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ON NICELY DEFINABLE FORCING NOTIONS

S. SHELAH

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Abstract. We prove that if $\mathbb{Q}$ is a nw-nep forcing then it cannot add a dominating real. We also show that amoeba forcing cannot be $\mathcal{P}(X)/I$ if I is an $\aleph_1$-complete ideal. Furthermore, we generalize the results of [12].

0. Introduction

Nicely definable forcing notions have been studied since the mid-eighties, especially for the case when "nicely definable" was interpreted as "Souslin" (see, e.g., [12], Judah and Shelah [8] or Goldstern and Judah [7]). Recently, in [14], we have initialized investigations of a wide class of "reasonably" definable forcing notions which satisfy the properness demand for countable models which are not necessarily elementary submodels of some $\mathcal{H}(\chi)$: the nep forcings. The present paper continues that research in two directions.

In the first section we introduce a very strong variant of nep forcing, where the the candidates (i.e., the models for which we postulate the existence of generic conditions) do not have to be well-founded (Definition 1.1).

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We show that those forcing notions, called nw-nep, cannot add dominating reals (Theorem 1.4). Then, by a similar proof, we show that a proper forcing notion which adds a dominating real and has sufficient amount of absoluteness for being predense, must force that $\mathfrak{b} = \aleph_1$ (see Theorem 1.6). This result applies to forcing notions like Amoeba for measure, the Hechler forcing and the Universal Meagre forcing (see Corollary 1.7), so in some sense it continues older work of Brendle, Judah and Shelah [3] and Brendle [2]. As a conclusion to 1.7 we get that a Boolean Algebra $\mathcal{P}(x)/J$, where J is a $\aleph_1$-complete ideal on x, cannot be isomorphic to the Boolean Algebra of the Amoeba forcing or the Universal Meagre forcing. This answers a question of Kamburelis (though this solution was obtained already in 1977 and discussed in [13, §4]).

In the second section we try to extend the results of [12] to nep forcing. There we showed that if a Souslin c.c.c. forcing notion $\mathbb{Q}$ adds an unbounded real, then it adds a Cohen real. Here we weaken the demands on $\mathbb{Q}$ (it is just nep c.c.c.), but the $\mathbb{Q}$-name for a Cohen real is constructed in $\mathbf{V}^{\mathbb{P}}$, where $\mathbb{P}$ is a forcing notion adding a dominating real and preserving $\mathbb{Q}$-candidates.

We refer the reader to [13, §4] and [14] for more background and references. This paper will be continued in [15].

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1. Nwnep forcing notions

In [14] we introduced nep (**n**on-**e**lementary **p**roper) forcing notions as the ones with reasonable definitions and such that the generic conditions exist over many countable models (not necessarily elementary submodels of $\mathcal{H}(\chi)$). Still, those models (called "candidates") were well-founded, see [14, §1] for details.

Here we consider a related property, allowing the candidates to be non-well founded (so the new notion has a flavour of a stronger property). The definition below is ad-hoc to simplify the presentation. The "nw" stands for "**n**on-**w**ell founded", of course.

Definition 1.1. 1) We say that N is $x-1$-nw-candidate <u>if</u>, fixing some strong limit χ, (a) or (b) holds where:

- (a) $N \prec (\mathcal{H}(\chi), \in)$ is countable, $x \in N$
- (b) for some N_1 as in (a), N is an elementary extension of N_1 not increasing ω^N; i.e., if $N \models$ "$g < \omega$" then $g \in N_1$, and if $N \models$ "y is a subset of $\mathcal{H}(\aleph_0)$" then $y = \{n \in N : N \models n \in y\}$ (so really it should have

one two-place relation E, E^N is the membership relation in N; but we shall write $N \models$ "$x \in y$").

2) We say that N is a standard $x-2$-nw-candidate if (for χ as above) (a) holds or

(b)' for some N_1 as in (a), N is a forcing extension of N_1 (and the demand in (b) on subsets of $\mathcal{H}(\aleph_0)$ holds).

3) Let $\mathbb{Q} \subseteq {}^{\omega}2$ or just $\mathbb{Q} \subseteq \mathcal{P}(\mathcal{H}(\aleph_0))$. We say that $\mathbb{Q}$ is a 1-nw-nep forcing notion if $\mathbb{Q}$ is a pair of formulas $\bar{\varphi} = (\varphi_0(x), \varphi_1(x, y))$, in the language of set theory (with parameter r) such that (below we write $\mathbb{Q}$-candidate instead of $r-1$-nw-candidate):

(a) $\varphi_0(x)$ defines a set of reals (= set of members of $\mathbb{Q}$)

(b) $\varphi_1(x, y)$ defines a set of pairs of reals, a quasi order on $\{x : \varphi_0(x)\}$, this is $\leq_{\mathbb{Q}}$

(c) φ_0, φ_1 are Σ^1_1-formulas; equivalently[1], are upward absolute from $\mathbb{Q}$-candidates, i.e., for $\mathbb{Q}$-candidates $N_1 \subseteq N_2$, we have $x \in \mathbb{Q}^{N_1} \Rightarrow x \in \mathbb{Q}^{N_2} \Rightarrow x \in \mathbb{Q}$ and $x \leq^{N_1}_{\mathbb{Q}} y \Rightarrow x \leq^{N_2}_{\mathbb{Q}} y \Rightarrow x \leq_{\mathbb{Q}} y$

(d) if N is a $\mathbb{Q}$-candidate and $p \in \mathbb{Q}^N$ (i.e. $N \models p \in \mathbb{Q}$) then there is q such that: $q \in \mathbb{Q}, p \leq_{\mathbb{Q}} q$ and q is $\langle N, \mathbb{Q}\rangle$-generic, i.e.

$$q \Vdash \text{"}\mathbb{Q}^N \cap G \text{ is a subset of } \mathbb{Q}^N, \text{ directed by } \leq^N_{\mathbb{Q}} \text{ and } G \cap \mathcal{I}^N \neq \emptyset\text{"}$$

whenever $N \models$ "$\mathcal{I} \subseteq \mathbb{Q}$ is predense".

4) We say $\mathbb{Q}$ is 2-nw-nep-forcing if above we replace $r-1$-nw-candidate by $r-2$-candidates.

5) Let nw-nep mean 1-nw-nep.

The examples of nw-nep forcing notions include all ${}^{\omega}\omega$-bounding forcing notions from [11], that is the class $\mathcal{K}$ defined in [1, Definition 0.4]. So, in particular, the Silver forcing notion and the Sacks forcing notion are nw-nep. In the realm of c.c.c. forcings, the natural examples of nw-nep are the Cohen forcing and the random real forcing. They both are nw-nep because they are very Souslin c.c.c., where:

Definition 1.2. A forcing notion $\mathbb{Q}$ is very Souslin c.c.c. if it is Souslin c.c.c. and the relation

$$\text{"}\langle r_n : n < \omega\rangle \text{ is a maximal antichain in } \mathbb{Q}\text{"}$$

is Σ^1_1.

Proposition 1.3. *Very Souslin c.c.c. forcing notions are nw-nep.*

[1] because the candidates here are not necessarily well founded

Rosłanowski and Shelah [10, 1.3.4(3), 1.5.15] give more examples of very Souslin c.c.c. (and thus also nw-nep) forcing notions.

Theorem 1.4. *Assume $\mathbb{Q}$ is nw-nep. Then forcing with $\mathbb{Q}$ does not add a dominating real.*

Proof. Toward contradiction assume $p^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta}^* \in {}^\omega\omega$ is a dominating real". Without loss of generality, $p^* \Vdash$ "$\underset{\sim}{\eta}^*$ is strictly increasing, $\underset{\sim}{\eta}^*(n) > n$". Let

$$\Gamma_0 = \{\eta \in {}^{\omega>}\omega : \eta \text{ strictly increasing and } \eta(\ell) > \ell \text{ for } \ell < \ell g(\eta)\};$$

so $p^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta}^* \in \lim(\Gamma_0)$". As $\mathbb{Q}$ is nw-nep there is p^{**} such that

$\otimes_1$ $p^* \leq_{\mathbb{Q}} p^{**}$ and for each n there is a countable $\mathcal{J}^*_n \subseteq \mathbb{Q}$ which is an antichain predense above p^{**}, such that each $p \in \mathcal{J}^*_n$ forces a value to $\underset{\sim}{\eta}^*(n)$ and is above p^* and above some $p' \in \mathcal{J}^*_m$ for each $m < n$.

[Why? Take a countable model $N \prec (\mathcal{H}(\chi), \in)$ such that $r, \underset{\sim}{\eta}^*, p^* \in N$ (so N is a $\mathbb{Q}$-candidate). Inside N by induction on $n < \omega$ we choose $\mathcal{J}^*_n \in N$ as above except countability. Let p^{**} be above p^* and be $\langle N, \mathbb{Q}\rangle$-generic.] Clearly above any $p \geq p^*$ there are two incompatible elements of $\mathbb{Q}$, so without loss of generality

$\otimes_2$ if $m < n$ and $p \in \mathcal{J}^*_m$ then there are infinitely many members of $\mathcal{J}^*_n$ which are above p.

Let Γ denote a subset of Γ_0 closed under initial segments such that $\langle\rangle \in \Gamma$ and $\eta \in \Gamma \;\Rightarrow\; (\exists^\infty n)(\eta^\frown\langle n\rangle \in \Gamma)$. We shall find $\Gamma, \bar{k}$ and choose $\bar{p}^*$ such that:

$\otimes_3$ (α) $\bar{p}^* = \langle p^*_\eta : \eta \in \Gamma\rangle$ and $p^*_\eta \in \mathbb{Q}$ (in fact $p^*_\eta \in \{p^*\} \cup \bigcup\limits_n \mathcal{J}^*_n$),

(β) $\nu \lhd \eta \Rightarrow \mathbb{Q} \models p^*_\nu \leq p^*_\eta$,

(γ) for $n \in [1, \omega)$ we have: $\langle p^*_\eta : \eta \in \Gamma \cap {}^n\omega\rangle$ is an antichain of $\mathbb{Q}$ predense above p^{**},

(δ) $\bar{k} = \langle k_\eta : \eta \in \Gamma\rangle$ where $k_\eta < \omega$,

(ε) if $\eta \in {}^{\omega>}\omega$, $\eta^\frown\langle m, n\rangle \in \Gamma$, then $p_{\eta^\frown\langle m,n\rangle}$ forces a value to $\underset{\sim}{\eta}^*(m)$, which we call $k_{\eta^\frown\langle m,n\rangle}$ and $n > k_{\eta^\frown\langle m,n\rangle} > m$,

(ζ) $p^*_{\langle\rangle} = p^*$ (= no information) and $\bar{k}_{\langle\rangle} = 1$.

So we choose $\Gamma \cap {}^n\omega$ and k_η, p_η (for $\eta \in \Gamma \cap {}^n\omega$) by induction on n with $\eta \neq \langle\rangle \Rightarrow p^*_\eta \in \{\mathcal{J}^*_m : m < \max \operatorname{Rang}(\eta)\}$.

For $n = 0$ let $p_{\langle\rangle} = p^*, k_{\langle\rangle} = 1$.

For $n = 1$ we declare that $\Gamma \cap {}^1\omega = \{\langle m\rangle : m > 0\}, \langle p^*_{\langle m\rangle} : m > 0\rangle$ is an enumeration of $\mathcal{J}^*_0$ and $k_{\langle m\rangle} = 1$.

For $n + 1, n \geq 1$, for each $\eta \in {}^n\omega$ let $m := m_\eta = \max \operatorname{Rang}(\eta)$ and let $\langle p_{\eta,j} : j < \omega\rangle$ list the members of $\mathcal{J}^*_m$ which are above p_η, so for some $k_{\eta,j}$ we have $p_{\eta,j} \Vdash_{\mathbb{Q}} \underset{\sim}{\eta}^*(m) = k_{\eta,j}$. Let $f_\eta : \omega \to \omega$ be strictly increasing such that $k_{\eta,j} < f_\eta(j)$ and $m_\eta < f_\eta(j)$ for $j < \omega$, and lastly, let $\Gamma \cap {}^{n+1}\omega = \{\eta^\frown\langle f_\eta(j)\rangle : \eta \in \Gamma \cap {}^n\omega$ and $j < \omega\}$ and $k_{\eta^\frown\langle f_\eta(j)\rangle} = k_{\eta,j}$ and $p_{\eta^1\langle f_\eta(j)\rangle} = p_{\eta,j}$.

Let $\underset{\sim}{\eta}'$ be the $\mathbb{Q}$-name of the ω-branch of Γ such that $p^{**} \Vdash_{\mathbb{Q}}$ "$p_{\underset{\sim}{\eta}' \restriction n} \in \underset{\sim}{G}_{\mathbb{Q}}$" for each $n < \omega$. We claim that:

$\boxtimes$ if $h : \Gamma \longrightarrow \omega$, then

$$p^{**} \Vdash_{\mathbb{Q}} \text{"for every large enough } n < \omega \text{ we have } \underset{\sim}{\eta}'(n) > h(\underset{\sim}{\eta}' \restriction n)\text{"}.$$

[Why? Let $f_h : \omega \longrightarrow \omega$ be

$$f_h(n) = \sup\{h(\eta) : \eta \in \Gamma \text{ and } \sup \text{Range}(\eta) \leq n\} + 1.$$

Note that the supremum is over a finite set as every $\eta \in \Gamma$ is strictly increasing. So assume $p^{**} \in G \subseteq \mathbb{Q}$, G is generic over $\mathbf{V}$, $\eta' = \underset{\sim}{\eta}'[G]$, $\eta^* = \underset{\sim}{\eta}^*[G]$, and we shall find n as required. Clearly for some $n^* > 2$ we have $m \in [n^*, \omega) \Rightarrow f_h(m) < \eta^*(m)$. We shall prove that $m \in [n^*, \omega) \Rightarrow h(\eta' \restriction (m+1)) < \eta'(m+1)$.

So assume $m \in [n^*, \omega)$, then:

$h(\eta' \restriction (m+1)) \leq f_h(\eta'(m))$ by the definition of f_h, η' being increasing,
$f_h(\eta'(m)) < \eta^*(\eta'(m))$ as $\eta'(m) \geq m \geq n^*$ (as $\eta' \restriction (m+1) \in \Gamma_0$) and the choice of n^*,
$\eta^*(\eta'(m)) = k_{\eta' \restriction (m+2)}$ by clause (ε) of $\otimes_3$ and
$k_{\eta' \restriction (m+2)} < \eta'(m+1)$ by clause (ε) of $\otimes_3$.]

For an ordinal $\alpha < \omega_1$, let $\Xi_\alpha = \{\bar\rho : \bar\rho = \langle \rho_\delta : \delta \leq \alpha$ where δ is limit$\rangle$ and each ρ_δ is a (strictly) increasing ω-sequence converging to $\delta\}$. For $\bar\rho \in \Xi_\alpha$, we define a function $g_{\bar\rho}$ from Γ to $\alpha+1$, defining $g_{\bar\rho}(\eta)$ by induction on $\lg(\eta)$ as follows:

(B) (a) $g_{\bar\rho}(\langle\rangle) = \alpha$,
(b) if $g_{\bar\rho}(\eta) = \beta + 1$ and $\eta^\frown\langle\ell\rangle \in \Gamma$, <u>then</u> $g_{\bar\rho}(\eta^\frown\langle\ell\rangle) = \beta$,
(c) if $g_{\bar\rho}(\eta) = \delta$, δ a limit ordinal and $\eta^\frown\langle\ell\rangle \in \Gamma$, <u>then</u> $g_{\bar\rho}(\eta^\frown\langle\ell\rangle) = \rho_\delta(\ell)$,
(d) if $g_{\bar\rho}(\eta) = 0$ and $\eta^\frown\langle\ell\rangle \in \Gamma$ then $g_{\bar\rho}(\eta^\frown\langle\ell\rangle) = \alpha$.

Let $A_{n,\bar\rho} = \{\eta \in \Gamma : g_{\bar\rho}(\eta) = \alpha$ and $|\{\ell < \lg(\eta) : g_{\bar\rho}(\eta \restriction \ell) = \alpha\}| = n\}$, so $A_{n,\bar\rho}$ is a front of Γ, and it is above $A_{m,\bar\rho}$ for $m < n$. Hence for each m we have $p^{**} \Vdash$"$\underset{\sim}{\eta}'$ has an initial segment in $A_{m,\bar\rho}$". Let a $\mathbb{Q}$-name $\underset{\sim}{h}_{\bar\rho}$ of a function from ω to ω be such that $v \in A_{n,\bar\rho} \Rightarrow p^*_v \Vdash \underset{\sim}{h}_{\bar\rho}(n) = \underset{\sim}{\eta}^*(\sup \text{Rang}(v))$. Clearly $\mathcal{I}_{n,\bar\rho} = \{p^*_\eta : \eta \in A_{n,\bar\rho}\}$ is predense above p^{**}, so [2] $p^{**} \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{h}_{\bar\rho} \in {}^\omega\omega$".

Now there is a $\mathbb{Q}$-candidate N, with $(\omega_1)^N$ not well ordered, and $p^*, p^{**}\bar p^* \in N$ such that $N \models$ "all the above statements on $p^{**}, \bar\rho \in \Xi_\alpha$ for every countable ordinal α".

[2] the $\underset{\sim}{h}_{\bar\rho}$-s are here for clarification only, but they will be necessary in the proof of 1.6 below.

[Why? Let $\mathcal{L}$ be a countable fragment of $\mathbb{L}_{\omega_1,\omega}(\{\in\})$ and let $N_1 \prec_{\mathcal{L}} (\mathcal{H}(\chi), \in)$ be such that $\mathbb{Q}, p^{**}, \bar{p}^* \in N_1$ and $N_1 \prec_{\mathcal{L}} N, N$ as above; $\omega^N = \omega^{N_1}$ as in the Definition 1.1. See Keisler [9]; alternatively, for a model N_1 with Skolem functions define

$$\begin{aligned} \mathbb{P}^{N_1} = \{Y : & \text{ for some } n = n(Y), Y \text{ is a set of decreasing sequences of} \\ & \text{countable cardinals of length } n \text{ such that if} \\ & n(Y) > 0 \text{ then } (\forall \alpha < \omega_1)(\exists \eta \in Y)(\alpha < \min \ \mathrm{Rang}(\eta))\} \\ & Y_1 \leq Y_2 \ \underline{\text{iff}}\ n(Y_1) \leq n(Y_2) \ \&\ (\forall \eta \in Y_2)(\exists \nu \in Y_1)(\nu \trianglelefteq \eta). \end{aligned}$$

If $G \subseteq \mathbb{P}^{N_1}$ is generic over N_1, we define N such that $N_1 \prec N, \alpha_n \in N, N = \mathrm{Sk}(N_1 \cup \{\alpha_n : n < \omega\})$ and $Y \in G \Rightarrow N \models \langle \alpha_0, \ldots, \alpha_{n(Y)-1} \rangle \in Y$.]

Choose α_n^* for $n < \omega$ such that $N \models$ "$\alpha_{n+1}^* < \alpha_n^*$ are countable ordinals". Without loss of generality $N \models$ "α_n^* is a limit ordinal", and so for some $\bar{\rho}$ we have $N \models$ "$\bar{\rho} \in \Xi_{\alpha_0^*}$". So clearly $N \models$ "$\mathcal{I}_{n,\bar{\rho}}$ is predense above p^{**}" for each n. By 1.1(3)(d), there is $r^* \in \mathbb{Q}$ above p^{**} which is $\langle N, \mathbb{Q} \rangle$-generic. Hence

$\boxtimes_1$ $\mathcal{I}_{n,\bar{\rho}}^N$ is predense above r^* for each $n < \omega$.

There is in $\mathbf{V}$ (not in N!) a function $f_0 : \Gamma \longrightarrow \omega$ such that, for $\eta \in \Gamma$

$$\bigvee_{n<\omega} \alpha_n^* \leq^N g_{\bar{\rho}}(\eta) \Rightarrow (\forall k)(k \geq f_0(\eta) \ \&\ \eta^\frown\langle k \rangle \in \Gamma \rightarrow \bigvee_{n<\omega} \alpha_n^* \leq^N g_{\bar{\rho}}(\eta^\frown\langle k \rangle)).$$

By $\boxtimes$ (as $p^* \leq_{\mathbb{Q}} r^*$) there are q and $\ell_0 < \omega$ such that $r^* \leq_{\mathbb{Q}} q$ and $q \Vdash_{\mathbb{Q}}$ "for every $\ell > \ell_0$, $\underset{\sim}{\eta}'(\ell) > f_0(\underset{\sim}{\eta}' \restriction \ell)$". So for some ℓ_1, q forces that

$$\ell_1 < \ell < \omega \ \Rightarrow\ \bigvee_n \alpha_n^* <^N g_{\bar{\rho}}(\underset{\sim}{\eta}' \restriction \ell)^N \ \Rightarrow\ (\underset{\sim}{\eta}' \restriction \ell) \notin \bigcup_n A_{n,\bar{\rho}}^N.$$

Hence

$\boxtimes_2$ $q \Vdash_{\mathbb{Q}}$ "the number of n such that $(\exists \ell)(\underset{\sim}{\eta}' \restriction \ell \in A_{n,\bar{\rho}}^N)$ is finite".

But $\boxtimes_1 + \boxtimes_2$ gives a contradiction. $\square_{1.4}$

Definition 1.5. Let $\mathbb{Q}$ be a forcing notion, $\mathcal{I}, \mathcal{J} \subseteq \mathbb{Q}$. We say that "$\mathcal{I}$ is predense above $\mathcal{J}$" whenever

($*$) if $p \in \mathbb{Q}$ is above every $q \in \mathcal{J}$ <u>then</u> p is compatible with some $r \in \mathcal{I}$.

Theorem 1.6. *Assume that*

(a) *$\mathbb{Q}$ is a (definable) forcing notion with set of elements $\subseteq {}^\omega 2$ which is proper (or at least the old countable sets of ordinals are cofinal in the new)*
(b) *$\underset{\sim}{\eta}^* \in {}^\omega\omega$ is a $\mathbb{Q}$–name,*
(c) *$p^* \in \mathbb{Q}$ forces that $\underset{\sim}{\eta}^*$ is a dominating real,*

(d) *"$\{p_1, \dots, p_n\}$ is predense over $\{q, q_1\}$, $n < \omega$" as well as "$p \in \mathbb{Q}$", "$p \leq_{\mathbb{Q}} q$" are upward absolute from $\mathbb{Q}$-candidates in the sense of Definition 1.1(1), so not necessarily well founded (i.e., these formulas are Σ^1_1).*

Then p^* forces that $\mathfrak{b} = \aleph_1$ in $\mathbf{V}^{\mathbb{Q}}$.

Proof. It is enough to prove that some condition above p^* forces $\mathfrak{b} = \aleph_1$.

By the properness of $\mathbb{Q}$ (or just assumption (a)) without loss of generality there are p^{**} and $\langle \mathcal{J}^*_n : n < \omega \rangle$ as in the beginning of 1.4. Let $\underset{\sim}{\eta}^*, \Gamma, \bar{p}^* = \langle p^*_\eta : \eta \in \Gamma \rangle, \underset{\sim}{\eta}', \Xi_\alpha$ for $\alpha < \omega_1$ and $g_{\bar{\rho}}, \underset{\sim}{h}_{\bar{\rho}}$ for $\bar{\rho} \in \Xi_\alpha, \alpha < \omega_1$ be as in the proof of 1.4. For limit $\delta < \omega_1$ choose $\bar{\rho}_\delta \in \Xi_\delta$ so that $\bar{\rho}_{\delta_0} = \bar{\rho}_{\delta_1} \restriction (\delta_0 + 1)$ whenever $\delta_0 < \delta_1 < \omega_1$ are limit. We are going to show that

$$p^{**} \Vdash \text{“}\{\underset{\sim}{h}_{\bar{\rho}_\delta} : \delta < \omega_1 \text{ limit}\} \text{ is not bounded”},$$

what will complete the proof. So assume not, hence for some $\underset{\sim}{h}^*$ and q^*, we have $p^{**} \leq_{\mathbb{Q}} q^*$ and

$$q^* \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{h}^* \in {}^\omega\omega \text{ dominates } \{\underset{\sim}{h}_{\bar{\rho}_\delta} : \delta < \omega_1 \text{ limit}\}\text{”}.$$

Without loss of generality $\underset{\sim}{h}^*$ is a hc (hereditarily countable) $\mathbb{Q}$-name above q^*, more specifically, for each $n < \omega$ we have an antichain $\langle r^*_{n,\ell} : \ell < \omega \rangle$ of $\mathbb{Q}$ predense over q^* and such that $r^*_{n,\ell} \Vdash_{\mathbb{Q}}$"$\underset{\sim}{h}^*(n) = k_{n,\ell}$". So $\underset{\sim}{h}^*$ is

$$\langle (n, \ell, \underset{\sim}{r}^*_{n,\ell}, k_{n,\ell}) : n, \ell < \omega \rangle.$$

Choose a countable model $M \prec (\mathcal{H}(\chi), \in)$ such that $\underset{\sim}{h}^*, p^*, p^{**}, \bar{p}^*, \langle \bar{\rho}_\delta : \delta < \omega_1 \text{ limit} \rangle \in M$, and choose a countable elementary extension N of M such that in N there are α^*_n for $n < \omega$ as in the proof of 1.4.

So in $N, \bar{\rho} = \bar{\rho}^N_{\alpha^*_0}$ is well defined, so as $M \prec N$ there are n^*, r^* such that

$(*)_0$ $N \models$ "$r^* \in \mathbb{Q}$ and $q^* \leq_{\mathbb{Q}} r^*$ and $n^* < \omega$ and $r^* \Vdash_{\mathbb{Q}}$"$\underset{\sim}{h}_{\bar{\rho}} \restriction [n^*, \omega) < \underset{\sim}{h}^* \restriction [n^*, \omega)$"".

Let $n \in [n^*, \omega)$ and $\ell < \omega$. Recalling that (in $\mathbf{V}$ hence in M hence in N) we have $r^*_{n,\ell} \Vdash_{\mathbb{Q}}$"$\underset{\sim}{h}^*(n) = k_{n,\ell}$" and recalling that every $\eta \in \Gamma$ is strictly increasing and the definition of $h_{\bar{\rho}}$ in N, clearly

$(*)^1_{n,\ell}$ the set $A_{n,\ell} = \{\nu \in A^N_{n,\bar{\rho}}\colon \max \operatorname{Rang}(\nu) < k_{n,\ell}\}$ is finite.

Also, by $(*)_0$,

$(*)^2_{n,\ell}$ in N the set $\{p^*_\nu : \nu \in A_{n,\ell}\}$ is predense (in $\mathbb{Q}^N$) above $\{r^*, r^*_{n,\ell}\}$.

But by the clause (d) of the assumption this amount of predensity is upward absolute (from N to $\mathbf{V}$) hence

$(*)^3_{n,\ell}$ in $\mathbf{V}$ the set $\{p^*_\nu : \nu \in A_{n,\ell}\}$ is predense (in $\mathbb{Q}$) above $\{r^*, r^*_{n,\ell}\}$.

But $q^* \leq_{\mathbb{Q}} r^*$ and $\{r^*_{n,\ell} : \ell < \omega\}$ is predense (in $\mathbb{Q}$) above q^*, hence

$(*)^4_n$ for each $n < \omega$ in $\mathbf{V}$ the set $\bigcup_{\ell<\omega}\{p^*_\nu : \nu \in A_{n,\ell}\}$ is predense in $\mathbb{Q}$ above r^*.

Now $\bigcup_{\ell<\omega}\{p^*_\nu : \nu \in A_{n,\ell}\} \subseteq \{p^*_\nu : \nu \in A^N_{n,\bar\rho}\} = \mathcal{I}^N_{n,\bar\rho}$. Hence

$(*)_5$ for every $n, \mathcal{I}^N_{n,\bar\rho}$ is predense in $\mathbb{Q}$ above r^*.

This means that in the proof of 1.4 the statement $\boxtimes_1$ holds and continues as in the proof of 1.4. $\square_{1.6}$

Corollary 1.7. *Amoeba forcing forces $\mathfrak{b} = \aleph_1$, and similarly dominating real forcing (= Hechler forcing) and universal meagre forcing.*

Proof. We will apply 1.6. The amoeba forcing $\mathbb{Q}$ is

$$\{T \subseteq {}^{\omega>}2 : T \text{ is non empty closed under initial segments and } \mathrm{Leb}(\lim(T)) > 1/2\},$$

ordered by inverse inclusion; note that for notational simplicity we allow trees with maximal nodes.

Clearly "$p \in \mathbb{Q}$","$p \le_{\mathbb{Q}} q$" are Borel relations and any $p, q \in \mathbb{Q}$ has a l.u.b.: $p \cap q$ and "p, q are compatible" is Borel. The main point is to show that "$\{p_\ell : \ell < n\}$ is predense above $\{q_1, q_2\}$" is upward absolute for nw-candidates; we can replace $\{q_1, q_2\}$ by $\{q\}$ where $q = q_1 \cap q_2$. Suppose that $0 < m, k < \omega$ and for $s \subseteq q_1 \cap q_2 \cap {}^m2$ define:

$$a_m(s) = \mathrm{Max}\{|s \cap p_\ell|/2^m : \ell < n\},$$

this is a real number $\in [0, 1]$, and we let

$$a_{m,k} = \mathrm{Min}\{a_m(s) : s \subseteq q_1 \cap q_2 \cap {}^m2 \text{ and } |s|/2^m \ge \frac{1}{2} + \frac{1}{k}\}.$$

We shall show that the following statements are equivalent:

(α) there is $r \in \mathbb{Q}$ above q incompatible with $p_0, \dots, p_{n-1}$

(β) for some $r \in \mathbb{Q}$ we have $\mathrm{Leb}(\lim(p_\ell \cap r)) \le \frac{1}{2} - \frac{1}{k}$ for $\ell < n$ and $\mathrm{Leb}(\lim(r)) > \frac{1}{2} + \frac{1}{k}$ for some $k \in (0, \omega)$

(γ) $\limsup\langle a_{m,k} : m < \omega\rangle \le \frac{1}{2} - \frac{1}{k}$ for some $k \in (0, \omega)$.

If (α) holds, let r exemplify it, so for some $\varepsilon_1 > 0$, $\mathrm{Leb}(\lim(r)) > 1/2 + \varepsilon_1$, and $\mathrm{Leb}(\lim(p_\ell \cap r)) \le 1/2$ for $\ell < n$. We can find, for $\ell < n$, a clopen subset B_ℓ of ${}^\omega2$ such that $\mathrm{Leb}(\lim(p_\ell \cap r) \cap B_\ell) > 0$, $\mathrm{Leb}(B_\ell) < \varepsilon_1/(n + 3)$. Let $r' = \{\eta \in r$: there is $\rho \in {}^\omega2 \setminus \bigcup_\ell B_\ell$ above $\eta\}$, and k be large enough, they exemplify (β).

If (β) holds, exemplified by r, k, then $\ell < n \Rightarrow$ $\mathrm{Leb}(\lim(p_\ell \cap r)) \le 1/2 - 1/k$. Hence by the definition of Lebesgue measure

$(*)$ $\dfrac{1}{2} - \dfrac{1}{k} \ge \lim\langle |p_\ell \cap r \cap {}^m 2|/2^m : m < \omega\rangle$ for each $\ell < n$,

and hence

$$\frac{1}{2} - \frac{1}{k} \ge \lim\langle \operatorname*{Max}_{\ell<n} |p_\ell \cap r \cap {}^m 2|/2^m : m < \omega\rangle.$$

But $a_{m,k} \le a_m(r \cap {}^m 2)$ because $|r \cap {}^m 2|/2^m \ge \mathrm{Leb}(\lim(r)) \ge 1/2 + 1/k$ and $a_m(r \cap {}^m 2) = \operatorname*{Max}_{\ell<n}(|p_\ell \cap r \cap {}^m 2|/2^m)$.

Putting together those inequalities and $(*)$ we have $1/2 - 1/k \ge \limsup\langle a_{m,k} : m < \omega\rangle$ as required, so (γ) holds, i.e. we have proved $(\beta) \Rightarrow (\gamma)$.

Lastly, assume (γ) and we shall prove (α). For each m let $s_m \subseteq q \cap {}^m 2$ be such that $a_m(s_m) = a_{m,k}$ and $|s_m|/2^m \ge 1/2 + 1/k$. Let m be large enough such that $a_{m,k} < 1/2 - 1/(4k)$ and $|q \cap {}^m 2|/2^m -$ $\mathrm{Leb}(\lim(q)) < 1/4k$. Let $r = \{\rho \in q$: if $\ell g(\rho) \ge m$ then $\rho \restriction m \in s_m\}$. Clearly $r \subseteq q$ is a subtree, and

$$\begin{aligned}
\mathrm{Leb}(\lim(r)) \ge\ & \mathrm{Leb}(\lim(q)) - \mathrm{Leb}\{\eta \in {}^\omega 2 : \eta \in \lim(q) \text{ but } \eta \restriction m \notin s_m\} \\
\ge\ & \mathrm{Leb}(\lim(q)) - (|q \cap {}^m 2|/2^m - |s_m|/2^m) \\
\ge\ & \mathrm{Leb}(\lim(q)) - ((\mathrm{Leb}(\lim(q)) + 1/4k) - |s_m|/2^m) \\
=\ & |s_m|/2^m - 1/4k \ge \frac{1}{2} + \frac{1}{k} - \frac{1}{4k} > \frac{1}{2}.
\end{aligned}$$

So $r \in \mathbb{Q}$, also for $\ell < n$, the conditions r, p_ℓ are incompatible as $\mathrm{Leb}(\lim(p_\ell \cap r)) \le$ $\mathrm{Leb}(\lim(\{\eta \in p_\ell\colon$ if $\ell g(\eta) \ge m$ then $\eta \restriction m \in s_m\})) \le$ $\mathrm{Leb}(\{\eta \in {}^\omega 2 : \eta \restriction m \in p_\ell \cap {}^m 2 \cap s_m\}) = |p_\ell \cap s_m|/2^m \le a_m(s_m) < 1/2 - 1/(4k) < 1/2$.

So we have finished proving $(\gamma) \Rightarrow (\alpha)$ hence proving $(\alpha) \Leftrightarrow (\beta) \Leftrightarrow (\gamma)$. $\square_{1.7}$

Conclusion 1.8. 1) *There is no $\aleph_1$-complete ideal J on a set X such that the Boolean algebra $\mathcal{P}(X)/J$ isomorphic to the Boolean algebra of the Amoeba forcing (or any other c.c.c. forcing satisfying the assumption of* 1.6*).*

2) *The following is impossible*

- (a) *J is a $(<\kappa)$-complete ideal on a set X, and*
- (b) *$\mathcal{P}(X)/J$ is isomorhpic to the Boolean algebra of the forcing notion $\mathbb{Q}$ which satisfies the κ^+-c.c.,*
- (c) *forcing with $\mathbb{Q}$ adds a dominating real, and*
- (d) *forcing with $\mathbb{Q}$ makes $\mathfrak{b} \le \kappa$ (κ as an ordinal).*

Proof. 1) Follows by part (2) for $\kappa = \aleph_1$ below and Corollary 1.7.
2) The proof is close to [4, 3.1] and [5], but we give a self contained proof.

Let κ_1 be maximal such that J is $(<\kappa_1)$-complete, so J is not $(<\kappa_1^+)$-complete, now replacing κ by κ_1, clauses (a)-(d) are still satisfied, so without loss of generality J is not $(<\kappa^+)$-complete.

Let $g : \mathbb{Q} \to \mathcal{P}(X)/J$ be a dense embedding (remember assumption (b) inverting the order).

Let $G \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$, and define $\underset{\sim}{D}[G] = \{Y \subseteq X : g(p) \subseteq Y \bmod J$ for some $p \in G\}$. Then $\underset{\sim}{D}[G]$ is an ultrafilter on X, i.e., on the Boolean algebra $\mathcal{P}(X)^{\mathbf{V}}$ disjoint to J. As $\mathcal{P}(X)/J$ satisfies the κ^+-c.c. and J is κ-complete (in $\mathbf{V}$) clearly the ultrapower $\mathbf{V}^X/\underset{\sim}{D}[G] = \{f/\underset{\sim}{D}[G] : f \in {}^X\mathbf{V}$ is from $\mathbf{V}\}$ is well founded so we identify it with its Mostowski collapse M. Let $\mathbf{j}$ be the natural elementary embedding of $\mathbf{V}$ into M. Clearly in $\mathbf{V}[G]$ the model M is closed under taking sequences of length $\leq \kappa$. In particular M contains all ω-sequences of natural numbers from $\mathbf{V}[G]$ hence $({}^\omega\omega)^M = ({}^\omega\omega)^{\mathbf{V}}[G]$.

As M contains all $(\leq \kappa)$-sequences of reals from $\mathbf{V}[G]$ and $\mathbf{V}[G] \models \mathfrak{b} < \kappa$, clearly in $\mathbf{V}[G]$ there are $\theta \leq \kappa$ and a sequence $\bar{f} = \langle f_\alpha : \alpha < \theta\rangle$ exemplifying $\mathfrak{b} = \theta \leq \kappa$, hence $\bar{f} \in M$. So necessarily $M \models$ "$\mathfrak{b} \leq \theta$" but $M \models$ "$\theta \leq \kappa < \mathbf{j}(\kappa)$" hence by Łoś theorem also $\mathbf{V} \models \mathfrak{b} < \kappa$. So let $\bar{f}'$ be such that $\mathbf{V} \models$ "$\bar{f}' = \langle f'_\alpha : \alpha < \theta\rangle$ exemplifies $\mathfrak{b} \leq \theta$", hence as $\theta < \kappa$, clearly $\mathbf{j}(\bar{f}) = \bar{f}$, so $\{f_\alpha : \alpha < \theta\} \subseteq ({}^\omega\omega)^M$ is unbounded in $({}^\omega\omega)^M$ but the latter is $({}^\omega\omega)^{\mathbf{V}[G]}$ in which there is a $\underset{\sim}{\eta} \in {}^\omega\omega$ dominating $({}^\omega\omega)^{\mathbf{V}}$ hence $\bar{f}$, a contradiction. $\square_{1.8}$

Remark 1.9. 1) Suppose that κ is a measurable cardinal and force with FS iteration of the Hecher forcing notions, κ in length, and then consider $\mathbb{Q} = \mathcal{P}(\kappa)/J$. Then $\Vdash_{\mathbb{Q}}$ "$\mathfrak{b} = \mathfrak{d} = \lambda$" where $\lambda = \operatorname{cf}(\kappa^\kappa/D)$ in $\mathbf{V}$.

The aim of the series of papers [4], [5], [6] is to show that the general situation is similar to this.

2) The original aim of 1.6 was to deal with c.c.c. simply defined forcing notions. For this the demands on $\mathbb{Q}$ in 1.6 seem to be reasonable.

2. Around "adding a Cohen real"

In [12] we have proved that if a Souslin-c.c.c. forcing notion $\mathbb{Q}$ adds an unbounded real, <u>then</u> it adds a Cohen real. Here, we try to extend the result to nep forcing. The proof here does not rely on [12] (and the results imply the results there).

More fully we use the following: let N be a countable elementary submodel of $(\mathcal{H}(\chi), \in)$ to which $\mathbb{P}, \mathbb{Q}$ belongs, G a subset of $\mathbb{P}^N$ generic over N <u>then</u> $N[G]$ is a $\mathbb{Q}$-candidate. It may be clearer to let M be the ordinal collapse of $N, \mathbf{j} : N \to M$ the isomorphism and demand $M[\mathbf{j}''G]$ is a $\mathbb{Q}$-candidate.

Definition 2.1. 1) Let $\mathfrak{K}$ be a class of countable submodels of $(\mathcal{H}(\chi), \in)$, all of a large part of ZFC, and let $\mathbb{Q}$ be a forcing notion with set of elements $\subseteq {}^{\omega}2$. We say that $\mathbb{Q}$ if $\mathfrak{K}$-nep <u>if</u> for some pair $\bar{\varphi} = (\varphi_0, \varphi_1)$ of Σ^1_1-formulas with a parameter $r \in {}^{\omega}2$ we have

(a) the set of elements of $\mathbb{Q}$ and $\leq^{\mathbb{Q}}$ are defined by $\varphi_0(x), \varphi_1(x, y)$,

(b) if $N \in \mathfrak{K}, \bar{\varphi} \in N$ and $p \in \mathbb{Q}^N$ <u>then</u> for some $q \in \mathbb{Q}$ we have

  (α) $\mathbb{Q} \models p \leq q$,

  (β) q is $(N, \mathbb{Q})$-generic, which means that for every $\mathcal{I}^* \in \operatorname{pd}(N, \mathbb{Q}) = \{\mathcal{I} \in N : N \models \text{“}\mathcal{I} \text{ is predense in } \mathbb{Q}\text{”}\}$, the set $\{r : N \models \text{“}r \in \mathcal{I}^*\text{”}\}$ is predense in $\mathbb{Q}$ above q and $q \Vdash \text{“}G \cap \mathbb{Q}^N \text{ is } \leq_{\mathbb{Q}^N}\text{-directed”}$.

2) Let $\mathbb{P}$ be a proper forcing notion. We define the class $\mathfrak{K}_{\mathbb{P}}$ as the collection of all countable models N such that

(a) either $N \prec (\mathcal{H}(\chi), \in)$,

(b) or for some countable model $M \prec (\mathcal{H}(\chi), \in)$ such that $\mathbb{P} \in M$ and some generic $G \subseteq \mathbb{P} \cap M$ over M we have $N = M[G]$.

Proposition 2.2. 1) *Assume*

(a) $\mathbb{P}$ is a proper forcing notion, $\mathbb{Q}$ is a $\mathfrak{K}_{\mathbb{P}}$-nep-forcing which is c.c.c., and

(b) $\Vdash_{\mathbb{Q}}$ “$\underset{\sim}{f} \in {}^{\omega}\omega$ is not dominated by any old $f \in {}^{\omega}\omega$”, and

(c) $\mathbb{P}$ adds a dominating real $\underset{\sim}{g} \in {}^{\omega}\omega$.

<u>Then</u> in $\mathbf{V}^{\mathbb{P}}$

$\circledast_1$ forcing with $\mathbb{Q}$ adds a Cohen real.

2) *Assume that we replace clause (b) above by*

(b)′ $\Vdash_{\mathbb{Q}}$ “$\underset{\sim}{\eta} \in {}^{\omega}2$ is not equal to any old member of ${}^{\omega}2$”.

<u>Then</u>, in $\mathbf{V}^{\mathbb{P}}$,

$\circledast_2$ there is a strictly increasing sequence $\langle n_i : i < \omega\rangle$ such that for every $q^* \in \mathbb{Q}$ for all $i < \omega$ large enough:
$2^i \leq |\{\eta \in {}^{n_i}2\text{: some } q' \text{ above } q^* \text{ forces that } \eta = \underset{\sim}{\eta} \restriction n_i\}|$.

Proof. Of course, it is enough to prove that for a dense set of $q \in \mathbb{Q}$, the result holds above q. For part (1) let $\mathbf{t}$ be 1, $\underset{\sim}{f}^{\mathbf{t}} = \underset{\sim}{f}$ and for part (2) let $\mathbf{t} = 2$ and $\underset{\sim}{f}^{\mathbf{t}} = \underset{\sim}{\eta}$. So $\underset{\sim}{f}^{\mathbf{t}}$ is actually $\langle (r^*_{n,\ell}, k_{n,\ell} : n < \omega, \ell < \omega\rangle$ where $r^*_{n,\ell} \Vdash$ “$\underset{\sim}{f}^{\mathbf{t}}(n) = k_{n,\ell}$” and, for each $n < \omega$, $\langle r^*_{n,\ell} : \ell < \omega\rangle$ is a maximal antichain of $\mathbb{Q}$; similarly for the $\mathbb{P}$-name $\underset{\sim}{g}$ as we can replace $\mathbb{P}$ by $\mathbb{P}_{\geq p}$ and $\mathbb{P}$ is proper. Without loss of generality $\underset{\sim}{f}^1, \underset{\sim}{g}$ are (forced to be) strictly increasing; note that for $\underset{\sim}{f}^1$ this just means that

$$n_1 < n_2 \;\&\; k_{n_1,\ell_1} \geq k_{n_2,\ell_2} \Rightarrow (r^*_{n_1,k_1}, r^*_{n_2,k_2} \text{ are incompatible})$$

so it is absolute enough. Suppose $q^* \in \mathbb{Q}$. Let (χ be strong limit and) $N \prec (\mathcal{H}(\chi), \in)$ be a countable model such that $\{\mathbb{P}, \underset{\sim}{g}, q^*, \mathbb{Q}\} \in N$ and $\underset{\sim}{f}^{\mathbf{t}} \in N$, i.e., $\langle (r^*_{n,\ell}, k_{n,\ell}) : n < \omega, \ell < \omega \rangle$ belongs to N. Now obviously

$(*)_1$ $N \models$ "$\mathbb{P}$ is a forcing notion, $\underset{\sim}{g}$ is a $\mathbb{P}$-name of an increasing member of ${}^\omega\omega$ dominating all old ones".

Observe:

$(*)_2$ if $M \in \mathfrak{K}_{\mathbb{P}}, \bar{r} = \langle r_\ell : \ell < \omega \rangle$ is a maximal antichain of $\mathbb{Q}$ and $\bar{r} \in M, r_\ell \in \mathbb{Q}^M$, <u>then</u> $M \models$ "$\bar{r}$ is a maximal antichain of $\mathbb{Q}$".

[Why? First, if $n < m < \omega, M \models$ "r_n, r_m are compatible in $\mathbb{Q}$" let $r \in \mathbb{Q}^M$ be a common upper bound by $\leq^M_{\mathbb{Q}}$, it is a common upper bound in $\mathbb{Q}$, contradiction. Second, if $M \models$ "$q \in \mathbb{Q}$ is incompatible with each r_ℓ", let $q_1 \in \mathbb{Q}$ be $(M, \mathbb{Q})$-generic such that $q \leq q_1$. But q_1 is necessarily $\leq_{\mathbb{Q}}$-compatible with r_n for some n so for some q_2 we have $r_n \leq_{\mathbb{Q}} q_2$ & $q_1 \leq_{\mathbb{Q}} q_2$, so $q_2 \Vdash$ "$\{q, r_n\} \subseteq \underset{\sim}{G}_{\mathbb{Q}} \cap M$". However $q_1 \leq_{\mathbb{Q}} q_2$ and $q_1 \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{G}_{\mathbb{Q}} \cap \mathbb{Q}^N$ is $\leq^M_{\mathbb{Q}}$-directed", a contradiction.]

Continuing $(*)_1$, for $\mathbf{t} = 1$:

$(*)_3$ $N \models$ "forcing with $\mathbb{P}$ preserves the property of $(\mathbb{Q}, \underset{\sim}{f}^{\mathbf{t}})$, i.e., $\underset{\sim}{f}^{\mathbf{t}}$ not dominated".

[Why? First being a $\mathbb{Q}$-name of a member of ${}^\omega\omega$ is preserved after forcing with $\mathbb{P}$ by $(*)_2$+ assumption (d): consider a generic $G \subseteq \mathbb{P}^N, G \in \mathbf{V}$ over N and let [3] $M = N[G]$ - it belongs to $\mathfrak{K}_{\mathbb{P}}$ and we can apply $(*)_2$. Second, assume toward contradiction that $(*)_3$ fails. Let $p^* \in \mathbb{P}^N$ force the negation (in N) and choose, in $\mathbf{V}$, a set $G \subseteq \mathbb{P}^N$ generic over N to which p^* belongs so $N \subseteq N[G] \in \mathbf{V}, N[G] \in \mathfrak{K}_{\mathbb{P}}$. As the conclusion of $(*)_3$ fails we can find $q_1 \in \mathbb{Q}^{N[G]}$ and $h \in ({}^\omega\omega)^{N[G]}$ such that $N[G] \models$ "$q_1 \in \mathbb{Q}, q^* \leq_{\mathbb{Q}} q_1$ and q_1 forces ($\Vdash_{\mathbb{Q}}$) that $\underset{\sim}{f}^{\mathbf{t}} \leq h$". Let q_2 be $(N[G], \mathbb{Q})$-generic condition satisfying $q_1 \leq_{\mathbb{Q}} q_2$ hence $q_2 \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{f}^{\mathbf{t}} \leq h \in {}^\omega\omega$", contradicting the choice of $q^*, \underset{\sim}{f}$.]

$(*)_4$ Also in $\mathbf{V}^{\mathbb{P}}, \underset{\sim}{f}^{\mathbf{t}}$ is not dominated if $\mathbf{t} = 1$.

[Why? As $N \prec (\mathcal{H}(\chi), \in)$.]

Without loss of generality

$(*)_5$ for every $h : {}^{\omega >}\omega \to \omega$ from $\mathbf{V}$ we have $\Vdash_{\mathbb{P}} (\forall^\infty n)\underset{\sim}{g}(n) > h(\underset{\sim}{g} \restriction n)$.

[Why? As, e.g., we can replace $\underset{\sim}{g}$ by $\underset{\sim}{\nu}$, where $\underset{\sim}{\nu}(0) = \underset{\sim}{g}(0)$ and $\underset{\sim}{\nu}(n+1) = \underset{\sim}{g}(\underset{\sim}{\nu}(n) + 1)$, note that $\underset{\sim}{\nu}$ is strictly increasing as $\underset{\sim}{g}$ is.]

Let the $\mathbb{Q}$-name $\underset{\sim}{\eta}^1 \in {}^\omega 2$ be such that for every $\ell < \omega$ we have $\underset{\sim}{\eta}^1(\ell) = 1 \Leftrightarrow \ell \in \text{Rang}(\underset{\sim}{f}^1)$ and let $\underset{\sim}{\eta}^2 = \underset{\sim}{\eta}$.

$(*)_6$ $N \models$ " $\Vdash_{\mathbb{P}} \Vdash_{\mathbb{Q}} \underset{\sim}{\eta}^{\mathbf{t}} \in {}^\omega 2$ is new".

[Why? For $\mathbf{t} = 1$ by $(*)_3$, for $\mathbf{t} = 2$ even easier immitating the proofs of $(*)_3$ and $(*)_4$.]

[3] can use the ord collapse of M

For $q \in \mathbb{Q}$ let

$$T_q[\underset{\sim}{\eta}^{\mathbf{t}}] = \{\nu \in {}^{\omega>}2 : q \nVdash_{\mathbb{Q}} \nu \ntrianglelefteq \underset{\sim}{\eta}^{\mathbf{t}}\},$$

so

$(*)_7$ $T_q[\underset{\sim}{\eta}^{\mathbf{t}}]$ is a non-empty subtree of ${}^{\omega>}2$ with no maximal nodes and no isolated ω-branches,

$(*)_8$ if $\mathbf{t} = 1$, then in $\mathbf{V}$ and also in $N[G]$ for each generic $G \subseteq \mathbb{P}^N$ over N, for every $q \in \mathbb{Q}$

(a) for some $n_* < \omega$:
for every $m \in (n_*, \omega)$ we have

$$q \nVdash_{\mathbb{Q}} (\exists \ell)(n_* \leq \ell < m \;\&\; \underset{\sim}{\eta}^{\mathbf{t}}(\ell) = 1);$$

therefore

(b) for every $m > n_*$ there is an $\nu \in {}^m2 \cap T_q[\underset{\sim}{\eta}^{\mathbf{t}}]$ such that for every $\ell \in [n_*, m)$, we have $\nu(\ell) = 0$,

(c) for some $\eta \in \lim(T_q[\eta^{\mathbf{t}}])$, the restriction $\eta \restriction [n_*, \omega)$ is constantly zero.

[Why? By $(*)_4$ and $(*)_3$.] Consequently,

$(*)_9$ $\underline{\mathbf{t} = 1}$: $q_1, q_2 \in \mathbb{Q}$ are incompatible if for no $n_* < \omega$ and $\eta \in \lim(T_{q_1}[\underset{\sim}{\eta}^{\mathbf{t}}]) \cap \lim(T_{q_2}[\underset{\sim}{\eta}^{\mathbf{t}}])$ do we have $\ell \in [n_*, \omega) \Rightarrow \eta(\ell) = 0$ and $(\exists^\infty \ell)(\eta \restriction \ell^\frown\langle 1\rangle \in T_{q_1}[\underset{\sim}{\eta}^{\mathbf{t}}] \cap T_{q_2}[\underset{\sim}{\eta}^{\mathbf{t}}])$

$\underline{\mathbf{t} = 2}$: $q_1, q_2 \in \mathbb{Q}$ are incompatible if $\lim(T_{q_1}[\underset{\sim}{\eta}^{\mathbf{t}}]) \cap \lim(T_{q_2}[\underset{\sim}{\eta}^{\mathbf{t}}])$ has finitely many members.

We say that $u \in [\omega]^{\aleph_0}$ is large for $(q, \underset{\sim}{\eta}^{\mathbf{t}})$ $\underline{\text{if}}$ for every $r \in \mathbb{Q}$ above q the following holds:

$\bigotimes_{r,u}$ $\underline{\text{Case } \mathbf{t} = 1}$: For some $n^* \in u$, for every $n, m \in u$ such that $n^* < n < m$, for some $\nu \in {}^m2 \cap T_r[\underset{\sim}{\eta}^{\mathbf{t}}]$ we have $\ell \in [n^*, m) \Rightarrow \nu(\ell) = 0$ but $(\exists \ell)(n \leq \ell < m \;\&\; \eta \restriction \ell^\frown\langle 1\rangle \in T_r[\underset{\sim}{\eta}^1])$.

$\underline{\text{Case } \mathbf{t} = 2}$: For some $n^* \in u$ if $n, m \in u$ and $n^* \leq n < m$ then for some $\nu_1, \nu_2 \in {}^m2 \cap T_r[\underset{\sim}{\eta}^{\mathbf{t}}]$ and $\ell \in [n, m)$ we have $\nu_1 \restriction \ell = \nu_2 \restriction \ell, \nu_1(\ell) \neq \nu_2(\ell)$.

Let a $\mathbb{P}$-name $\underset{\sim}{g}^* \in {}^\omega\omega$ be defined by $\underset{\sim}{g}^*(0) = 0, \underset{\sim}{g}^*(n+1) = \underset{\sim}{g}(n+1+\underset{\sim}{g}^*(n))$.

$\underline{\text{Subclaim}}$: Let $q^* \in \mathbb{Q}^N$. Then $N \models \Vdash_{\mathbb{P}}$ "some $u \in [\omega]^{\aleph_0}$ is large for $(q^*, \underset{\sim}{\eta}^{\mathbf{t}})$".

It will take us awhile. Let $\underset{\sim}{u} = \operatorname{Rang}(\underset{\sim}{g}^*)$, it is a $\mathbb{P}$-name in N, and assume that $\underset{\sim}{u}$ is not as required, so for some $p^* \in \mathbb{P}^N$ and $\mathbb{P}$-names $\underset{\sim}{q}, n$ we have

$N \models$"$p^* \Vdash_{\mathbb{P}} [\underset{\sim}{q} \in \mathbb{Q}$ is above q^* and $\neg\otimes_{\underset{\sim}{q},\underset{\sim}{u}}]$".

Let $G (\in \mathbf{V})$ be a subset of $\mathbb{P}^N$ generic over N and such that $p^* \in G$.

Now, in $N[G]$, we choose inductively a sequence $\langle (k_i, n_i, m_i) : i < \omega\rangle$ so that:

Case $\mathbf{t} = 1$: $k_i, n_i, m_i \in \underset{\sim}{u}[G], k_i < n_i < m_i < k_{i+1}$ and for each $i < \omega$, there is no $\eta \in {}^{m_i}2 \cap T_{\underset{\sim}{q}[G]}[\underset{\sim}{\eta}^{\mathbf{t}}]$ satisfying

$$(\forall \ell \in [k_i, m_i))(\eta(\ell) = 0) \quad \text{and} \quad (\exists \ell \in [n_i, m_i))(\eta \restriction \ell^\frown\langle 1\rangle \in T_{\underset{\sim}{q}[G]}[\underset{\sim}{\eta}^{\mathbf{t}}]);$$

Case $\mathbf{t} = 2$: $k_i, n_i, m_i \in \underset{\sim}{u}[G]$, $k_i < n_i < m_i = \mathrm{Min}(\underset{\sim}{u}[G] \setminus (n_i + 1)) < n_{i+1}$ and for each $i < \omega$, there are no $\nu_1 \neq \nu_2 \in {}^{m_i}2 \cap T_{\underset{\sim}{q}[G]}[\underset{\sim}{\eta}^{\mathbf{t}}]$ satisfying $\nu_1 \restriction n_i = \nu_2 \restriction n_i$.

Note that, in both cases, the choice is possible by our assumption on p^* (and by $p^* \in G$).

Let $n_i = \underset{\sim}{n}_i[G], m_i = \underset{\sim}{m}_i[G], k_i = k_i[G]$ for some sequence $\langle \underset{\sim}{k}_i, \underset{\sim}{n}_i, \underset{\sim}{m}_i : i < \omega\rangle \in N$ of $\mathbb{P}$-names. Without loss of generality $p^* \in \mathbb{P}$ is such that it forces all the above. So

$(*)_{10}$ (a) if $\mathbf{t} = 1$ then

$$N \models \text{“}p^* \Vdash_{\mathbb{P}} \text{ if } i < \omega, \eta \in T_{\underset{\sim}{q}}[\underset{\sim}{\eta}^1] \cap {}^{\underset{\sim}{m}_i}2, \ell \in [\underset{\sim}{k}_i, \underset{\sim}{m}_i) \Rightarrow \eta(\ell) = 0$$

$$\text{then for no } \ell \in [\underset{\sim}{n}_i, \underset{\sim}{m}_i) \text{ do we have } \eta \restriction \ell^\frown\langle 1\rangle \in T_{\underset{\sim}{q}}[\underset{\sim}{\eta}^1]\text{”}$$

(b) if $\mathbf{t} = 2$ then

$$N \models \text{“}p^* \Vdash_{\mathbb{P}} \text{ if } i < \omega, m \in [\underset{\sim}{n}_i, \underset{\sim}{m}_i) \text{ and } \eta \in T_{\underset{\sim}{q}}[\underset{\sim}{\eta}^2] \cap {}^m2 \text{ then}$$

$$\eta \text{ has a unique successor in } T_{\underset{\sim}{q}}[\underset{\sim}{\eta}^2] \cap {}^{m+1}2\text{”}.$$

Let $\langle \mathcal{I}_n : n < \omega\rangle$ list the dense open subsets of $\mathbb{P}$ which belong to N. Let $\langle J_k, \rho_k : k < \omega\rangle$ be such that: J_k is a finite front of ${}^{\omega>}2$, $J_0 = \{<>\}, \rho_k \in J_k, J_{k+1} = (J_n \backslash \{\rho_k\}) \cup \{\rho_k{}^\frown <0>, \rho_k{}^\frown <1>\}$ and $n < \omega$ & $\rho \in J_n \Rightarrow (\exists m \geq n)(\rho_m = \rho)$ and $k < n < \omega \Rightarrow \ell g(\rho_k) \leq \ell g(\rho_n)$. Moreover, we require that if $\ell < \ell g(\rho_k) = \ell g(\rho_n), \rho_k \restriction \ell = \rho_n \restriction \ell$ and $\rho_k(\ell) = 0, \rho_n(\ell) = 1$, then $k < n$. We choose $\bar{p}^k = \langle p_\rho, k_\rho, m_\rho, n_\rho : \rho \in J_k\rangle$ by induction on k such that (after we choose $\bar{p}^k$ we have already chosen $\bar{p}^{k+1} \restriction (J_{k+1} \backslash \{\rho_k{}^\frown <0>, \rho_k{}^\frown <1>\})$):

$\circledast_0$ (a) $p^* \leq_{\mathbb{P}} p_\rho \in \mathbb{P}^N$

(b) $m_\rho < n_\rho$

(c) if $m > n_\rho$ then for some $q \in \mathbb{P}^N$ we have $p_\rho \leq_{\mathbb{P}} q$ and $q \Vdash_{\mathbb{P}} (\exists i)(\underset{\sim}{n}_i \leq n_\rho \wedge \underset{\sim}{m}_i > m)$

(d) $p_{\rho_k} \leq_{\mathbb{P}} p_{\rho_k{}^\frown <\ell>} \in \mathcal{I}_{\ell g(\rho_k)}$ for $\ell = 0, 1$

(e) $p_{\rho_k{}^\frown <\ell>} \Vdash_{\mathbb{P}}$ “$(\exists i)(\underset{\sim}{n}_i \leq n_{\rho_k}$ & $\underset{\sim}{m}_i > m_{\rho_k{}^\frown <\ell>})$”

(f) $m_{\rho_k{}^\frown <\ell>} > \sup\{n_\nu : \nu \in J_k\}$.

Let us carry out the induction.

In step $k = 0$ let $p_{<>} = p^*, n_{<>}$ is chosen as below, $m_{<>}$ is immaterial. If we have defined $\bar{p}^k$, first choose $m_{\rho_k{}^\frown <\ell>}$ to satisfy clause (f), then choose

$p'_{\rho_k{}^\frown<\ell>} \geq p_{\rho_k}$ to satisfy clause (e) (possible by clause (c)) and choose $p_{\rho_k{}^\frown<\ell>} \geq p'_{\rho_k{}^\frown<\ell>}$ to satisfy clause (d). Lastly, choose $n_{\rho_k{}^\frown<\ell>}$ to satisfy clause (c); this is possible by Oservation 2.3 below.

For each $\rho \in {}^\omega 2$ let $G_\rho = \{p \in \mathbb{P}^N : p \leq_{\mathbb{P}} p_{\rho\restriction\ell}$ for some $\ell < \omega\}$, clearly $G_\rho (\in \mathbf{V})$ is a subset of $\mathbb{P}^N$ generic over N (by clause (d)). Now $q_\rho = \underset{\sim}{q}[G_\rho]$ and $T_\rho = T_{q_\rho}[\underset{\sim}{\eta}^{\mathbf{t}}]$, are well defined in $N[G_\rho]$ hence in $\mathbf{V}$. It is easy to see the following.

$\circledast_1$ Assume that $\nu_1 \neq \nu_2 \in {}^\omega 2$ and $\nu_1 \restriction k \neq \nu_2 \restriction k$. Then

- (α) if $n > n_{\nu_1\restriction k}, n_{\nu_2\restriction k}$ then for some $i \geq k$ we have $n \in [n_{\nu_1\restriction i}, m_{\nu_1\restriction(i+1)})$ or $n \in [n_{\nu_2\restriction i}, m_{\nu_2\restriction(i+1)})$
- (β) if $\mathbf{t} = 2$ and $\eta \in T_{\nu_1} \cap T_{\nu_2}$ and $\ell g(\eta) > n_{\nu_1\restriction k}, n_{\nu_2\restriction k}$, <u>then</u> η has at most one successor in $T_{\nu_1} \cap T_{\nu_2}$.

[Why? Clause (α) follows from clauses (b) + (f) of $\circledast_0$. Clause (β) follows from clause (α) and $(*)_{10}(b)$.]

Hence

$\circledast_2$ (α) if $\mathbf{t} = 1$ and $\nu_1 \neq \nu_2 \in {}^\omega 2$ and $\eta \in \lim(T_{\nu_1}) \cap \lim(T_{\nu_2})$ and $\eta(\ell) = 0$ for every $\ell < \omega$ large enough then the set

$$\{\ell < \omega : \eta \restriction \ell^\frown\langle 1\rangle \in \lim(T_{\nu_1}) \cap \lim(T_{\nu_2})\}$$

is finite

(β) if $\mathbf{t} = 2$ and $\nu_1 \neq \nu_2 \in {}^\omega 2$ then $\lim(T_{\nu_1}) \cap \lim(T_{\nu_2})$ is finite.

Now for each $\nu \in {}^\omega 2, N[G_\nu] \in \mathfrak{K}_{\mathbb{P}}$ and $N[G_\nu] \models$ "$q_\nu \in \mathbb{Q}$", hence there is $q^+_\nu \in \mathbb{Q}$ which is $\langle N[G_\nu], \mathbb{Q}\rangle$-generic and $\mathbb{Q} \models$"$q_\nu \leq q^+_\nu$". Hence really $q^+_\nu \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta}^{\mathbf{t}} \in \lim(T_\nu)$", so by $\circledast_2$ and $(*)_9$ we have

$\circledast_3$ if $\nu_1 \neq \nu_2 \in {}^\omega 2$ <u>then</u> $q^+_{\nu_1}, q^+_{\nu_2}$ are incompatible in $\mathbb{Q}$.

But this contradicts assumption (a) of 2.2, i.e., the c.c.c. $\square_{\text{Subclaim}}$

Observation 2.3. *Assume*

$(*)$ $p^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{n}_i < \underset{\sim}{n}_{i+1} < \omega, \underset{\sim}{n}_i < \underset{\sim}{m}_i < \omega$ *for every* $i < \omega$ *and for every* $h \in {}^\omega\omega \cap \mathbf{V}$ *for infinitely many* i *we have* $h(\underset{\sim}{n}_i) < \underset{\sim}{m}_i$".

<u>Then</u> we can find $n^* < \omega$ *such that for every* $m \in [n^*, \omega)$ *there is* q *satisfying* $p^* \leq_{\mathbb{P}} q$ *and* $q \Vdash (\exists i)(\underset{\sim}{n}_i \leq n^* \ \&\ \underset{\sim}{m}_i \geq m)$.

Proof. We define a function $h : \omega \to (\omega + 1)$ by

$$h(n) = \sup\{m : n < m \text{ and for some } q \in \mathbb{P} \text{ we have } p \leq_{\mathbb{P}} q \text{ and } q \Vdash_{\mathbb{P}} \text{“}(\exists i)(\underset{\sim}{n}_i \leq n \wedge m \leq \underset{\sim}{m}_i)\text{”}\}.$$

If for some $n, h(n) = \omega$ we are done. Otherwise $h \in {}^\omega\omega$ and hence $p^* \Vdash_{\mathbb{P}}$ $(\exists i)(h(\underset{\sim}{n}_i) < \underset{\sim}{m}_i)$. So there are n, m, q, i such that $p^* \leq_{\mathbb{P}} q$ and

$$q \Vdash \text{“}h(\underset{\sim}{n}_i) < \underset{\sim}{m}_i \ \&\ \underset{\sim}{n}_i = n \ \&\ \underset{\sim}{m}_i = m\text{”}.$$

But by the definition of q, q witnesses that $h(n) \geq m$, a contradiction. $\square_{\text{Observation}}$

Continuation of the proof of 2.2:
The conclusion of the subclaim holds in $\mathbf{V}$ as $N \prec (\mathcal{H}(\chi), \in)$, and this gives the conclusion of part (2) of 2.2 when $\mathbf{t} = 2$, and the conclusion of part (1) of 2.2 when $\mathbf{t} = 1$; the proof is similar to [12, 1.12, p. 168] but we give details. By the subclaim, as $N \prec (\mathcal{H}(\chi), \in)$, clearly in $\mathbf{V}^{\mathbb{P}}$ we have: for every $q^* \in \mathbb{Q}$ some infinite $u \subseteq w$ is large for $(q^*, \underset{\sim}{\eta}^{\mathbf{t}})$. Fix such q^*, u. We concentrate on $\mathbf{t} = 1$ as the case $\mathbf{t} = 2$ is obvious by this point.

Let $u \backslash \{0\} = \{n_i : 1 \leq i < \omega\}$ be such that $n_0 =: 0 < n_1 < n_2 < \ldots$, let $\langle k(i, \ell) : \ell < \omega \rangle$ be such that $i = \Sigma_\ell k(i, \ell) 2^\ell$ where $k(i, \ell) \in \{0, 1\}$, so $k(i, \ell) = 0$ when $2^\ell > i$. For $m < \omega$ let $\rho^*_m = \langle k(i, \ell) : \ell \leq [\log_2(i+1)] \rangle$ where $i = i_u(m)$ is the unique i such that $n_i \leq m < n_{i+1}$. We define a $\mathbb{Q}$-name $\underset{\sim}{\rho}$ (of a member of $({}^\omega 2)^{\mathbf{V}^{\mathbb{Q}}}$): let $\{\underset{\sim}{k}_i : i < \omega\}$ list in the increasing order the elements of the set $\{k < \omega : \underset{\sim}{\eta}^{\mathbf{t}}(k) = 1\}$ and $\underset{\sim}{\rho}$ be $\rho^*_{\underset{\sim}{k}_0} {}^\frown \rho^*_{\underset{\sim}{k}_1} {}^\frown \rho^*_{\underset{\sim}{k}_2} {}^\frown \ldots$.

Clearly for every $p \in \mathbb{Q}$ and $n < \omega$ we have $p \nVdash$ "$\underset{\sim}{\eta}^{\mathbf{t}}(k) = 0$ for every $k \geq n$". Hence $\Vdash_{\mathbb{Q}}$ "$\{k < \omega : \underset{\sim}{\eta}^{\mathbf{t}}(k) = 1\}$ is infinite", hence $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\rho} \in {}^\omega 2$". It is enough to prove that $q^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\rho}$ is a Cohen real over $\mathbf{V}^{\mathbb{P}}$". So let $T \in \mathbf{V}^{\mathbb{P}}$ be a given subtree of ${}^{\omega >}2$ which is nowhere dense, i.e., $(\forall \eta \in T)(\exists \nu)[\eta \triangleleft \nu \in {}^{\omega >}2 \backslash T]$, and we should prove $q^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\rho} \notin \lim(T)$". So assume $q^* \leq q \in \mathbb{Q}$ and we shall find $q', q \leq q' \in \mathbb{Q}$ such that $q' \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\rho} \notin \lim(T)$", this suffices. We apply the choice of u so for some $n_* \in u$, if $n, m \in u, n_* < m < n$ then for some $\nu \in {}^n 2 \cap T_q[\underset{\sim}{\eta}^{\mathbf{t}}]$ we have $\ell \in (n_*, m) \Rightarrow \nu(\ell) = 0$ but $(\exists \ell)(m \leq \ell < n \ \& \ \underset{\sim}{\eta}^*(\ell) = 1]$. As $n_* \in u$ for some $i(*)$ we have $n_* = n_{i(*)}$ and $\Xi =: \{\rho_m : m < n_*\}$ is finite hence $\Xi' =: \{\rho^*_{k_0} {}^\frown \ldots {}^\frown \rho^*_{k_1} : k_0 < \ldots < k_\ell < n_*\}$ is finite. As T is nowhere dense we can find a sequence $\rho^* \in {}^{\omega >}2$ such that: $\rho \in \Xi' \Rightarrow \rho {}^\frown \rho^* \notin T$ and choose $i > i(*)$ such that $\rho^* \triangleleft \rho^*_{n_i}$. This is possible by the definition of $\rho^*_{n_i}$, i.e., it is enough that $i > 2^{\ell g(\rho^*)}$ and $i = \Sigma\{\rho^*(\ell) 2^\ell : \ell < \ell g(\rho^*)\} \bmod 2^{\ell g(\rho^*)}$.

As said above we can find $q' \geq q$ such that $q' \Vdash$ "if $n_* \leq \ell < n_i$ then $\underset{\sim}{\eta}^{\mathbf{t}}(\ell) = 0$ but for some $\ell \in [n_i, n_{i+1})$ we have $\underset{\sim}{\eta}^{\mathbf{t}}(\ell) = 1$". So q' forces that ρ_{n_i} appears in the choice of $\underset{\sim}{\rho}$ and before it we have a concatenation of finite sequences which belong to $\underset{\sim}{\Xi}'$, so we are done. $\square_{2.2}$

References

[1] Bartoszyński, T., Rosłanowski, A., *Towards Martin's minimum*, Arch. Math. Logic, **41** (2002), 65–82. `math.LO/9904163`.[4]

[2] Brendle, J., *Combinatorial properties of classical forcing notions*, Ann. Pure Appl. Logic, **73** (1995), 143–170.

[3] Brendle, J., Judah, H., Shelah, S., *Combinatorial properties of Hechler forcing.* Ann. Pure Appl. Logic, **58** (1992), 185–199. `math.LO/9211202`.

[4] Gitik, M., Shelah, S., *Forcings with ideals and simple forcing notions*, Israel J. Math., **68** (1989), 129–160.

[5] Gitik, M., Shelah, S., *More on simple forcing notions and forcings with ideals*, Ann. Pure Appl. Logic, **59** (1993), 219–238.

[6] Gitik, M., Shelah, S., *More on real-valued measurable cardinals and forcing with ideals*, Israel J. Mathematics, **124** (2001), 221–242. `math.LO/9507208`.

[7] Goldstern, M., Judah, H., *Iteration of Souslin forcing, projective measurability and the Borel conjecture.* Israel J. Math. **78** (1992), 335–362.

[8] Ihoda, J. (Judah Haim), Shelah, S., *Souslin forcing*, J. Symbolic Logic **53** (1988), 1188–1207.

[9] Keisler, J. H., *Model Theory for Infinitary Logic. Logic with Countable Conjunctions and Finite Quantifiers*, Stud. Logic Found. Math. **62**, North-Holland Publishing Co., Amsterdam-London, 1971.

[10] Rosłanowski, A., Shelah, S., *Sweet & sour and other flavours of ccc forcing notions*, Arch. Math. Logic **43** (2004), 583–663. `math.LO/9909115`.

[11] Rosłanowski, A., Shelah, S., *Norms on Possibilities I: Forcing with Trees and Creatures*, Mem. Amer. Math. Soc., **141**(671), Providence, RI, 1999. `math.LO/9807172`.

[12] Shelah, S., *How special are Cohen and random forcings i.e. Boolean algebras of the family of subsets of reals modulo meagre or null*, Israel J. Math., **88** (1994), 159–174. `math.LO/9303208`.

[13] Shelah, S., *On what I do not understand (and have something to say): Part I*, Fund. Math. **166** (2000), 1–82. `math.LO/9906113`.

[14] Shelah, S., *Properness without elementaricity.* J. Appl. Anal. **10**(2) (2004), 169–289. `math.LO/9712283`.

[15] Shelah, S., *More on nw-nep forcing notions.*

SAHARON SHELAH
THE HEBREW UNIVERSITY OF JERUSALEM
EINSTEIN INSTITUTE OF MATHEMATICS
EDMOND J. SAFRA CAMPUS, GIVAT RAM
JERUSALEM 91904, ISRAEL
E-MAIL: SHELAH@MATH.HUJI.AC.IL

DEPARTMENT OF MATHEMATICS
HILL CENTER — BUSCH CAMPUS
RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY
110 FRELINGHUYSEN ROAD
PISCATAWAY, NJ 08854-8019 USA

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