Since all we need of the regularity of X is just the existence of such a $\varkappa$-complete subalgebra $\mathcal{B}$ of $RO(X)$, we assume in what follows that X has the fine topology w.r.t. $\{X_\alpha: \alpha\in\lambda\}$.

But then, in view of (i), Lemma 2′ applies and yields us $\chi(p,\mathcal{B})\leqslant\varkappa$ for all $p\in X$. On the other hand since (ii) is satisfied Lemma 3′ can also be applied and this gives us $\chi(p,\mathcal{B})>\varkappa$ for some $p\in X$. This contradiction then finishes the proof. ■

Corollary. *If X is T_4 and $X=\bigcup_\alpha^{\uparrow} X_\alpha$ with $wL(X_\alpha)\cdot\chi(X_\alpha)\leqslant\varkappa$ for all α then $|X|\leqslant 2^\varkappa$.*

Proof. Assume, indirectly, that $X=\bigcup^{\uparrow}\{X_\alpha: \alpha\in\lambda\}$ and $|X|=\lambda=(2^\varkappa)^+$. Similarly as in the above proof we can see that $|\overline{X}_\alpha|\leqslant 2^\varkappa$ for each α, consequently $wL(\overline{X}_\alpha)\cdot\chi(\overline{X}_\alpha)\leqslant\varkappa$ is also valid because $\overline{X}_\alpha\subset X_\beta$ holds for some $\beta\in\lambda$. But $\overline{X}_\alpha$ is also T_4 and thus by $X=\bigcup^{\uparrow}\{\overline{X}_\alpha: \alpha\in\lambda\}$ we get a contradiction with Theorem 2.

Note that this corollary does not follow immediately from Theorem 2 because a subspace of a T_4 space is not necessarily T_4.

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Modules over arbitrary domains II

by

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Abstract. Let R be a commutative ring and $S\subseteq R$ a multiplicatively closed subset of R. Defining torsion-free modules with respect to S, we derive new results of this category extending from $|S|=\aleph_0$. In §8 we realize any R-algebra A with torsion-free, reduced R-module structure on modules G as

$$\operatorname{End} G = A\oplus \operatorname{Ines} G$$

where $\operatorname{Ines} G$ are all endomorphisms on G with ω-complete image in G. In §9 we determine $\operatorname{Ines} G$ more explicitely and derive properties of G from the given algebra A.

§ 1. Introduction. We will discuss right R-modules $G=G_R$ over nonzero commutative rings R. The ring R will have a fixed multiplicatively closed subset S such that R as an R-module is S-reduced and S-torsion-free. These well-known conditions on a module G are $\bigcap_{s\in S} Gs=0$ respectively ($gs=0\Rightarrow g=0$) for all $g\in G$, $s\in S$.

Many questions on the existence of R-modules with prescribed properties can be reduced to representation theorems of R-algebras A as endomorphism algebras — in many cases modulo some "small" or "inessential" endomorphisms. Well-known examples for such problems are decomposition-properties related with the Krull–Remak–Schmidt Theorem — respectively related with Kaplansky's test problems, other derive from questions on prescribed automorphism groups or topologies. The investigation of classical problems in module theory in this sense goes back to a number of fundamental papers by A. L. S. Corner; see [CG] for further references.

In the recent years these investigations have been extended to R-modules of arbitrary large size, however under the restriction that S is essentially countable; see [DG 1,2], [GS 1], [S 2,3] and [CG] for a uniform treatment and further extensions, including torsion, mixed and torsion-free R-modules.

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In particular, we have a very satisfying picture for abelian groups. We also want to point out that this progress in algebra is due to recent results in model theory; see [S1], Chap. VIII. In the mentioned papers we are able to use topological methods and to construct the required modules as dense submodules of some appropriate "S-completion" of a free A-module. The great advantage of such constructions is due to the fact that an endomorphism by continuity extends *uniquely* to its completion. In fact, also Corner's classical papers of this subject depend heavily on this. More recently modules over arbitrary valuation domains came into the focus of investigations; see e.g. the new textbook by [FS]. In order to cover these and more general classes of modules, we are forced to omit the countability of S. One way to do this is to replace the underlying free A-module (see above) by a more general object and to preserve the topological methods. The result which is a representation theorem for a certain type of R-algebras including R itself can be found in [GS2]. If we want to derive a more general representation theorem, we have to give up the topological methods, which is carried out in this paper.

The rudiment of the topological arguments will be the requirement of a "countable support" of all elements of the constructed modules. The unique extension of homomorphisms will be replaced by the existence of solutions of certain infinite systems of linear equations. Again, the ideal $\operatorname{Ines} G$ of all inessential endomorphisms, the classes of cotorsion (= ω-cotorsion) and cotorsion-free modules will play a crucial role. These notions will need natural extension to this larger category of modules G. We say that G is ω-complete (= ω-cotorison) if any countable sequence of elements $a_n \in G$ and divisors $s_{n+1} \in Ss_n$ give rise to an element $x \in G$ such that $x \equiv \sum_{i<m} a_i s_i \mod Gs_m$ for all $m \in \omega$. If G is any module, then the set $\operatorname{Ines} G$ is the collection of all *inessential* endomorphisms of G which are those with ω-complete image. The set $\operatorname{Ines} G$ is a two-sided ideal of G. For $\varkappa$ a cardinal, $\operatorname{Ines}_\varkappa G$ denotes the ideal of all $\gamma \in \operatorname{Ines} G$ with $|G\gamma| < \varkappa$. The module G is (ω-)cotorsion-free if G contains no ω-complete submodule $\neq 0$.

We will say that a cardinal λ is admissible (for A and S) if $\lambda \geqslant |A|^{\aleph_0}$, and $\operatorname{cf}\lambda > |S|^{\aleph_0}$ in case S does not contain a divisor chain $(s_n)_{n\in\omega}$ with $\bigcap_{n\in\omega} Rs_n = 0$. Then our main result (8.1) is as follows:

Let A be an R-algebra with S-reduced and S-torsion-free R-module structure A_R and λ any admissible cardinal. Then there exists an S-reduced and S-torsion-free R-module of cardinal $\lambda^{\aleph_0}$ such that

$$(*)\qquad \operatorname{End}_R G = A \oplus \operatorname{Ines} G = A \oplus \operatorname{Ines}_{|A|^+} G .$$

The proof of this result is based on a substantial simplification of the algebraic argument due to a more suitable "stationary Black Box" (4.3) which takes better care of some combinatorial arguments. However, it should be said that also the "classical" Black Box (4.2) or those used in [S2], [CG] or [DG2] are successful in proving (*) for certain cardinals. We will show in § 9 that every complete R-submodule is contained in an A-submodule of finite rank; and every A-submodule of finite rank is contained in a pure and finitely generated free A-submodule of G. This has immediate consequences on the structure of G. The module G is $\aleph_1$-free if and only if A_R is $\aleph_1$-free, and G is cotorsion-free if and only if A_R is cotorsion-free. If G is cotorsion-free, obviously $\operatorname{Ines} G = 0$ and (*) reduces to an exact realization theorem:

If A is an R-algebra with A_R cotorsion-free and λ is admissible, then there exists a cotorsion-free R-module G with $\operatorname{End}_R G = A$ and $|G| = \lambda^{\aleph_0}$.

In the case of proper Dedekind domains R (= neither fields nor complete discrete valuation domains) this result was derived in [DG2] for certain admissible cardinals λ. In this earlier paper some local-global arguments appeared to be necessary in order to deal with Dedekind domains having uncountable many primes. Unfortunately this leads to a representation of a larger algebra A^* which, in the case of Dedekind domains turns out to be A using a result of E. Matlis. This kind of difficulty has been abolished by our *ad hoc* assumption that S does not need to contain a chain of cofinality ω (in the lattice of principle ideals).

§ 2. Algebraic preliminaries. Let R be a nonzero commutative ring with 1 and S a multiplicatively closed subset of nonzero divisors such that $\bigcap_S Rs = 0$. Modules will be right R-modules and a module M is S-*reduced*, if $\bigcap_S Ms = 0$. The module M is S-*torsion-free* if $ms = 0$ ($m \in M$, $s \in S$) implies $m = 0$, i.e. multiplication by s is injective. If multiplication by s is surjective, we say that M is S-divisible. A submodule U of M is S-*pure* if

$$Ms \cap U = Us \quad \text{for all } s \in S .$$

If M is S-torsion-free, this amounts to say that M/U is S-torsion-free as well. If $s, s' \in S$, $r \in S$ with $s' = rs$, we will also write $s|s'$ and (s_n) will denote a divisor-chain, i.e. $s_0 = 1$, $s_n | s_{n+1} \in S$ for all $n \in \omega$. We will reserve s_n^* for $s_n s_{n-1}^{-1}$.

The set S will be fixed throughout, and we will omit the prefixed S in the definitions. Let A be a fixed R-algebra which is reduced and torsion-free as an R-module. We also choose an admissible cardinal λ; i.e. a cardinal $\lambda \geqslant |A|^{\aleph_0}$ with $\operatorname{cf}\lambda > |S|^{\aleph_0}$ in case that the lattice defined by S is not cofinal with ω; cf. § 1. W.L.O.G. we will require that $\lambda^{\aleph_0} = \lambda$; otherwise apply one further combinatorial step similar to [S3]. If M is an A-module and X a subset of M, then $\langle X\rangle_R$ respectively $\langle X\rangle_A$ denote the R-submodule respectively the A-submodule generated by X. If it is clear from the context, we will omit R respectively A.

Recall that an R-module M is *algebraically compact* if any system of equations $\sum_{j\in J} x_j r_{ij} = m_i \in M$, $i \in I$, $r_{ij} \in R$ and $r_{ij} = 0$ for almost all j (if i is fixed) which has solutions for all finite subsystems also has a global solution $x_j \in M$. There are various ways to characterize algebraically compact modules; see R. Warfield [W] or M. Ziegler [Z]. In particular, any module is purely contained in a "unique" algebraically compact hull. We will need only a weak version of these results, which is outlined in (2.1). Our systems of equations $\sum_{j\in J} x_j r_{ij} = m_i$ $(i\in I)$ are very special and all index sets are countable.

In § 4 we will consider a support function [ ] on a module M; i.e. we assign to each $m\in M$ a countable set $[m]$ in a fixed set T (which is a tree). The support function [ ] will have the following (obvious) properties.

(a) $[0]=\emptyset$, $[ma]\subseteq[m]$ for all $a\in A$, $m\in M$,

$[m_1+m_2]\subseteq[m_1]\cup[m_2]$ for $m_i\in M$.

(b) For the sake of completeness we mention (i) and (ii)

(i) $\eta\in[m]$ implies $m\restriction\eta\in\hat{A}$ (= algebraically compact hull),

(ii) $\eta\in T\setminus[m]$ implies $m\restriction\eta=0$,

(iii) If $m+g\in Ms$ and $[m]\cap[g]=\emptyset$, then $m\in Ms$ and $g\in Ms$.

LEMMA 2.1. *Let M be an A-module and $S\subseteq R$ as above. Let [ ] be a support function on M satisfying* (a) *and* (b) (iii), *$a_n\in M$ $(n\in\omega)$ and (s_n) a divisor chain. Then there exists an extension M' of M such that*

(i) *$M' = \bigcup_{n\in\omega} M\oplus y^nA$ where $y^n s_n^* = a_{n-1}+y^{n-1}$. In particular, M is a pure submodule of M'.*

(ii) *$x = y^0 s_0 \in M'$ satisfies the equations*

$$x-\sum_{i<n} a_i s_i = -y^n s_n \quad \text{for all } n\in\omega .$$

We denote x by the suggestive symbol $x=\sum_{i\in\omega} a_i s_i$, moreover

$$y^n = \sum_{i\geq n} a_i \frac{s_i}{s_n}.$$

(iii) *$[m+y^n a] = [m]\cup\bigcup_{i\geq n}[a_i a]$ extends the support function from M to M'. In particular we have $[\sum_{i\in\omega} a_i s_i] = \bigcup_{i\in\omega}[a_i]$.*

(iv) *If $f,g\in M'$ and $[f]\cap[g]=\emptyset$, then*

(a) *$g\in M$ and $s|(f+g)$ imply $s|f$ and $s|g$,*

(b) *f and g are A-independent.*

Proof. (i) Let $E = M\oplus\bigoplus_{n\in\omega} y_n A$ and $D=\langle y^n s_n^* - y^{n-1} - a_{n-1}\colon n\in\omega\rangle_A$ an A-submodule of E. If $d\in D\cap M$, then $d=\sum_{n\in\omega}(y^n s_n^* - y^{n-1}-a_{n-1})c_n = m\in M$ and $c_n=0$ for almost all $n\in\omega$. Therefore

$$y^0 s_0^* c_1 + \sum_{n>1} y^n(s_n^* c_n - c_{n+1}) - \Big(\sum_{n\geq 1} a_{n-1}c_n + m\Big) = 0 ,$$

which implies $c_1=0$ and inductively $c_n=0$ for all $n\in\omega$. Since $m=-\sum_{n\in\omega} a_{n-1}c_n$, also $m=0$ and $D\cap M=0$. If $M'=E/D$, the map $M\to M'$ $(m\to m+D)$ is an injection and we identify M and its image. Moreover, $(y^nA+D)/D\cong y^nA$ is a complement of M in $(M\oplus y^nA)+D/D$ and (i) follows with the natural identifications.

(ii) The relations in (i) are equivalent with $a_{i-1}s_{i-1} = y^i s_i - y^{i-1}s_{i-1}$. Summation $\sum_{i<n}$ leads to $\sum_{i<n} a_i s_i = y^n s_n + y^0 s_0$, hence

$$y^0 s_0 - \sum_{i<n} a_i s_i = -y^n s_n \in M' \quad (n\in\omega)$$

as required.

(iii) If $f\in M'$ then $f=m+y_n a$ for some $m\in M$, $n\in\omega$, $a\in A$. Using (i) inductively, we have

(*) $$f=(m-a_n a - a_{n+1}s_{n+1}^* a - a_{n+2}s_{n+2}^* s_{n+1}^* a - \ldots) + (y^{n+k}s_{n+k}^*\ldots s_{n+1}^* a).$$

The first bracket belongs to M the second to $y_{n+k}A$. Using (*), it is easy to see that (iii) is a well-defined extension of the support function [ ].

(iv) (a) Let $f=m+y_n a$ be as in (iii) and $g\in M$. Then (iii) implies $[m]\cap[g] = [a_i a]\cap[g]=\emptyset$ for all $i\geq n$. Since

$$f+g = (m+g)+y^n a = (m^*+y^n a^*)s \quad \text{for some } a^*\in A,\ m^*\in M ,$$

we have

$$(m+g-m^*s)+y^n(a-a^*s)=0$$

and (*) implies $\prod_{1+n\leq i\leq k} y^{n+k}s_i^*(a-a^*s)=0$, hence $a=a^*s$. Therefore $m+g = m^*s$ in M with $[m]\cap[g]=\emptyset$, which leads to $m,g\in Ms$ by property (b) (iii) of support functions. We derive $f\in M's$ and $g\in Ms$.

(b) follows by a similar argument.

§ 3. Combinatorial preliminaries.

We will consider the tree $T={}^{\omega>}\lambda$ with a cardinal λ from § 2. Then T consists of all finite sequences $\tau\colon n\to\lambda$ $(n<\omega)$ which we also denote by $\tau=\langle\tau_0\rangle^\wedge\langle\tau_1\rangle^\wedge\ldots{}^\wedge\langle\tau_{n-1}\rangle$. We call $n=l(\tau)$ the *length* of τ.

If $\sigma,\tau\in T$ we have the natural order on T given by containment, i.e.

$$\sigma\leq\tau \Leftrightarrow \sigma\subseteq\tau\Leftrightarrow(\operatorname{dom}\sigma\subseteq\operatorname{dom}\tau \text{ and } \sigma=\tau\restriction\operatorname{dom}\sigma).$$

We also fix a strictly increasing continuous map $\varrho\colon \mathrm{cf}(\lambda)+1\to\lambda+1$ with $\varrho(0)=0$ and $\varrho(\mathrm{cf}(\lambda))=\lambda$. This is used to define the *norm* $\|\eta\|$ for $\eta\in T$ which is $\|\eta\| = \min\{\alpha<\mathrm{cf}(\lambda)\colon \eta\in{}^{\omega>}\varrho(\alpha)\}$ and can be extended to subsets X of T by $\|X\| = \sup\{\|\eta\|\colon\eta\in X\}$.

A maximal linearly ordered subset $v\subset T$ is called a *branch* and we will write $v=\{v_n = v\restriction n\colon n\in\omega\}$ for some $v\colon\omega\to T$ where $l(v_n)=n$. If $X\subseteq T$, then $\mathrm{Br}(X)$ denotes the set of all branches $v\subseteq X$. We call a subset w of T a branch $\mathrm{Br}(w)=v$ *with leaves* $\{\alpha_n<\lambda\colon n\in\omega\}$ if $w=\{w_n\colon n\in\omega\}$ and $w_n = v\restriction n\cup(v\restriction n-1)^\wedge\langle\alpha_{n-1}\rangle$ such that $\alpha_{n-1}<v(n-1)$; i.e. leaves grow only on one side of the branch and if we strip them off, the naked branch $\mathrm{Br}(w)$ remains. We call v a *constant branch* at the ordinal $\alpha<\lambda$ if $v(i)=\alpha$ for all $i\in\omega$.

Finally we call a sequence $\{x_n\in T\colon n\in\omega\}$ an *antibranch* if $\|x_n\|<\|x_{n+1}\|$ for all $n\in\omega$ and there is no pair $\{x_n,x_m\}$ which is comparable in T. If $(T,\leq,l)$ and $(T',\leq,l)$ are two trees, a map $f\colon T\to T'$, is a tree embedding if f is injective and if $\sigma,\tau\in T$ and $\sigma<\tau$ then $l(\sigma)=l(f(\sigma))$ and $f(\sigma)<f(\tau)$. We will reserve f^k for tree embeddings $f^k\colon {}^{k\geq}\omega\to T={}^{\omega>}\lambda$ $(k\in\omega)$ such that $f^{k-1}\subseteq f^k$.

If $X \subseteq T$ and $\nu < \mathrm{cf}(\lambda)$, then we write

$$_{\nu}X = \{\alpha \in X, \|\alpha\| > \nu\}.$$

§ 4. The black box. Let A be the reduced, torsion-free R-algebra from § 2 and $T = {}^{\omega>}\lambda$ be the tree from § 3. Then we consider the cartesian product $C = \prod_{\eta \in T} \eta A$ and let $[c] = \{\eta \in T, c \restriction \eta \neq 0\}$ be the support of $c \in C$. The norm $\|c\| = \|[c]\|$ is an ordinal $\leqslant \mathrm{cf}(\lambda)$. Let $B = \bigoplus_{\eta \in T} \eta A$ denote the free A-submodule of C. We now define certain elements, their support and norm, which are the candidates of generators of our final module G. They are defined inductively and we call such *elements potential* (over B):

(i) If $u \in B$, then u is potential and $[u]$, $\|u\|$ are defined above.

(ii) Let u_n $(n \in \omega)$ be potential, v a branch or a branch with leaves in T such that $\|\{u_n: n \in \omega\}\| < \|v\|$ and (s_n) a divisor-chain, then $y^k = \sum_{n \geqslant k} u_n s_n / s_k$ is defined in $\langle u_n: n \in \omega\rangle'_A$ according to (2.1) and $[y^k] = \bigcup_{n \geqslant k} [u_n]$. The elements $w^k = y^k + \sum_{n \geqslant k} (v \restriction n) s_n / s_k$ $(k \in \omega)$ are potential and $[w^k] = [y^k] \cup \{v_n: n \geqslant k\}$ and $\|w^k\| = \|v\|$.

(iii) Elements of an A-module generated by potential elements are potential. In this case we say that the *module is potential.*

A *homomorphism is potential* if its domain and image are potential. In our constructions we only deal with potential homomorphisms, and it is essential that their cardinality is bounded by some power of $|B|$. It will be convenient to say that U is a *canonical module* if U is a potential module with countable $[U]$ contained in U.

DEFINITION 4.1. A *trap* will be a sequence $(f, P, \Phi, a_n, s_n: n \in \omega)$ with $f: {}^{\omega>}\omega \to T$ a tree-embedding, P a canonical module, $\Phi \in \mathrm{End}\, P$, $a_n \in P$ $(n \in \omega)$ and (s_n) a divisor-chain such that

(a) $\mathrm{Im} f \subseteq [P]$;

(b) $[P]$ is a subtree of T;

(c) $\mathrm{cf}(\|P\|) = \omega$;

(d) $\|v\| = \|P\|$ for all $v \in \mathrm{Br}(\mathrm{Im} f)$.

Using modifications of [S2], [S3] or [CG] we obtain

THE BLACK BOX 4.2. There exists a sequence of traps

$$(f^\alpha, P^\alpha, \Phi^\alpha, a_n^\alpha, s_{\alpha n}: n \in \omega) \ (\alpha < \lambda^*)$$

for some ordinal λ^* such that the following holds for $\alpha, \beta < \lambda^*$:

(i) $\beta \leqslant \alpha \Rightarrow \|P^\beta\| \leqslant \|P^\alpha\|$,

(ii) $\beta \neq \alpha \Rightarrow \mathrm{Br}(\mathrm{Im} f^\alpha \cap \mathrm{Im} f^\beta) = \emptyset$,

(iii) $\beta + 2^{\aleph_0} \leqslant \alpha \Rightarrow \mathrm{Br}(\mathrm{Im} f^\alpha \cap [P^\beta]) = \emptyset$.

(iv) If P is a canonical module, $\Phi \in \mathrm{End}\, K$ potential, $B \subseteq K$ and $P \subseteq K$, $X \leqslant K$, $|X| \leqslant \aleph_0$, $a_n \in P$ $(n \in \omega)$, (s_n) a divisor-chain then

(a) there exists $\alpha < \lambda^*$ such that $(*)$ holds:

$(*)$
$$P \cup X \subseteq P^\alpha, \ \|P\| < \|P^\alpha\| \ \ \Phi \restriction P^\alpha = \Phi^\alpha,$$
$$a_n^\alpha = a_n, \ s_{\alpha n} = s_n \ (n \in \omega).$$

(b) if in addition $\|B\Phi\| < \lambda$, then there exists $\alpha < \lambda^*$ such that $(*)$ holds and

$(**)$ For all $v \in \mathrm{Br}(\mathrm{Im} f^\alpha)$ there are leaves $\alpha_i^v < \lambda$ $(i \in \omega)$ such that $(v \restriction i)^\wedge\langle \alpha_i^v\rangle \in P^\alpha$ and $(v \restriction i)\Phi^\alpha = ((v \restriction i-1)^\wedge\langle \alpha_{i-1}^v\rangle \Phi^\alpha$ for all $i \in \omega$.

Proof. Define *partial traps* of length $m \in \omega$ to be

$$\mathscr{P}^m = (f^m, P^m, \Phi^m, a_n^m, s_n^m: n \in \omega)$$

consisting of a tree-embedding $f^m: {}^{m>}\omega \to T$, a canonical module P^m, a potential homomorphism $\Phi^m: P^m \to \mathrm{Im}(\Phi^m)$, elements $a_n^m \in P^m$ $(n \in \omega)$ and a divisibility chain (s_n^m).

The number of partial traps is $\lambda^{\aleph_0}$, because each coordinate in $\mathscr{P}^m$ takes at most $\lambda^{\aleph_0}$ values. Hence we may code all partial traps by $\{\mathscr{P}_\xi: \xi < \lambda^{\aleph_0}\}$ where each partial trap $\mathscr{P}^m$ appears $\lambda^{\aleph_0}$ many times in the list.

For the sake of simplicity we now assume $\lambda^{\aleph_0} = \lambda$ to be regular; the case $\lambda^{\aleph_0} > \lambda$ can be treated similar using some ultra-filter argument; the changes are relevant only for $\mathrm{cf}\lambda = \omega$ and can be found in [S3]. Hence we conclude

$(+)$ $\{\mathscr{P}_\xi: \xi < \lambda\}$ and for each partial trap $\mathscr{P}^m$
$$|\{\xi: \xi < \lambda, \mathscr{P}_\xi = \mathscr{P}^m\}| = \lambda.$$

Next we choose an injection $h: \omega \times \lambda \times \mathrm{cf}(\lambda) \to \lambda$ which is strictly increasing in each argument. Given a partial trap $\mathscr{P}^m = \mathscr{P}_\xi$ then we define

$$H(\mathscr{P}_\xi) = f^{m+1}: {}^{m+1\geqslant}\omega \to T \quad \text{such that } f^{m+1} \restriction {}^{m\geqslant}\omega = f^m$$

and if $\sigma \in {}^{m+1\geqslant}\omega$, $l(\sigma) = m+1$, then

$$f^{m+1}(\sigma) = f^m(\sigma \restriction m)^\wedge h(\sigma(m), \xi, \|P^m\|).$$

Now we define an H-trap $\mathscr{P} = (f, P, \Phi, a_n, s_n: n \in \omega)$ consisting of a sequence of partial traps $\mathscr{P}_{\xi(m)} = (f^m, P^m, \Phi^m, a_n^m, s_n^m: n \in \omega)$ for all length $m \in \omega$ with $f^{m+1} = H(\mathscr{P}_{\xi(m)})$, $\mathrm{Im} f^{m+1} \cup P^m \cup [P^m]_\leqslant \subseteq P^{m+1}$ (where $[X]_\leqslant$ denotes the subtree generated by $[X] \subseteq T$), $P^m \Phi^m \subseteq P^{m+1}$ and $\Phi^m \subseteq \Phi^{m+1}$ such that

$$f = \bigcup_{m \in \omega} f^m, \quad P = \bigcup_{m \in \omega} P^m, \quad \Phi = \bigcup_{m \in \omega} \Phi^m, \quad a_n^m = a_n, \quad s_n^m = s_n \ (n \in \omega).$$

It is now easy to see that the conditions force $\mathscr{P}$ to be a genuine trap. The required sequence of traps $\mathscr{P}^\alpha = (f^\alpha, P^\alpha, \Phi^\alpha, a_n^\alpha, s_{\alpha n}: n \in \omega)$ is just the set of all H-traps with a suitable well-ordering. The conditions (i) to (iv) (a) can be derived similar to [CG], Theorem A.7, or [S2]. Hence we will concentrate on (iv)(b), and suppose $\Phi \in \mathrm{End}\, K$, $\|B\Phi\| < \mathrm{cf}\lambda$. We must find an H-trap $\mathscr{P}$ with the required property $(**)$, i.e. we must find the associated sequence of partial traps. This will be carried out by induction on $m \in \omega$.

Suppose the partial trap $\mathscr{P}^m$ has been constructed. In particular we know $f^m: {}^{m\geqslant}\omega \to T$ and let $\{v_i: i \in \omega\}$ be the set of all elements $v \in \mathrm{Im} f^m$ of length $l(v)$

$= m$. Then we define $J = \bigcup_{i\in\omega} \{\alpha<\lambda:\ \exists\gamma<\alpha;$ such that $(v_i\hat{}\langle\alpha\rangle)\Phi = (v_i\hat{}\langle\gamma\rangle)\Phi\}$. Since $\{v_i\hat{}\langle\alpha\rangle:\ \alpha<\lambda\} \subseteq T \subseteq B$ and $\|B\Phi\| < \mathrm{cf}\lambda$, also $|J|<\lambda$.

Let $Y_\xi = \{h(\sigma(m), \xi, \|P_\xi\|):\ \sigma \in {}^{m+1\geq}\omega\setminus{}^{m\geq}\omega\} = \{h(n,\xi,\|P_\xi\|):\ n\in\omega\}$ for $\xi<\lambda$ such that $\mathscr{P}_\xi = \mathscr{P}^m$. By (+) there are λ many such ξ. If always $Y_\xi \cap J \neq \emptyset$, there is one $n\in\omega$ with $h(n,\xi,\|P_\xi\|)\in J$ for λ many $\xi<\lambda$. Since h is injective in the second co-ordinate, this implies $|J|=\lambda$, which was excluded. So we find $\xi<\lambda$ with $\mathscr{P}_\xi = \mathscr{P}^m$ and $Y_\xi\cap J=\emptyset$ takes care of the requirement (**) at stage $m+1$; to be definite we may choose a leaf α_m^v of the initial part $v\restriction m+1$ of a later branch in $\mathrm{Im}f$ to be $\alpha_m^v = \min\{\gamma\in\lambda:\ \gamma\neq v_{m+1},\ (v\restriction m+1)\Phi = ((v\restriction m)\hat{}\langle\gamma\rangle)\Phi\}$. Finally we extend P^m to P^{m+1} to satisfy the remaining condition of an H-trap at stage $m+1$, including $(v\restriction m)\hat{}\langle\alpha_m^v\rangle \in P^{m+1}$ for all initial parts of branches v. Observe that these are countably many "requirements". From this induction we derive the required trap $\mathscr{P}$ satisfying (**).

The Stationary Black Box 4.3. If V, V' and V'' are any disjoint stationary subsets of $\{\alpha\in\mathrm{cf}\lambda:\ \mathrm{cf}\alpha=\omega\}$, the traps in (4.2) $(f^\alpha, P^\alpha, \Phi^\alpha, a_n^\alpha, s_{\alpha n}:\ n\in\omega)$ $(\alpha<\lambda^*)$ satisfy the additional properties

(siv) (a) Condition (iv) (a) holds with $\|P^\alpha\|\in V$

(b) Condition (iv) (b) holds with $\|P^\alpha\|\in V'$.

Remark. This follows from the Black Box (4.2) using an additional argument coming from model theory. A model theoretic proof is given in [S3] and a direct proof is contained in [FG]. Since $\{\alpha\in\mathrm{cf}\gamma:\ \mathrm{cf}\alpha=\omega\}$ is a stationary set, the existence of V, V' and V'' follows from Solovay's Theorem; see [J], p. 58, p. 433.

§ 5. The construction of the modules. The Black Box (4.3) will now be used to construct the required modules by a transfinite induction on $\alpha<\lambda^*$. For transparency we will construct only one candidate G for each λ. However, with this method we can easily build a "rigid system" of 2^λ modules. This natural extension can be copied from [CG] using [G2].

With the Black Box (4.3) we have disjoint stationary sets $V\cup V'\cup V''\subseteq \mathrm{cf}(\lambda)$ and we let $\lambda_c^* = \{\alpha<\lambda^*:\ \|P^\alpha\|\in V'\}$ and $\lambda^0 = \{\alpha\in\lambda^*:\ \|P^\alpha\|\in V\cup V'\}$. We want to define an increasing and continuous chain $G_\mu(\mu<\lambda^*)$ of A-modules such that $G_0 = B$ (see § 4) and $G = \bigcup_{\mu<\lambda^*} G_\mu$. If $\mu=\alpha+1$ is not a limit, we also define potential elements a_α^k $(k\in\omega)$ as follows. Suppose G_α has been constructed and $\mathscr{P}^\alpha = (f^\alpha, P^\alpha, \Phi^\alpha, a_n^\alpha, s_{\alpha n}:\ n\in\omega)$ is taken from the Black Box (4.3). We want to find a suitable branch $v_\alpha\in\mathrm{Br}(\mathrm{Im}f^\alpha)$ and if $\alpha\in\lambda_c^*$ also leaves α_i^v $(i\in\omega)$ such that $a_{\alpha n}^1 = v_\alpha\restriction n$ respectively $(\alpha\in\lambda_c^*)$ $a_{\alpha n}^1 = v_\alpha\restriction n-(v_\alpha\restriction n-1)\hat{}\langle\alpha_{n-1}^v\rangle$. Then we choose a suitable $\delta_\alpha\in\{0,1\}$ and let

$$(*)\qquad a_{\alpha n} = a_n^\alpha\delta_\alpha + a_{\alpha n}^1 \quad\text{and}\quad a_\alpha^k = \sum_{i\geq k} a_{\alpha i}\frac{s_{\alpha i}}{s_{\alpha k}},$$
$$G_{\alpha+1} = \langle G_\alpha, a_\alpha^k:\ k\in\omega\rangle_A.$$

The letters a_α^k, $a_{\alpha n}$ are reserved for these generators, and we will use $a_\alpha = a_\alpha^0$.

The choice of v_α, δ_α depends on the following circumstances. We say that *an ordinal $\beta\leq\alpha$ is strong at stage $\alpha+1$* if there does not exist $x\in G_{\alpha+1}$ with

$$\sum_{i<m} a_{\beta i}s_{\beta i}\Phi^\beta \equiv x \bmod G_{\alpha+1}s_{\beta m}\quad \text{for all } m\in\omega.$$

If $\alpha\in\lambda^*\setminus\lambda_c^*$, we say that α *is strong*, if we find a branch $v_\alpha\in\mathrm{Br}(\mathrm{Im}f^\alpha)$ and $\delta_\alpha\in\{0,1\}$ such that $a_{\alpha n}^1 = v_\alpha\restriction n$ and (*) imply that α is strong at stage $\alpha+1$ *and* if $\beta<\alpha$ was strong at earlier stages (it is enough to say "at stage α"), then β remains strong at stage $\alpha+1$.

If $\alpha\in\lambda_c^*$, we say that α is *complete* if there is a branch $v_\alpha\in\mathrm{Br}(\mathrm{Im}f^\alpha)$ with leaves α_i^v $(i\in\omega)$ and $\delta_\alpha = 1$ such that

$$a_{\alpha n}^1 = v_\alpha\restriction n-(v_\alpha\restriction n-1)\hat{}\langle\alpha_{n-1}^v\rangle \text{ and } a_{\alpha n}^1\Phi^\alpha = 0 \quad\text{for all } n\in\omega.$$

Furthermore we require that $\beta<\alpha$ remains strong at stage $\alpha+1$ in (*) if β was strong at earlier stages, e.g. at α.

If $\alpha\in\lambda^*$ is neither complete nor strong, we say that α is *weak* if we find at least a branch $v_\alpha\in\mathrm{Br}(\mathrm{Im}f^\alpha)$ and $\delta_\alpha\in\{0,1\}$ with $a_{\alpha n}^1 = v_\alpha\restriction n$ and using (*) any $\beta<\alpha$ remains strong at $\alpha+1$ if it was strong. Otherwise we call $\alpha<\lambda^*$ to be *useless*.

We make our choice v_α, δ_α always best possible for α. Naturally, "strong or complete" is better than "weak" and "weak" is still better than being "useless". To complete the induction step (and hence the construction), we may use $a_\alpha^k = 0$ $(k\in\omega)$ if α is useless. However, we will show in (5.3) that this case does not occur because *all* ordinals are quite useful.

Remark. Complete ordinals will serve for completions (in the topological/algebraical sense) of certain submodules of G. If $\alpha\in\lambda^0$, we will also use $\alpha+1$ to denote the successor of α in λ^0.

Recognition-Lemma 5.1. *Let $g\in G\setminus B$. Then $\|g\|\in V\cup V'$ and*

(a) *there is a unique $\alpha\in\lambda^0$ such that $g\in G_{\alpha+1}\setminus G_\alpha$ and there are ordinals $\alpha=\alpha_1>\ldots>\alpha_r$ in λ^* and branches v_{α_j} with leaves α_i^j $(i\in\omega)$ if $\alpha_j\in\lambda_c^*$ or without leaves if $\alpha_j\in\lambda^*\setminus\lambda_c^*$ such that $\|v_\alpha\| = \|v_{\alpha_j}\|$. If $w_{\alpha_j} = v_{\alpha_j}$ respectively $w_{\alpha_j} = v_{\alpha_j}\cup \bigcup\{(v_{\alpha_j}\restriction i)\hat{}\langle\alpha_i^j\rangle:\ i\in\omega\}$, there exists $\nu<\|v_\alpha\|$ such that*

$$_\nu[g] = F\cup\bigcup_{j\leq r} {}_\nu[w_{\alpha_j}]$$

where $F\subseteq T$ is finite and $\|v_\alpha\|<\|\eta\|$ for all $\eta\in F$.

(b) *If $\beta<\lambda^0$ and $\|v_\beta\| = \|v_\alpha\|$ there are $a\in A$ and $k\in\omega$ such that $g\restriction\sigma = \sigma\dfrac{s_{\beta l(\sigma)}}{s_{\beta k}}\cdot a$ for almost all $\sigma\in v_\beta$.*

Proof. It is clear from the construction that

$$g = b+x+\sum_{j\leq r} a_{\alpha_j}^k a_j \quad\text{where } b\in B,\ a_j\in A,$$
$$\|x\|<\|v_{\alpha_j}\| = \|v_\alpha\| \text{ and } \alpha_j>\alpha_{j+1}\quad \text{for all } j\leq r.$$

Observe that we use k simultaneously for all summands which is no loss of generality. We will always do this, mainly to simplify notations. Since $\|v_{\alpha_j}\|$ is a limit ordinal and $\|b\|$ a successor, condition (a) is immediate by (4.3).

(b) If $\beta = \alpha_j$ we claim $g \restriction \sigma = \sigma \frac{s_{\beta l(\sigma)}}{s_{\beta k}} a_j$ for almost all $\sigma \in v_\beta$. Choose $j = 1$ and suppose $\sigma \in [a^k_{\alpha_2} a_2]$ for infinitely many $\sigma \in v_\beta$. From the Black Box (4.2) (ii) we derive $\sigma_n = (v_{\alpha_2} \restriction n-1)^\wedge \alpha^2_{n-1} \in \operatorname{Im} f^\beta$ for infinitely many $n \in \omega$. Since f^β is a tree-embedding also $(v_{\alpha_2} \restriction n-1) \in \operatorname{Im} f^\beta$ for infinitely many $n \in \omega$. Therefore $v_{\alpha_2} \in \operatorname{Br}(\operatorname{Im} f^\beta)$ and $\beta = \alpha_2$ by (4.2)(ii). This contradicts $\beta = \alpha_1 > \alpha_2$. Hence $g \restriction \sigma$ has no contribution from $a^k_{\alpha_2} a_2$ for almost all $\sigma \in v_{\alpha_1}$, and the claim follows by induction. Now (b) follows easily; observe that $a = 0$ if $\beta \neq a_j$ for all $j \leqslant r$.

No-Useless-Ordinals-Lemma 5.2. *Let $\alpha < \lambda^*$ and suppose we have naked branches w or branches with leaves for each $\operatorname{Br}(w) \in \operatorname{Br}(\operatorname{Im} f^\alpha)$ and elements $a^w_{\alpha n}$ $(n \in \omega)$, a^{kw}_α $(k \in \omega)$ as in the construction (*) (replacing v_α by $\operatorname{Br}(w)$). Then there exist $2^{\aleph_0}$ such w with the following property:*

If $G^w_{\alpha+1} = \langle G_\alpha, a^{kw}_\alpha : k \in \omega \rangle_A$ then any $\beta < \alpha$ which was strong at stage α remains strong at stage $\alpha+1$.

Proof. We will find one such w; then it is easy to find $2^{\aleph_0}$ many. Suppose w with (5.2) does not exist. Then for each w from the Lemma there is an ordinal $\beta = \beta(w) < \alpha$ which is strong at stage α but not a stage $\alpha+1$. There exists $x \in G^w_{\alpha+1}$ such that

(a) $$\sum_{i<m} a_{\beta i} s_{\beta i} \Phi^\beta \equiv x \bmod G^w_{\alpha+1} s_{\beta m} \quad \text{for all } m \in \omega .$$

Since $x \in G^w_{\alpha+1}$ we find $g_\alpha \in G_\alpha$, $k \in \omega$ and $r \in A$ with $x = g_\alpha + a^{kw}_\alpha r$. Suppose that we can find some $t \in \omega$ with $r s_{\alpha t} \in \bigcap_{m \in \omega} A s_{\beta m}$. If $g_m = \sum_{i<m} a_{\beta i} s_{\beta i} \Phi^\beta$, then $a_{\beta i} \in P^\beta$ and $P^\beta \Phi^\beta \subseteq P^\beta$ imply

(b) $$g_m \in P^\beta,$$

and (a) becomes

$$g_m \equiv g_\alpha + a^{kw}_\alpha r \bmod G^w_{\alpha+1} s_{\beta m} .$$

Since $a^{kw}_\alpha r = b' + a^{qw}_\alpha s_{\alpha t} r$ for some $b' \in B$, $q \in \omega$, we also have

$$g_m \equiv g_\alpha + b' + a^{qw}_\alpha s_{\alpha t} r \equiv g_\alpha + b' \bmod G^w_{\alpha+1} s_{\beta m}$$

by assumption on r. Hence we can find $g^m_\alpha \in G_\alpha$, $r_m \in A$ and $k(m) \in \omega$ such that

$$g_m = g_\alpha + b' + g^m_\alpha s_{\beta m} + a^{k(m)w}_\alpha r_m s_{\beta m}$$

If m is fixed, choose $\sigma \in \operatorname{Br}(w)$ large enough and restrict this equation to σ. From (b) and the Recognition Lemma 5.1 (which we use silently in the future) we have $\frac{s_{\alpha l(\sigma)}}{s_{\alpha k(m)}} r_m s_{\beta m} = 0$ for $l(\sigma) \geqslant k(m)$ and in particular $r_m = 0$. The last equation turns into $g_m = g_\alpha + b' + g^m_\alpha s_{\beta m}$ or equivalently

$$\sum_{i<m} a_{\beta i} s_{\beta i} \Phi^\beta \equiv (g_\alpha + b') \bmod G_\alpha s_{\beta m} \quad \text{for all } m \in \omega .$$

Since β was strong at stage α, this is a contradiction. We conclude

(c) If $t \in \omega$, there exists $m \in \omega$ such that $r s_{\alpha t} \notin A s_{\beta m}$.

Using the notation above, we have $g_m \equiv g_\alpha + a^{kw}_\alpha r \bmod G^w_{\alpha+1} s_{\beta m}$ and there exists $y_m \in G^w_{\alpha+1}$ with $g_m = g_\alpha + a^{kw}_\alpha r + y_m s_{\beta m}$. Choose $\sigma \in v = \operatorname{Br}(w)$ large enough and suppose $g_m \restriction \sigma = 0$ for all $m \in \omega$. Then $\frac{s_{\alpha l(\sigma)}}{s_{\alpha k}} \cdot r \in A s_{\beta m}$ for all $m \in \omega$ which contradicts (c). Therefore $g_m \restriction \sigma \neq 0$ for some m. From (b) we conclude $\sigma \in [P^\beta]$. Since $[P^\beta]$ is a subtree, also $v \in \operatorname{Br}([P^\beta])$. On the other hand, $v \in \operatorname{Br}(\operatorname{Im} f^\alpha)$ by hypothesis. The Black Box (4.2) (iii) implies $\beta < \alpha < \beta + 2^{\aleph_0}$. We can summarize:

For each w as in the lemma with $\operatorname{Br}(w) \in \operatorname{Br}(\operatorname{Im} f^\alpha)$ there are $\beta(w) < \lambda^*$, $g^w_\alpha \in G_\alpha$, a^{kw}_α and r_w such that

(d) $$\beta(w) < \alpha < \beta(w) + 2^{\aleph_0} \quad \text{and for all } m \in \omega$$
$$\sum_{i<m} a_{\beta(w)i} s_{\beta(w)i} \Phi^{\beta(w)} \equiv g^w_\alpha + a^{kw}_\alpha r_w \bmod G^w_{\alpha+1} s_{\beta(w)m} .$$

Let β_0 be the minimal $\beta(w) < \lambda^*$ in (d). Then $\beta_0 < \beta(w) < \beta_0 + 2^{\aleph_0}$. Since $|\operatorname{Br}(\operatorname{Im} f^\alpha)| = 2^{\aleph_0}$, there are two distinct branches $\operatorname{Br}(w) \neq B(w')$ in $\operatorname{Im} f^\alpha$ such that $\beta(w) = \beta(w') = \beta$. Subtracting the corresponding equations in (d) leads to

$$0 \equiv g^w_\alpha - g^{w'}_\alpha + a^{kw}_\alpha r_w - a^{kw'}_\alpha r_{w'} \bmod G^{ww'}_{\alpha+1} s_\beta ,$$

where $G^{ww'}_{\alpha+1} = \langle G_\alpha, a^{kw}_\alpha, a^{kw'}_\alpha : k \in \omega \rangle_A$.

If $g = g^{w'}_\alpha - g^w_\alpha \in G_\alpha$, we find some $g^m \in G^{ww'}_{\alpha+1}$ such that

(e) $$g = a^{kw}_\alpha r_w - a^{kw'}_\alpha r_{w'} + g^m s_{\beta m} .$$

Choose $\sigma \in \operatorname{Br}(w)$ large enough, such that $\sigma \notin w'$, $g \restriction \sigma = 0$ and $a^{kw}_\alpha \restriction \sigma = \frac{s_{\alpha l(\sigma)}}{s_{\alpha k}}$.

From (e) we derive $\frac{s_{\alpha l(\sigma)}}{s_{\alpha k}} r_w + (g^m \restriction \sigma) s_{\beta m} = 0$ and $r_w \frac{s_{\alpha l(\sigma)}}{s_{\alpha k}} \in \bigcap_{m \in \omega} A s_{\beta m}$ which contradicts (c). Hence the Lemma is shown.

We have two immediate consequences

Corollary 5.3. *There are no useless ordinals.*

Corollary 5.4. *Let $\Phi \in \operatorname{End} G$, $\|B\Phi\| < \operatorname{cf} \lambda$, $a_n \in G$ $(n \in \omega)$ and (s_n) a divisor-chain. Then we can find a complete ordinal $\alpha < \lambda^*$ such that*

$$\mathscr{P}^\alpha = (f^\alpha, P^\alpha, \Phi \restriction P^\alpha, a_n, s_n : n \in \omega) .$$

Proof of 5.4. By the Black Box (4.3) we find $\alpha \in \lambda^0_c$ such that

$$\mathscr{P}^\alpha = (f^\alpha, P^\alpha, \Phi \restriction P^\alpha, a_n, s_n : n \in \omega)$$

and (4.2) (b) (**) hold. By (5.2) we can find a branch in $\operatorname{Im} f^\alpha$ with leaves such that any ordinal $<\alpha$ which was strong at α remains strong at the next step $\alpha+1$. Therefore α is complete.

§ 6. Basic properties of the ideal Ines$G$ and the module G. The following is a natural extension of a definition from [DG 1]; it also extends well-known ideals in the case of abelian p-groups respectively of mixed abelian groups; see [CG].

DEFINITION 6.1. If G is an R-module, then we say that R-submodule U of G is ω-complete, if for all $a_n \in U$ and divisor-chains s_n $(n \in \omega)$ there exists $x \in U$ such that $x \equiv \sum_{i<m} a_i s_i \bmod U s_m$ for all $m \in \omega$. Then $\operatorname{Ines} G = \operatorname{Ines}_R G$ is the set of all endomorphisms Φ of G with ω-complete $\operatorname{Im}\Phi$. If $\varkappa$ is a cardinal, then $\operatorname{Ines}_\varkappa G = \{\Phi \in \operatorname{Ines} G, |\operatorname{Im}\Phi| < \varkappa\}$.

This set $\operatorname{Ines}_\varkappa G$ is always a two-sided ideal of $\operatorname{End} G$ and $\operatorname{Ines}_\varkappa G = \operatorname{Ines}_R G$ for $\varkappa > |G|$. In addition we have the

THEOREM 6.2. *Let G be the R-module constructed in § 5.*

(a) *$\operatorname{Ines}_\varkappa G$ is a pure submodule of $\operatorname{End} G$, i.e. $s\operatorname{End} G \cap \operatorname{Ines}_\varkappa G = s\operatorname{Ines}_\varkappa G$ for all $s \in S$.*

(b) *$A \cap \operatorname{Ines}_\varkappa G = 0$.*

Remark. G is an A-module. Therefore A acts by scalar multiplication on G. Since this action is faithful we can identify $A \subset \operatorname{End} G$ in a natural way.

Proof. (a) Let $\Phi \in \operatorname{End} G$ and $s\Phi \in \operatorname{Ines} G$. Therefore $\operatorname{Im} s\Phi$ is ω-complete. If $a_n s\Phi \in Gs\Phi$ and (s_n) is any divisor-chain, there is $xs \in Gs\Phi$ such that

$$\sum_{i<m} a_i s_i s\Phi \equiv xs \bmod G\Phi s s_m \quad (m \in \omega)$$

by (6.1). We find $g^m \in G\Phi$ such that

$$\Big(\sum_{i<m} a_i s_i \Phi\Big)s = xs + g^m s_m s.$$

Since G is torsion-free, we cancel s and derive

$$\sum_{i<m} a_i s_i \Phi \equiv x \bmod G\Phi s_m \quad (m \in \omega),$$

i.e. $\Phi \in \operatorname{Ines} G$.

If $|\operatorname{Im} s\Phi| < \varkappa$, also $|\operatorname{Im}\Phi| < \varkappa$ and (a) follows immediately.

(b) Suppose $0 \neq a \in \operatorname{Ines} G$. Since $\bigcap_S As = 0$, we find $s \in S$ such that

(*) $$a \notin As$$

Let v be a constant branch in T, $\sigma_n \in v$ of length n and $s_n = s^n$ $(n \in \omega)$. By (6.1) we find $x \in G$ such that

(**) $$\sum_{i<m} \sigma_i s_i a + g^m s_m = x \quad (m \in \omega)$$

for some $g^m \in G\Phi$. Since $x \in G$, we may write

$$x = \sum_{j=1}^{t} a_{\alpha_j}^k r_j + b \quad \text{with } \alpha_1 > \ldots > \alpha_t,\ r_j \in A \text{ and } b \in B.$$

Therefore (**) becomes

(***) $$\sum_{i<m} \sigma_i s_i a + g^m s_m = \sum_{j=1}^{t} a_{\alpha_j}^k r_j + b.$$

Suppose $\|v_{\alpha_1}\| > \|v\|$. Choose $\sigma \in \operatorname{Br}(v_{\alpha_1})$ large enough such that $\sigma \notin v$ and $a_{\alpha_1}^k \upharpoonright \sigma = \frac{s_{\alpha_1 n}}{s_{\alpha_1 k}} b \upharpoonright \sigma = 0$. Restrict (***) to σ and derive $\frac{s_{\alpha_1 n}}{s_{\alpha_1 k}} r_1 = (g^m \upharpoonright \sigma)\cdot s^m$. If we vary m, then $\frac{s_{\alpha_1 n}}{s_{\alpha_1 k}} r_1 \in \bigcap_{m\in\omega} As^m$. Since $a_{\alpha_1}^k r_1 = b' + a_{\alpha_1}^n s_{\alpha_1 n} r_1$ for some $b' \in B$, we can "absorb $a_{\alpha_1}^n s_{\alpha_1 n} r_1$ into $g^m s^m$ for all $m \in \omega$", i.e.

$$\sum_{i<m} \sigma_i s_i a + (g^m - a_{\alpha_1}^n s_{\alpha_1 n} r_1 s^{-m}) s^m = \sum_{j=2}^{t} a_{\alpha_j}^k r_j + (b+b').$$

The first bracket is in G because $s_{\alpha_1 n} r_1 s^{-m} \in A$. Choose t minimal and conclude $\|v_{\alpha_1}\| \leqslant \|v\|$. The elements with finite support are treated similarly:

Let $b = \sum_{j=1}^{d} \varrho_j u_j$ with $\varrho_j \in T$, and $u_j \in A$. Equation (***) becomes

$$\sum_{j<m} \sigma_j s_j a + g^m s_m = \Big(\sum_{j=1}^{t} a_{\alpha_j}^k r_j + \sum_{j=2}^{d} \varrho_j u_j\Big) + \varrho_1 u_1.$$

If $\varrho_1 \in v$, say $\varrho_1 = \sigma_n$ we may "absorb $\varrho_1 u_1$ into $\sum_{i<m} \sigma_i s_i a$"; i.e. consider only $m > n$ and replace the first sum by $\sum_{n \neq i < m} \sigma_i s_i a + \sigma_n(s_n a - u_1)$. In general consider any $m > \max_{j=1}^{d} l(\varrho_i) + 1$ with $\sigma_{m-1} \notin \bigcup_{j=1}^{t} v_{\alpha_j}$ and restrict the last equation to σ_{m-1}. Therefore $s^{m-1} a + (g^m \upharpoonright \sigma_{m-1}) s^m = 0$ and $a = -(g^m \upharpoonright \sigma_{m-1})\cdot s \in As$ contradicts (*). Hence $a = 0$ and (b) holds. ■

Remark. In the last proof we "absorbed certain elements into certain elements". This concept is stated explicitely above and it will be used again in later proofs without going into detail.

The next result will become trivial from the view of our main result (8.1). However, it is based on a few algebraic arguments only and it also holds for more general modules. It might be a starting point for different proofs of the main theorem. Therefore, we include the

THEOREM 6.3. *$A \oplus \operatorname{Ines}_\varkappa G$ is a pure submodule of $\operatorname{End} G$.*

Proof. Let $\Phi \in \operatorname{End} G$ such that $s\Phi = a + \psi \in A \oplus \operatorname{Ines}_\varkappa G$. We want to show

(*) $$a \in As$$

Then $a = a's$ $(a' \in A)$ implies $\psi = s\Phi - a = s(\Phi - a') \in \operatorname{Ines}_\varkappa G \cap s\operatorname{End} G$. By (6.2) (a) we have $\Phi - a' \in \operatorname{Ines}_\varkappa G$, hence $\Phi \in A \oplus \operatorname{Ines}_\varkappa G$ and (6.3) follows from (6.2) (b).

Suppose (*) does not hold, i.e.

(a) $a \notin As$, $\psi = s\Phi - \acute{a} \in \operatorname{Ines}_\varkappa G$.

Define inductively a sequence $\sigma_n \in T$ such that

(b) $l(\sigma_n) = n$.

(c) distinct pairs σ_m, σ_n are not comparable in T.

(d) $\max_{i\leqslant n} \|\sigma_i\psi\| < \|\sigma_{n+1}\|$

for all $n\in\omega$. Apply (6.1) to (σ_n, s^n) and find $x\in G\psi$ such that

$$\sum_{i<m} \sigma_i s^i\psi \equiv x \bmod G\psi s^m$$

for all $m\in\omega$. Therefore we find $g^{m+1}\in G\psi$ with

(e) $\sum_{i\leqslant m} \sigma_i s^i + g^{m+1}s^{m+1} = x$ for all $m\in\omega$.

From (d) we have

$$(\sigma_i\psi)\restriction\sigma_m \equiv \begin{cases} 0 & i\neq m,\\ -a & i=m,\end{cases} \quad \bmod As.$$

Restricting (e) to σ_m we derive $-as^m \equiv (x\restriction\sigma_m) \bmod As^{m+1}$. If $x\restriction\sigma_m \equiv 0 \bmod As^{m+1}$, then $-as^m = a's^{m+1}$ and $a = -a's\in As$ because A is torsion-free. Since the last equation contradicts (a), we have

(f) $\lim_{k\in\omega}\|\sigma_k\| = \gamma \leqslant \|x\|$.

Let $x = \sum_{j=1}^{t} a^k_{\alpha_j} r_j + b$ with $b\in B$, $\alpha_1 > \ldots > \alpha_t$ and suppose $\|v_{\alpha_1}\| > \gamma$. We can pick $\sigma\in v_{\alpha_1}\setminus\{\sigma_i\colon i\in\omega\}$ such that $\|\sigma\|>\gamma$ and $a^k_{\alpha_1}\restriction\sigma = \frac{s_{\alpha_1 n}}{s_{\alpha_1 k}}$. Restrict (e) to σ and derive $\frac{s_{\alpha_1 n}}{s_{\alpha_1 k}} r_1 = (g^{m+1}\restriction\sigma)s^{m+1}$. The element $a^k_{\alpha_1}s_{\alpha_1 n}r_1$ can be absorbed into $g^{m+1}s^{m+1}$. Hence $x = y+b$ and $\|y\|\geqslant\gamma$. The element b can be treated similarly; see proof of (6.2). We derive $\|x\|\leqslant\gamma$, and together with (f) we have $\|x\| = \gamma$. This is impossible by (c), because $\{\sigma_k\colon k\in\omega\}$ is an antibranch. ■

In order to determine the *inessential* endomorphisms, i.e. the elements of $\operatorname{Ines} G$, we have to derive further properties of G. It is easy to see that G is torsion-free and reduced; cf. (6.5) (a). However, we will derive stronger results towards cotorsion-free. Hence a definition from [CG] seems to be useful.

Definition 6.4. Let $g\in G$, then

$$[g]^* = \begin{cases}\{\sigma\in T\colon \|\sigma\| = \|g\|,\ \sigma\in[g]\}\ (\|g\| \text{ not a limit})\\ \alpha<\lambda^*\colon \|v_\alpha\| = \|g\|,\ \sigma\in[g] \text{ for almost all } \sigma\in v_\alpha\ (\|g\| \text{ a limit})\end{cases}$$

is called the *top* of g. If $U\subseteq G$, let $[U]^* = \bigcup_{g\in G}[g]^*$.

Lemma 6.5. (a) *G is S-torsion-free and S-reduced.*

(b) *If $U\subseteq G$ is ω-complete, then $[U]^*$ is finite.*

Proof. (a) Since G is visibly S-torsion-free, we want to show that $\bigcap_{s\in S} Gs = 0$. Let $0\neq x\in Gs$; then $x = ys$ and either $y\in B$ or $y = \sum_{j=1}^{t} a^k_{\alpha_j} r_j + b$ with $t\geqslant 1$. In the first case choose any $\sigma\in[y]$ and remember $\bigcap_S As = 0$. Therefore $(x\restriction\sigma) = (b\restriction\sigma)s\in As$ for all $s\in S$ is a contradiction. In the second case choose any $\sigma\in v_{\alpha_1}$ large enough such that $(x\restriction\sigma) = (y\restriction\sigma)s = \frac{s_{\alpha_1 n}}{s_{\alpha_1 k}}s$. Therefore $(x\restriction\sigma)\in\bigcap_{s\in S} As = 0$ contradicts $(x\restriction\sigma)\neq 0$. ■

Proof by contradiction. (b) Suppose $[U]^*$ is infinite. Then we find a sequence $x_n\in U$ $(n\in\omega)$ such that

(a) $$X_n = [x_n]^*\setminus\bigcup_{k<n}[x_k]^* \neq \varnothing.$$

Observe that $\bigcup_{k<n}[x_k]^*$ is finite by the Recognition Lemma 5.1. Next we choose a subsequence of x_n $(n\in\omega)$ and change the enumeration such that we get a normalization

(b) $$\|x_n\|\ (n\in\omega) \text{ is not decreasing}$$

and there are two cases

(i) all $\|x_n\|$ are non-limits or

(ii) all $\|x_n\|$ are limit ordinals.

In the first case choose $\sigma_n\in X_n = X_n\cap T$ such that

(c) $$\begin{cases}\sup\|\sigma_n\| = \sup\|x_n\| = \alpha^*.\\ x_n\restriction\sigma_n\neq 0,\\ x_k\restriction\sigma_n = 0 \quad \text{for all } k<n\end{cases}$$

In the second case choose $\alpha_n\in X_n = X_n\cap\lambda^*$. Then all α_n are different and we can pick $\sigma_n\in v_{\alpha_n}$ such that (c) holds as well.

Passing to a subsequence of the σ_n we may introduce two further very convenient normalizations.

(d) If infinitely many σ_n lie on a single branch, then all of them do so.

(e) $$l(\sigma_n)\leqslant l(\sigma_{n+1}) \quad \text{for all } n\in\omega.$$

First we want to show

(f) There is a branch v_α with $\|v_\alpha\| = \alpha^*$ and $\sigma_n\in v_\alpha$ for all $n\in\omega$.

Choose $s_{n+1}\in S$ such that $s_n | s_{n+1}$ and

(g) $$(x_n\restriction\sigma_n)s_n\not\equiv 0 \bmod As_{n+1}.$$

This follows immediately from (c) and $\bigcap_{s\in S} As = 0$. Since U is ω-complete, we find $x\in U$ such that

$$\sum_{i<m} x_i s_i \equiv x \bmod Us_m \quad \text{for all } m\in\omega.$$

Therefore $x = \sum_{i\leqslant m} x_i s_i + x^{m+1} s_{m+1}$ for some $x^{m+1} \in U$. From (c) and (g) we have

$$x \restriction \sigma_m = (x_m \restriction \sigma_m) s_m + (x^{m+1} \restriction \sigma_m) s_{m+1} \equiv (x_m \restriction \sigma_m) s_m \equiv 0 \bmod A s_{m+1}.$$

Therefore $x \restriction \sigma_m \neq 0$ for all $m \in \omega$ and $\sigma_m \in [x]$. We derive $\alpha^* \leqslant ||x||$ from (c).

Using the concept of absorbing elements, it is easy to see that also $||x|| \leqslant \alpha^*$. We conclude $||x|| = \alpha^*$. From the Recognition Lemma 5.1 we have a branch v_α with $\sigma_n \in v_\alpha$ for infinitely many $n \in \omega$. The normalization (d) implies $\sigma_n \in v_\alpha$ for all $n \in \omega$, and (f) follows.

From (f) and (e) we also derive

(h) $$l(\sigma_n) = n^* < (n+1)^* \quad \text{for all } n \in \omega.$$

Now we choose $n \in \omega$ and better elements $s_n \in S$ such that

(k) $$s_{\alpha n^*} \cdot s_{n-1} | s_n \text{ and } (x_n \restriction \sigma_n) s_n \not\equiv 0 \bmod A s_{n+1}.$$

This is possible by (c). We also consider a fresh $x \in U$ such that

$$\sum_{i\leqslant m} x_i s_i \equiv x \bmod U s_m \quad \text{for all } m \in \omega$$

using completeness of U. Therefore $x = \sum_{i\leqslant m} x_i s_i + x^{m+1} s_{m+1}$ for some $x^{m+1} \in U$. Condition (k) implies

$$x \restriction \sigma_m = (x_m \restriction \sigma_m) s_m + (x^{m+1} \restriction \sigma_m) s_{m+1} \equiv (x_m \restriction \sigma_m) s_m \not\equiv 0 \bmod A s_{m+1};$$

(l) $$x \restriction \sigma_m \equiv (x_m \restriction \sigma_m) s_m \not\equiv 0 \bmod A s_{m+1}.$$

Therefore $\sigma_m \in [x]$ and $||x|| = \alpha^*$ as above.

From (f) we see that v_α "contributes" to the representation of x; we have from (5.1) $x = a_\alpha^k a + y$ with

(m) There exist $a \in A$, $k \in \omega$ such that $x \restriction \sigma_m = \frac{s_{\alpha m^*}}{s_{\alpha k}} a$ for all $m \geqslant k'$ and some $k' \geqslant k$.

If $\frac{a}{s_{\alpha k}} \in \bigcap_{n\in\omega} A s_n$, we can absorb $a_\alpha^k a$ into $U s_m$ for all $m \in \omega$. Hence

(n) $$\frac{a}{s_{\alpha k}} \notin \bigcap_{n\in\omega} A s_m.$$

Condition (k) implies $s_m \in A s_{\alpha m^*} s_{m-1}$, and with (l) we get $x \restriction \sigma_m \equiv (x_m \restriction \sigma_m) s_m \not\equiv 0 \bmod A s_{\alpha m^*} s_{m-1}$, i.e. $x \restriction \sigma_m \notin A s_{\alpha m^*} s_{m-1}$. From (m) we conclude $\frac{s_{\alpha m^*}}{s_{\alpha k}} a \in A s_{\alpha m^*} s_{m-1}$.

Since A is torsion-free, we derive $\frac{a}{s_{\alpha k}} \in A s_{m-1}$ for all $m \in \omega$, which contradicts (n). Therefore (6.5) holds. ∎

From (6.5) we have the immediate

Corollary 6.6. *If $\Phi \in \operatorname{Ines} G$, then $|\operatorname{Im} \Phi| \leqslant |A|$.*

Proof. The module $U = \operatorname{Im}\Phi$ is ω-complete and $[U]^*$ is finite by (6.5). If $|U| > |A|$, then $[U]^*$ must be infinite by definition (6.4). ∎

From (6.6) we already see $\operatorname{Ines} G = \operatorname{Ines}_{|A|^+} G$. In order to derive a stronger result, we will need further algebraic properties of G which are of independent interest. Recall the following well-known definition. An R-module U has *finite* (*R-*)*rank*, if there exist $x_i \in U$ $(i \leqslant m)$ such that $U \subseteq x_1 S^{-1} R + \ldots + x_m S^{-1} R$. This is equivalent to say that $\langle x_1, \ldots, x_m \rangle_* = U$ where $*$ denotes the purification in U. The notion of countable rank is similar.

Lemma 6.7. *Every A-submodule of finite A-rank in G is contained in a pure A-submodule of G, which is freely generated as A-submodule by finitely many elements from $T \cup \{a_\alpha^k \colon \alpha \in \lambda^*, k \in \omega\}$.*

Remark. (6.7) can be applied to R-submodules: Every finite rank R-submodule of G is contained in a free finite rank A-submodule of G.

Proof. Let U be an A-submodule in G of finite A-rank. We find $x_i \in U$ $(i \leqslant m)$ such that $U \subseteq \langle x_i A : i \leqslant m \rangle_*$. From (5.1) we have representations $x_j = \sum_{i=1}^{n(j)} a_{\alpha_{ij}}^{k(ij)} a_{ij} + b_j$ with $a_{ij} \in A$ and $b_j \in B$. Let $\alpha \in \lambda^*$ be minimal with $x_j \in G_\alpha$ for $j \leqslant m$. Therefore $\alpha = 0$ or α is a successor ordinal. Since G_α is a pure A-submodule of G, also $U \subseteq \langle x_j A \colon j \leqslant m \rangle_* \subseteq G_\alpha$. If $\alpha = 0$, then $G_0 = B = \bigoplus_T \sigma A$ is free; see § 5. Therefore we may assume $\alpha > 0$ and consider $C = \{\alpha_{ij} \colon i \leqslant n(i), j \leqslant m\}$. Then $\alpha \in C$ by the minimality of α. It is easy to find a natural number k with

(i) $k \geqslant \max\{k(ij) \colon \alpha_{ij} \in C\}$.
(ii) $[v_\alpha^k] \cap [v_\beta] = \varnothing$ for all $\beta \in C \setminus \{\alpha\}$.
(iii) $[v_\alpha^k] \cap [b_j] = \varnothing$ for all $j \leqslant m$.
(iv) If $||v_\beta|| \neq ||v_\alpha||$, $\beta \in C$, then $||v_\beta|| < ||v_\alpha(k)||$.

Now we enlarge the module U using our notation from § 5 (*). If $\beta \in \lambda^*$, then $a_\beta^k \in G$ $(k \in \omega)$ and therefore $a_\beta^i - \frac{s_{\beta i+1}}{s_{\beta i}} a_\beta^{i+1} = a_{\beta i} \in G$. Since $a_{\beta i} = a_i^\beta \delta_\beta + a_{\beta i}^1$ and $a_{\beta i}^1 \in B \subseteq G$, also $a_i^\beta \delta_\beta \in G$. Therefore the enlarged module U' (below) will be a submodule of G. In order to define U' consider all "finite parts" of U and all "segments of branches (with leave) below k", i.e. let

$$F = \bigcup_{j=1}^{m} [b_j] \cup \bigcup_{\beta\in C} (v_\beta \setminus v_\beta^k) \cup \bigcup_{\beta \in C \cap \lambda_\omega^*} \{(v_\beta \restriction n)^\wedge \langle \beta_n^\nu \rangle \colon n < k\}.$$

Then F is a finite subset of T, the A-module

$$U' = \langle \sigma A, a_\beta^k A, a_i^\beta \delta_\beta A \colon \sigma \in F, \beta \in C, i \leqslant k \rangle_*$$

has finite A-rank and $U' \subseteq G$. The choice of the generators is made to ensure $x_j \in U'$ $(j \leqslant m)$. Since U' is a pure A-submodule, also $U \subseteq \langle x_j A, j \leqslant m \rangle_* \subseteq U'$ and visibly $U' \subseteq G_\alpha$. Next we want to split off a free summand from U' and determine its complement U'' in terms of generators:

$$U'' = \langle \sigma A, a^k_\beta A, a^\gamma_i \delta_\gamma A: \sigma \in F, \|\sigma\| < \|v_\alpha\|, \beta \in C \setminus \{\alpha\}, \gamma \in C, i \leqslant k \rangle_* .$$

If $E = \{\sigma \in \bigcup_{j=1}^{m} [b_j]: \|\sigma\| > \|v_\alpha\|\}$, then $U' = U'' \oplus a^k_\alpha A \oplus \bigoplus_{\sigma \in E} \sigma A$. Since α is a successor ordinal > 0, we conclude $U'' \subseteq G_{\alpha-1}$. Ordinals are well-founded hence (6.7) follows by finite induction.

We add an extension of (6.7) which is not needed to derive the main result.

Lemma 6.8. *Every A-submodule of countable A-rank in G is contained in a freely generated A-submodule of G with generators from $T \cup \{a^k_\alpha: \alpha \in \lambda^*, k \in \omega\}$. In particular, G is a $\aleph_1$-free A-module.*

Proof (similar).

§ 7. **Endomorphisms of G are B-bounded.** If X, Y are subsets of the module G from (§ 5), then $X - Y$ denotes the complex $\{x-y: x \in X, y \in Y\}$ and $XA = \{xa: x \in X, a \in A\}$. Since $T \subseteq G$, also $T-T$ has *algebraic* meaning (distinguish from $X \setminus X = \emptyset$!). Let

$$G^\alpha = \langle \eta, a^k_\beta: \eta \in T, k \in \omega, \beta < \lambda^* \text{ and } \|\eta\| < \alpha, \|v_\beta\| < \alpha \rangle_A .$$
$$= \{x \in G: \|x\| < \alpha\} \quad \text{for } \alpha < \mathrm{cf}(\lambda) .$$

Recall from the Black Box (4.3), that we defined disjoint and unbounded subsets V, V', V'' of $\mathrm{cf}(\lambda)$. The following Definition 7.1 (b) serves internally for this section only.

Definition 7.1. If $\Phi \in \mathrm{End}\, G$, say that

(a) Φ is *B-bounded*, if there are $a \in A$ and $\alpha < \mathrm{cf}(\lambda)$ such that $\|B(\Phi - a)\| < \alpha$.

(b) Φ is *too large*, if for all $\alpha \in \mathrm{cf}\,\lambda$ there are sequences $e_n \in TA - TA$, (s_n) and $\alpha < \alpha_n < \alpha_{n+1} \in V''$ such that $\alpha_n < \|e_n\|$, $\|e_n \Phi\| < \alpha_{n+1}$ and there does not exist $r \in A$ with solutions $x_n \in G^{\alpha_n}$ of the equations $e_n s_n^2 \Phi - e_n r \equiv x_n \bmod G s_{n+1}$ for all $n \in \omega$.

First we want to show the

Lemma 7.2. *If $\Phi \in \mathrm{End}\, G$ is too large, we can find sequences $e_n \in G$ and (s_n) $(n \in \omega)$ such that the module $G^* = \langle G, y^m: m \in \omega \rangle_A$ with $y^m = \sum_{j \geqslant m} e_i \frac{s_i}{s_m}$ does not have a solution $x \in G^*$ of the equations*

$$\sum_{i<m} e_i s_i \Phi \equiv x \bmod G^* s_m \quad \textit{for all } m \in \omega .$$

Proof. Let $\Phi \in \mathrm{End}\, G$ be a fixed too large endomorphism. For any divisor chain (s_n) we define

$C_\Phi(s_n) = \{\beta \in \mathrm{cf}\,\lambda$: There are sequences of ordinals α_n with $\sup \alpha_n = \beta$, elements $e_n \in G$ and a divisor chain (s'_n) contained in (s_n) such that (α_n, e_n, s'_n) satisfy (7.1) (b) for (α_n, e_n, s_n) and some $\alpha\}$.

This set $C_\Phi(s_n)$ is obviously ω-closed, i.e. countable limits of elements in $C_\Phi(s_n)$ belong to $C_\Phi(s_n)$; observe that Φ and (s_n) are fixed and a diagonal argument will help.

Let $\alpha(s_n) = \sup C_\Phi(s_n)$ for all sequences (s_n). We want to show that

(b′) there exists a divisor sequence (s_n) such that $C_\Phi(s_n)$ is unbounded in $\mathrm{cf}\,\lambda$.

(Condition (b′) strengthens (7.1) (b)).

If S has a divisor sequence (s_n) with $\bigcap_{n \in \omega} R s_n = 0$, we fix this sequence and replace S by (s_n). Then (b′) follows immediately. In the opposite case $\mathrm{cf}\,\lambda > |S^{\aleph_0}|$, because λ is admissible. If (b′) does not hold, then $\alpha(s_n) < \mathrm{cf}\,\lambda$ for all (s_n), and also $\alpha = \sup_{(s_n)} \alpha(s_n) < \lambda$ by $|S^{\aleph_0}| < \mathrm{cf}\,\lambda$. Since Φ is too large, we find some (s_n), $e_n \in G$, $\alpha_n > \alpha$ $(n \in \omega)$ which satisfy (7.1) (b). Therefore $\alpha^* = \sup_{n \in \omega} \alpha(s_n) \in C_\Phi(s_n)$ and $\alpha^* > \alpha = \sup_{(s'_n)} C_\Phi(s'_n) \geqslant \sup C_\Phi(s_n) \geqslant \alpha^*$ is a contradiction and (b′) holds.

Fix $C_\Phi(s_n)$ with (b′) and let C be its closure in the interval topology on $\mathrm{cf}\,\lambda$. The set C is unbounded by (b′) and hence a cub. Therefore $C \cap V'' \neq \emptyset$ as V'' is a stationary set.

We find $\alpha^* \in C \cap V''$ with $\mathrm{cf}\alpha^* = \omega$ by definition $V'' \subseteq \{\alpha \in \mathrm{cf}\,\lambda: \mathrm{cf}\alpha = \omega\}$. Therefore $\sup \beta_n = \alpha^*$ for some strictly increasing sequence β_n. We want to show that $\alpha^* \in C_\Phi(s_n)$, and hence we will construct an increasing sequence $\alpha_n \in C_\Phi(s_n)$ with $\alpha^* = \sup_{n \in \omega} \alpha_n$. Then $\alpha^* \in C_\Phi(s_n)$ such that $\max\{\beta_n, \alpha_{n-1}\} \leqslant \alpha_n \leqslant \alpha^*$. Obviously, $\sup_{n \in \omega} \alpha_n = \sup_{n \in \omega} \beta_n = \alpha^*$. This implies

(b″) Condition (7.1) (b) holds with the further requirements $\sup_{n \in \omega} \alpha_n \in V''$.

Now we use elements e_n, s_n, α_n from (b″) and suppose that they do not satisfy (7.2). Then we can find $x \in G^*$ such that $x \equiv \sum_{i<m} e_i s_i \Phi \bmod G^* s_m$ for all $m \in \omega$. Since $x \in G^*$, we may write $x = y^k r + \sum_{i=1}^{t} a^k_{\beta_i} r_i + b$ with $r, r_i \in A$, $b \in B$. If $\frac{s_{\beta_1 n}}{s_{\beta_1 k}} r_1 \in \bigcap_{m \in \omega} A s_m$ for some $n \geqslant k$, then by the now standard argument we can absorb $a^k_{\beta_1} r_1$ into $G^* s_m$ for all m. Hence there is no loss of generality to assume

(i) If $n \geqslant k$ there exists $m \in \omega$ with $\frac{s_{\beta_1 n}}{s_{\beta_1 k}} r_1 \notin A s_m$.

If $\alpha^* = \sup_{n \in \omega} \alpha_n$ and $u_m = -y^k r + \sum_{i<m} e_i s_i \Phi$, then $\|u_m\| \leqslant \alpha^*$ follows by hypothesis (7.1) (b). The above equalities lead to

$$\sum_{i=1}^{t} a^k_{\beta_i} r_i + b - u_m \in G^* s_m .$$

If $\|v_{\beta_1}\| > \alpha^*$ we conclude from the last observation and (2.1) (iv) that $a^k_{\beta_1} r_1 \in G^* s_m$ which contradicts (i). Therefore $\|v_{\beta_1}\| \leqslant \alpha^*$ and also $\|v_{\beta_1}\| < \alpha^*$ from (b″) which is $\alpha^* \in V''$ and $\|v_{\beta_1}\| \notin V''$. We can find $n_0 \in \omega$ such that $\|v_{\beta_1}\| < \alpha_{n_0}$ and if we write $w = \sum_{i=1}^{t} a^k_{\beta_i} r_i$, then $\|w\| < \alpha_{n_0}$ and $x = y^k r + w + b$.

Now we treat b in a similar way; let $b = \sum_{i=1}^{t} \sigma_i r_i$ $(\sigma_i \in T)$. If $\sigma_1 r_1 \in \bigcap_{n\in\omega} Gs_m$, then $\sigma_1 \cdot r_1$ can be absorbed into $G^* s_m$ for all $m \in \omega$. Hence we choose $m \in \omega$ such that

(ii) $\sigma_1 r_1 \notin Gs_m$.

If $\|\sigma_1\| > \alpha^*$, the same argument as above contradicts (ii). Therefore $\|\sigma_1\| \leqslant \alpha^*$ and $\|\sigma_1\| < \alpha^*$ since $\mathrm{cf}(\alpha^*) = \omega$ and $\|\sigma_1\|$ is a successor ordinal. We can find $n_1 > n_0$ such that $\|\sigma_1\| < \alpha_{n_1}$ and by induction we can show that $x = y^k r + u$ for some $n \in \omega$ and $u \in G^{\alpha_n}$. The above equalities become

(iii) $\sum_{i\leqslant m} e_i s_i \Phi \equiv y^k r + u \mod G^* s_{m+1}$, $\|u\| < \alpha_{\bar n}$ $(m \in \omega)$.

If $m > \max(n, k)$ we can decompose

$$y^k r + u = e_m \frac{s_m}{s_k} r + u' + \Big(\sum_{i>m} e_i \frac{s_i}{s_{m+1}}\Big)\frac{s_{m+1}}{s_k} r,$$

where $u' = u + \sum_{k\leqslant i<m} e_i \frac{s_i}{s_k} r$. Since $\sum_{i>m} e_i \frac{s_i}{s_{m+1}} = y^{m+1} \in G^*$, (iii) can be changed into

$$\sum_{i\leqslant m} e_i s_i \Phi \equiv e_m \frac{s_m}{s_k} r + u' \mod G^* \frac{s_{m+1}}{s_k}, \quad \text{where } \|u'\| < \alpha_m.$$

By hypothesis also $\|\sum_{i<m} e_i s_i \Phi\| < \alpha_m$ and we can find $r' \in A$, $u'' \in G^{\alpha_m}$ with $e_m s_m \Phi - -e_m r' \equiv u'' \mod G^* \frac{s_{m+1}}{s_k}$. Multiply the last equation with s_m and change names, then

$$e_m s_m^2 \Phi - e_m r' \equiv u'' \mod G^* s_{m+1} \text{ with } r' \in A,\ u'' \in G^{\alpha_m}.$$

Therefore $e_m s_m^2 \Phi - e_m r' - u'' = g_{m+1} s_{m+1} + y^{t(m)} a_m s_{m+1}$ for some $g_{m+1} \in G$, $a_m \in A$, $t(m) \in \omega$. If $\|g_{m+1}\| > \alpha^*$ we restrict the last equation to large enough $\sigma \in [g_{m+1}]$ such that $\sigma \notin [e_m s_m^2 \Phi - e_m r' - u'' - y^{t(m)} a_m s_{m+1}]$. The "standard argument of absorbing elements" leads to a contradiction. Therefore $\|g_{m+1}\| \leqslant \alpha^*$ and also $\|g_{m+1}\| < \alpha^*$. We find $n' \in \omega$ such that $\|g_{m+1}\| < \alpha_{n'}$. Hence we can choose $\sigma \in [y^{t(m)}] \setminus [g^{m+1}]$ $(\|\sigma\| > \alpha_{n'})$, and the last equation leads to $(y^{t(m)} \restriction \sigma) a_m s_{m+1} = 0$ and $a_m s_{m+1} = 0$. Therefore

$$e_m s_m^2 \Phi - e_m r' \equiv u'' \mod G s_{m+1} \ (m \in \omega)$$

from the last equation. This contradicts our hypothesis (7.1) (b). ■

Next we will show that our module G does not allow too large endomorphisms in fact we prove the

LEMMA 7.3. *If $\Phi \in \mathrm{End}\, G$, we cannot find sequences $e_n \in G$ (s_n) $(n \in \omega)$ such that* (*) *holds.*

(*) *The extended module $G^* = \langle G, y^m\colon m \in \omega\rangle_A$ with $y^m = \sum_{i\geqslant m} e_i \frac{s_i}{s_m}$ does not have a solution $x \in G^*$ of the equations $\sum_{i<m} e_i s_i \equiv x \mod G^* s_m$ for all $m \in \omega$.*

Proof. Suppose $\Phi \in \mathrm{End}\, G$ satisfies (*) for certain $e_n \in G$, (s_n). From the Black Box 4.2 we can choose $\alpha \in \lambda^* \setminus \lambda_c^*$ such that $e_n \in P^\alpha$, $\|e_n\| < \|v_\alpha\|$, $\|e_n \Phi\| < \|v_\alpha\|$, $\Phi \restriction P^\alpha = \Phi^\alpha$ and $s_n = s_{\alpha n}$ for all $n \in \omega$. If α turns out to be strong, let $x = a_\alpha \Phi^\alpha = a_\alpha \Phi \in P^\alpha \cap G$. Since $\sum_{i<m} a_{\alpha i} s_{\alpha i} = a_\alpha - a_\alpha^m s_{\alpha m}$ and $a_\alpha^m \in G$, we have

$$\sum_{i<m} a_{\alpha i} s_{\alpha i} \Phi^\alpha = \sum_{i<m} a_{\alpha i} s_{\alpha i} \Phi = x - a_\alpha^m \Phi s_{\alpha m} \equiv x \mod G s_{\alpha m} \quad \text{for all } m \in \omega.$$

By the "standard method of absorbing elements", conclude $x \in G_{\alpha+1}$ from $x \in G$. This is a contradiction since α is supposed to be strong. Therefore it is enough to show that α is strong, and again, we assume that this is not the case. From the "No-useless-ordinals"-Lemma (5.2) we obtain a branch (in fact $2^{\aleph_0}$ branches) $v \in \mathrm{Br}(\mathrm{Im} f^\alpha)$ with $v \neq v_\alpha$ and the additional property that any ordinal $\beta < \alpha$ which was strong at earlier stages remains strong if we add $z^k = \sum_{i\geqslant k} (v \restriction i) \frac{s_i}{s_k}$ $(k \in \omega)$ to the module G_α; see (5.2) (and use $s_{\alpha n} = s_n$). Since α is not strong, the construction "will go wrong" at stage α. There must be $x' \in G' = \langle G, z^k\colon k \in \omega\rangle_A$ such that

(i) $$\sum_{i<m} (v \restriction i) s_i \Phi \equiv x' \mod G' s_m \quad \text{for all } m \in \omega.$$

By the same argument we can choose $w^k = \sum_{i\geqslant k} (e_i + v \restriction i) \frac{s_i}{s_k}$ and $x'' \in G'' = \langle G, w^k\colon k \in \omega\rangle_A$ such that

(ii) $$\sum_{i<m} (e_i + v \restriction i) s_i \Phi \equiv x'' \mod G'' s_m \quad \text{for all } m \in \omega.$$

The modules G' and G'' are contained in

$$G^{**} = \langle G, z^k, w^k\colon k \in \omega\rangle_A = \langle G, z^k, y^k\colon k \in \omega\rangle_A,$$

and since $\|e_n\| < \|v\|$, the elements z^k and y^k are independent; see (2.1). Subtracting (ii) from (i), we obtain $x = x' - x'' \in G^{**}$ such that $x \equiv \sum_{i<m} e_i s_i \Phi \mod G^{**} s_m$ for all $m \in \omega$. Since $x \in G^{**}$, we may write $x = c + y^k r_1 + z^k r_2$ $(k \in \omega,\ r_i \in A,\ c \in G)$ and the last equations turn into

(iii) $$\sum_{i<m} e_i s_i \Phi - y^k r_1 - z^k r_2 = c + g_m s_m \quad \text{for some } g_m \in G^{**}.$$

Now we repeat the standard argument; see (6.2) or (7.2).

If $\|c\| > \|v\|$, we may write $c = \sum_{i=1}^{t} a_{\alpha_i}^d r_i + b$ with $b \in B$ and either $\|v\| < \|v_{\alpha_1}\|$ or $\|v\| < \|b\|$. In the first case choose $\sigma \in v_{\alpha_1}$, $\|\sigma\| > |v\|$ and restrict (iii) to σ. Therefore $a_{\alpha_1}^d r_1$ can be absorbed into $G^{**} s_m$ and together with a similar argument for b we derive $\|c\| \leqslant \|v\|$. Since $v \neq v_\alpha$, we can choose $\sigma \in v$, $\sigma \notin [c] \cup \bigcup_{t\in\omega} [e_t]$.

Restrict (iii) to σ and derive $\frac{s_n}{s_k} r_2 = (g_m \restriction \sigma) s_m$ for all $m \in \omega$ and some fixed n.

Therefore $z^k r_2$ can also be absorbed into $G^{**}s_m$ and we use the structure of G^{**} to change (iii) into

(iv) $$\sum_{i<m} e_i s_i \Phi - y^k r_1 - z^{k(m)} a_m s_m - y^{k(m)} a'_m s_m - c = g'_m s_m$$

for certain elements $a_m, a'_m \in A$, $k(m) \in \omega$, $g'_m \in G$. Absorbing further elements we obtain $\|g'_m\| \leqslant \|v\|$ and $g'_m = \sum_{i=1}^{t} a^d_{\alpha_i} r_i + b$ with $\|v_{\alpha_1}\| \leqslant \|v\|$. Since $v_{\alpha_1} \neq v$, the equation (iv) restricted to large enough $\sigma \in v$ forces $\frac{s_n}{s_{k(m)}} s_m a_m = 0$. Therefore $a_m = 0$ and (iv) reduces to $\sum_{i<m} e_i s_i \Phi - y^k r_1 - y^{k(m)} a'_m s_m - c = g'_m s_m$. Since $y^k \in G^*$, we derive $\sum_{i<m} e_i s_i \Phi \equiv y^k r_1 + c \mod G^* s_m$ for all $m \in \omega$. This contradicts our hypothesis (*). ■

We have an immediate consequence from (7.2) and (7.3).

COROLLARY 7.4. *The endomorphisms of G are not too large.*

This will be used to show a

THEOREM 7.5. *The endomorphisms of G are B-bounded.*

Proof. Let $\Phi \in \operatorname{End} G$ and assume that Φ is not B-bounded. We want to show that Φ is too large, which is impossible by (7.4).

Suppose for contradiction that Φ is not too large, say that the construction of sequences in (7.1) (b) breaks down at some $\alpha_n \in V''$. If $\alpha = \alpha_n$, we may assume

(i) For all $\eta a \in TA \cup (T-T)A$ with $\|\eta\| > \alpha$, $s \in S$ $(s_{n-1}|s)$ there are $r = r_{a\eta s} \in A$, $x = x_{a\eta s} \in G^\alpha$ such that $\eta a s^2_{n-1}\Phi - \eta a r \equiv x \mod Gs$.

We want to show that r in (i) is independent of s, a, η and let $s^2_{n-1} = q$. For the moment we also fix $a \in A$ and let $\eta \in T$, $\|\eta\| > \alpha$. Then $\eta a \Phi = \sum_{i=1}^{t} a^k_{\alpha_i} r_i + b$ with $r_i \in A$ and $b \in B$. We derive from (i)

(ii) $$\sum_{i=1}^{t} a^k_{\alpha_i} r_i q + b' \equiv x \mod Gs \quad \text{for some } b' \in B\,.$$

If $\|v_{\alpha_1}\| > \alpha$, we can choose $\sigma \in v_{\alpha_1}$ large enough such that $\sigma \notin [x] \cup [b']$. Since $\bigcap_S As = 0$ and $(v_{\alpha_1} \upharpoonright \sigma) r_1 \neq 0$ we find $s \in S$ with $(v_{\alpha_1} \upharpoonright \sigma) r_1 q \notin As$. On the other hand, (ii) implies $(v_{\alpha_1} \upharpoonright \sigma) r_1 q \in As$ and therefore $\|v_{\alpha_1}\| \leqslant \alpha$. Since $\alpha \in V''$, $\|v_{\alpha_1}\| \notin V''$ also $\|v_{\alpha_1}\| < \alpha$ and $\sum_{i=1}^{t} a^k_{\alpha_i} r_i q \in G^\alpha$. Equations (i) and (ii) reduce to

(iii) $$\eta(r^1 q - ar) + \sum_{i=2}^{t} \varrho_i r^i \equiv x \mod Gs,\ x \in G^\alpha \quad \text{and}$$
$$b = \sum_{i=1}^{t} \varrho_i r^i,\ \varrho_i \in T,\ \varrho_1 = \eta \quad \text{and } \|\varrho^i\| > \alpha\,.$$

We have from (2.1) that $(r^1 q - ar) \in As$ for $i = 1$ and $r^i \in As$ for $i > 1$. Therefore $r^1 q = ar$ and $r^1 = r^a_\eta$ depends only on η and a by definition. Equations (iii) become

(iv) For any $a \in A$ there is $r^a_\eta \in A$ such that $\eta a \Phi q - \eta r^a_\eta q \equiv x_{a\eta s} \mod Gs$ and $x_{a\eta s} \in G^\alpha$ for all $s \in S$.

If $\eta \neq \eta' \in T$, $\|\eta\|, \|\eta'\| > \alpha$, we consider $(\eta - \eta')a$ and a similar argument will show that there exists $r^a_{\eta-\eta'} \in A$ which does not depend on s such that $(\eta-\eta')a\Phi q - (\eta-\eta') r^a_{\eta-\eta'} q \equiv x' \mod Gs$ for some $x'_s \in G^\alpha$. Since $(\eta-\eta')a\Phi = \eta a \Phi - \eta' a \Phi$, we can subtract the last equation from (iv) and derive

$$\eta r^a_\eta q + \eta' r^a_{\eta'} q - (\eta-\eta') r^a_{\eta-\eta'} q \equiv y \mod Gs \quad \text{and} \quad y \in G^\alpha.$$

This equation restricted to η respectively η' forces $r^a_\eta = r^a_{\eta-\eta'}$ and $r^a_{\eta'} = r^a_{\eta-\eta'}$ and therefore $r^a_\eta = r^a_{\eta'} = r^a$ which is now independent of η as well. The claim (iv) reduces to (v), i.e.

(v) For any $a \in A$ there is r^a such that
$$\eta a \Phi q - \eta r^a q \equiv x_{a\eta s} \mod Gs,\ x_{a\eta s} \in G^\alpha$$

Using this notation, we suppose that there exists $a \in A$ such that $ar^1 \neq r^a$ We find $t \in S$ with

(vi) $$(ar^1 - r^a) q \notin At$$

It is easy to choose $\eta_i \neq v_i \in T$, $\alpha_i < \alpha_{i+1} \in V''$ $(i > n)$ such that $\alpha_i < \|\eta_i\|, \|v_i\|$, $\alpha < \alpha_i$ and

$$\max\{\|\eta_i\|, \|v_i\|, \|\eta_i \Phi\|, \|\eta_i a \Phi\|\} < \alpha_{i+1}\,.$$

Setting $s_n = t^{n^2}$ and $e_n = \eta_n - v_n a$ we will show that (*) in (7.3) is fulfilled. Therefore $\Phi \notin \operatorname{End} G$ by (7.3), which implies $ar^1 = r^a$ for all $a \in A$.

Suppose there exists a solution $x \in G^* = \langle G, y^m \colon m \in \omega \rangle$ with $y^m = \sum_{i \geqslant m} e_i t^{i^2 - m^2}$ which satisfies the equations

$$\sum_{i<m} e_i t^{i^2} \equiv x \mod G^* t^{m^2} \quad \text{for all } m \in \omega\,.$$

Using the standard argument (see e.g. proof of (7.2); (iii)) we can write $x \equiv y^k \cdot r + u$ such that $\|u\| < \alpha_{n'}$ for some $n' > n$. Since $y^k r \equiv e_m t^{m^2 - k^2} r \mod G^* t^{(m+1)^2 - k^2}$ we derive $e_m t^{m^2} \Phi \equiv e_m t^{m^2 - k^2} r + u' \mod G^* t^{(m+1)^2 - k^2}$ for $m \geqslant n'$ and $u' \in G^*$ with $\|u'\| < \alpha_m$. Therefore

$$(\eta_m t^{m^2} \Phi - \eta_m t^{m^2 - k^2} r) + (v_m a t^{m^2} \Phi - v_m a t^{m^2 - k^2} r) \equiv u'$$

and by (v) this becomes

$$\eta_m (t^{m^2} r^1 q - t^{m^2 - k^2} r) + v_m (t^{m^2} r^a q - a r t^{m^2 - k^2}) \equiv u$$

for some $u'' \in G^*$ with $\|u''\| < \alpha_m$. We conclude $(t^{m^2} r^1 q - t^{m^2 - k^2} r) \in A t^{(m+1)^2 - k^2}$ and $(t^{m^2} r^a q - t^{m^2 - k^2} a r) \in A t^{(m+1)^2 - k^2}$; see (2.1).

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Combining the two statements, we derive $(ar^1-r^a)q\in At^{2m+1-k^2}$ which contradicts (vi) if we choose $m>k^2$. Therefore $ar^1=r^a$ and (v) turns into

(vii) There is $r=r^1\in A$ such that $\eta a\Phi q-\eta arq\equiv x_{a\eta s} \bmod Gs$ and $x_{a\eta s}\in G^\alpha$.

Since $\eta aq(\Phi-r)\in G^\alpha \bmod Gs$ for all $s\in S$, we have $\eta aq(\Phi-r)\in G^\alpha$ by our standard argument. Hence $B(\Phi-r)\subseteq G^\alpha$ or equivalently $||B(\Phi-r)||<\alpha$ and Φ is B-bounded. Hence (7.5) follows from this contradiction. ■

§ 8. The realization-theorem. If G is the R-module from § 5, then we have a first

THEOREM 8.1. $\operatorname{End} G = A\oplus \operatorname{Ines} G$.

Proof. We will show that $\Phi\in \operatorname{End} G\setminus A\oplus\operatorname{Ines} G$ leads to a contradiction. Since $\Phi\in\operatorname{End} G$, we apply (7.5) to ensure that Φ is B-bounded. By Definition 7.1 (a) we find $a\in A$ and $\alpha<\operatorname{cf}(\lambda)$ such that $||B(\Phi-a)||<\alpha$. If $\psi=\Phi-a$, then $\psi\notin\operatorname{Ines} G$ by hypothesis. From Definition 6.1 there are $a_n\in G$ and (s_n) such there is no solution $x\in G\psi$ of the equations

(i) $$\sum_{i<m} a_i s_i\psi \equiv x \bmod G\psi s_m \quad \text{for all } m\in\omega.$$

Since ψ is B-bounded, we also have $|B\psi|<\lambda$ (W.L.O.G. we restrict to regular λ). The Black Box 4.3 provides $\alpha\in\lambda_c^*$ which satisfies (4.2) (b) and α is complete by (5.4). Hence $s_{\alpha n}=s_n$, $a_n^\alpha=a_n\in P^\alpha$, $\psi\restriction P^\alpha=\Phi^\alpha$ and there is a branch $v\in\operatorname{Br}(\operatorname{Im} f^\alpha)$ with leaves $\alpha_n^v\in P^\alpha$ such that $a_{\alpha n}^1=(v\restriction n)-(v\restriction n-1)^\wedge a_{n-1}^v$ and $a_{\alpha n}^1\psi=0$ for all $n\in\omega$. By construction of G we have

$$a_\alpha^m=\sum_{i\geq m}(a_i+a_{\alpha i}^1)\frac{s_i}{s_m}\in G$$

and therefore $s_m a_\alpha^m\psi\in G\psi s_m$. On the other hand

$$x=a_\alpha\psi=\sum_{i\in\omega}(a_i+a_{\alpha i}^1)s_i\psi=\sum_{i<m}(a_i+a_{\alpha i}^1)s_i\psi+a_\alpha^m\psi s_m=\sum_{i<m}a_is_i\psi+a_\alpha^m\psi_m$$
$$\equiv\sum_{i<m}a_is_i\psi \bmod G\psi s_m \quad\text{for all } m\in\omega.$$

This contradicts (i) and (8.1) follows. ■

§ 9. Characterization of Ines G and consequences for the realization theorem. From (6.6) we know already $\operatorname{Ines} G=\operatorname{Ines}_{|A|^+}G$, i.e. $|\operatorname{Im}\Phi|\leq|A|$ for all $\Phi\in\operatorname{Ines} G$. This will be improved in (9.2) using (6.7). Instead of $||x||$ as in earlier sections, we will use the precise definition of $[x]$; see (2.1) and § 5 (*). Moreover, it will be convenient to define the history $H(x)$ of $x\in G$. Any $x\in G$ has a representation $x=\sum_{i=1}^n a_{\alpha_i}^k r_i+b$ with $r_i\neq 0$, $b\in B$, $[b]\cap[v_{\alpha_i}]=\emptyset$ by the Recognition Lemma 5.1.

We require $a_{\alpha_i}^k\in H(x)$, $[v_{\alpha_i}^k]\subseteq H(x)$ and $[b]\subseteq H(x)$. If

$$a_\alpha^k=\sum_{i\geq k}a_{\alpha i}\frac{s_{\alpha i}}{s_{\alpha k}}+\sum_{i\geq k}(v_\alpha\restriction i)\frac{s_{\alpha i}}{s_{\alpha k}}\in H(x)$$

from § 5 (*), then $a_{\alpha i}\in H(x)$, $[v_\alpha^k]\subseteq H(x)$ $(i\geq k)$. Next we unravel $a_{\alpha i}$ $(i\geq k)$ in the same way as x and add the components to $H(x)$. This process is completed after finitely many steps. Obviously $[x]\subseteq H(x)$; and this can be used to determine $[x]$ as well. A finite set of A-free generators from $T\cup\{a_\alpha^k\colon \alpha\in\lambda^*,\ k\in\omega\}$ will be called *basically* and an A-module generated by a basically set will be called *basically* as well. Hence every A-submodule of finite rank in G is contained in a basically submodule by (6.7).

We have a lemma which follows by induction from (2.1).

LEMMA 9.1. *Let $a_\alpha^k\notin H(x)$ and $x\in G$, $a\in A$, $s\in S$ such that $a_\alpha^k a\notin Gs$, then $a_\alpha^k a+x\notin Gs$.*

The main result of this section is a

THEOREM 9.2. *Every ω-complete R-submodule of G is contained in a pure A-free finitely generated A-submodule of G.*

Proof. Let U be an ω-complete R-submodule. In view of Lemma 6.7 it is enough to show that U is contained in an A-submodule of finite rank. Suppose, this is not the case. Using (6.7) we can define inductively a quadruple $(V_n, x_n, \sigma_n, s_n)$ such that

(1) (a) $V_{n-1}\subseteq V_n$ is a finite basically set.

(b) If $V^n=\langle V_n\rangle_A$, $H_n=\{H(x)\colon x\in V^n\}$ and $\{h_n^i\colon i\in\omega\}=H_n\cup\{\sigma_1,\dots,\sigma_k\}$ an enumeration with $h_n^i=\sigma_i$ $(i\leq k)$, then $h_k^i\in V_n$ for all $i,k<n$.

(c) $x_{n-1}\in V^n$.

(2) There exists $x_n\in U\setminus V^n$.

(3) Choose $v\in V^n$ such that $x_ns_n-v=x_n'\in G_\alpha$ with α minimal. Hence $\alpha-1$ exists.

Case 1. $x_n'=y+b$ with $||y||=||v_\alpha||$ and $||\sigma||>||v_\alpha||$ for $\sigma\in[b]$. Then we require $|[b]|$ to be minimal and choose $\sigma_n\in[b]^*$, $s_{n+1}\in s_nS$ such that $b\restriction\sigma_n\notin s_{n+1}A$.

Case 2. If $x_n'=y$, $||y||=||v_\alpha||$, let $y=y'+a_\alpha^k a$ with $a\neq 0$, $y'\in G_{\alpha-1}$. Choose $\sigma_n\in v_\alpha$ large enough such that $\sigma_n\notin V^n$, $\sigma_n\notin\{\sigma_1,\dots,\sigma_{n-1}\}$ and $y\restriction\sigma_n=a_\alpha^k a\restriction\sigma_n=\frac{s_{\alpha n}}{s_{\alpha k}}a\neq 0$. Choose $s_{n+1}\in s_nS$ with $a_\alpha^k a\restriction\sigma_n\notin As_{n+1}$.

The choice of σ_n in case 2 is possible because V^n is finitely generated. Since $\sigma_1,\dots,\sigma_{n-1}\in V^n$ we have also in case 1 that $\sigma_n\notin\{\sigma_1,\dots,\sigma_{n-1}\}$. Furthermore let $V_\omega=\bigcup_{n\in\omega}V_n$ and $V=\langle V_\omega\rangle_A$. Then $V^{n-1}\subseteq V^n$, $H_{n-1}\subseteq H_n$ and $H_n\subseteq V$ by (1). Hence V is "historically closed" in an obvious sense. In particular

(i) $$\sigma\in[v],\ v\in V\Rightarrow\sigma\in V.$$

(ii) $$x_ns_n\not\equiv 0 \bmod V^n+Gs_{n+1}\quad\text{for all } n\in\omega.$$

Since $x_n\in U$ and (s_n) is a divisor-chain, we find $x\in U$ such that

$$x\equiv\sum_{i<m}x_is_i \bmod Us_{n+1}$$

by ω-completeness of U. In particular, $x \equiv \sum_{i<m} x_i s_i \bmod G s_{n+1}$. Then (1)(c) is $x_i \in V^n$ $(i<n)$ which implies $x \equiv x_n s_n \bmod V^n + G s_{n+1}$. Let $x' = x+v$ $(v \in V)$ such that $x' \in G_\alpha$ with α minimal. Furthermore, if $x' = y+b$, $\|y\| \leqslant \|v_\alpha\|$ and $\|\sigma\| > \|v_\alpha\|$ for $\sigma \in [b]$, then require $|[b]|$ to be minimal. Since $v \in V^n$ for almost all $n \in \omega$, we have

(iii) $$x' \equiv x_n s_n \bmod V^n + G s_{n+1} \quad \text{for almost all } n \in \omega .$$

Let $x' = \sum_{i=1}^{k} a^k_{\alpha_i} a_i + b$ and suppose $k \geqslant 1$. If $a^k_{\alpha_1} a_1 \in \bigcap_{n\in\omega} G s_n$, we can absorb $a^k_{\alpha_1}$ into V^n. Therefore we may assume

(iv) $$a^k_{\alpha_1} a_1 \notin G s_n \quad \text{for almost all } n \in \omega,\ a^k_{\alpha_1} \notin V.$$

Suppose $a^k_{\alpha_1} \in H(v_n)$ with v_n from

(*) $$x' + v_n \equiv x_n s_n \bmod G s_{n+1} \quad \text{and } v_n \in V^n \quad \text{for almost all } n \in \omega;$$

see (iii). Then $a^k_{\alpha_1} \in V$ by (1) and $a^k_{\alpha_1} \in V^n$ for almost all $n \in \omega$. We can absorb $a^k_{\alpha_1}$ into V^n; hence $a^k_{\alpha_1} \notin H(v_n)$. From (9.1), the last remark and (iv) we derive $a^k_{\alpha_1} a_1 + v_n \notin G s_n$. A fortiori $a^k_{\alpha_1} a_1 + v_n \notin G s_{n+1}$ and trivially $a^k_{\alpha_1} a_1 - x_n s_n + v_n \in G s_{n+1}$ as well as $x' - x_n s_n + v_n \in G s_{n+1}$, which contradicts (*). Therefore $x' = b = \sum_{i=1}^{k} \varrho_i r_i$; let $r_1 = r$, $\varrho_1 = \varrho$. If $r \in \bigcap_{n\in\omega} A s_n$, we can absorb ϱr into $G s_n$; and (*) still holds. If $\varrho \in V$ we can replace x' by $x' - \varrho r$, which contradicts the minimality of $|[b]|$. Therefore

(v) $$\varrho \notin V \text{ and } r \notin A s_n \quad \text{for almost all } n \in \omega .$$

From (*) we have $x' + v_n - x_n s_n + g_n s_{n+1} = 0$ for some $g_n \in G$. Restricting to ϱ gives $\varrho r - (x_n \restriction \varrho) s_n + (g_n \restriction \varrho) s_{n+1} = 0$ which contradicts (v). Therefore $x' = 0$ and hence $0 \equiv x_n s_n \not\equiv 0 \bmod V^n + G s_{n+1}$ for almost all $n \in \omega$ by (ii) and (iii) is a contradiction. Theorem 9.2 follows. ■

DEFINITION 9.3 (cf. [DG 1,2]). An R-module is (ω-)cotorsion-free if 0 is the only ω-complete submodule.

COROLLARY 9.4. *If A^+ is cotorsion-free, then G is cotorsion-free.*

Proof. If $U \subseteq G$ is ω-complete, then $U \subseteq \bigoplus_{i=1}^{n} \alpha_i A = F$ by (9.2). Since A is cotorsion-free U must be 0.

COROLLARY 9.5. *If A_R is cotorsion-free, then* $\operatorname{End}_R G = A$.

Proof. If $\Phi \in \operatorname{Ines} G$, then $U = \operatorname{Im} \Phi$ is complete. Since G is cotorsion-free by (9.4), we have $U = 0$ by definition. Therefore $\operatorname{Ines} G = 0$ and (9.5) follows from (8.1). ■

COROLLARY 9.6. *If A_R is $\aleph_1$-free, then G is $\aleph_1$-free.*

Proof. This follows from (6.8).

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