[a]Topological Partition Relations of the Form $\omega^* \to (Y)^1_2$

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ABSTRACT: THEOREM. The topological partition relation $\omega^* \to (Y)^1_2$

(a) Fails for every space Y with $|Y| \geq 2^{\mathfrak{c}}$;
(b) Holds for Y discrete if and only if $|Y| \leq \mathfrak{c}$;
(c) Holds for certain nondiscrete P-spaces Y;
(d) Fails for $Y = \omega \cup \{p\}$ with $p \in \omega^*$;
(e) Fails for Y infinite and countably compact.

1. INTRODUCTION

For topological spaces X and Y we write $X \approx Y$ if X and Y are homeomorphic, and we write $f: X \approx Y$ if f is a homeomorphism of X onto Y. The "topological inclusion relation" is denoted by $\subseteq_h$; that is, we write $Y \subseteq_h X$ if there is $Y' \subseteq X$ such that $Y \approx Y'$.

The symbol ω denotes both the least infinite cardinal and the countably infinite discrete space; the Stone–Čech remainder $\beta(\omega)\backslash\omega$ is denoted ω^*.

For a space X we denote by wX and dX the weight and density character of X, respectively. Following [7], for $A \subseteq \omega$ we write $A^* = (\mathrm{cl}_{\beta(\omega)}A)\backslash\omega$.

For proofs of the following statements, and for other basic information on topological and combinatorial properties of the space ω^*, see [7], [3], [12].

THEOREM 1.1: (a) $\{\mathrm{cl}_{\beta(\omega)}A : A \subseteq \omega\}$ is a basis for the open sets of $\beta(\omega)$; thus $w(\beta(\omega)) = \mathfrak{c}$.

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(b) There is an (almost disjoint) family $\mathscr{A}$ of subsets of ω such that $|\mathscr{A}| = \mathbf{c}$ and $\{A^* : A \in \mathscr{A}\}$ is pairwise disjoint.

(c) ω^* contains a family of $2^{\mathbf{c}}$-many pairwise disjoint copies of $\beta(\omega)$.

(d) Every infinite, closed subspace Y of ω^* contains a copy of $\beta(\omega)$, so $|Y| = |\beta(\omega)| = 2^{\mathbf{c}}$.

For cardinals κ and λ and topological spaces X and Y, the symbol $X \rightarrow (Y)^{\kappa}_{\lambda}$ means that if the set $[X]^{\kappa}$ of all κ-membered subsets of X is written in the form $[X]^{\kappa} = \bigcup_{i<\lambda} P_i$, then there are $i < \lambda$ and $Y' \subseteq X$ such that $Y \approx Y'$ and $[Y']^{\kappa} \subseteq P_i$. Our present primary interest is in topological arrow relations of the form $X \rightarrow (Y)^1_2$ (with $X = \omega^*$). For spaces X and Y, the relation $X \rightarrow (Y)^1_2$ reduces to this: if $X = P_0 \cup P_1$, then either $Y \subseteq_h P_0$ or $Y \subseteq_h P_1$.

The relation $X \rightarrow (Y_0, Y_1)^1$ indicates that if $X = P_0 \cup P_1$, then either $Y_0 \subseteq_h P_0$ or $Y_1 \subseteq_h P_1$.

It is obvious that if X and Y are spaces such that $Y \subseteq_h X$ fails, then $X \rightarrow (Y)^1_2$ fails.

By way of introduction it is enough here to observe that the classical theorem of F. Bernstein, according to which there is a subset S of the real line $\mathbf{R}$ such that neither S nor its complement $\mathbf{R}\backslash S$ contains an uncountable closed set, is captured by the assertion that the relation $\mathbf{R} \rightarrow (\{0, 1\}^{\omega})^1_2$ fails; in the positive direction, it is easy to see that the relation $\mathbf{Q} \rightarrow (\mathbf{Q})^1_2$ holds for $\mathbf{Q}$ the space of rationals.

For a report on the present-day "state of the art" concerning topological partition relations, and for references to the literature and open questions, the reader may consult [14–16].

This paper is organized as follows. Section 2 shows that $\omega^* \rightarrow (Y)^1_2$ fails for every infinite compact space Y. Section 3 characterizes those discrete spaces Y for which $\omega^* \rightarrow (Y)^1_2$, and Section 4 shows that $\omega^* \rightarrow (Y)^1_2$ holds for certain nondiscrete spaces Y. Section 5 shows that $\omega^* \rightarrow (Y)^1_2$ fails for spaces of the form $Y = \omega \cup \{p\}$ with $p \in \omega^*$, hence fails for every infinite countably compact space Y. The results of Sections 2–5 prompt several questions, and these are given in Section 6.

We announced some of our results in the abstract [2]. See also [1] for related results.

2. $\omega^* \nrightarrow (Y)^1_2$ FOR $|Y| \geq 2^{\mathbf{c}}$

LEMMA 2.1: If $Y \subseteq_h \omega^*$, then $|\{A \subseteq \omega^* : A \approx Y\}| = 2^{\mathbf{c}}$.

Proof: The inequality $\geq$ is immediate from Theorem 1.1(c). For $\leq$, it is enough to fix (a copy of) $Y \subset \omega^*$ and to notice that since $dY \leq wY \leq w(\omega^*) = \mathbf{c}$ [by Theorem 1.1(a)], the number of continuous functions from Y into ω^* does not exceed $|(\omega^*)^{dY}| \leq (2^{\mathbf{c}})^{\mathbf{c}} = 2^{\mathbf{c}}$. □

THEOREM 2.2: If Y is a space such that $|Y| \geq 2^{\mathbf{c}}$, then $\omega^* \nrightarrow (Y)^1_2$.

Proof: We assume $Y \subseteq_h \omega^*$ (in particular we assume $|Y| = |\omega^*| = 2^{\mathbf{c}}$), since otherwise $\omega^* \nrightarrow (Y)^1_2$ is obvious. Following Lemma 2.1 let $\{A_\xi : \xi < 2^{\mathbf{c}}\}$ enumerate $\{A \subseteq \omega^* : A \approx Y\}$, choose distinct $p_0, q_0 \in A_0$, and recursively, if $\xi < 2^{\mathbf{c}}$ and p_η, q_η have been chosen for all $\eta < \xi$, choose distinct

$$p_\xi, q_\xi \in A_\xi \backslash (\{p_\eta : \eta < \xi\} \cup \{q_\eta : \eta < \xi\}).$$

It is then clear, writing

$$P_0 = \{p_\xi : \xi < 2^{\mathfrak{c}}\} \qquad \text{and} \qquad P_1 = \omega^* \backslash P_0,$$

that the relations $Y \subseteq_h P_0$ and $Y \subseteq_h P_1$ both fail. □

The following statement is an immediate consequence of Theorems 2.2 and 1.1(d).

COROLLARY 2.3: The relation $\omega^* \to (Y)^1_2$ fails for every infinite compact space Y.

By less elementary methods we strengthen Corollary 2.3 in Theorem 5.14 below.

3. CONCERNING THE RELATION $\omega^* \to (Y)^1_2$ FOR Y DISCRETE

The very simple result of this section, included in the interest of completeness, shows for discrete spaces Y that $\omega^* \to (Y)^1_2$ if and only if $Y \subseteq_h \omega^*$.

THEOREM 3.1: For a discrete space Y, the following conditions are equivalent.

(a) $|Y| \leq \mathfrak{c}$;
(b) $\omega^* \to (Y)^1_{\mathfrak{c}}$;
(c) $\omega^* \to (Y)^1_2$;
(d) $Y \subseteq_h \omega^*$.

Proof: (a) ⇒ (b). [Here we profit from a suggestion offered by the referee.] Given $\omega^* = \bigcup_{i<\mathfrak{c}} P_i$, recall from [10,(2.2)] or [12,(3.3.2)] this theorem of Kunen: there is a matrix $\{A^\xi_i : \xi < \mathfrak{c}, i < \mathfrak{c}\}$ of clopen subsets of ω^* such that

(i) For each $i < \mathfrak{c}$ the family $\{A^\xi_i : \xi < \mathfrak{c}\}$ is pairwise disjoint;
(ii) Each $f \in \mathfrak{c}^{\mathfrak{c}}$ satisfies $\bigcap_{i<\mathfrak{c}} A^{f(i)}_i \neq \varnothing$

Now if one of the sets P_i meets A^ξ_i for each $\xi < \mathfrak{c}$ (say $p_\xi \in A^\xi_i$), then the discrete set $D = \{p_\xi : \xi < \mathfrak{c}\}$ satisfies $Y \subseteq_h D \subseteq P_i$; otherwise, for each $i < \mathfrak{c}$ there is $f(i)$ such that $P_i \cap A^{f(i)}_i = \varnothing$, so $\varnothing \neq \bigcap_{i<\mathfrak{c}} A^{f(i)}_i \subseteq \omega^* \backslash \bigcup_{i<\mathfrak{c}} P_i$.

That (b) ⇒ (c) and (c) ⇒ (d) are clear.

(d) ⇒ (a). Theorem 1.1(a) gives $|Y| = wY \leq w(\beta(\omega)) = \mathfrak{c}$. □

4. $\omega^* \to (Y)^1_2$ FOR CERTAIN NONDISCRETE Y

For an infinite cardinal κ we denote by P_κ the ordinal space $\kappa + 1 = \kappa \cup \{\kappa\}$ topologized to be "discrete below κ" and with a neighborhood base at κ the same as in the usual interval topology. That is, a subset U of $\kappa + 1$ is open in P_κ if and only if either $U \subseteq \kappa$ or some $\xi < \kappa$ satisfies $(\xi, \kappa] \subseteq U$.

THEOREM 4.1: For cardinals $\kappa \geq \omega$ and $m_0, m_1 < \omega$, the space P_κ satisfies $P^{m_0+m_1}_\kappa \to (P^{m_0}_\kappa, P^{m_1}_\kappa)^1$

Proof: Let $P^I = X_0 \cup X_1$ and $|I| = m_0 + m_1$ and suppose without loss of generality that the point $c = \langle c_i \rangle_{i\in I}$ with $c_i = \kappa$ (all $i \in I$) satisfies $c \in X_0$. Let $I = I_0 \cup I_1$ with

$|I_0| = m_0$, $|I_1| = m_1$, and set $D = P_\kappa \backslash \{\kappa\}$, and for $x \in D^{I_0}$ define

$$S(x) = \{x\} \times \{y \in P_\kappa^{I_1} : \max\{x_i : i \in I_0\} < \min\{y_i : i \in I_1\}\}.$$

If some $x \in D^{I_0}$ satisfies $S(x) \subseteq X_1$, we have $P_\kappa^{m_1} \approx S(x) \subseteq X_1$ and the proof is complete. Otherwise for each $x \in D^{I_0}$ there is $p(x) \in S(x) \cap X_0$ and then

$$P_\kappa^{m_0} \approx \{p(x) : x \in D^{I_0}\} \cup \{c\} \subseteq X_0,$$

as required. □

COROLLARY 4.2: Every infinite cardinal κ satisifes $P_\kappa \times P_\kappa \to (P_\kappa)^1_2$.

We say as usual that a topological space $X = \langle X, \mathscr{T}\rangle$ is a P-space if each $\mathscr{U} \subseteq \mathscr{T}$ with $|\mathscr{U}| \leq \omega$ satisfies $\bigcap \mathscr{U} \in \mathscr{T}$. Since (clearly) P_κ is a nondiscrete P-space if and only if $\mathrm{cf}(\kappa) > \omega$, the following theorem shows the existence of a nondiscrete Y such that $X \to (Y)^1_2$.

THEOREM 4.3: Let $\omega_1 \leq \kappa \leq \mathfrak{c}$ satisfy $\mathrm{cf}(\kappa) > \omega$. Then $\omega^* \to (P_\kappa)^1_2$.

Proof: It is a theorem of E. K. van Douwen that every P-space X such that $wX \leq \mathfrak{c}$ satisfies $X \subseteq_h \omega^*$. (For a proof of this result, see [4] or [12]). Thus for κ as hypothesized we have $P_\kappa \times P_\kappa \subseteq_h \omega^*$, so the relation $\omega^* \to (P_\kappa)^1_2$ is immediate from Corollary 4.2. □

REMARKS 4.4: (a) The following simple result, suggested by the proof of Theorem 4.1, is peripheral to the principal thrust of our paper. Here as usual for a space $X = \langle X, \mathscr{T}\rangle$ we denote by $PX = \langle PX, P\mathscr{T}\rangle$ the set X with the smallest topology $P\mathscr{T}$ such that $P\mathscr{T} \supseteq \mathscr{T}$ and PX is a P-space; thus, $\{\bigcap \mathscr{U} : \mathscr{U} \subseteq \mathscr{T}, |\mathscr{U}| \leq \omega\}$ is a base for $P\mathscr{T}$.

THEOREM. For a P-space Y, the following conditions are equivalent.

(i) $\omega^* \to (Y)^1_2$;
(ii) $\{0, 1\}^\mathfrak{c} \to (Y)^1_2$;
(iii) $P(\omega^*) \to (Y)^1_2$;
(iv) $P(\{0, 1\}^\mathfrak{c}) \to (Y)^1_2$.

Proof: The implications (iii) ⇒ (iv) ⇒ (i) ⇒ (ii) follow, respectively, from the inclusions $P(\omega^*) \subseteq_h P(\{0, 1\}^\mathfrak{c}) \subseteq_h \omega^* \subseteq_h \{0, 1\}^\mathfrak{c}$. (Of these three inclusions the third follows from Theorem 1.1, the first from the third, and the second from van Douwen's theorem cited earlier.) That (ii) ⇒ (iii) follows from $P(\{0, 1\}^\mathfrak{c}) \subseteq_h \omega^*$ (whence $P(\{0, 1\}^\mathfrak{c}) \subseteq_h P(\omega^*)$) and the case $A = \{0, 1\}^\mathfrak{c}$, $B = Y = PY$ of this general observation: if $A \to (B)^1_2$, then $PA \to (PB)^1_2$. □

(b) We note in passing the following result, from which (with Theorem 4.1) it follows that for $\kappa \geq \omega$ the space P_κ satisfies $P_\kappa^{2^n} \to (P_\kappa)^1_{n+1}$.

THEOREM: Let S be a space such that $S^{m_0+m_1} \to (S^{m_0}, S^{m_1})^1$ for $m_0, m_1 < \omega$. Then

$$S^{2^n} \to (S)^1_{n+1} \quad \text{for} \quad n < \omega. \tag{*}$$

Proof: Statement (*) is trivial when $n = 0$, and is given by the case $m_0 = m_1 = 1$ of the hypothesis when $n = 1$.

Now suppose (*) holds for $n = k$, and let $S^{2^{k+1}} = \bigcup_{i=0}^{k+1} X_i$. With $Y_0 = X_0$ and $Y_1 = \bigcup_{i=1}^{k+1} X_i$, it follows from $S^{2^k+2^k} \to (S^{2^k}, S^{2^k})$ that there is $T \subseteq S^{2^{k+1}}$ such that $T \approx S^{2^k}$ and either $T \subseteq Y_0$ or $T \subseteq Y_1$. In the first case we have $S \subseteq_h T \subseteq X_0$, and in the second case from $T \subseteq \bigcup_{i=1}^{k+1} X_i$ and (*) at k there exists i such that $1 \leq i \leq k + 1$ and $S \subseteq_h X_i$, as required. □

(c) The method of proof of Theorem 4.1 and Corollary 4.2 applies to many spaces other than those of the form P_κ. The reader may easily verify, for example, denoting by C_κ the one-point compactification of the discrete space κ, that $C_\kappa \times C_\kappa \to (C_\kappa)^1_2$, and hence $\{0, 1\}^\kappa \to (C_\kappa)^1_2$, for all $\kappa \geq \omega$. For a proof due to S. Todorčević of a much stronger topological partition relation, namely $\{0, 1\}^\kappa \to (C_\kappa)^1_{\mathrm{cf}(\kappa)}$, see Weiss [15].

5. $\omega^* \nrightarrow (Y)^1_2$ FOR Y INFINITE AND COUNTABLY COMPACT

To prove this result, we show first that the relation $\omega^* \to (\omega \cup \{p\})^1_2$ fails for every $p \in \omega^*$. While this can be proved directly by combinatorial arguments, we find it convenient (given $p \in \omega^*$) to introduce and use as a tool a new topology $\mathscr{T}(p)$ on ω^*.

Given $f : \omega \to \omega^*$, we denote by $\bar{f} : \beta(\omega) \to \omega^*$ the Stone extension of f. For $X \subseteq \omega^*$ we set

$$X^p = X \cup \{\bar{f}(p) : f : \omega \approx f[\omega] \subseteq X\};$$

that is, X^p is X together with its "p-limits through discrete countable sets."

LEMMA 5.1: There is a topology $\mathscr{T}(p)$ for ω^* such that each $X \subseteq \omega^*$ satisfies: X is $\mathscr{T}(p)$-closed if and only if $X = X^p$.

Proof: It is enough to show

(a) $\emptyset = \emptyset^p$;
(b) $\omega^* = (\omega^*)^p$;
(c) $X_0 \cup X_1 = (X_0 \cup X_1)^p$ if $X_i = X_i^p$ $(i = 0, 1)$; and
(d) $\bigcap_{i \in I} X_i = (\bigcap_{i \in I} X_i)^p$ if each X_i satisfies $X_i = X_i^p$.

Now (a) and (b) are obvious, as are the inclusions $\subseteq$ of (c) and (d).

(c) ($\supseteq$) If $f : \omega \approx f[\omega] \subseteq X_0 \cup X_1$ satisfies $\bar{f}(p) = x \in (X_0 \cup X_1)^p$, then with $A_i = \{n < \omega : f(n) \in X_i\}$ we have $A_0 \cup A_1 \in p$, and hence $A_{\bar{i}} \in p$ for suitable $\bar{i} \in \{0, 1\}$; changing the values of f on $\omega \backslash A_{\bar{i}}$ if necessary (to ensure $f[\omega] \subseteq A_{\bar{i}}$), we conclude that $x = \bar{f}(p) \in X_{\bar{i}}^p = X_{\bar{i}} \subseteq X_0 \cup X_1$.

(d) ($\supseteq$). If $x = \bar{f}(p)$ with $f : \omega \approx f[\omega] \subseteq \bigcap_i X_i$, then $x \in \bigcap_i (X_i^p) = \bigcap_i X_i$. □

REMARKS 5.2: (a) In the terminology of Lemma 5.1, the topology $\mathscr{T}(p)$ is defined by the relation

$$\mathscr{T}(p) = \{\omega^* \backslash X : X \subseteq \omega^*, X \text{ is } \mathscr{T}(p)\text{-closed}\}.$$

(b) For notational convenience we denote by $I(p)$ the set of $\mathscr{T}(p)$-isolated points of ω^*, and we write $A(p) = \omega^* \backslash I(p)$. Clearly, $x \in I(p)$ if and only if x is not a "discrete limit" of points in $\omega^* \backslash \{x\}$, that is, if and only if every $f : \omega \approx f[\omega] \subseteq \omega^* \backslash \{x\}$ satisfies $\bar{f}(p) \neq x$. The fact that $I(p) \neq \emptyset$ has been known for many years. Indeed, Kunen [10] has shown that there exist $2^{\mathfrak{c}}$-many points $x \in \omega^*$ such that $x \notin \mathrm{cl}_{\beta(\omega)} A$ whenever $A \subseteq \omega^* \backslash \{x\}$ and $|A| \leq \omega$. (These are the so-called weak-P-points of ω^*.)

As a mnemonic device one may think of $A(p)$ and $I(p)$ as the sets of p-accessible and p-inaccessible points, respectively.

(c) For $X \subseteq \omega^*$ the set X^p may fail to be closed. Indeed, the $\mathcal{T}(p)$-closure of $X \subseteq \omega^*$ is determined by the following iterative procedure (cf. also [1]).

LEMMA 5.3: Let $X \subseteq \omega^*$. For $\xi \leq \omega^+$ define X_ξ by:

$$X_0 = X;$$

$$X_\xi = \cup_{\eta < \xi} X_\eta \text{ if } \xi \text{ is a limit ordinal;}$$

$$X_{\xi+1} = X_\xi^p.$$

Then $X_{\omega^+} = \mathcal{T}(p) - \mathrm{cl}\, X$.

The following fact, noted in [8], [5], [6], is crucial to many studies of ω^* (see also [3, (16.13)] for a proof). One may capture the thrust of this lemma by paraphrasing the picturesque terminology of Frolík [6]: "No type produces itself."

LEMMA 5.4: No homeomorphism form $\beta(\omega)$ into ω^* has a fixed point.

LEMMA 5.5: Let A and B be countable, discrete subsets of ω^*, with $A \subseteq B^*$. Then $A^p \cap B^p = \emptyset$.

Proof: If $x \in A^p \cap B^p$, we may suppose without loss of generality that there are $f : \omega \approx A$ and $g : \omega \approx B$ such that $x = \bar{f}(p) = \bar{g}(p)$. The function $h = f \circ g^{-1} : B \approx A \subseteq B^*$ satisifies

$$\bar{f} \circ \overline{g^{-1}} = \bar{h} : \beta(B) \approx \beta(A) \subseteq B^*$$

and $\bar{h}(x) = x \in B^*$, contrary to Lemma 5.4. □

COROLLARY 5.6: Let A and B be countably infinite, discrete subsets of ω^* such that $A \cap B = \emptyset$. Then $A^p \cap B^p = \emptyset$.

Proof: Let $x \in A^p \cap B^p$ and let $f : \omega \to f[\omega] \subseteq A$ and $g : \omega \to g[\omega] \subseteq B$ satisfy $x = \bar{f}(p) = \bar{g}(p)$. Leaving f and g unchanged on suitably chosen elements of p, but making modifications elsewhere if necessary, we assume without loss of generality that either $f[\omega] \subseteq (g[\omega])^*$ or $g[\omega] \subseteq (f[\omega])^*$ or $f[\omega] \cap (g[\omega])^* = (f[\omega])^* \cap g[\omega] = \emptyset$. By Lemma 5.5 the first of these possibilities, and by symmetry the second, cannot occur. We conclude that $f[\omega] \cup g[\omega]$ is a countable, discrete subset of ω^* such that $f[\omega] \cap g[\omega] = \emptyset$; it follows that $(f[\omega])^* \cap (g[\omega])^* = \emptyset$, since every countable (discrete) subset of ω^* is C^*-embedded (cf. [7, (14.27, 14N.5)], [3, (16.15)]). This contradicts the relation $x \in (f[\omega])^* \cap (g[\omega])^*$. □

COROLLARY 5.7: If $\omega^* \supseteq X \in \mathcal{T}(p)$, then $X^p \in \mathcal{T}(p)$.

Proof: If $\omega^* \backslash X^p$ is not $\mathcal{T}(p)$-closed, then there is $f : \omega \approx f[\omega] = A \subseteq \omega^* \backslash X^p$ such that $x = \bar{f}(p) \in X^p$. Since $X \in \mathcal{T}(p)$, we have $x \in X^p \backslash X$, so there is $g : \omega \approx g[\omega] = B \subseteq X$ such that $x = \bar{g}(p)$. From $A \cap B = \emptyset$ and Corollary 5.6 now follows $x \in A^p \cap B^p = \emptyset$, a contradiction. □

COROLLARY 5.8: If $\omega^* \supseteq X \in \mathscr{T}(p)$, then $\mathscr{T}(p) - \mathrm{cl}\, X \in \mathscr{T}(p)$.

Proof: This is immediate from Lemma 5.3 and Corollary 5.7. □

Our goal is to 2-color the points of ω^* in such a way that every copy of $\omega \cup \{p\}$ receives two colors. First we consider how to extend a given coloring function.

LEMMA 5.9: Let $\omega^* \supseteq X \in \mathscr{T}(p)$ and let $c: X \to 2 = \{0, 1\}$ be a function with no monochromatic copy of $\omega \cup \{p\}$ (that is, if $X \supseteq Y \approx \omega \cup \{p\}$, then $c^{-1}(\{i\}) \cap Y \neq \varnothing$ for $i \in \{0, 1\}$). Then c extends to $\bar{c} : X^p \to 2$ with no monochromatic copy of $\omega \cup \{p\}$.

Proof: Set $X_i = c^{-1}(\{i\})$ for $i \in 2 = \{0, 1\}$, so that $X^p = X_0^p \cup X_1^p$ by Lemma 5.1(c), and

$$(X_0^p \backslash X) \cap (X_1^p \backslash X) = \varnothing$$

by Corollary 5.6. Since $\{X, X_0^p \backslash X, X_1^p \backslash X\}$ is a partition of X^p, the function $\bar{c} : X^p \to 2$, given by the rule

$$\begin{aligned} \bar{c}(x) &= c(x) \text{ if } x \in X \\ &= 1 \text{ if } x \in X_0^p \backslash X \\ &= 0 \text{ if } x \in X_1^p \backslash X, \end{aligned}$$

in well-defined. To see that $\bar{c}$ is as required, let $h : \omega \cup \{p\} \approx A \cup \{x\} \subseteq X^p$ with $h : \omega \approx A$, $h(p) = x$. Modifying h (as before) if necessary, we assume without loss of generality that either (i) $A \subseteq X_0$ or (ii) $A \subseteq X_0^p \backslash X$ (the cases $A \subseteq X_1$, $A \subseteq X_1^p \backslash X$ are treated symmetrically). In case (i) we have $\bar{c} \equiv 0$ on A and $\bar{c}(x) = 1$ (since either $x \in X$ or $x \in X_0^p \backslash X$); case (ii) cannot arise, since $x \in X$ violates $X \in \mathscr{T}(p)$, while $x \in X^p \backslash X$ violates Corollary 5.6. □

Combining Lemmas 5.9 and 5.3 yields this.

LEMMA 5.10: Let $\omega^* \supseteq X \in \mathscr{T}(p)$, and let $c : X \to \{0, 1\}$ be a function with no monochromatic copy of $\omega \cup \{p\}$. Then c extends to $\bar{c} : \mathscr{T}(p) - \mathrm{cl}\, X \to \{0, 1\}$ with no monochromatic copy of $\omega \cup \{p\}$.

The preceding lemma indicates how to extend a coloring function from $X \in \mathscr{T}(p)$ over $\mathscr{T}(p) - \mathrm{cl}\, X$, but it remains to initiate the coloring procedure. For this purpose it is convenient to consider a particular base $\mathscr{S}(p)$ for the topology $\mathscr{T}(p)$. We call the elements of $\mathscr{S}(p)$ the *p-satellite sets.*

DEFINITION 5.11: Let $x \in \omega^*$. A set $S = S(x)$ is a *p-satellite set* based at x if there are a tree $T \subseteq \omega^{<\omega} = \bigcup_{n<\omega} \omega^n$ (ordered by containment), and for $s \in T$ a point $x_s \in S$ and $U_s \subseteq \omega^*$ such that

(i) U_s is open-and-closed in the usual topology of ω^*;
(ii) $x = x_{\langle\rangle}$, with $\langle\ \rangle$ the empty sequence;
(iii) $U_{\langle\rangle} = \omega^*$;
(iv) If $x_s \in S(x)$ and $x_s \in A(p)$, then $\{x_{s^\frown n} : n < \omega\}$ enumerates the range of a function f such that $f : \omega \approx f[\omega] \subseteq \omega^*$ with $\bar{f}(p) = x_s$, and $\{U_{s^\frown n} : n < \omega\}$ is a pairwise disjoint family such that $x_{s^\frown n} \in U_{s^\frown n} \subseteq U_s$;

(v) If $x_s \in S(x)$ and $x_s \in I(p)$, then s is a maximal node in T (and $x_{s^\frown n}$, $U_{s^\frown n}$ are defined for no $n < \omega$).

REMARK 5.12: It is not difficult to see that for every $x \in X \in \mathscr{T}(p)$ there is $S = S(x) \in \mathscr{S}(p)$ such that $x \in S \subseteq X$. (If $x \in I(p)$, one takes $S = \{x\}$; if $x_s \in S \cap X$ has been defined, one uses (iv) and $X \in \mathscr{T}(p)$ to choose $x_{s^\frown n} \in S \cap X$ if $x_s \in A(p)$.) That each of the sets $S(x)$ is $\mathscr{T}(p)$-open is immediate from Corollary 5.6. It follows that $\mathscr{S}(p)$ is indeed a base for $\mathscr{T}(p)$.

THEOREM 5.13: Every $p \in \omega^*$ satisfies $\omega^* \not\to (\omega \cup \{p\})^1_2$.

Proof: Let $\{S(x(i)) : i \in I\}$ be a maximal pairwise disjoint subfamily of $\mathscr{S}(p)$. For each $i \in I$ define $c_i : S(x(i)) \to 2$ by

$$\begin{aligned} c_i(x(i)_s) &= 0, \qquad \text{if length of } s \text{ is even} \\ &= 1, \qquad \text{if length of } s \text{ is odd.} \end{aligned}$$

It is clear from Corollary 5.6 that not only each function c_i on $S(x(i))$, but also the function

$$c = \cup_{i \in I} c_i : \cup_{i \in I} S(x(i)) \to 2$$

is monochromatic on no copy of $\omega \cup \{p\}$. Since $\cup_{i \in I} S(x(i))$ is $\mathscr{T}(p)$-open and $\mathscr{T}(p)$-dense in ω^*, the desired result follows from Lemma 5.10. □

THEOREM 5.14: The relation $\omega^* \to (Y)^1_2$ fails for every infinite, countably compact space Y.

Proof: Given infinite $Y \subseteq \omega^*$ there is $f : \omega \approx f[\omega] \subseteq Y$, and if Y is countably compact, there is $p \in \omega^*$ such that $\bar{f}(p) \in Y$. Since $f[\omega]$ is C^*-embedded in ω^* we have

$$\omega \cup \{p\} \approx f[\omega] \cup \{\bar{f}(p)\} \subseteq Y,$$

so $\omega^* \not\to (Y)^1_2$ follows from $\omega^* \not\to (\omega \cup \{p\})^1_2$. □

REMARKS 5.15: (a) We cite three facts that (taken together) show that the index set I used in the proof of Theorem 5.13 satisfies $|I| = 2^{\mathfrak{c}}$: (i) The set W of weak-P-points of ω^* introduced by Kunen [10] satisifes $|W| = 2^{\mathfrak{c}}$; (ii) each $S(x) \in \mathscr{S}(p)$ satisfies $|S(x)| \leq \omega$; (iii) $W \subseteq I(p)$, so $W \subseteq \cup_{i \in I} S(x(i))$.

(b) With no attempt at a complete topological classification, we note five elementary properties enjoyed by each of our topologies $\mathscr{T}(p)$ on ω^*.

(i) $\mathscr{T}(p)$ refines the usual topology of ω^*, so $\mathscr{T}(p)$ is a Hausdorff topology.
(ii) $\mathscr{T}(p)$ has $2^{\mathfrak{c}}$-many isolated points. (Indeed, we have noted already that the set W of weak-P-points satisfies $|W| = 2^{\mathfrak{c}}$ and $W \subseteq I(p)$.)
(iii) Since $\mathscr{S}(p)$ is a base for $\mathscr{T}(p)$ and each $S(x) \in \mathscr{S}(p)$ satisfies $|S(x)| \leq \omega$, the topology $\mathscr{T}(p)$ is locally countable.
(iv) From Theorem 1.1(b) it is easy to see that if $S(x) \in \mathscr{S}(p)$ and $|S(x)| = \omega$, then $|\mathscr{T}(p) - \text{cl}\, S(x)| = \mathfrak{c}$. Thus $\mathscr{T}(p)$ is not a regular topology for ω^*.

(v) According to Corollary 5.8, the $\mathscr{T}(p)$-closure of each $\mathscr{T}(p)$-open subset of ω^* is itself $\mathscr{T}(p)$-open. Such a topology is said to be extremally disconnected.

(c) In our development of $\mathscr{T}(p)$ and its properties we did not introduce explicitly the Rudin–Frolík preorder $\sqsubseteq$ on ω^* (see [5], [6], or [13], or [3] for an expository treatment), since doing so does not appear to simplify the arguments. We note, however (as in [1]), that the relation $\sqsubseteq$ lies close to our work. For $x, p \in \omega^*$ one has p x if and only if some $f : \omega \approx f[\omega] \subseteq \omega^*$ satisfies $\bar{f}(p) = x$.

6. QUESTIONS

Perhaps this paper is best viewed as establishing some boundary conditions that may help lead to a solution of the following ambitious general problem.

PROBLEM 6.1: Characterize those spaces Y such that $\omega^* \to (Y)^1_2$.

There are P-spaces Y such that $|Y| = 2^c$ and $Y \subseteq_h \omega^*$. (For example, according to van Douwen's theorem cited earlier, one may take $Y = P(\omega^*)$.) According to Theorem 2.2, the relation $\omega^* \to (Y)^1_2$ fails for each such Y. This situation suggests the following question.

QUESTION 6.2: Does $\omega^* \to (Y)^1_2$ for every P-space Y such that $Y \subseteq_h \omega^*$ and $|Y| < 2^c$? What if $|Y| = c$?

We have no example of a non-P-space Y such that $\omega^* \to (Y)^1_2$, so we are compelled to ask the following.

QUESTION 6.3: If Y is a space such that $\omega^* \to (Y)^1_2$, must Y be a P-space?

For $|Y| = \omega$, Question 6.3 takes the following simple form.

QUESTION 6.4: If Y is a countable space such that $\omega^* \to (Y)^1_2$, must Y be discrete?

REMARK 6.5: In connection with Question 6.4 it should be noted that there exists a countable, dense-in-itself subset C of ω^* such that every $x \in C$ satisfies

$$x \notin \mathrm{cl}_{\beta(\omega)} D \text{ whenever } D \text{ is discrete and } D \subseteq C \backslash \{x\} \qquad (*)$$

(equivalently, $\omega \cup \{p\} \subseteq_h C$ fails for every $p \in \omega^*$). To find such C we follow the construction of van Mill [11, (3.3), pp. 53–54]. Let E be the absolute (i.e., the Gleason cover) of the Cantor set $\{0, 1\}^\omega$, let $\pi : E \to \{0, 1\}^\omega$ be perfect and irreducible, and embed E into ω^* as a c-OK set; then every countable $F \subseteq \omega^* \backslash E$ satisfies $E \cap \mathrm{cl}_{\beta(\omega)} F = \emptyset$. Now by the method of [11, (3.3)] for $t \in \{0, 1\}^\omega$ choose $x_t \in \pi^{-1}(\{t\})$ such that every discrete $D \subseteq E \backslash \{x_t\}$ satisfies $x_t \notin \mathrm{cl}_{\beta(\omega)} D$, and take $C = \{x_t : t \in C_0\}$ with C_0 a countable, dense subset of $\{0, 1\}^\omega$. Since π is irreducible, the set C is dense in E and is dense-in-itself, and it is easy to see that condition (*) is satisfied.

Of course no element of C is a P-point of ω^*. The existence in ZFC of non-P-points $x \in \omega^*$ such that $x \notin \mathrm{cl}_{\beta(\omega)} D$ whenever D is a countable, discrete, subspace of $\omega^* \backslash \{x\}$ is given explicitly by van Mill [11]; see also Kunen [9] for a construction in ZFC + CH (or, in ZFC + MA) of a set C as above.

For the set C constructed earlier, the relation $\omega \cup \{p\} \subseteq_h C$ fails for every $p \in \omega^*$, so the following question, closely related to Question 6.4, is apparently not answered by the methods of this paper.

QUESTION 6.6: Let C be a countable, dense-in-itself subset of ω^* such that $\omega \cup \{p\} \subseteq_h C$ fails for every $p \in \omega^*$. Is the relation $\omega^* \rightarrow (C)^1_2$ valid?

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