup_{\beta \leq \alpha} B_\beta.$$

Let $z \in \bigcup_{\beta \leq \alpha} B_\beta$. If

$$\left| E(z) \Big\backslash \Big(A_\alpha \Big\backslash \bigcup_{\beta \leq \alpha} B_\beta \Big) \right| \leq \left| \bigcup_{\beta \leq \alpha} B_\beta \right| = |\alpha| + \omega,$$

then

$$E(z) \Big\backslash \Big(A_\alpha \Big\backslash \bigcup_{\beta \leq \alpha} B_\beta \Big) \subseteq \bigcup_{\beta \leq \alpha} B_\beta \qquad \text{by (e);}$$

if

$$\left|E(z)\Big\backslash\Big(A_\alpha\Big\backslash\bigcup_{\beta\leqslant\alpha}B_\beta\Big)\right| > \left|\bigcup_{\beta\leqslant\alpha}B_\beta\right|$$

then

$$\left|\left[E(z)\Big\backslash\Big(A_\alpha\Big\backslash\bigcup_{\beta\leqslant\alpha}B_\beta\Big)\right]\cap B_\alpha\right| = \left|\bigcup_{\beta\leqslant\alpha}B_\beta\right|.$$

Thus the conditions of the lemma are satisfied. Therefore, there is a function $g_{\alpha+1}\colon A_{\alpha+1}\to\{0,1,2\}$ extending g'_α which satisfies both (f) and (g).

This defines the g_α for $\alpha\leqslant\mu$. It remains to show that $g=g_\mu$ is satisfactory for every element of B. Let $z\in B$. If $z\notin\bigcup_{\alpha<\mu}B_\alpha$, then $z\in A_{\alpha+1}\backslash(A_\alpha\cup B_\alpha)$ for some $\alpha<\mu$. Since $A_{\alpha+1}$ is the closure of $A_\alpha\cup B_\alpha$, it follows that $|E(z)|=|E(z)\cap A_{\alpha+1}|$. Since $g_{\alpha+1}$ is satisfactory for z, it is completely satisfactory and, hence, g is also satisfactory for z. Suppose now that $z\in B_\alpha$ for some non-limit $\alpha<\mu$. Let $\beta\leqslant\mu$ be minimal such that $|E(z)|=|E(z)\cap A_\beta|$. Then $\beta>\alpha$ since A_α is closed and $z\notin A_\alpha$. In order to show that g is satisfactory for z we shall consider separately the following cases.

Case 1 $\beta=\gamma+1$ is a successor ordinal.

Case 1(i) $|E(z)|=|E(z)\cap B_\gamma|$.

For non-limit ξ ($\alpha\leqslant\xi<\gamma$), there holds $|E(z)\backslash A_\xi|>|\xi|+\omega$, otherwise $E(z)\subseteq A_\xi\cup B_\xi\subseteq A_\gamma$, which contradicts the choice of β. If $\gamma>\alpha$, then

$$|E(z)\cap A_\gamma|\geqslant\sum\{|\xi|+\omega:\alpha\leqslant\xi<\gamma\}=|\gamma|+\omega=|B_\gamma|\geqslant|E(z)|.$$

This again is a contradiction, and so $\gamma=\alpha$. Therefore,

$$|E(z)|=|E(z)\cap B_\alpha|.$$

Since g'_α is satisfactory for z, it is completely satisfactory and, hence, so is g.

Case 1(ii) $|E(z)|>|E(z)\cap B_\gamma|$.

Since $|E(z)|>|E(z)\cap A_\gamma|$, it follows that

$$\begin{aligned}|E(z)|=|E(z)\cap A_{\gamma+1}|&=|E(z)\cap[A_{\gamma+1}\backslash(A_\gamma\cup B_\gamma)]|\\&>|E(z)\cap(A_\gamma\cup B_\gamma)|.\end{aligned}$$

Therefore, $g_{\gamma+1}$ is completly satisfactory for z, and so is g.

Case 2 β is a limit ordinal.

Case 2(i) $|E(z)| \leqslant |\beta|$.

For non-limit ξ $(\alpha \leqslant \xi < \beta)$, we have $|E(z) \backslash A_\xi| > |\xi| + \omega$ (else $E(z) \subseteq A_{\xi+1}$) and, hence, $E(z) \cap B_\xi \neq \emptyset$. It follows that $|\beta| = \beta$ and

$$|\{\xi : \alpha \leqslant \xi < \beta, \xi \in I_\kappa(z) \text{ for some } \kappa < \beta\}| = |\beta|.$$

Since $g_\xi(z) \neq g_\xi(x_\xi)$ if $\xi \in I_\kappa(z)$, it follows that g_β is completely satisfactory for z, and therefore so is g.

Case 2(ii) $|E(z)| > |\beta|$.

In this case $|E(z)| = \lambda$ is singular and there are an increasing sequence of cardinals λ_ι $(\iota < \text{cf}(\lambda))$ and an increasing sequence of ordinals β_ι $(\iota < \text{cf}(\lambda))$ such that $\lambda = \sup \lambda_\iota$, $\beta = \sup \beta_\iota$ and

$$|E(z) \cap [A_{\beta_\iota+1} \backslash (A_{\beta_\iota} \cup B_{\beta_\iota})]| = \lambda_\iota > |E(z) \cap (A_{\beta_\iota} \cup B_{\beta_\iota})|.$$

But this implies that $g_{\beta_\iota+1}$ $(\iota < \text{cf}(\lambda))$ is satisfactory for z, i.e.

$$|\{y \in E(z) \cap A_{\beta_\iota+1} : g_{\beta_\iota}(y) \neq g_{\beta_\iota}(z)\}| = \lambda_\iota \qquad (\iota < \text{cf}(\lambda)).$$

From this it follows that g is satisfactory for z, and this completes the proof. □

References

[1] R. Aharoni, E. C. Milner & K. Prikry, Unfriendly partitions of a graph (to appear in *J. Combin. Theory*)

[2] R. Cowen & W. Emerson, Proportional colorings of graphs (unpublished)