An Extension of the Ehrenfeucht-Fraïssé Game for First Order Logics Augmented with Lindström Quantifiers

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Dedicated to Yuri Gurevich on the occasion of his 75th birthday.

Abstract. We propose an extension of the Ehrenfeucht-Fraïssé game able to deal with logics augmented with Lindström quantifiers. We describe three different games with varying balance between simplicity and ease of use.

1 Introduction

The Ehrenfeucht-Fraïssé game [7–10] is an important tool in contemporary model theory, allowing to determine whether two structures are elementarily equivalent up to some quantifier depth. It is one of the few model theoretic machineries that survive the transition from general model theory to the finite realm.

There are quite a few known extensions of the Ehrenfeucht-Fraïssé game and in the following we mention a few (this is far from bring a comprehensive list). In [12] Immerman describes how to adapt the Ehrenfeucht-Fraïssé game in order to deal with finite variable logic, first dealt with in Poizat's article [20]. Infinitary logic has a precise characterization by a similar game [2,11]. An extension for fixpoint logic and stratified fixpoint logic was provided by Bosse [3].

Lindström quantifiers were first introduced and studied by Lindström in the sixties [16–19] and may be seen as precursors to his theorem. There are several extensions and modifications of the Ehrenfeucht-Fraïssé game for logics augmented with Lindström quantifiers. We give a partial description of the history of the subject. Perhaps the first treatment of this subject was provided by Krawczyk and Krynicki, [15], who introduced a game capturing $\mathcal{L}_{\omega\omega}(Q)$ equivalence for monotone simple unary quantifier Q. A back-and-forth technique was given by Caicedo in [5], who considered also fragments of bounded quantifier degree. Weese, in [21], gave a sufficient condition for equivalence relative to first-order logic with Lindström quantifiers in the form of a game. This condition is also necessary in the case of monotone quantifiers.

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Games dealing with the special case of *counting quantifiers* were also investigated [4,13].

Probably the most relevant for our work is the work of Kolaitis and Väänänen, [14]. In their paper the authors describe four similar games, all for unary Lindström quantifiers. Of the four variants, one game is similar, but not identical, to the first game we propose here – the *definable* (k, Q)*-Pebble game*. This game captures $\mathcal{L}^k_{\infty\omega}(Q)$-equivalence, and also $\mathcal{L}^k_{\omega\omega}(Q)$-equivalence for finite models. However, as the authors mention, describing a winning strategy for this game may be difficult in practice since this requires an analysis of definability on the structures forming the game-board. Hence the authors go on to describe two more games: the *invariant* (k, Q)*-Pebble game* and the *monotone* (k, Q)*-Pebble game*. While all games are equivalent for monotone quantifiers, this is not the case in general. This leaves the task of finding a game avoiding definability requirements but capturing extensions by general Lindström quantifiers as an open problem.

The main aim of this paper is to present several related extensions of the Ehrenfeucht-Fraïssé game adapted to logics augmented with Lindström quantifiers. Our main contribution is a description of an Ehrenfeucht-Fraïssé like game capturing general Lindström quantifiers without forcing the players to chose definable structures by the game rules.

2 The Game

Notation 1. 1. Let τ denote a vocabulary. We assume τ has no function symbols, but that is purely for the sake of clearer presentation. τ may have constant symbols.

2. First order logic will be denoted by $\mathcal{L}_{\mathcal{FO}}$. In the course of this paper we will consider extensions of first order logic; therefore the logic under discussion will change according to our needs. Of course, we always assume closure under substitution. We shall denote the logic currently under discussion by $\mathcal{L}$, and we will explicitly redefine $\mathcal{L}$ whenever needed.
3. Given a vocabulary τ, we use $\mathcal{L}(\tau)$ to denote the *language* with logic $\mathcal{L}$ and vocabulary τ. We will use this notation only when clarity demands, so in fact we may abuse notation and use $\mathcal{L}$ also for the unspecified language under discussion.
4. For even further transparency, all the examples in this work (in particular, all cases of pairs of models to be proved equivalent) will be dealing with simple[1] graphs. Hence (only in examples) we further assume that τ is the vocabulary of graphs denoted henceforth by τ_{GRA}. Explicitly, $\tau_{\text{GRA}} = \{\sim\}$ where $\sim$ is a binary, anti-reflexive and symmetric relation. For the Lindström quantifiers given in examples, we may use vocabularies other than τ_{GRA}.

   A few basic graph theoretic notions will be used in the examples and are defined here (with the standard notation): In the context of graphs we will

[1] An undirected graph with no loops and no double edges is called *a simple graph*.

refer to the relation $\sim$ as *adjacency*. Given a graph $G = (V, E)$ we will denote the *neighborhood* of a vertex x in G by $N_G(x)$, defined as the set $N_G(x) = \{y \in V \mid x \sim y\}$. The *degree* of x will be denoted by $d_G(x) = |N_G(x)|$. We may omit the subscript G when it is clear which is the graph under discussion. We shall denote the *graph induced on* U by $G[U] = (U, \{e \in E \mid e \subset U\})$ where $U \subseteq V$ is a set of vertices. Finally, we will denote the *complement graph of* G by $\overline{G} = (V, \binom{V}{2} \setminus E)$, where $\binom{V}{2}$ is the set of all subsets of V of size two.

5. Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ be classes of models, each closed under isomorphism. The models in $\mathbf{A}_i$ are all τ_i-structures in some relational vocabulary $\tau_i = \{P_{i,1}^{a_{i,1}}, \ldots, P_{i,t_i}^{a_{i,t_i}}\}$, where $P_{i,j}$ is the jth relation in τ_i and $a_{i,j}$ is the arity of $P_{i,j}$. We assume $\mathbf{A}_0$ represents the existential quantifier and is always defined.
6. For simplicity, we will assume that each τ_i has an additional relation, $P^1_{i,0}$. This will serve for the formula defining the universe of the model. Formally, our models may have any set as their domain (perhaps some set used as the entire universe for the discussion), and the first relation will define a subset serving as the domain de facto. See also Remark 6 where we mention other flavors of Lindström quantifiers.
7. We set $a_{i,0} = 1$ for every i.
8. For each i, the class $\mathbf{A}_i$ corresponds to a Lindström quantifier Q_i, binding $a_i = \sum_{j=0}^{t_i} a_{i,j}$ variables.

Remark 2. The games and the claims associating them with logics remain valid even with the absence of the existential quantifier. Still, it seems that the main interesting case is when first order logic is properly *extended*, and so we focus on this case.

Example 3. 1. $\mathbf{A}_1$ may be the class of commutative groups, in which case τ_1 consists of a constant symbol encoding zero[2] and a ternary relation encoding the group operation.
2. Another example may be finite Hamiltonian graphs, in which case the vocabulary is the vocabulary of graphs and the class $\mathbf{A}_1$ will be the class of all finite Hamiltonian graphs.

Notation 4. Given a vector $\bar{x}$, we denote the number of elements in the vector by $\mathrm{len}(\bar{x})$.

Definition 5. *We define the quantifier Q_i corresponding to the class $\mathbf{A}_i$ as follows: Let G be a τ-structure with domain V. For any sequence of formulae $\varphi_0(x_0, \bar{y}), \varphi_1(\bar{x}_1, \bar{y}), \ldots, \varphi_{t_i}(\bar{x}_{t_i}, \bar{y})$ such that* $\mathrm{len}(\bar{x}_j) = a_{i,j}$ *and denoting the elements of $\bar{x}_j$ are by subscript l (i.e., $x_{j,l}$), we determine the satisfaction of the sentence $Q_i\, x_0, \bar{x}_1, \ldots, \bar{x}_{t_i}(\varphi_0(x_0, \bar{b}), \varphi_1(\bar{x}_1, \bar{b}), \ldots, \varphi_{t_i}(\bar{x}_{t_i}, \bar{b}))$ according to the satisfaction of*

[2] Of course, one may encode zero using the relation.

$$G \models Q_i \quad x_0, \bar{x}_1, \ldots, \bar{x}_{t_i}(\varphi_0(x_0, \bar{b}), \varphi_1(\bar{x}_1, \bar{b}), \ldots, \varphi_{t_i}(\bar{x}_{t_i}, \bar{b})) \iff$$
$$(\{x_0 \in V \mid G \models \varphi_0(x_0, \bar{b})\},$$
$$\{\bar{x}_1 \in V^{a_{i,1}} \mid \textstyle\bigwedge_{l=1}^{a_{i,1}} G \models \varphi_0(x_{1,l}, \bar{b}) \wedge G \models \varphi_1(\bar{x}_1, \bar{b})\}, \ldots,$$
$$\{\bar{x}_{t_i} \in V^{a_{i,t_i}} \mid \textstyle\bigwedge_{l=1}^{a_{i,t_i}} G \models \varphi_0(x_{t_i,l}, \bar{b}) \wedge G \models \varphi_{t_i}(\bar{x}_{t_i}, \bar{b})\}) \in \mathbf{A}_i,$$

where $\bar{b}$ are parameters.

Remark 6. Definition 5 requires φ_0 to have exactly one free variable, x_0 (excluding $\bar{y}$, saved for parameters). However there is no real reason to avoid sets of vectors of any fixed length from serving as the domain of the model defined in the quantifier. We will not discuss this here, but the generalization of the proposed games to this case are straightforward. Sometime Lindström quantifiers are defined over *equivalence classes* of such vectors. See, e.g., [6]. Again, we will not discuss this generalization here.

Definition 7. *Let φ be a formula in $\mathcal{L}$. The* quantifier depth *of φ, denoted* $\mathrm{QD}(\varphi)$*, is defined as follows:*

1. *If φ is an atomic formula, in our examples this means φ is of the form $x = y$ or $x \sim y$, then we define $\mathrm{QD}(\varphi) = 0$.*
2. *If $\varphi = \neg\psi$ then $\mathrm{QD}(\varphi) = \mathrm{QD}(\psi)$.*
3. *If $\varphi = \psi_1 \vee \psi_2$ or $\varphi = \psi_1 \wedge \psi_2$, then $\varphi = \max(\mathrm{QD}(\psi_1), \mathrm{QD}(\psi_2))$.*
4. *If*[3]

$$\varphi = Q_i\, x_0, \bar{x}_1, \ldots, \bar{x}_{t_i}(\psi_0(x_0, \bar{b}), \psi_1(\bar{x}_1, \bar{b}), \ldots, \psi_{t_i}(\bar{x}_{t_i}, \bar{b})),$$

then

$$\varphi = 1 + \max(\mathrm{QD}(\psi_1), \mathrm{QD}(\psi_2), \ldots, \mathrm{QD}(\psi_{t_i})).$$

Definition 8. *An important role in the following is played by the notion of* k-equivalency*:*

1. *Let τ be a vocabulary and $\mathcal{L} = \mathcal{L}(\tau)$ be a language. Given two τ-structures G_1, G_2 (not necessarily with distinct universe sets) and two sequences of elements $\bar{x}_1 \in G_1$, $\bar{x}_2 \in G_2$ of equal length a, we say that $(G_1, \bar{x}_1)$ and $(G_2, \bar{x}_2)$ are* k-equivalent with respect to $\mathcal{L}$ *if for any formula $\varphi(\bar{x}) \in \mathcal{L}$ of quantifier depth at most k one has*

$$G_1 \models \varphi(\bar{x}_1) \iff G_2 \models \varphi(\bar{x}_2).$$

2. *When considering only one model, that is, when we take $G = G_1 = G_2$, we refer to the equivalence classes of this relation in the domain of G simply by* the (a, k, G) -equivalence classes *where a is the length of tuples in the equivalence class. As usual, when the parameters are clear enough from the context we shall simple use "equivalence classes" for (a, k, G)-equivalence classes.*

[3] Notice that by our definition in 1. item (5) above, Q_0 is always the existential quantifier, and so our definition coincides with the standard definition when relevant.

Remark 9. Notice that unions of (a, k, G)-equivalence classes are exactly the definable sets of a-tuples of elements in $\mathrm{dm}(G)$ using $\mathcal{L}$-formulas of quantifier depth at most k.

Example 10. Let $\mathcal{L}$ be the first order language of graphs, $\mathcal{L} = \mathcal{L}_{\mathcal{FO}}(\tau_{\mathrm{GRA}})$, and let $G = (V, E)$ be a graph. If G is simple then the $(1, 0, G)$-equivalence classes are V and $\emptyset$. If $|V| > 1$ then the $(1, 1, G)$-equivalence classes are[4] the set of isolated vertices in G, the set of vertices adjacent to all other vertices and the set of vertices having at least one neighbor and one non-neighbor (some of which may be empty of course).

Notation 11. We denote the logic obtained by augmenting the first order logic with the quantifiers $Q_1, Q_2, \ldots, Q_n$ by $\mathcal{L} = \mathcal{L}[Q_1, Q_2, \ldots, Q_n]$.

Example 12. Consider the language $\mathcal{L} = \mathcal{L}[Q_{\mathrm{HAM}}](\tau_{\mathrm{GRA}})$, where Q_{HAM} stands for the "Hamiltonicity quantifier" (corresponding to the class of graphs containing a Hamiltonian cycle — a cycle visiting each vertex precisely once). Let G be a graph. Then the set of all vertices x for which all of the graphs $G[N_G(x)], \overline{G}[N_G(x)], G[N_{\overline{G}}(x)], \overline{G}[N_{\overline{G}}(x)]$ are Hamiltonian is an example of a $(1, 1, G)$-equivalence class with respect to $\mathcal{L}[Q_{\mathrm{HAM}}]$. The set of vertices with degree exactly two is a union of $(1, 1, G)$-equivalence classes, as can be seen by[5]

$$\varphi(x) = Q_{\mathrm{HAM}} x_0, x_1, x_2(x_0 \sim x, x_1 \neq x_2).$$

2.1 Description of the First Game

Before describing the game, we need the following definition:

Definition 13. *Let τ be a vocabulary, $\mathcal{L} = \mathcal{L}(\tau)$ a language over that vocabulary (not necessarily first order) and G a structure of vocabulary τ. Additionally, let $M = (S', R'_1, \ldots, R'_t)$ be a structure of another vocabulary τ'. A* copy of M in G *is a tuple $(S, R_1, \ldots, R_t)$ such that*

1. *S is a subset of $\mathrm{dm}(G)$ with the same cardinality as $\mathrm{dm}(M) = S'$ (where $\mathrm{dm}(G)$ is the universe or underlying set of G).*
2. *$R_1, \ldots, R_t$ are relations over S, such that each R_j has the same arity as R'_j.*
3. *$(S, R_1, \ldots, R_t)$ is isomorphic to $(S', R'_1, \ldots, R'_t)$.*

For our first (and simplest) game we will need a more restrictive notion, defined below:

Definition 14. *In the setting of Definition 13, if in addition to requirements* (1) – (3) *of Definition 13 the following holds*

4. *S is a union of $(1, k, G)$-equivalence classes, and each relation R_j of arity a_j is a union of (a_j, k, G)-equivalence classes,*

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[4] The atomic sentences appearing in $\varphi(x)$ are $x = y$ and $x \sim y$.

[5] φ expresses: "the complete graph $K_{d(x)}$ is Hamiltonian" which is true when $d(x) > 2$ and false when $d(x) = 2$ (we may treat K_0 and K_1 separately, if needed).

*we say that a copy of M in G is k-*induced by $\mathcal{L}$*. When k or $\mathcal{L}$ can be clearly determined by the context, we may omit mentioning it.*

We are now ready to define the first game.

Definition 15. *Consider Lindström quantifiers $Q_1, Q_2, \dots, Q_n$ and define $\mathcal{L} = \mathcal{L}[Q_1, Q_2, \dots, Q_n](\tau)$ to be a language over some relational vocabulary τ defined as in definition 5 above. Let G_1 and G_2 be two τ-structures with domains V_1 and V_2 respectively. Let $k \geq 0$ an integer and $\bar{c}_\ell = (c^1_\ell, \dots, c^r_\ell) \in V^r_\ell$ for $\ell \in \{1,2\}$ two finite sequences. We define the game*[6] $\mathrm{EFL}_1[G_1, G_2, \bar{c}_1, \bar{c}_2\,;\,k]$*. There are two players, named by the (by now) traditional names* Duplicator *and* Spoiler*, as suggested by Spencer. The game board is the models G_1 and G_2 plus the sequences $\bar{c}_\ell$ and there are k rounds. Each round is divided into two parts, and each part consists of two sub-rounds. The game is defined recursively. If $k = 0$, then if the mapping $c^i_1 \to c^i_2$ is a partial isomorphism, then* Duplicator *wins, otherwise* Spoiler *wins.*

When $k > 0$ then first Spoiler *plays. He picks one of the models G_1 or G_2 (denoted henceforth by G_ℓ) and a quantifier Q_i (or the existential quantifier*[7]*). Next* Spoiler *picks a model $M \in \mathbf{A}_i$, and embeds it into G_ℓ in a manner that preserves $(k-1, G_\ell)$-equivalence classes. That is,* Spoiler *picks a tuple $(S_\ell, R_{\ell,1}, \dots, R_{\ell,t_i})$ that is a copy of M in G which is $(k-1)$-induced by $\mathcal{L}$ enriched with r constants having values $\bar{c}_\ell$. If* Spoiler *can not find such an embedding, he loses*[8]*. Implicitly* Spoiler *claims that* Duplicator *can not find a matching induced copy of a model from $\mathbf{A}_i$.*

Second, Duplicator *responds by choosing a model M' from $\mathbf{A}_i$ (M' may not necessarily be the same as M), and then picking an induced copy of M' in $G_{3-\ell}$ which we naturally denote by $(S_{3-\ell}, R_{3-\ell,1}, \dots, R_{3-\ell,t_i})$. She is implicitly claiming that her choices match the picks of* Spoiler*, that is, each $R_{3-\ell,j}$ (or $S_{3-\ell,j}$) is a union of $(a_{i,j}, k-1, G_{3-\ell})$-equivalence classes defined by the same formulas as the formulas defining the $(a_{i,j}, k-1, G_\ell)$-equivalence classes of which $R_{\ell,j}$ is made. If* Duplicator *can not complete this part she loses. This ends the first part of the round.*

In the second part of the round Spoiler *chooses $m \in \{1,2\}$ and $0 \leq j \leq t_i$. He then picks $(c^{r+1}_m, \dots, c^{r+a_{i,j}}_m) \in R_{m,j}$ (implicitly challenging* Duplicator *to do the same). Finally* Duplicator *picks $(c^{r+1}_{3-m}, \dots, c^{r+a_{i,j}}_{3-m}) \in R_{3-m,j}$ and they move on to play*

$$\begin{aligned}\mathrm{EFL}_1[G_1, G_2, \quad &(c^1_1, \dots, c^r_1, c^{r+1}_1, \dots, c^{r+a_{i,j},}_1),\\ &(c^1_2, \dots, c^r_2, c^{r+1}_2, \dots, c^{r+a_{i,j},}_2); k-1].\end{aligned}$$

This ends the second part and the round. Since k goes down every round, the game ends when $k = 0$, as described above.

[6] We will describe a few variants, hence the subscript.

[7] In this case, $\mathbf{A}_\exists = P(V) \setminus \{\emptyset\}$, so Spoiler may choose any non-empty subset S_ℓ of V_ℓ.

[8] We will consider only logics stronger than first-order, hence the existential quantifier is always assumed to be at Spoiler' disposal and he will never lose in this manner.

Given the description above, the following should be self-evident:

Lemma 16. *Let $\mathcal{L} = \mathcal{L}[Q_1, Q_2, \ldots](\tau)$ be a language over some vocabulary τ where $Q_1, Q_2, \ldots$ are Lindström quantifiers, and let G_1, G_2 be two structures with vocabulary τ. Then,* Duplicator *has a winning strategy for* $\mathrm{EFL}_1[G_1, G_2, \emptyset, \emptyset\,;\, k]$ *if and only if for any sentence $\varphi \in \mathcal{L}$ of quantifier depth at most k*

$$G_1 \models \varphi \iff G_2 \models \varphi.$$

2.2 A Game Where Definability is Not Forced

While the claim of Lemma 16 may seem satisfying, in practice it may be hard to put this lemma into use since it takes finding unions of $(a, k-1, G)$-equivalence classes for granted, being a rule of the game. This might hinder strategy development and we would like to describe another game with looser rules, denoted EFL_2.

In this version the players are not bound to choosing unions of $(a, k-1, G)$-equivalence classes when picking a copy of the chosen model (hence we call their action "picking a copy of M in G_ℓ", omitting the "induced" part). That is, we omit requirement 4 in Definition 13. It falls to the other player to challenge the claim that indeed every relation is a union of the relevant equivalence classes.

Definition 17. *Our definition of the game* $\mathrm{EFL}_2[G_1, G_2, \bar{c}_1, \bar{c}_2\,;\, k]$ *is based on the definition of* $\mathrm{EFL}_1[G_1, G_2, \bar{c}_1, \bar{c}_2\,;\, k]$*. The setting is the same, but now a round goes as follows:*

Spoiler *picks a structure $G_\ell \in \{G_1, G_2\}$ and a quantifier Q_i (or, as before, the existential quantifier). Next* Spoiler *picks a model $M \in \mathbf{A}_i$ and picks a copy of M in G_ℓ. His implicit claim now includes the claim that each of the relations he chose is a union of $(a_{i,j}, k-1, G_\ell)$-equivalence classes with respect to $\mathcal{L}$ enriched with r constants having values $\bar{c}_\ell$.*

Duplicator *can respond in two different ways — she can "accept the challenge" (as she did in* EFL_1*), or attack the second part of the claim of* Spoiler*. That is, she can do one of the following:*

1. Duplicator *accepts the challenge. In this case she chooses $M' \in \mathbf{A}_i$ and picks a copy of M' in $G_{3-\ell}$. Implicitly she is claiming that her choices match the choices of* Spoiler*. That is, the set of vertices $S_{3-\ell}$ and each of the relations defined on it are a union of the $(a_{i,j}, k-1, G_{3-\ell})$-equivalence classes corresponding*[9] *to the ones that* Spoiler *picked. This ends the first part of the round.*

   Spoiler *may continue in a two different ways.*

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[9] We say that E_1, an (a, k, G_1)-equivalence class of a-tuples in G_1 corresponds to E_2 — a set of a-tuples in G_2 if for any $\bar{x}_1 \in E_1$ and $\bar{x}_2 \in E_2$ one has

$$G_1 \models \varphi(\bar{x}_1) \Leftrightarrow G_2 \models \varphi(\bar{x}_2)$$

for any $\varphi \in \mathcal{L}$ of quantifier depth at most k.

(a) Spoiler *rejects the fact that the set* $S_{3-\ell}$ *or one of the relations picked by* Duplicator *is a union of equivalence classes. In order to settle this, we recursively use* EFL_2*:*

This part of the round begins with Spoiler *picking* $j \in \{0, \ldots, t\}$*, presumably according to the relation that is not a union of equivalence classes. Again, we let* $a = a_{i,j}$ *be the arity of the allegedly invalid relation* $R_{3-\ell,j}$*. Next,* Spoiler *picks two a-tuples of elements* from the same structure,

$$(c^{r+1}, \ldots, c^{r+a}) \in R_{3-\ell,j} \quad \textit{and} \quad (c'^{r+1}, \ldots, c'^{r+a}) \in V^a_{3-\ell} \setminus R_{3-\ell,j},$$

and they move on to play

$$\mathrm{EFL}_2[G_{3-\ell}, G_{3-\ell}, \quad (c^1_{3-\ell}, \ldots, c^r_{3-\ell}, c^{r+1}, \ldots, c^{r+a}), (c^1_{3-\ell}, \ldots, c^r_{3-\ell}, c'^{r+1}, \ldots, c'^{r+a}); k-1].$$

with exchanged roles (since this time Spoiler *claims the two tuples are actually* $(a, k-1, G_{3-\ell})$*-equivalent).*

(b) Spoiler *rejects the fact that* Duplicator*'s choice matches his choice (as he did in* EFL_1*). In this case* Spoiler *picks a relation* $P_j \in \tau_i$ *and an* $a_{i,j}$*-tuple of elements from* S_ℓ *(or one element if he challenges* Duplicator*'s choice of* $S_{3-\ell}$*). Denote the choices of* Spoiler *by* $(c^{r+1}_\ell, \ldots, c^{r+a_{i,j}}_\ell) \in S_\ell$. Duplicator *responds by picking another a-tuple* $(c^{r+1}_{3-\ell}, \ldots, c^{r+a_{i,j},}_{3-\ell}) \in S_{3-\ell}$*, and they move on to play*

$$\mathrm{EFL}_2[G_1, G_2, \quad (c^1_1, \ldots, c^r_1, c^{r+1}_1, \ldots, c^{r+a_{i,j},}_1), (c^1_2, \ldots, c^r_2, c^{r+1}_2, \ldots, c^{r+a_{i,j},}_2); k-1].$$

2. Duplicator *rejects* Spoiler*'s claim. In this case* Duplicator *wants to prove that* S_ℓ *or one of the relations picked by* Spoiler *is not a union of equivalence classes. We continue similarly to case 1.(b):*

As before, we begin this move with Duplicator *picking* $j \in \{0, \ldots, t\}$*, presumably according to the relation that is not a union of equivalence classes. Again, we let* $a = a_{i,j}$ *be the arity of the allegedly invalid relation* $R_{\ell,j}$ *splitting an equivalence class. Next,* Duplicator *picks two a-tuples of elements* from the same structure,

$$(c^{r+1}, \ldots, c^{r+a}) \in R_{\ell,j} \quad \textit{and} \quad (c'^{r+1}, \ldots, c'^{r+a}) \in V^a_\ell \setminus R_{3-\ell,j},$$

and they move to play

$$\mathrm{EFL}_2[G_\ell, G_\ell, \quad (c^1_\ell, \ldots, c^r_\ell, c^{r+1}, \ldots, c^{r+a}), (c^1_\ell, \ldots, c^r_\ell, c'^{r+1}, \ldots, c'^{r+a}); k-1].$$

this time keeping their original roles.

For any two models G_1 and G_2, constants $\bar{c}_1, \bar{c}_2$ of elements from the domains of G_1 and G_2 respectively, and $k \in \mathbb{N}$, whoever has a winning strategy for

$\mathrm{EFL}_1[G_1, G_2, \bar{c}_1, \bar{c}_2; k]$ has a winning strategy for $\mathrm{EFL}_2[G_1, G_2, \bar{c}_1, \bar{c}_2; k]$. Hence the parallel of Lemma 16 is true for EFL_2 as well.

While we got the benefit of in-game validation of the equivalence classes integrity claims, EFL_2 is not easy to analyze in applications because the game-board and players' role change over time. We amend this in the last suggested version of the game.

2.3 A Game with Fixed Game-Board and Fixed Roles

The last version, denoted EFL_3, forks from EFL_2 in two places.

Definition 18. *We define* EFL_3 *like* EFL_2 *except that:*

1. *First, assume the game reaches step 2., where* Duplicator *wants to prove that* Spoiler *has chose a relation $R_{\ell,j}$ splitting an equivalence class. In this case the first part of the round ends immediately and the second part goes as follows:*
   Duplicator *picks $j \in \{0, \dots, t\}$, as before. Next,* Duplicator *chooses two $a_{i,j}$-tuples, $\bar{c}_{\ell,1}$ from $R_{\ell,j}$ and $\bar{c}_{\ell,2}$ from the complement of $R_{\ell,j}$. She then picks another $a_{i,j}$-tuple from the universe set of $G_{3-\ell}$, denoted $\bar{c}_{3-\ell}$.* Spoiler *than picks one of $\bar{c}_{\ell,1}$ or $\bar{c}_{\ell,2}$ and they move on to play* EFL_3 *with $\bar{c}_{3-\ell}$ concatenated to the constants of $G_{3-\ell}$ and* Spoiler*'s choice concatenated to the constants of G_ℓ, and $k-1$ moves. They keep their roles and the game-board remains G_1 and G_2.*
   If Duplicator *cannot find a matching tuple in $G_{3-\ell}$, she cannot disprove the integrity claim of* Spoiler*, but it does not matter as G_1 and G_2 are not k-equivalent and she is bound to lose anyway.*
   Notice that in this case the first part of the round had only Spoiler *playing, and in the second part* Duplicator *played first.*
2. *The second (and last) change from* EFL_2 *happens when the game is in step 1a. In this case* Spoiler *wants to prove that* Duplicator*'s choice of at least one relation $R_{3-\ell,j}$ is splitting an equivalence relation. As always, here also the move begins with* Spoiler *choosing j. Then* Spoiler *picks a tuple $\bar{c}_{3-\ell}$ (from the suspicious equivalence class) in $G_{3-\ell}$ that is not in $R_{3-\ell,j}$ and challenges* Duplicator *to find a matching tuple $\bar{c}_\ell$ in $G_{3-\ell}$ that is not in $R_{\ell,j}$. They move on to play* EFL_3 *with these choices and $k-1$ moves. Again both roles and game-board remain as they were. Notice that the game flow in this case is actually the same as the game flow in 1b.*

As before, it is easy to convince oneself that the claim of Lemma 16 is still valid. We repeat it here:

Lemma 19. Duplicator *has a winning strategy for* $\mathrm{EFL}_3[G_1, G_2, \emptyset, \emptyset\,; k]$ *if and only if for any sentence $\varphi \in \mathcal{L}$ of quantifier depth at most k*

$$G_1 \models \varphi \Longleftrightarrow G_2 \models \varphi.$$

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3 Summary

We have presented three equivalent variants of the celebrated Ehrenfeucht-Fraïssé game adapted to deal with logics extended by Lindström quantifiers. We believe EFL_3 may be easier to analyse than direct quantifier elimination and it is our hope that it will find applications.

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References

1. Arruda, A.I., Chuaqui, R., Da Costa, N.C.A. (eds.): Mathematical Logic in Latin America, Studies in Logic and the Foundations of Mathematics, vol. 99. North-Holland Publishing Company, Amsterdam (1980). Proceedings of the IV Latin American Symposium on Mathematical Logic held in Santiago, December 1978
2. Barwise, J.: On moschovakis closure ordinals. J. Symbolic Log. **42**(2), 292–296 (1977)
3. Bosse, U.: An "Ehrenfeucht-Fraïssé game"for fixpoint logic and stratified fixpoint logic. In: Martini, S., Börger, E., Kleine Büning, H., Jäger, G., Richter, M.M. (eds.) CSL 1992. LNCS, vol. 702, pp. 100–114. Springer, Heidelberg (1993)
4. Cai, J.Y., Fürer, M., Immerman, N.: An optimal lower bound on the number of variables for graph identification. Combinatorica **12**(4), 389–419 (1992)
5. Caicedo, X.: Back-and-forth systems for arbitrary quantifiers. In: Arruda et al. [1], Proceedings of the IV Latin American Symposium on Mathematical Logic held in Santiago, pp. 88–102, December 1978
6. Ebbinghaus, H.D., Flum, J.: Finite Model Theory. Springer Monographs in Mathematics, 2nd edn. Springer, Heidelberg (2006)
7. Ehrenfeucht, A.: An application of games to the completeness problem for formalized theories. Fundamenta Mathematicae **49**, 129–141 (1961)
8. Fraïssé, R.: Sur une nouvelle classification des systèmes de relations. Comptes rendus hebdomadaires des séances de l'Académie des sciences **230**, 1022–1024 (1950). French
9. Fraïssé, R.: Sur quelques classifications des systèmes de relations. Ph.D. thesis, University of Paris (1953), published in [10]. (French)
10. Fraïssé, R.: Sur quelques classifications des systèmes de relations. Publications Scientifiques de l'Université d'Alger, Série A, Alger-mathématiques, vol. 1, pp. 35–182 (1954). (French)
11. Immerman, N.: Upper and lower bounds for first order expressibility. J. Comput. Syst. Sci. **25**(1), 76–98 (1982)
12. Immerman, N.: Descriptive Complexity. Graduate Texts in Computer Science. Springer, New York (1998)
13. Immerman, N., Lander, E.: Describing graphs: a first-order approach to graph canonization. In: Selman, A.L. (ed.) Complexity Theory Retrospective, pp. 59–81. Springer, New York (1990). Chapter 4
14. Kolaitis, P.G., Väänänen, J.A.: Generalized quantifiers and pebble games on finite structures. Ann. Pure Appl. Log. **74**(1), 23–75 (1995)

15. Krawczyk, A., Krynicki, M.: Ehrenfeucht games for generalized quantifiers. In: Marek, W., Srebrny, M., Zarach, A. (eds.) Set Theory and Hierarchy Theory. Lecture Notes in Mathematics, vol. 537, pp. 145–152. Springer-Verlag, Berlin (Mar (1976)
16. Lindström, P.: First order predicate logic with generalized quantifiers. Theoria **32**(3), 186–195 (1966). http://onlinelibrary.wiley.com/doi/10.1111/j.1755-2567.1966.tb00600.x/abstract
17. Lindström, P.: On characterizability in $L_{\omega_1 \omega_0}$. Theoria **32**(3), 165–171 (1966). http://onlinelibrary.wiley.com/doi/10.1111/j.1755-2567.1966.tb00598.x/abstract
18. Lindström, P.: On relations between structures. Theoria **32**(3), 172–185 (1966)
19. Lindström, P.: On extensions of elementary logic. Theoria **35**(1), 1–11 (1969). http://onlinelibrary.wiley.com/doi/10.1111/j.1755-2567.1969.tb00356.x/pdf
20. Poizat, B.: Deux ou trois choses que je sais de l_n. J. Symbolic Log. **47**(3), 641–658 (1982). http://projecteuclid.org/euclid.jsl/1183741092
21. Weese, M.: Generalized ehrenfeucht games. Fundamenta Mathematicae **109**(2), 103–112 (1980). http://eudml.org/doc/211151