On Failure of 0-1 Laws

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Dedicated to Yuri Gurevich on the occasion of his 75th birthday.

Abstract. Let $\alpha \in (0,1)_{\mathbb{R}}$ be irrational and $G_n = G_{n,1/n^\alpha}$ be the random graph with edge probability $1/n^\alpha$; we know that it satisfies the 0-1 law for first order logic. We deal with the failure of the 0-1 law for stronger logics: $\mathbb{L}_{\infty,\mathbf{k}}, \mathbf{k}$ a large enough natural number and the inductive logic.

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Let $G_{n,p}$ be the random graph with set of nodes $[n] = \{1,\dots,n\}$, each edge of probability $p \in [0,1]_{\mathbb{R}}$, the edges being drawn independently, (see $\boxplus_1$ below). On 0-1 laws (and random graphs) see the book of Spencer [6] or Alon-Spencer [1], in particular on the behaviour of the random graph $G_{n,1/n^\alpha}$ for $\alpha \in (0,1)_{\mathbb{R}}$ irrational. On finite model theory see Flum-Ebbinghaus [2], e.g. on the logic $\mathbb{L}_{\infty,\mathbf{k}}$ and on inductive logic, also called LFP logic (i.e. least fix point logic). A characteristic example of what can be expressed in this logic is "in the graph G there is a path from the node x to the node y", this is closed to what we shall use. We know that $G_{n,p}$ (i.e. the case the probability p is constant), satisfies the 0-1 law for first order logic (proved independently by Fagin [3] and Glebskii-et-al [4]). This holds also for many stronger logics like $\mathbb{L}_{\infty,\mathbf{k}}$ and the inductive logic. If $\alpha \in (0,1)_{\mathbb{R}}$ is irrational, the 0-1 law holds for $G_{n,(1/n^\alpha)}$ and first order logic.

The question we address is whether this holds also for stronger logics as above. Though our real aim is to address the problem for the case of graphs, the proof seems more transparent when we have two random graph relations (with appropriate probabilities; we make them directed graphs just for simplicity). So

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we shall deal with two cases A and B. In Case A, the usual graph, we have to show that there are (just first order) formulas $\varphi_\ell(x,y)$ for $\ell = 1,2$ with some special properties, (actually we have also $\varphi_0(x,y)$). For Case B, those formulas are $R_\ell(x,y), \ell = 1,2$, the two directed graph relations. Note that (for Case B), the satisfaction of the cases of the R_ℓ are decided directly by the drawing and so are independent, whereas for Case A there are (small) dependencies for different pairs, so the probability estimates are more complicated.

Recall

$\boxplus_1$ a 0-1 context consists of:

- *(a)* a vocabulary τ, here just the one of graphs or double directed graphs,
- *(b)* for each n, K_n is a set of τ-models with set of elements = nods $[n]$, in our case graphs or double directed graphs,
- *(c)* a distribution μ_n on K_n, i.e. $\mu_n : K \to [0,1]_{\mathbb{R}}$ satisfying $\Sigma\{\mu_n(G) : G \in K_n\} = 1$
- *(d)* the random structure is called $G_n = G_{\mu_n}$ and we tend to speak on G_{μ_n} or G_n rather than on the context.

Note that in this work "for every random enough G_n ..." is a central notion, where:

$\boxplus_2$ for a given 0-1 context, let "for every random enough G_n we have $G_n \models \psi$, i.e. G satisfies ψ" <u>means</u> that the sequence $\langle \text{Prob}(G_n \models \psi) : n \in \mathbb{N}\rangle$ converge to 1; of course, $\text{Prob}(G_n \models \psi) = \Sigma\{\mu_n(G) : G \in K_n \text{ and } G \models \psi\}$.

But

$\boxplus_3$ $G_{n,p}$ is the case K_n = graph on $[n]$ and we draw the edges independently,

- (a) with probability p when p is constant, e.g. $\frac{1}{2}$, and
- (b) with probability $p(n)$ or probability p_n when p is a function from $\mathbb{N}$ to $[0,1]_{\mathbb{R}}$.

In the constant p case, the 0-1 law is strong: it is done by proving elimination of quantifiers and it works also for stronger logics: $\mathbb{L}_{\infty,\mathbf{k}}$ and so also for inductive logic $\mathbb{L}_{\text{ind}}$. Another worthwhile case is:

$\boxplus_4$ $G_{n,1/n^\alpha}$ where $\alpha \in (0,1)_{\mathbb{R}}$; so $p_n = 1/n^\alpha$.

Again the edges are drawn independently but the probability depends on n.

The 0-1 law holds if α is irrational, but we have elimination of quantifiers only up to (Boolean combination of) existential formulas. Do we have 0-1 law also for those stronger logics? We shall <u>show that not</u> by proving that for some so called scheme $\bar\varphi$ of interpretation, for any random enough G_n, $\bar\varphi$ interpret an initial segment of number theory, say up to $m(G_n)$ where $m(G_n)$ is not too small; e.g. at least $\log_2(\log_2(n))$.

For the probabilistic argument we use estimates; they are as in the first order case (see [1], so we do not repeat them).

For the full version see the author website or the mathematical arXive. The statements for which we need more estimates will probably be further delayed; those are the ones proving that:

$\boxplus_5$
- using n^ε instead of $\log_2(\log_2(n))$ in the proof for Case 1 so the value of "Prob$(G_{n,1/n^\alpha}) = \psi$" may change more quickly,
- we can define "n even" (i.e. Lim(Prob$(G_{n,1/n^\alpha} \models \psi$ iff n is even) exists and is one; this is done by defining a linear order on $G_{n,\bar\alpha}$.
- we may formalize the quantification on paths, so getting a weak logic failing the 0-1 law, but its naturality is not so clear.

A somewhat related problem asks whether for some logic the 0-1 law holds for $G_{n,p}$ (for constant $p \in (0,1)_\mathbb{R}$, e.g. $p = \frac{1}{2}$) but does not have the elimination of quantifier, see [5].

We now try to informally describe the proof, naturally concentrating on case B.

Fix reals $\alpha_1 < \alpha_2$ from $(0, \frac{1}{4})_\mathbb{R}$, so $\bar\alpha = (\alpha_1, \alpha_2)$ letting $\alpha(\ell) = \alpha_\ell$;

$\boxplus_6$ let the random digraph $G_{n,\bar\alpha} = ([n], R_1, R_2) = ([n], R_1^{G_{n,\bar\alpha}}, R_2^{G_{n,\bar\alpha}})$ with R_1, R_2 irreflexive relations drawn as follows:

(a) for each $a \neq b$, we draw a truth value for $R_2(a,b)$ with probability $\frac{1}{n^{1-\alpha_2}}$ for yes

(b) for each $a \neq b$, we draw a truth value for $R_1(a,b)$ with probability $\frac{1}{n^{1+\alpha_1}}$ for yes

(c) those drawings are independent.

Now for random enough digraph $G = G_n = G_{n,\bar\alpha} = ([n], R_1, R_2)$ and node $a \in G$ we try to define the set $S_k = S_{G,a,k}$ of nodes of G not from $\cup\{S_m : m < k\}$ by induction on k as follows:

For $k = 0$ let $S_k = \{a\}$. Assume $S_0, \ldots, S_k$ has been chosen, and we shall choose S_{k+1}.

$\boxplus_7$ For $\iota = 1, 2$ we ask: is there an R_ι-edge (a,b) with $a \in S_k$ and b not from $\cup\{S_m : m \leq k\}$?

If the answer is no for both $\iota = 1, 2$ we stop and let height$(a, G) = k$. If the answer is yes for $\iota = 1$, we let S_{k+1} be the set of b such that for some a the pair (a,b) is as above for $\iota = 1$., If the answer is no for $\iota = 1$ but yes for $\iota = 2$ we define S_{k+1} similarly using $\iota = 2$.

Let the height of G be max$\{$height$(a, G) : a \in G\}$. Now we can prove that for every random enough G_n, for $a \in G_n$ or easier- for most $a \in G_n$, for not too large k we have:

$\boxplus_8$ $S_{G_n,a,k}$ is on the one hand not empty and on the other hand with $\leq n^{2\alpha_2}$ members.

This is proved by drawing the edges not all at once but in k stages. In stage $m \leq k$ we already can compute $S_{G_n,a,0}, \dots S_{G_n,a,m}$ and we have already drawn all the R_1-edges and R_2-edges having at least one node in $S_{G_n,a,0} \cup \dots \cup S_{G_n,a,m-1}$; that is for every such pair (a, b) we draw the truth values of $R_1(a, b), R_2(a, b)$. So arriving to m we can draw the edges having a nod in S_m and not dealt with earlier, and hence can compute S_{m+1}.

The point is that in the question $\boxplus_7$ above, if the answer is yes for $\iota = 1$ then the number of nodes in S_{m+1} will be small, essentially smaller than in S_m. Further, if the answer for $\iota = 1$ the answer is no but for $\iota = 2$ the answer is yes then necessarily S_m is smaller than say $n^{(\alpha_1+\alpha_2)/2}$ but it is known that the R_2-valency of any nod of G_n is near n^{α_2}. So the desired inequality holds.

By a similar argument, if we stop at k then in $S_0 \cup \dots \cup S_k$ there are many nodes- e.g. at least near n^{α_2} by a crud argument. As each S_m is not too large necessarily the height of G_n is large.

The next step is to express in our logic the relation $\{(a_1, b_1, a_2, b_2)$: for some k_1, k_2 we have $b_1 \in S_{G_n,a_1,k_1}, b_2 \in S_{G_n,a_2,k_2}, k_1 \leq k_2\}$.

By this we can interpret a linear order with height(G_n) members. Again using the relevant logic this suffice to interpret number theory up to this height. Working more we can define a linear order with n elements, so can essentially find a formula "saying" n is even (or odd).

For random graphs we have to work harder: instead of having two relations we have two formulas; one of the complications is that their satisfaction for the relevant pairs are not fully independent.

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