PROPOSITION 4. *The space of all infinite subsets of natural numbers with the Ellentuck topology contains a closed and separable subspace which contains a closed and discrete subset of the cardinality continuum.*

Proof. Let V be the set of odd numbers and let

$$F = \{M \in T\colon V \subset M\} = T \setminus \bigcup \{\langle \emptyset, N \setminus \{t\}\rangle\colon t \in V\}.$$

The subspace F is obviously closed. It is separable because the set $\{V \cup x\colon x \in H\}$ is dense in F. Let h be a one-to-one mapping of the Cartesian product of natural numbers by itself onto the set of even numbers. We set

$$A^* = \{z = h(t, s)\colon t \in A \text{ and } s \notin A\}$$

and

$$U = \{V \cup A^*\colon \emptyset \neq A \subsetneq N\}.$$

The set U has the cardinality continuum and it is contained in F. Since

$$T \setminus U = \bigcup \{\langle \{h(t, s), h(r, t)\}, N\rangle\colon s, r, t \in N\} \cup \bigcup \{\langle \emptyset, V \cup (A^* \setminus \{z\})\rangle\colon A \subset N \text{ and } z \in A^* \cup V\}$$

the set U is closed. It is discrete in the Ellentuck topology because

$$U \cap \langle \emptyset, V \cup A^*\rangle = \{V \cup A^*\}. \blacksquare$$

Proposition 4 implies Keesling's result [5], which says, in our terms, that the Ellentuck topology is not normal. To see this it is enough to note that the subspace F from the above proposition cannot be normal. Note also that, since there is a closed discrete subset of the cardinality continuum, there is one of arbitrary cardinality less than the continuum.

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INSTYTUT MATEMATYKI
UNIWERSYTETU ŚLĄSKIEGO
ul. Bankowa 14
40-007 Katowice
Poland

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Squares with diamonds and Souslin trees with special squares

by

U. Abraham (Beer Sheva), **S. Shelah** (Jerusalem) and **R. M. Solovay** (Berkeley)

Abstract. The squares and the diamonds are useful set-theoretic axioms used in construction of infinite objects. Here we introduce and study different versions of such combinatorial principles on successor of singular cardinals. We prove some implications (in ZFC), inquire the situation in L, and give an application.

Introduction. One feature of the work of Jensen and Johnsbråten [J&J] is the construction in L of a Souslin tree T such that its square — minus the diagonal, of course — is a special tree (that is, embeddable into the rationals). We present in § 4 a generalization of this result to higher cardinals. In L:

For any cardinal $\varkappa$ there is a Souslin tree of height $\varkappa^+$ such that its square — minus the diagonal — is special.

The proof of [J&J] can be generalized to successors of regular cardinals — but successor of singulars seem to require a different approach. A new kind of a diamond sequence is used to construct the trees; it is called a "square sequence with built-in diamond". In fact there are several kinds of square sequences with diamonds. Such a sequence was first presented by C. Gray in his thesis [G]. We present here, essentially three other forms which are discussed in §§ 1–3. The forms in §§ 2 and 3 hold in L and require the fine structure for their proof (the proof of § 2 is simpler than that of § 3); the form in § 1 seems weaker than that of § 2 but it holds in a very general setting — in fact it is a consequence of GCH + usual kind of squares. (So reading of § 1 does not require knowledge of fine structure.) Each section can be read independently of the others (the construction in § 4 uses the square sequence of § 1 but the reader can see that the form of § 2 yields a slightly simpler proof).

In § 1 ideas of K. Kunen (the proof that $\diamondsuit^* \rightarrow \diamondsuit$), and of J. Gregory [Gr] and [S] are used.

We would like to thank L. Harrington and M. Stanley who raised in conversation the question about the Souslin trees settled here.

1. Implications in ZFC.

1.1. DEFINITION. In what follows $\varkappa$ is a singular cardinal. A *square* for $\varkappa^+$ is a sequence $\langle C_\alpha |\ \alpha \in \lim \varkappa^+ \rangle$ such that:

(1) $C_\alpha \subseteq \alpha$ is closed unbounded in α,

(2) $\operatorname{otp}(C_\alpha) < \varkappa$,

(3) If $\delta \in C'_\alpha$ then $C_\delta = C_\alpha \cap \delta$. ($C'$ is the set of limit points of C.)

We denote the existence of such a square sequence by $\square(\varkappa^+)$.

1.2. DEFINITION. A square sequence with built-in diamond on $\varkappa^+$ (denoted $\boxed{\Diamond}(\varkappa^+)$) consists of a square sequence $\langle C_\alpha |\ \alpha \in \lim \varkappa^+ \rangle$ and of a diamond sequence $\langle S_\alpha |\ \alpha \in \lim \varkappa^+ \rangle$ such that

(1) $S_\alpha \subseteq \alpha$.

(2) For $X \subseteq \varkappa^+$ let $G(X) = \{\zeta |\ X \cap \zeta = S_\zeta\}$, then not only $G(X)$ is stationary but, for any closed unbounded $C \subseteq \varkappa^+$, $C_\alpha \subseteq G(X) \cap C$ for C_α with order type as high as we wish (below $\varkappa$).

In this section we prove

1.3. THEOREM. *If $\varkappa$ is strong limit and singular, and if $2^\varkappa = \varkappa^+$, then*

$$\square(\varkappa^+) \Rightarrow \boxed{\Diamond}(\varkappa^+).$$

Proof. First we describe a general thinning procedure for square sequences. Suppose $\bar{C} = \langle C_\alpha |\ \alpha \in \lim \varkappa^+ \rangle$ is a square sequence. Let $\bar{E} = \langle E_\zeta |\ \zeta \in \lim \varkappa \rangle$ be a sequence which is like a square sequence in this sense:

(1) $E_\zeta \subseteq \zeta$ is closed unbounded.

(2) For $\tau \in E'_\zeta$, $E_\tau = E_\zeta \cap \tau$.

Then the thinning of $\bar{C}$ via $\bar{E}$ is the square sequence $\langle C^1_\alpha |\ \alpha \in \lim \varkappa^+ \rangle$ defined thus:

For α, let $\zeta = \operatorname{otp}(C_\alpha)$ and $g_\alpha\colon \zeta \to C_\alpha$ be continuous and increasing. We set

$$C^1_\alpha = g_\alpha[E_\zeta].$$

Turning now to the proof let $\bar{C} = \langle C_\alpha |\ \alpha \in \lim \varkappa^+ \rangle$ be a square sequence. Let $D = \{d_\xi |\ \xi \in \operatorname{cf}(\varkappa)\}$ be closed unbounded in $\varkappa$ such that $d_0 = 0$, $\operatorname{cf}(d_{\xi+1}) > d_\xi$, $\operatorname{cf}(d_1) > \operatorname{cf}(\varkappa)$ and

$$\{\alpha |\ \operatorname{otp}(C_\alpha) = d_{\xi+1}\} \text{ is stationary in } \varkappa^+.$$

Now fix $A_\xi \subseteq d_{\xi+1} - d_\xi$ closed unbounded of order type $\operatorname{cf}(d_{\xi+1})$. Let $\bar{E} = \langle E_\zeta |\ \zeta \in \lim \varkappa \rangle$ be defined as follows: If $\zeta \in D'$ put

$$E_\zeta = D \cap \zeta.$$

Otherwise $\zeta \in (d_\xi, d_{\xi+1}]$ for some $\xi \in \operatorname{cf}(\varkappa)$; we let

$$E_\zeta = A_\xi \cap \zeta \quad \text{if } \zeta \in A'_\xi, \quad \text{and}$$
$$E_\zeta = \zeta - \operatorname{Sup}(A_\xi \cap \zeta) \quad \text{otherwise}.$$

Now we let $\bar{C}^1$ be the thinning of $\bar{C}$ via E.

The sequence $\langle C^1_\alpha |\ \alpha \in \lim \varkappa^+ \rangle$ thus obtained is a square sequence with the following advantage. For any $d_{\xi+1}$ let $S(d_{\xi+1})$ be the collection of all β's which are limit points of some C^1_α for α such that $\operatorname{otp}(C_\alpha)$ is in $(d_\xi, d_{\xi+1}]$. Then

(1) the $S(d_{\xi+1})$'s are pairwise disjoint.

(2) $\langle C^1_\beta |\ \beta \in S(d_{\xi+1}) \rangle$ is a $(S(d_{\xi+1}), \operatorname{cf}(d_{\xi+1}))$-square sequence, where:

1.4. DEFINITION. $\langle C_\alpha |\ \alpha \in S \rangle$ is a (S, χ)-*square sequence*, for $S \subseteq \lim \varkappa^+$, and $\chi < \varkappa$ a regular uncountable cardinal, bigger than $\operatorname{cf}(\varkappa)$, if:

(1) $C_\alpha \subseteq \alpha$ is closed unbounded and $\operatorname{otp}(C_\alpha) \leqslant \chi$ for $\alpha \in S$.

(2) If $\alpha \in S$ and β is a limit point of C_α, then $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$.

(3) $\{\alpha |\ \operatorname{otp}(C_\alpha) = \chi\}$ is stationary in $\varkappa^+$.

Similarly to Definition 2.2 we can say

1.5. DEFINITION. A square with *built-in diamond* on (S, χ) consists of a (S, χ)-square sequence $\langle C_\alpha |\ \alpha \in S \rangle$ and of a diamond sequence $\langle S_\alpha |\ \alpha \in S \rangle$ such that

(1) $S_\alpha \subseteq \alpha$

(2) For $\chi \subseteq \varkappa^+$ and $C \subseteq \varkappa^+$ a club set, there is $\alpha \in S$ with $\operatorname{otp}(C_\alpha) = \chi$ such that for β a limit point of C_α (and hence for $\beta = \alpha$), $\beta \in C$ and $X \cap \beta = S_\beta$.

In view of the above, the following lemma proves Theorem 1.3.

1.6. LEMMA. *$\varkappa$ is singular strong limit cardinal and $2^\varkappa = \varkappa^+$. Assume S and the regular χ are such that a (S, χ)-square sequence exists. Then there is a (S, χ)-sequence with built-in diamond.*

Proof. Let $\langle C_\alpha |\ \alpha \in S \rangle$ be the given (S, χ)-square sequence. Let $g_\alpha\colon \operatorname{otp}(C_\alpha) \to C_\alpha$ be the increasing and continuous enumeration of C_α. (So g_α^{-1} is the collapsing map of C_α.)

Using the square sequence, we can define a one-to-one map $d_\alpha\colon \alpha \to \varkappa$, for $\alpha \in S - \varkappa$, such that for $\beta \in C'_\alpha$, $d_\alpha \upharpoonright \beta = d_\beta$. (Assume $d_\alpha(0) = 0$.)

We also fix an enumeration $(B_\zeta |\ \zeta \in \varkappa^+\}$ of all bounded subsets of $\varkappa^+ \times \varkappa$. $B_0 = \emptyset$.

Let

$$I = \{(A, f) |\ A \text{ is closed unbounded subset of } \chi \text{ and}$$
$$f\colon A \to \varkappa \text{ is a bounded function.}\}$$

As $\varkappa$ is strong limit, there are only $\varkappa$ many bounded functions $f\colon \chi \to \varkappa$, and so $|I| = \varkappa$. Let us enumerate $I = \{I_i = (A_i, f_i) |\ i \in \varkappa\}$.

Given $I_i \in I$ we are going to define an (S, χ)-square sequence $\langle C^i_\alpha |\ \alpha \in S \rangle$ and a sequence $\langle S^i_\alpha |\ \alpha \in S \rangle$ which we hope is an (S, χ)-square with built-in diamond. If we fail for all possible i's, a contradiction is derived; thence the proof of Lemma 1.6.

Well, let $I_i = (A, f)$ be given; enumerate $A = \{a_\zeta |\ \zeta \in \chi\}$. C^i_α is defined:

(a) if $\tau = \operatorname{otp}(C_\alpha)$ is a limit point of A, we let

$$C^i_\alpha = g_\alpha[A \cap \tau].$$

(b) If $\tau = \mathrm{otp}(C_\alpha) \in (a_\zeta, a_{\zeta+1}]$, we get C_α^i by cutting the tail of C_α contained in $g_\alpha(a_\zeta)$.

The definition of S_α^i now follows. If case (b) above holds, then set $S_\alpha^i = \emptyset$. Suppose case (a) holds, and so $\tau = \mathrm{otp}(C_\alpha) \in A'$. Let

$$H = d_\alpha^{-1}(f[A \cap \tau]), \qquad M = \bigcup \{B_\zeta \mid \zeta \in H\}.$$

M is a set of ordered pairs.

Now define

$$S_\alpha^i = M(i) = \{\mu \in \alpha \mid (\mu, i) \in M\}$$

What happens if every $i < \varkappa$ fails? Then we have sets $\chi^i \subseteq \varkappa^+$ and closed unbounded $D^i \subseteq \varkappa^+$ such that for any $\alpha \in S$ with $\mathrm{otp}(C_\alpha) = \chi$ there is β a limit point of C_α with $X^i \cap \beta \neq S_\beta^i$ or $\beta \notin D^i$.

Let $D = \bigcap_{i<\varkappa} D^i$, D is a closed unbounded subset of $\varkappa^+$. Let

$$X = \bigcup \{X^i \times \{i\} \mid i < \varkappa\}.$$

There is a closed unbounded $C \subseteq D$ such that for $\alpha \in C$

$$\beta < \alpha \Rightarrow X \cap \beta \times \beta = B_\eta \quad \text{for some } \eta < \alpha.$$

Pick now $\alpha \in C' \cap S$ such that $\mathrm{otp}(C_\alpha) = \chi$. $C_\alpha^* = C_\alpha \cap C'$ is a closed unbounded subset of α, since χ is a regular uncountable cardinal. For any $\mu \in C_\alpha^*$, $X \cap \mu \times \mu = B_{\varphi(\mu)}$ for some $\varphi(\mu)$ which is below the successor of μ in C_α^*. Since $\chi > \mathrm{cf}(\varkappa)$ we can find an unbounded $U \subseteq C_\alpha^*$ such that $F = \{d_\alpha(\varphi(\mu)) \mid \mu \in U\}$ is bounded in $\varkappa$.

Let A consist of the closure in χ of $g_\alpha^{-1}[U] = A^*$. A is a closed unbounded subset of χ. Now we know that, whatever f is, if $I_i = (A, f)$ then C_α^i is the closure of U in α. Define $f\colon A \to \varkappa$ by letting $f(\varrho) = 0$, for $\varrho \notin A^*$; and for $\varrho \in A^*$ let $f(\varrho) = d_\alpha(\varphi(g_\alpha(\varrho)))$. Now $f[A]$ is bounded in $\varkappa$, since either $f(\varrho) = 0$ or $f(\varrho) \in F$.

So, for some $i \in \varkappa$, $I_i = (A, f)$. We claim that for any β a limit point of C_α^i, $\beta \in D^i$ and $X^i \cap \beta = S_\beta^i$. Indeed, $C_\alpha^i \subseteq C_\alpha^* \subseteq D^i$. How was S_β^i defined? Clearly case (a) holds: As $\beta = g_\alpha(\tau)$ for some $\tau \in A'$, $\mathrm{otp}(C_\beta) \in A'$.

Now

$$H = d_\beta^{-1}(f[A \cap \tau]).$$

But since $f[A \cap \tau] \subseteq \mathrm{range}(d_\beta)$, and since $d_\alpha \restriction \beta = d_\beta$,

$$H = d_\alpha^{-1}[f[A \cap \tau]) = \{0\} \cup \varphi[U \cap \beta].$$

By the definition of φ, and as $U \cap \beta$ is unbounded in β,

$$M = \bigcup \{B_\xi \mid \zeta \in H\} = \bigcup \{B_{\varphi(\mu)} \mid \mu \in U \cap \beta\}$$
$$= \bigcup \{X \cap \mu \times \mu \mid \mu \in U \cap \beta\} = X \cap \beta \times \beta.$$

Now, $S_\beta^i = M(i) = X \cap \beta \times \beta(i) = X^i \cap \beta$.

This gives the desired contradiction.

We shall state now some further results which can be achieved using similar methods. No proofs are given, only some outlines.

1.7. Definition. Let λ be a regular uncountable cardinal, $S \subseteq \lambda$. $\delta < \lambda$. $\bar{C}$ is called a *weak* $(S, <\delta)$*-square sequence* if

(1) $\bar{C} = \langle C_\alpha \mid \alpha \in S\rangle$, S is a stationary set of limit ordinals in λ and $C_\alpha \subseteq \alpha$ is closed unbounded for $\alpha \in S$.

(2) If $\beta \in C_\alpha' \cap S$ then $C_\beta = C_\alpha \cap \beta$

(3) For every $\alpha \in S$ $\mathrm{otp}(C_\alpha) < \delta$.

We say that $\bar{C}$ is a weak (S, δ)-square sequence if δ is regular and (3) is replaced by

(3′) For every $\alpha \in S$ $\mathrm{otp}(C_\alpha) \leqslant \delta$ and $\{\alpha \mid \mathrm{otp}(C_\alpha) = \delta\}$ is stationary in λ. $\bar{C}$ is a weak square sequence if it is a weak (S, δ) or $(S, <\delta)$-sequence for some S, δ.

So, if $\bar{C}$ is a weak square sequence we are not requiring that $C_\alpha' \subset S$ for $\alpha \in S$.

1.8. Definition. Suppose $\bar{C}^l = \langle C_\alpha^l \mid \alpha \in S^l\rangle$ $l = 1, 2$ are two $(S^l, <\delta^l)$ or (S^l, δ^l) weak square sequences. We define

(a) $\bar{C}^1 \leqslant \bar{C}^2$ if $S^1 = S^2$ and $C_\alpha^1 \subseteq C_\alpha^2$ for $\alpha \in S^1$.

(b) $\bar{C}^1 \leqslant^* \bar{C}^2$ if $S^1 \subseteq S^2$ and, for $\alpha \in S^1$,

$$C_\alpha^1 \subseteq C_\alpha \quad \text{but } C_\alpha^1 \cap S^2 \subseteq S^1.$$

Remark. If $\bar{C}^2$ is a $(S^2, <\delta^2)$ or (S^2, δ^2)-square sequence (i.e., not weak) and $\bar{C}^1 \leqslant \bar{C}^2$ or $\bar{C}^1 \leqslant^* \bar{C}^2$ then $\bar{C}^1$ is also non weak. Hence "weak" can be omitted from the assumptions and conclusions of all subsequent results.

1.9. Definition. For a weak square sequence $\bar{C}$ let

$$SP(\bar{C}) = \{\gamma \mid \{\alpha \in \lambda \mid \mathrm{otp}(C_\alpha) = \gamma \text{ is stationary}\}.$$

1.10. Theorem. *Suppose $\bar{C} = \langle C_\alpha \mid \alpha \in S\rangle$ is a weak $(S, <\delta)$-square, $\Gamma \subseteq SP(\bar{C})$ is discrete (in the ordinal topology). For $\gamma \in \Gamma$ let δ_γ be the order-type of the ordinal interval $[\sup(\gamma \cap \Gamma), \gamma)$. There are then a weak $(S, <\delta)$-square sequence $\bar{C}^1 \leqslant \bar{C}$ and a partition of S to S_γ $(\gamma \in \Gamma)$ and S_∞ such that*

(1) $\langle C_\alpha^1 \mid \alpha \in S_\infty\rangle \leqslant^* \bar{C}$ *is a weak $(S_\infty, <\delta)$-square,*

(2) $\langle C_\alpha^1 \mid \gamma \in S_\gamma\rangle \leqslant^* \bar{C}$ *is a weak $(S_\gamma, \delta_\gamma)$-square sequence for $\gamma \in \Gamma$.*

We can instead ask in (2) that $\langle C_\alpha^1 \mid \alpha \in S_\gamma\rangle \leqslant^* \bar{C}$ is a weak $(S_\gamma, \mathrm{cf}(\delta_\gamma))$-square sequence.

The use of this theorem is when $\boxed{\diamond}$ is constructed; only weak (S, δ_γ)-square sequences have to be dealt with.

1.11. Theorem. *Suppose $\chi < \lambda$ are regular and uncountable. Assume $\mu = \mu^\chi < \lambda$. Let $S \subseteq \lambda$ and $\bar{C}$ be a weak (S, χ)-square sequence. Suppose further that $\langle P_\alpha \mid \alpha \in S\rangle$ satisfies:*

(a) *P_α is a family of $\leqslant \mu$ subsets of α.*

(b) *For every $A \subseteq \lambda$, $G(A) = \{\alpha \mid \alpha \in S$ and $A \cap \alpha \in P_\alpha\}$ is stationary in λ;*

moreover, there is $\alpha \in G(A)$ *with* $\mathrm{cf}(\alpha) = \chi$ *such that for some closed unbounded* $C \subseteq \alpha$, $C \cap S \subseteq G(A)$.

Then there is $\bar{C}' \leqslant \bar{C}$ *and* $\langle A_\alpha |\ \alpha \in S\rangle$ *such that for every* $A \subseteq \lambda$ *and closed unbounded* $C \subseteq \lambda$, *for some* $\alpha \in S$, $\mathrm{cf}(\alpha) = \chi$ *and* $C'_\alpha \subseteq C$ *and*

$$(\forall \beta \in S \cap (C^1_\alpha \cup \{\alpha\}))\ (A \cap \beta = A_\beta)\,.$$

The proof of this theorem is similar to that of Theorem 1.3.

1.12. THEOREM. *Suppose* $\bar{C} = \langle C_\alpha |\ \alpha \in S\rangle$ *is a weak* $(S, <\delta)$*-square sequence, where* S *is stationary in* λ *and* $|\delta|^+ < \lambda$. *Suppose also that* $S_i \subseteq S$, $i < \delta$, *are stationary sets. Then there are* $\bar{C}^1 \leqslant \bar{C}$ *and stationary subsets* $S^*_i \subseteq S_i$, *such that*

$$E_i = \{\tau \in S|\ \tau \in (C^1_\alpha \cup \{\alpha\})' \text{ for some } \alpha \in S^*_i\}, \qquad i < \delta\,,$$

are pairwise disjoint.

Proof. Firstly, we can use Theorem 6.2 of [BHM] to find stationary subsets which are pairwise disjoint; hence we assume tha tthe S_i's are pairwise disjoint.

For any $i < \lambda$ there is $\theta_i < \lambda$ and a stationary subset of S_i such that, for α in that stationary set, if $(C_\alpha)' \cap S$ is bounded in α then it is bounded by θ (good for all $i < \delta$).

By shrinking the S_i's and redefining C_α to be $C_\alpha - \theta$ for $\alpha > \theta$, we can assume now that for each $i < \delta$, $S_i \subseteq \lambda - \theta$ and either

(1) $\forall \alpha \in S_i\ (C_\alpha)' \cap S$ is unbounded in α or
(2) $\forall \alpha \in S_i\ (C_\alpha)' \cap S \cap \alpha = \emptyset$.

We make a further change of the C_α's and define a sequence C^*_α: If $X = (C_\alpha)' \cap S \cap \alpha - \theta \neq \emptyset$ then we let $C^*_\alpha = C_\alpha - \mu$ for the first μ in X. If $X = \emptyset$ we leave $C^*_\alpha = C_\alpha$.

Now we know that if i satisfies (2) above then for any $j \neq i$, $E_i \cap E_j = \emptyset$.

So we can assume without loss of generality that all i satisfy (1) above.

Now, $\varrho \in S$ is called a *good point* for S_i if $\varrho \in (C^*_\alpha)'$ for stationary many $\alpha \in S_i$. There must be λ many good points for S_i (λ is regular).

Our aim is to find $\varrho(i)$ good for S_i for $i < \delta$ such that for $i \neq j$

(*) $\varrho(i) \notin (C^*_{\varrho(j)})^1 \cup \{\varrho(j)\}$.

If we succeed, then $S^*_i = \{\alpha \in S_i|\ \varrho(i) \in (C^*_\alpha)'\}$ are as required. Because for any $\alpha \in S$ we can define C^1_α thus: If there is i such that $\varrho(i) \in (C^*_\alpha)'$, then there is a unique such i and we let

$$C^1_\alpha = C^*_\alpha - \varrho(i)\,.$$

Otherwise we let

$$C^1_\alpha = C^*_\alpha\,.$$

We try to define the $\varrho(i)$'s by induction on $i < \delta$ so that (*) holds. If we fail, i.e. we arrive at some $i < \delta$ such that, for any ϱ good for S_i, there is some $j < i$ which destroy (*); then we try again, but this time choosing ϱ's which are different from all those picked before. We try λ many trials. Suppose we always fail. Then λ many trials fail at the same $i < \delta$. Look at the first $|\delta|^+$ many trials which fail for that i. There are only $< \lambda$ many points which were chosen as good points so far or which are in C^*_α for some good point α chosen so far. Pick any ϱ good for S_i which is not among those $< \lambda$ many; this ϱ could not serve as $\varrho(i)$ for no trial among those $|\delta|^+$ many which failed at i. Hence for some $j < i$, at $|\delta|^+$ many trials, the good point $\varrho(j)$ chosen for j satisfies

$$\varrho(j) \in (C^*_\varrho)'\,.$$

It follows that $|C'_\varrho| > \delta$, a contradiction.

1.13. COROLLARY. *Assume* $\bar{C}$ *is a weak* $(S, <\delta)$*-square sequence.* S *is stationary in the regular cardinal* λ *and* $2^{|\delta|} < \lambda$. *Let* $S_i \subseteq S$, $i < \delta$, *be given stationary sets consisting of uncountable cofinality ordinals. Then there is* $\bar{C}^1 \leqslant \bar{C}$ *such that for any closed unbounded* $E \subseteq \lambda$ *and* $i < \delta$ *there is* $\alpha \in S_i$ *such that*

$$\lim C^1_\alpha \subseteq E\,.$$

Proof. We can assume that $\bar{C}$ and S_i, $i < \delta$, are already as given by the theorem. Let us define $\alpha \in \bar{S}_i$ iff for some $\tau \in S_i$ and $\beta \in S \cap (C'_\alpha \cup \{\alpha\})$, $\beta \in C'_\tau \cup \{\tau\}$. Now if $\alpha \in S$ and $\gamma \in C'_\alpha$ then $\gamma \in \bar{S}_i$ iff $\alpha \in \bar{S}_i$. $S_i \subseteq \bar{S}_i$ and the $\bar{S}_i$'s are pairwise disjoint. Let $R = S - \bigcup_{i<\delta} \bar{S}_i$, then $S \cap C'_\alpha \subseteq R$ for $\alpha \in R$.

Now for each $\bar{S}_i$ separately we can find $\bar{C}^i \leqslant^* \bar{C}$ (as in Theorem 1.3) and then recombine to get the desired $\bar{C}^1$.

2. The combinatorial principle $SD_\varkappa$.

2.1. Let $\varkappa$ be an uncountable cardinal. In this section, we shall introduce a new combinatorial principle $SD_\varkappa$. We shall also show that if $V = L$, then $SD_\varkappa$ holds for every uncountable cardinal $\varkappa$.

The principle $SD_\varkappa$ is best thought of as a combination of $\Box_\varkappa$ and $\diamondsuit^+_\varkappa$. Like these principles $SD_\varkappa$ asserts the existence of certain systems of sets satisfying certain requirements. The cast of characters is as follows:

(1) For α a limit ordinal less than $\varkappa^+$, C_α is a subset of α;
(2) S is a subset of $\varkappa^+$ consisting of limit ordinals greater than $\varkappa$;
(3) for $\gamma \in S$, X_γ is a subset of γ.

The axioms which we will put on these data will imply, in particular, that the C_α's are a $\Box_\varkappa$ sequence, and that the X_γ's are a $\diamondsuit^+_\varkappa(S)$ sequence.

(i) C_α is a closed unbounded subset of α.
(ii) The order type of C_α is $\leqslant \varkappa$.
(iii) Let γ be a limit point of C_α. Then $C_\gamma = C_\alpha \cap \gamma$.
(iv) Let γ be a limit point of C_α. Then $\gamma \in S$ iff $\alpha \in S$.
(v) Let $\alpha \in S$. Let γ be a limit point of C_α. Then $X_\gamma = X_\alpha \cap \gamma$.
(vi) Let A be a subset of $\varkappa^+$. Let K be a closed unbounded subset of $\varkappa^+$. Then there is a $\gamma \in S$ such that: (a) the order type of C_γ is $\varkappa$; (b) $X_\gamma = A \cap \gamma$; (c) The limit points of C_γ are in K.

Discussion: Keeping the notation of (vi), it follows that $\gamma \in K$. Axiom (vi) readily implies that S is stationary, and that $\langle X_\gamma;\ \gamma \in S\rangle$ is a $\diamondsuit_\varkappa^+(S)$ sequence.

2.2. From now on, we assume $V = L$, and prove $SD_\varkappa$. Our proof will rely heavily on the material in Chapter 5 of [J1], and we shall need to assume the reader is familiar with this material.

The system of C_α's that we use is essentially the one constructed in [J1]. However, we find it important to use a slightly different notion of parameter. We only use finite sets of ordinals as parameters. (It is clear that there is no loss of generality in doing this since, for any β, every element of J_β is $\Sigma_1(J_\beta)$ definable from some finite set of ordinals less than $\omega\beta$.) We use the following standard well-ordering of finite sets of ordinals: Let $A = \{\alpha_1, \ldots, \alpha_r\}$ and $B = \{\beta_1, \ldots, \beta_s\}$ be finite sets of ordinals listed in strictly decreasing order. Then A is less than B if, for some k, we have $\alpha_i = \beta_i$ for $i < k$, and either (a) α_k is undefined, and β_k is defined; or (b) $\alpha_k < \beta_k$. (The following alternative description makes it clear that this is a well-ordering: If A, as above, is a finite set of ordinals, let $f(A)$ be the ordinal:

$$\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}.$$

Then A is less than B is the ordering described above iff $f(A) < f(B)$ *qua* ordinals.)

The construction of square-sequences in Section 5 of [J1] can be easily adapted to this new convention, and we leave it to the reader to carry this out.

The construction in [J1] contains a phase (centered in Lemma 5.14 and its proof) designed to insure that if $\varkappa$ is singular, no C_α has order type $\varkappa$. Four our current purposes, it is quite important to omit this phase of the construction. That is, the square-sequence will extend the partial square-sequence provided by the proof of Lemma 5.13 [J1].

We take as our C_α's the ones provided by the proof of Theorem 5.2 of [J1] with the two changes just indicated. It is then evident that axioms (i) through (iii) of $SD_\varkappa$ are satisfied.

2.3. Before giving the definition of S, it is necessary to recall some of the notation from [J1]. Let α be an ordinal with $\varkappa < \alpha < \varkappa^+$. Following [J1], p. 272–273, we let $\beta(\alpha)$ be the least $\beta \geqslant \alpha$ such that there is a $\Sigma_\omega(J_\beta)$ map of $\varkappa$ onto α; we let $n(\alpha)$ be the least integer $n \geqslant 1$ such that there is a $\Sigma_n(J_\beta)$ function which maps a subset of $\varkappa$ onto α.

We now define the set S. It will consist of those α's such that:

(a) $\varkappa < \alpha < \varkappa^+$.

(b) $n(\alpha) = 1$.

(c) Let β be $\beta(\alpha)$. Let p be the canonical parameter p_β^1. Recall that by the conventions of § 1.2, p is a finite set of ordinals. We require that p have a single member, $\delta(\alpha)$.

(d) Let δ be $\delta(\alpha)$. Then J_δ is a model of ZF-. The ordinal α is the largest cardinal of J_δ. Finally, $\alpha = \varkappa^+$ holds in J_δ.

2.4. We now verify axiom (iv) of $SD_\varkappa$. Let α be a limit ordinal less than $\varkappa^+$, and let $\bar\alpha$ be a limit point of C_α. We shall show that $\alpha \in S$ iff $\bar\alpha \in S$.

There is a class of ordinals named S which is referred to during the proof of Theorem 5.2 of [J]. To avoid confusion, we shall refer to this class as S^J. S^J consists of those ordinals α such that

(a) $\varkappa < \alpha < \varkappa^+$;

(b) α is closed under the Gödel pairing function;

(c) each $\nu < \alpha$ has cardinality $\leqslant \varkappa$ in J_α (i.e. some $f \in J_\alpha$ maps $\varkappa$ onto ν if $\nu \neq 0$).

It is evident from clause (d) of the definition of S that $S \subseteq S^J$. Moreover, it is clear from the proof of Theorem 5.2 of [J1] that $\alpha \in S^J$ iff $\bar\alpha \in S^J$. Thus we may assume, without loss of generality that $\alpha \in S^J$.

In the proof of Theorem 5.2 of [J1] there is a division into 4 cases. Since $\alpha \in S^J$, it follows that α does not come under Cases 1 or 2a. Since C_α has the limit point $\bar\alpha$, it follows that α does not come under Cases 2b or 3. Hence, α comes under Case 4.

Since α and $\bar\alpha$ lie in S^J, they both satisfy clause (a) of the definition of S.

Lemma 5.9 of [J1] shows that clause (b) of the definition of S holds of α iff it holds of $\bar\alpha$. Hence, we may assume that $n(\alpha) = n(\bar\alpha) = 1$.

Since $n(\alpha) = 1$, there is a canonical Σ_0 map $\tilde\pi$: $J_{\beta(\bar\alpha)} \to J_{\beta(\alpha)}$. The proof in [J1] that $\tilde\pi$ preserves parameters applies equally well with our revised notion of parameter. It follows that clause (c) of the definition of S holds of α iff it holds of $\bar\alpha$. Hence, we may assume that clause (c) holds of α.

We set $\delta = \delta(\alpha)$ and $\bar\delta = \delta(\bar\alpha)$. Since $\tilde\pi$ preserves parameters, we have $\tilde\pi(\bar\delta) = \delta$. Since $\tilde\pi$ is Σ_0, it follows that $\tilde\pi(J_{\bar\delta}) = J_\delta$. Moreover the restriction of $\tilde\pi$ to $J_{\bar\delta}$ is a Σ_ω embedding (which we call π_*) of $J_{\bar\delta}$ into J_δ.

Recall that for α coming under Case 4 of the proof of Theorem 5.2 of [J1], the ordinal $\varrho(\alpha)$ is defined to be ϱ_β^{n-1}. Since $n(\alpha) = 1$ in the case under consideration, we have $\varrho = \beta$. The construction in [J1] is designed so that $\varrho = \alpha$ holds iff $\varrho(\bar\alpha) = \bar\alpha$. (Cf. the discussion of the following paragraph.) Moreover, in the case that $\alpha < \varrho$, then $\tilde\pi(\zeta) = \zeta$, for $\zeta < \bar\alpha$, and $\tilde\pi(\bar\alpha) = \alpha$. It follows readily that α satisfies clause (d) of the definition of S iff $\bar\alpha$ does. The verification of axiom (iv) of $SD_\varkappa$ is complete.

Discussion. Actually, there is a slight bug in the proof of Theorem 5.1 of [J1] (on which the proof of Theorem 5.2 of [J1] is based) in that no steps are taken in the construction to insure that if $\varrho = \alpha$, then $\varrho(\bar\alpha) = \bar\alpha$. This bug is easily repaired by a minor modification of the definition of the function t on page 279 of [J1].

2.5. We now define the "diamond sequence" $\langle X_\alpha;\ \alpha \in S\rangle$. The definition is by induction on α for $\alpha \in S$. The definition we give will be absolute for models of ZF$^-$ in which α has cardinality $\leqslant \varkappa$. The following definition of X_α will take place in the model J_δ (where δ is as in clauses (c) and (d) of the definition of S). (Recall that J_δ thinks that $\alpha = \varkappa^+$.) In making this definition we shall use the sequence $\langle X_\gamma;\ \gamma \in S \cap \alpha\rangle$ which is uniformly definable in J_δ.

Case 1. There is a subset of $\varkappa^+$, A, and a club subset C of $\varkappa^+$ such that for every $\gamma \in A$, one of the following three conditions does not hold: (a) the order type of C_γ is $\varkappa$; (b) $X_\gamma = A \cap \gamma$; (c) The limit points of C_γ are in C.

Let $\langle A, C\rangle$ be the least such pair in the canonical well-ordering of L. Set $X_\alpha = A$.

Case 2. Otherwise.

Then set $X_\alpha = \emptyset$.

2.6. It is now quite easy to verify axiom (v) of $SD_\varkappa$. Let $\alpha \in S$, and let $\bar\alpha$ be a limit point of C_α. Let $\bar\delta = \delta(\bar\alpha)$ and $\delta = \delta(\alpha)$. Then $X_{\bar\alpha}$ is definable in $J_{\bar\delta}$ be the same definition that defines X_α in J_δ. It follows that $\pi_*(X_{\bar\alpha}) = X_\alpha$. But $\pi_*(\zeta) = \zeta$ for $\zeta < \bar\alpha$. Hence $X_{\bar\alpha} = X_\alpha \cap \bar\alpha$. The verification of axiom (v) is complete.

2.7. We turn to the verification of axiom (vi). This will be by far the most difficult axiom to verify. Towards a contradiction, assume that (vi) fails. Let $\langle A, C\rangle$, be the L-least pair that counterinstances (vi). Let M be the Σ_1-elementary submodel of $J_{\varkappa^{+++}}$ generated by $\varkappa \cup \{\varkappa^{++}\}$. Let ψ: $M \to J_\beta$ be the transitive collapse map. Let δ be $\psi(\varkappa^{++})$.

If is clear that any set which is definable in $J_{\varkappa^{++}}$ from parameters $<\varkappa$ lies in M. In particular, $\varkappa$ and $\varkappa^+$ lie in M. Let α be $\psi(\varkappa^+)$. (Of course, $\psi(\varkappa) = \varkappa$.) We shall show that $\alpha \in S$, and eventually that α meets the three conditions that prevent $\langle A, C\rangle$ from being a counterexample. This will contradict the choice of $\langle A, C\rangle$, and show that axiom (vi) of $SD_\varkappa$ is satisfied.

Since $\varkappa^+$ is a cardinal in $J_{\varkappa^{+++}}$, it is clear that α is a cardinal in J_β. Moreover, it is clear from the definition of β that there is a $\Sigma_1(J_\beta)$ map of a subset of $\varkappa$ onto J_β. It follows that $\beta(\alpha) = \beta$ and that $n(\alpha) = 1$.

We next must argue that $p(\alpha) = \{\delta\}$. It is clear from the definitions of β and δ that there is a $\Sigma_1(J_\beta)$ map of a subset of $\varkappa$ onto J_β with parameter $\{\delta\}$. In view of the ordering of parameters used, (cf. § 1.2), it is clear that if $p(\alpha) \neq \{\delta\}$, then $p(\alpha) \subseteq \delta$. It follows that δ is Σ_1 definable from ordinals less than δ. Employing $\tilde\pi$, we see that $\varkappa^{++}$ is Σ_1 definable from ordinals less than $\varkappa^{++}$. But this contradicts a well-known consequence of the Mostowski collapse theorem to the effect that $J_{\varkappa^{++}}$ is a Σ_1 elementary submodel of L. Thus $p(\alpha) = \{\delta\}$.

2.8. We have verified that α satisfies the first three clauses of the definition of S. It is clear that the inverse of the collapse map ψ restricts to a Σ_ω-elementary embedding of J_δ into $J_{\varkappa^{++}}$ which maps α onto $\varkappa^+$. Hence clause (d) of the definition of S also clearly holds for α, and we have completed the verification that $\alpha \in S$.

2.9. We next show that $X_\alpha = A \cap \alpha$. Indeed, it is clear that the pair $\langle A, C\rangle$ is definable in $J_{\varkappa^{++}}$ and hence lies in M. Since M contains all the ordinals $\leqslant \varkappa$, it is clear that the intersection of M with $\varkappa^+$ is an initial segment of $\varkappa^+$ (and therefore must be α). It follows that the map ψ is the identity on the ordinals $<\alpha$. But then $\psi(\langle A, C\rangle) = \langle A \cap \alpha, C \cap \alpha\rangle$ satisfies the same definition in J_δ that $\langle A, C\rangle$ satisfies in $J_{\varkappa^{++}}$. If we now recall the definition of X_α, it follows readily that $X_\alpha = A \cap \alpha$. I.e., the pair $\langle A, C\rangle$ satisfies condition (vi–b) at α.

2.10. We next verify that $\langle A, C\rangle$ satisfies condition (vi–c) at α. Let $\bar\alpha$ be a limit point of C_α, and let π_*: $J_{\bar\delta} \to J_\delta$ be the canonical elementary embedding discussed in § 1.4. In view of the properties recalled there, it is clear that $\pi_*(C \cap \bar\alpha) = C \cap \alpha$. It follows that $C \cap \bar\alpha$ is closed unbounded in $\bar\alpha$, and thus that $\bar\alpha \in C$ (since C is closed). Thus $\langle A, C\rangle$ satisfies condition (vi–c) at α.

2.11. It remains to verify that the order type of C_α is precisely $\varkappa$. Recall that the construction of C_α in [J1] requires the construction of three auxiliary functions k, l, and m defined on some ordinal $\theta \leqslant \varkappa$. We shall prove first that $\theta = \varkappa$.

We begin by recalling some notation from the Jensen proof. Let h be the canonical Σ_1 Skolem function for $\langle J_\beta, \emptyset\rangle$. Recall that $p = \{\delta\}$. Set $\tilde h(i, x) = h(i, \langle x, p\rangle)$. Define a map g: $u \to \alpha$ where $u \subset \varkappa$ by

$$g(\omega\nu+i) \simeq \tilde h(i, \nu)$$

if $i<\omega$, $\nu<\varkappa$, and $\tilde h(i, \nu)<\alpha$; otherwise $g(\omega\nu+i)$ is undefined. Then g maps $\varkappa$ cofinally into α. Let γ be the least ordinal $\leqslant\varkappa$ such that g maps γ cofinally into α. The following lemma implies that $\gamma = \varkappa$.

LEMMA. *Let $\eta<\varkappa$. Let β_1 be the* sup *of $\tilde h''(\omega\times J_\eta)$. Let α_1 be the* sup *of $g''(\eta)$. Then $\alpha_1<\alpha$ and $\beta_1<\beta$.*

Proof. The proofs of the two claims are entirely similar. We show that $\beta_1<\beta$ and leave the proof that $\alpha_1<\alpha$ to the reader.

Let $D \subseteq \omega\times J_\eta$ be the precise domain of the restriction of $\tilde h$ to $\omega\times J_\eta$. Then $D \in J_\varkappa$. Let π: $J_\beta \to J_{\varkappa^{+++}}$ be the Σ_1 elementary embedding that is the inverse of the transitive collapse map ψ. The ordinal β_1 is Σ_1 definable from the parameters δ and D as follows: There is an ordinal ζ such that in J_ζ the function $\tilde h$ is defined at all points in D and the sup of its range is β_1. (To see that such a $\zeta<\beta$ exists, employ the map π and the fact that the cofinality of $\varkappa^{+++}$ is greater than $\varkappa$.) It follows that $\pi(\beta_1)<\varkappa^{+++}$, and hence that $\beta_1<\beta$.

2.12. It is now clear that the map g has domain an unbounded subset of $\varkappa$. If $\varkappa$ is regular, the fact that $\gamma = \varkappa$ readily implies that the cofinality of α is $\varkappa$, and thus that C_α has order type $\varkappa$. So from now on, we may assume that $\varkappa$ is singular, and therefore that $\varkappa$ is a limit cardinal.

Let $F \subseteq \omega\times J_\varkappa$ be the precise domain of $\tilde h$. We define a function k_*: $\varkappa\to\varkappa$ by transfinite induction as follows: Let $\zeta<\varkappa$. Then $k_*(\zeta)$ is the least ordinal η such that:

(1) $\sup\{k_*(\zeta_1)\colon \zeta_1<\zeta\}<\eta$.

(2) J_η is a model of ZF-.

(3) Let $\gamma<\eta$. Then γ is a cardinal in J_η iff γ is a cardinal in L.

(4) The structure $\langle J_\eta; F\cap J_\eta\rangle$ is amenable.

It is clear that $k_*(\zeta)$ is defined for every $\zeta<\varkappa$ (since any successor cardinal will satisfy requirements (2) through (4)). One can check easily by induction on $\zeta<\varkappa$ that $k_*(\zeta)<\max(\aleph_1, \zeta^+)$. (Here ζ^+, for any ordinal ζ is the least cardinal greater than ζ.) It follows that k_* maps $\varkappa$ into $\varkappa$.

We define functions $l_*: \varkappa \to \alpha$ and $m_*: \varkappa \to \beta$ as follows:

$$l_*(\zeta) \text{ is the sup of } g''k_*(\zeta).$$

$$m_*(\zeta) \text{ is the sup of } \tilde{h}''\omega \times J_{k_*(\zeta)}.$$

Lemma 1.11 is needed to see that $l_*(\zeta) < \alpha$ and $m_*(\zeta) < \beta$ for all $\zeta < \varkappa$. Moreover, the proof of Lemma 1.11 and the definition of k_* imply easily that $l_*(\zeta_1) < l_*(\zeta_2)$ and $m_*(\zeta_1) < m_*(\zeta_2)$ if $\zeta_1 < \zeta_2$.

We shall prove by induction on $\zeta < \varkappa$ that $k(\zeta) < k_*(\zeta)$, $l(\zeta) < l_*(\zeta)$, and that $m(\zeta) < m_*(\zeta)$. This will establish, in particular, that the common domain, θ, of k, l, and m is precisely $\varkappa$. Before beginning this inductive proof, we shall need to establish several lemmas. We remark that it is crucial for the lemmas that follow that $n(\alpha) = 1$, so that the canonical master code (A in Jensen's notation) is simply $\emptyset$.

2.13. Lemma. *Let ζ be a limit ordinal $< \theta$ (so that $m(\zeta)$ is defined), and let η be a primitive recursively closed ordinal such that $m(\zeta) < \eta < \beta$. Then $m(\zeta)$ and $l(\zeta)$ are $\Sigma_1(J_\beta)$ definable in the parameters ζ, η, and δ.*

Proof. We give the proof for $m(\zeta)$. The proof for $l(\zeta)$ is totally analogous.

If one carries out the definition of k, l, and m, on ordinals less than ζ *within the model* J_η one gets the same answers as when the definitions are carried out in J_β. This is easily checked by induction. The key point is that g_η agrees with g on ordinals of the form $k(\xi)$ (for $\xi < \zeta$) since $\eta > m(\zeta)$. (Here g_η is the analogue of g defined in J_η.) It follows that the sequence $\langle m(\xi);\ \xi < \zeta\rangle$ is Σ_1 definable from ζ, η, and δ in J_β. Since $m(\zeta)$ is the sup of this sequence, the lemma follows.

2.14. Lemma. *Let $\zeta < \varkappa$. Let $\delta < \eta < m_*(\zeta)$. Let g_η be the analogue of g defined in J_η. Let α_1 be the* sup of *$g_\eta''\varkappa$. Then $\alpha_1 < l_*(\zeta)$.*

Proof. By increasing η if necessary, we may assume that η is $\Sigma_1(J_\beta)$ definable from δ and some $\gamma_1 < k_*(\zeta)$. It suffices to see that α_1 is $\Sigma_1(J_\beta)$ definable from δ and γ_1. But the definition of α_1 in the statement of the lemma provides such a Σ_1 definition, if we replace η by a Σ_1 definition of η from δ and γ_1. The only possible problem is that the sup used to define α_1 might yield α. However, applying π to this sup, we get a sup of at most $\varkappa$ ordinals less than $\varkappa^+$. Thus π maps the sup into an ordinal less than $\varkappa^+$, and it follows that the sup is less than α.

2.15. It is now quite straightforward to prove the stated bounds on $k(\zeta)$, $l(\zeta)$, and $m(\zeta)$, in terms of $k_*(\zeta)$, $l_*(\zeta)$, and $m_*(\zeta)$ by induction on ζ. Lemma 2.14 is used to get the estimate on $l(\zeta)$, and Lemma 2.13 is used to handle clause (iii) in the definition of $m(\nu+1)$ on p. 284 of [J1]. The proof that $\theta = \varkappa$ is complete.

2.16. In the Jensen construction of C_α, the sequence $\langle l(\zeta);\ \zeta < \theta\rangle$ is subjected to a further thinning in order to obtain C_α. We shall show that in the particular case at hand, no further thinning takes place. It will follow then that the order type of C_α is precisely $\varkappa$ since we have already established that $\theta = \varkappa$.

In the first place, it is easily seen that $m(\theta) = \delta+1$. It follows (in the terminology of p. 279 of [J1] that $\alpha \in X_0$ (since α is definable in J_δ). Also, $l(0)$ is clearly $\geqslant \varkappa$. Hence $t(0) = 0$. (Cf. [J1] loc. cit. for the definition of t.)

We continue to use X_ν in the sense of [J1] It is clear that $X_\nu \cap J_\delta$ is an elementary submodel of J_δ. Since $\varkappa \subseteq X_\nu$, and $\varkappa \in X_\nu$, it follows that $\alpha \cap X_\nu$ is an initial segment of α. But then $\alpha \cap X_\nu$ must be $l(\nu)$. It follows that the transitive collapse of $X_\nu \cap J_\delta$ is a model of ZF-, and $l(\nu)$ is the $\varkappa^+$ of this model. Hence $l(\nu)$ is a limit point of Q. (Here Q is the set of ordinals greater than ω that are closed under the Gödel pairing function.) It is now quite easy to check that $t(\nu) = \nu$ by induction on $\nu < \varkappa$.

In view of our previous remarks, this establishes that C_α has order type $\varkappa$. I.e. the pair $\langle A, C\rangle$ satisfies condition (vi–a) at α. We have checked that $\langle A, C\rangle$ satisfies all the requirements of (vi) at α, and that $\alpha \in S$. This contradicts our choice of $\langle A, C\rangle$. Since the assumption that $SD_\varkappa$ is false has led to a contradiction, our proof of $SD_\varkappa$ (from $V = L$) is complete.

3. A strengthening of the principle $\square_\varkappa$.

3.1. Throughout this section, we assume $V = L$. Let $\varkappa$ be an uncountable cardinal, fixed once for all. Our work will rely heavily on the first sections of [J1], and we refer the reader to that paper for any unexplained notation. We recall the statement of the combinatorial principle $\square_\varkappa$: There is a sequence C_α defined for α a limit ordinal $< \varkappa^+$ such that

(i) C_α is closed and unbounded in α;
(ii) the order-type of C_α is $\leqslant \varkappa$;
(iii) if γ is a limit point of C_α, then $C_\gamma = \gamma \cap C_\alpha$.

3.2. Discussion. In [J1], condition (ii) is replaced by the following stronger condition:

(ii*) if $\mathrm{cf}(\lambda) < \varkappa$, then C_λ has order-type less than $\varkappa$.

It is not hard to show that if there is a sequence of C's satisfying conditions (i) to (iii), then there is a sequence which satisfies, in addition, the condition (ii*). However, we shall be interested in sequences which satisfy, in addition to (i) through (iii) a certain further condition (iv). If $\varkappa$ is singular and of uncountable cofinality, then condition (iv) is incompatible with condition (ii*).

3.3. Our next task is to describe the additional condition (iv) that we shall impose on our square-sequences. Roughly speaking, for certain α, there is a natural choice of C_α. Condition (iv) says that we make this natural choice whenever we can.

Let α be an ordinal greater than $\varkappa$ such that for some ordinal δ greater than α, J_δ is a model of ZF^- which thinks that α is the least cardinal greater than $\varkappa$. In this situation, we define a sequence of elementary submodels of J_δ, as follows: M_0 is the elementary submodel of J_δ generated by the ordinals $\leqslant \varkappa$. $M_{\xi+1}$ is defined iff M_ξ is defined and is a member of J_δ. In that case, $M_{\xi+1}$ is the elementary submodel of J_δ generated by $M_\xi \cup \{M_\xi\}$. Finally, if λ is a limit ordinal, and M_ξ is defined for all $\xi < \lambda$, then

$$M_\lambda = \bigcup_{\xi<\lambda} M_\xi.$$

We say that $\alpha \in A$, if for some δ as above, and some limit ordinal $\lambda \leqslant \varkappa$, we have $M_\lambda = J_\delta$. It is not hard to see that the δ which serves to put α into A is completely determined by α. For suppose that δ_1 and δ_2 can both play the role of δ in the preceding definition, and $\delta_1 < \delta_2$. Then α has cardinality $\varkappa$ in the model J_{δ_2}, (cf. the argument of the following paragraph) contradicting one of the requirements imposed on δ.

Let $\alpha \in A$, and let δ, λ, be as in the definition of A. It is easy to prove by induction on ξ that M_ξ has cardinality $\varkappa$ in J_δ for each $\xi < \lambda$. (The proof uses the fact that $M_{\xi+1} \in J_\delta$.) It follows that $M_\xi \cap \alpha$ is an ordinal less than α. We set

$$C_\alpha^A = \{M_\xi \cap \alpha |\ \xi < \lambda\}.$$

We leave it as an exercise to the reader to verify that C_α^A is a closed unbounded subset of α of order-type $\leqslant \varkappa$, and that if γ is a limit point of C_α^A, then $\gamma \in A$, and $C_\gamma^A = C_\alpha^A \cap \gamma$.

We can now formulate precisely condition (iv): Let $\alpha \in A$. Then $C_\alpha = C_\alpha^A$. The following result is the main theorem of this section.

3.3. Theorem. *Assume $V = L$. Let $\varkappa$ be an uncountable cardinal. Then there is a sequence C_α defined for α a limit ordinal $< \varkappa^+$ which satisfies conditions* (i) *through* (iv).

We remark that if δ is admissible, then $J_\delta = L_\delta$. This certainly happens if either J_δ or L_δ is a model of ZF$^-$.

3.4. The obvious first attempt at proving Theorem 3.3 would be to use the Jensen square-sequence C_α^J for $\alpha \notin A$, and our given sequence C_α^A for $\alpha \in A$. This runs into the following difficulty. It might happen that α is not in A, but some limit point of C_α^J lies in A. In this case, we would have a conflict between conditions (iv) and (iii) at this limit point. We get around this difficulty by the following approach: We shall partition the limit ordinals less than $\varkappa^+$ into a finite number of pieces $E_1, \ldots, E_n$. (In particular, we will have $A = E_n$.) We shall, at stage i of our construction, define C_α for $\alpha \in E_i$. We shall do this in such a way that if γ is a limit point of C_α, and $\alpha \in E_i$, then $\gamma \in E_i$. We shall also arrange that C_α satisfies (i) through (iii) "on E_i". If we can do this, then it is clear that the union of the different square systems (on the various E_i's) will satisfy conditions (i) through (iv) provided we use C_α^A on the piece E_n.

Discussion. In [J2, Chapter 6] an alternative proof of $\square_\varkappa$ is given, which selects, for many α's a "natural choice" of C_α. It is natural to conjecture that for $\alpha \in A$, this natural choice is just C_α^A. This conjecture, if true, would imply the main theorem of this section. Unfortunately, the conjecture is false.

3.5. We now outline the proof of Theorem 3.3 and make some comments on the ideas underlying the proof. The first phase of our proof will consist of a detailed study of the "Jensen invariants" (such as $\beta(\alpha)$ and $n(\alpha)$) of an α in A. We shall succeed in getting a complete characterization of membership in A in terms of these invariants and one other concept which we shall describe presently. The classes E_i will correspond to those α's failing to meet one of the clauses of the characterization.

In the second phase of the proof, we shall define the E_i's, and define the set C_α for $\alpha \in E_i$. In defining C_α for $\alpha \in E_i$, a delicate point will be to insure that if $\bar\alpha$ is a limit point of C_α, then $\bar\alpha \in E_i$. The C_α's we define will be derived from the C_α^J's of [J1], and an important tool in handling this point will be the naturality properties established in [J1] for the C_α^J construction. (We have in mind the following sorts of facts: If $\bar\alpha$ is a limit point of C_α^J, then $n(\bar\alpha) = n(\alpha)$, (equals n, say) and there is a canonical Σ_{n-1} map $\tilde\pi$ of $J_{\beta(\bar\alpha)}$ into $J_{\beta(\alpha)}$. Unfortunately, simple examples show that $\tilde\pi$ need not map the projecta of $\beta(\bar\alpha)$ onto the corresponding projecta of $\beta(\alpha)$. We get around this difficulty in the following way. The map $\tilde\pi$ does preserve the canonical parameters. In [J1] the parameter is an arbitrary member of the appropriate J_ϱ, and the "preservation of parameters" is not particularly useful. Instead, we take as possible parameters the finite sets of ordinals. Again, the map $\tilde\pi$ will preserve parameters, (if the theory of [J1] is redone with this new notion of parameter), and the ordinals which are members of the new parameters will frequently have concrete significance. (For example, if $\alpha \in A$, one of the canonical parameters associated with α will turn out to be $\{\delta\}$.) The fact that $\tilde\pi$ preserves these new parameters will be an adequate substitute for the missing "preservation of projecta".

This completes our sketch of the ideas underlying the proof of Theorem 3.3. We turn now to the details of the formal proof.

3.6. Let α be an ordinal with $\varkappa < \alpha < \varkappa^+$. Following [J1], p. 272–273, we let $\beta(\alpha)$ be the least $\beta \geqslant \alpha$ such that there is a $\Sigma_\omega(J_\beta)$ map of $\varkappa$ onto α; we let $n(\alpha)$ be the least integer $n \geqslant 1$ such that there is a $\Sigma_n(J_\beta)$ function which maps a subset of $\varkappa$ onto α.

Our next goal is to prove:

Lemma. *Let $\alpha \in A$. Then $\beta(\alpha) = \delta+1$. (The ordinal δ was defined when the class A was defined, and it was shown there that δ depends only on α.) Moreover, $n(\alpha) = 2$.*

Before proving this lemma, we shall have to develop some preliminary material on Σ_ω elementary submodels.

3.7. Lemma. *Let β be an ordinal. Then the following two sets are in natural* 1—1 *correspondence.*

(a) *The set of M such that M is a Σ_ω elementary submodel of J_β.*

(b) *The set of N such that $\beta \in N$ and N is a Σ_1 elementary submodel of $J_{\beta+1}$. The* 1—1 *correspondence associates to an M as in* (a), *the Σ_1 elementary submodel of $J_{\beta+1}$ generated by $M \cup \{\beta\}$. The inverse map assigns to an N as in* (b), *the set $N \cap J_\beta$.*

Proof. It is necessary to verify three points:

(1) Let N be a Σ_1 elementary submodel of $J_{\beta+1}$ such that $\beta \in N$. Let $M = N \cap J_\beta$. Then M is a Σ_ω elementary submodel of J_β.

(2) Let M, N be as in (1). Let N^* be the Σ_1 elementary submodel of $J_{\beta+1}$ generated by $M \cup \{\beta\}$. Then $N = N^*$.

(3) Let M_0 be a Σ_ω elementary submodel of J_β. Let N_0 be the Σ_1 elementary submodel of $J_{\beta+1}$ generated by $M_0 \cup \{\beta\}$. Let M_1 be $N_0 \cap J_\beta$. Then $M_0 = M_1$.

We first consider (1). It is clear that M is closed under ordered pairs, and that $J_\beta \in N$. Let $y \in J_\beta$ be Σ_ω definable from $x \in M$, we must show that $y \in M$. But y is clearly Σ_1 definable from J_β and x in $J_{\beta+1}$. Hence $y \in N$. It follows that $y \in M$.

We turn now to (2). It is clear that $N^* \subseteq N$. Let $y \in N$. We have to show that $y \in N^*$. Pick an integer k so large that $y \in S_{\omega\beta+k}$. There is a map f of J_β onto $S_{\omega\beta+k}$ such that $f \in J_{\beta+1}$ and f is Σ_1 definable in $J_{\beta+1}$ from the parameter β. Let $x \in N$ be such that $f(x) = y$. Then clearly $x \in M$. It follows readily that $y \in N^*$.

We turn to the proof of (3). It is clear that $M_0 \subseteq M_1$. Let $x \in M_1$. We must show that $x \in M_0$. Since $x \in M_1$, we have $x \in N_0$. Hence there is a $y \in M_0$ and a sufficiently large integer k such that x is Σ_ω definable from y, β in $S_{\omega\beta+k}$. It follows that there is a rudimentary function r such that $x = r(J_\beta, y)$. Using Lemma 1.2 of [J1], we see that x is Σ_0 definable from y, J_β in $J_\beta \cup \{J_\beta\}$. It follows readily that x is Σ_ω definable from y in J_β. Hence $x \in M_0$ as was to be proved.

3.8. We return to the situation of § 3.6. Thus we are given $\alpha \in A$, and an ordinal $\delta > \alpha$ as in the definition of the set of ordinals A. It is clear that $\beta(\alpha) > \delta$ since α is a cardinal in J_δ. Now let β be $\delta+1$. We shall show:

(1) There is no $\Sigma_1(J_\beta)$ map of $\varkappa$ onto α;

(2) There is a $\Sigma_2(J_\beta)$ map of $\varkappa$ onto α.

From (1) and (2), it will follow immediately that $\beta(\alpha) = \beta$ and that $n(\alpha) = 2$ as claimed in Lemma 3.6.

We begin with the proof of (1). Since $\alpha \in A$, there is a limit ordinal λ, and a chain of elementary submodels of J_δ, $\langle M_\xi\colon \xi < \lambda\rangle$ with union J_δ as discussed in § 3.3. For $\xi < \lambda$, we let N_ξ be the Σ_1 submodel of $J_{\delta+1}$ generated by $M_\xi \cup \{\delta\}$. Clearly $J_{\delta+1}$ is the union of the N_ξ's.

If (1) is false, then there is a parameter $p \in J_{\delta+1}$ such that the Σ_1 submodel of $J_{\delta+1}$ generated by $\varkappa \cup \{p\}$ contains all ordinals less than α. But we can find a $\xi < \lambda$ such that $p \in N_\xi$. Since N_ξ contains all the ordinals less than $\varkappa$, it follows that it must contain all the ordinals less than α. From the discussion in § 3.7, it follows that $N_\xi \cap J_\delta = M_\xi$. Since $\alpha \subseteq J_\delta$, we must have $\alpha \subseteq M_\xi$. But we have already seen in § 3.3 that $M_\xi \cap \alpha$ is bounded in α. Our assumption that (1) is false has led to a contradiction.

3.9. Our proof of (2) will be somewhat more involved. Let δ be an ordinal such that J_δ is a model of ZF-. Let E be the set of $\gamma < \delta$ such that $\varkappa < \gamma$ and J_γ is an elementary submodel of J_δ. The set E will play an important role both in our proof of (2) and in the remainder of the proof of the main theorem of this section.

Lemma. *Let M be an elementary submodel of J_δ such that $\varkappa \in M$ and the ordinals of M are not cofinal in δ. (This will certainly happen if $M \in J_\delta$.) Let γ be the sup of the ordinals in M. Then $\gamma \in E$.*

Proof. Since cofinally many ordinals in M are closed under the Gödel pairing functions, it suffices to establish the following: Let ζ_1, ζ_2, and η be ordinals such that $\zeta_1 < \zeta_2$, $\zeta_2 \in M$, and η is definable from ζ_1 in J_δ. Then $\eta < \gamma$.

Fix $n \in \omega$ such that η is Σ_n definable from ζ_1 in J_δ. Let η_1 be the sup of those ordinals which are Σ_n definable in J_δ from an ordinal less than ζ_2. Clearly $\eta \leqslant \eta_1$. Using the fact that satisfaction for Σ_n formulae is expressible in the language of set-theory, it is easy to see that η_1 is definable in J_δ from ζ_2. Hence $\eta_1 \in M$. It follows that $\eta \leqslant \eta_1 < \gamma$, as was to be shown.

Corollary. *Let $\alpha \in A$ and let $\delta = \delta(\alpha)$. Then if E defined as above, then E is cofinal in δ.*

Proof. This is immediate from the preceding Lemma and the definition of the set A.

3.10. Lemma. *Suppose that J_δ is a model of ZF$^-$. Let $E \subseteq \delta$ be defined as in the preceding paragraph. Then E is a $\Pi_1(J_{\delta+1})$ subset of δ (in the parameters $\varkappa$ and δ.)*

Proof. The ordinal ζ lies in E iff $\varkappa < \zeta < \delta$ and for every $\eta < \zeta$ and every ordinal η_1 which is $\Sigma_1(J_{\delta+1})$-definable from the parameters η and δ, we have $\eta_1 < \zeta$. (This follows readily from the results in § 3.7.) The lemma is now clear.

3.11. Suppose that $\alpha \in A$, that $\delta = \delta(\alpha)$, and that E is as defined above. Let M_ξ be one of the chain of elementary submodels that witnesses $\alpha \in A$. Finally, let $\zeta \in E$ be such that ζ is greater than every ordinal in M_ξ. Since $\varkappa < \zeta$, and J_ζ is an elementary submodel of J_δ, it is easy to see that if the sequence M^*_η is defined for $\eta \leqslant \xi$ from J_ζ the way that the sequence M_η was defined from J_δ, then $M^*_\eta = M_\eta$ for all $\eta \leqslant \xi$. Moreover, the whole process (which generates the sequence of M^*_η's from J_ζ) can clearly be carried out within J_δ.

It is now easy to see that the function with domain λ which assigns to the ordinal ξ the model M_ξ is $\Sigma_2(J_{\delta+1})$. Indeed $N = M_\xi$ iff there is an ordinal $\zeta \in E$ such that if M^*_ξ is computed from J_ζ as described in the preceding paragraph, then the computation works and has output N. It follows readily that the function that lists C^A_α in increasing order is also $\Sigma_2(J_{\delta+1})$. The proof of (2) now follows easily from the facts that $\lambda \leqslant \varkappa$ and that α is the least cardinal greater than $\varkappa$ in J_δ. As we remarked previously, Lemma 3.6 follows immediately from the claims (1) and (2).

3.12. Our treatment of parameters differs in some minor, but essential, ways from that of [J1]. In the first place, we only use finite sets of ordinals as parameters. (It is clear that there is no loss of generality in doing this since, for any β, every element of J_β is $\Sigma_1(J_\beta)$ definable from some finite set of ordinals less than $\omega\beta$.) We use the following standard well-ordering of finite sets of ordinals: Let $A = \{\alpha_1, \ldots, \alpha_r\}$ and $B = \{\beta_1, \ldots, \beta_s\}$ be finite sets of ordinals listed in strictly decreasing order. Then A is less than B if, for some k, we have $\alpha_i = \beta_i$ for $i < k$, and either (a) α_k is undefined, and β_k is defined; or (b) $\alpha_k < \beta_k$. (The following alternative description makes it clear that this is a well-ordering: If A, as above, is a finite set of ordinals, let $f(A)$ be the ordinal:

$$\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}.$$

Then A is less than B in the ordering described above iff $f(A) < f(B)$ *qua* ordinals.)

We shall also adopt a slightly different definition of the ordinal $p(\alpha)$ that appears in the proof of $\square_\varkappa$ in [J1]. Let $\alpha \in S$, and let $\beta(\alpha)$, $n(\alpha)$ be as previously defined. Then $\varrho_\beta^n = \varkappa$. We set $\varrho = \varrho(\alpha) = \varrho_\beta^{n-1}$, and let $A = A(\alpha) = A_\beta^{n-1}$ be the corresponding master code. We let $p(\alpha)$ be the least parameter p such that there is a $\Sigma_1(\langle J_\varrho, A\rangle)$ map of a subset of $\varkappa$ onto α with parameters $\varkappa$, p. The construction of square-sequences in Section 5 of [J1] can easily be adapted to these new conventions, and we leave it to the reader to carry this out. (In particular, in various places where in the original proof $p(\alpha)$ appears, in the revised proof the pair $\langle \varkappa, p(\alpha)\rangle$ will appear.)

3.13. We return to our study of what the Jensen invariants amount to for $\alpha \in A$. Let $\beta = \beta(\alpha) = \delta+1$. Let $\varrho = \varrho_\beta^1$ be the Σ_1 projectum of β, and let $p^* = p_\beta^1$ be the corresponding canonical parameter. (We use, of course, the conventions on parameters just indicated in the preceding paragraph.)

LEMMA. *With notations as above, we have $\varrho = \delta$ and $p^* = \{\delta\}$.*

Proof. Since ϱ is an admissible ordinal less than $\delta+\omega$, we clearly have $\varrho \leqslant \delta$. Towards a contradiction, assume that $\varrho < \delta$.

We first wish to argue that all the members of p^* are ordinals $\leqslant \delta$. Indeed the minimality of p^*, and the particular well-ordering of finite sets of ordinals employed insure that any ordinal which is $\Sigma_1(J_{\delta+1})$ in a finite set of smaller ordinals cannot be a member of p^*. In particular, p^* must consist of limit ordinals, and hence (since it is a member of $J_{\delta+1}$) of ordinals $\leqslant \delta$.

Now let $\zeta \in E$ be chosen so that (1) $\varrho < \zeta$; (2) ζ is greater than any member of $p^* \cap \delta$. (This choice is possible since E is cofinal in δ.) Let M be the Σ_1 elementary submodel of $J_{\delta+1}$ generated by $J_\zeta \cup \{\delta\}$. On the one hand, since $\varrho \cup \{p^*\}$ is a subset of M, we must have $M = J_{\delta+1}$. On the other hand, the results of § 3.7 show that $M \cap J_\delta = J_\zeta$. This contradiction establishes that $\varrho = \delta$.

It is clear that there is a $\Sigma_1(J_{\delta+1})$ map of δ onto $J_{\delta+1}$ with parameter $\{\delta\}$. Taking account of the well-ordering of parameters we are using, we see that if $p^* \neq \{\delta\}$, then $p^* \subseteq \delta$. Using the fact that δ is closed under the Gödel pairing function, it is quite easy to see that if there is a Σ_1 map of δ onto $J_{\delta+1}$ with parameter a finite subset of δ, then there is one with the parameter $\varnothing$. Thus towards a contradiction, we may assume that there is a $\Sigma_1(J_{\delta+1})$ map, f, of δ onto $J_{\delta+1}$ with parameter $\varnothing$.

Suppose then that $f(x) = J_\delta$. By the proof of Lemma 3.7, this is equivalent to a certain Σ_ω formula $\varphi(x)$ holding of x in J_δ. Let $\zeta \in E$ be such that $x \in J_\zeta$. Since J_ζ is an elementary submodel of J_δ, $\varphi(x)$ holds of x in J_ζ. But then $f(x) = J_\zeta$ holds in $J_{\zeta+1}$ and hence (since Σ_1 assertions are upward absolute) in $J_{\delta+1}$. This contradicts our assumption that $f(x) = J_\delta$ holds in $J_{\delta+1}$. The proof of the current lemma is complete.

3.14. We have to establish the value of one last parameter of α for $\alpha \in A$. Let $A_* = A_\beta^1$ be the canonical master code. (Thus $A_* \subseteq J_\varrho$, and for $B \subseteq J_\varrho$, B is $\Sigma_2(J_{\delta+1})$ iff it is $\Sigma_1(\langle J_\varrho, A_*\rangle)$.) Recall that $p = p(\alpha)$ is the least parameter such that there is a $\Sigma_1(\langle J_\varrho, A_*\rangle)$ map of a subset of $\varkappa$ onto J_ϱ with parameters $\varkappa$, p.

LEMMA. *Let $\alpha \in A$. Then $p(\alpha) = \varnothing$.*

Proof. We have seen in the proof of Lemma 3.6 that there is a $\Sigma_2(J_{\delta+1})$ map of a subset of $\varkappa$ onto $J_{\delta+1}$ with parameters $\varkappa$ and δ. It follows readily that there is a $\Sigma_1(\langle J_\varrho, A_*\rangle)$ map of $\varkappa$ onto J_ϱ with parameter $\varkappa$. This clearly implies that $p(\alpha) = \varnothing$.

3.15. We can now describe the characterization of A in terms of the Jensen invariants. The proof that this characterization is correct will be developed at the same time that we define the E_i's and the C_α's.

For α to be a member of A, it is necessary and sufficient that:

(a) $\varkappa < \alpha < \varkappa^+$;
(b) α is closed under Gödel's pairing function;
(c) each $\nu < \alpha$ has cardinality $\leqslant \varkappa$ in J_α;
(d) $n(\alpha) = 2$ and $\beta(\alpha)$ is the successor of an ordinal δ;
(e) the canonical parameter p_β^1 is equal to $\{\delta\}$;
(f) J_δ is a model of ZF^-;
(g) the canonical parameter $p(\alpha)$ (cf. § 3.14) is equal to $\varnothing$;
(h) the set E (cf. § 3.9) is cofinal in δ and has order-type $\leqslant \varkappa$.

3.16. We now turn to the main part of the proof where we shall define the sets $E_1, \ldots, E_n$; when we define E_i, we shall simultaneously define C_α for $\alpha \in E_i$. We shall do this so that the following requirements are met:

(i) If $\bar\alpha$ is a limit point of C_α (where $\alpha \in E_i$) then $\bar\alpha \in E_i$.
(ii) If $\bar\alpha$ is a limit point of C_α, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.
(iii) C_α will be a closed unbounded subset of α of order type at most $\varkappa$.
(iv) Either $E_i = A$ or E_i and A are disjoint.

We let R_i be the set of limit ordinals less than $\varkappa^+$ which are not members of $E_1 \cup \ldots \cup E_i$. We require of our construction that the E_i's are pairwise disjoint, and that for $i<n$, we have $A \subseteq R_i$. We will eventually succeed in getting $R_j = A$. We will then terminate the construction at the next stage, by setting $n = j+1$, $E_n = A$, and setting $C_\alpha = C_\alpha^A$ for $\alpha \in A$.

The following metaphor may perhaps be useful. We think of the E_i's as the different layers of an onion whose innermost layer is A. We strip away different subsets of the set of limit ordinals less than $\varkappa^+$ disjoint from A, until we are finally left with just the set A.

3.17. Following [J1, p. 283], we let S be the set of all limit ordinals α such that
(a) $\varkappa < \alpha < \varkappa^+$;
(b) α is closed under Gödel's pairing function;
(c) each $\nu < \alpha$ has cardinality $\leqslant \varkappa$ in J_α (i.e., some $f \in J_\alpha$ maps $\varkappa$ onto ν, provided $\nu \neq 0$).

Then S is closed unbounded in $\varkappa^+$, and it is easy to see that $A \subseteq S$. We shall take E_1 to consist of those limit ordinals $< \varkappa^+$ which are not in S. It is easy using the result of Lemma 5.15 of [J1] to define $C_\alpha^{E_1}$ meeting the requirements (i) through (iv) of our outline.

3.18. We let E_2 consist of those elements α of S which fall under Cases 2b and 3 of the Jensen procedure. (Cf. pages 272–273 of [J1].) That is, $\alpha \in E_2$ iff $\alpha \in S$ and either (1) α is an element of Q but not a limit point of Q or (2) $n(\alpha) = 1$ and $\beta(\alpha)$ is not a limit ordinal. It is clear that no α in E_2 is a member of A. Moreover, it is shown in section 5 of [J1], that all the α's in E_2 are of cofinality ω. We take C_α (for $\alpha \in E_2$) to be some set of order type ω cofinal in α.

By C_α^J we mean the square-sequence obtained by following the procedure of the proof of Theorem 5.2 of [J1] after the modifications in the treatment of parameters suggested in § 3.12 have been carried out. We also wish to omit those steps taken in [J1] to insure that if $\varkappa$ is singular, then the order-type of C_α is always $< \varkappa$. (Cf. Lemma 5.14 of [J1] where the phase of the argument we wish to omit is centered.) It is clear from the proof of Theorem 5.2 of [J1] that if $\alpha \in R_2$ and $\bar{\alpha}$ is a limit point of C_α^J, then $\bar{\alpha} \in R_2$.

3.19. We let E_3 consist of those elements α of R_2 such that $n(\alpha) \neq 2$. For $\alpha \in E_3$, we take C_α to be C_α^J. It is evident from Lemma 1.6 that E_3 is disjoint from A. In the course of the proof of Theorem 5.2 of [J1], it is established that if $\bar{\alpha}$ is a limit point of C_α^J (where $\alpha \in R_2$) then $n(\bar{\alpha}) = n(\alpha)$. It is easy now to check that requirements (i) through (iv) are satisfied on E_3. It is also clear that if $\alpha \in R_3$, and $\bar{\alpha}$ is a limit point of C_α^J, then $\bar{\alpha} \in R_3$.

3.20. Before introducing the class E_4, we need to establish some notation. Let $\alpha \in R_3$. Let $\beta = \beta(\alpha)$ be the least β such that α is singular in $J_{\beta+1}$. Let $\varrho = \varrho(\alpha)$ be the Σ_1 projectum of J_β. Finally, let $p^*(\alpha)$ be the least parameter p such that there is a $\Sigma_1(J_\beta)$ map of ϱ onto J_β with parameter p. We have already computed the value of $p^*(\alpha)$ in the case that $\alpha \in A$, and seen that in that case, $p^*(\alpha) = \{\delta\}$.

We put α into E_4 iff $\alpha \in R_3$ and one of the following three conditions is met:

(1) $p^*(\alpha)$ (which is a finite set of ordinals) has cardinality different from 1. (If (1) does not obtain, we let $\delta(\alpha)$ be the unique member of $p^*(\alpha)$.)

(2) $\delta(\alpha)$ is not the largest limit ordinal in $J_{\beta(\alpha)}$.

(3) $J_{\delta(\alpha)}$ is not a model of ZF^-.

We set, for $\alpha \in E_4$, $C_\alpha = C_\alpha^J$. It should be clear from Lemma 3.13 that E_4 is disjoint from A. Thus the only requirement for E_4 that is not evident is requirement (i). This requirement will follow immediately from the facts about the C_α^J (in the case that $n(\alpha) = 2$) that we now recall.

Suppose that $\alpha \in R_3$, and that $\bar{\alpha}$ is a limit point of C_α^J. Then there is a canonical Σ_0 map, π, of $\langle J_{\varrho(\bar{\alpha})}, A(\bar{\alpha})\rangle$ into $\langle J_{\varrho(\alpha)}, A(\alpha)\rangle$. Moreover, π has a canonical prolongation, $\tilde{\pi}$, to a Σ_1 map of $J_{\beta(\bar{\alpha})}$ into $J_{\beta(\alpha)}$ which sends the parameter $p^*(\bar{\alpha})$ into $p^*(\alpha)$.

It is also clear from the facts just recalled that if $\alpha \in R_4$, and $\bar{\alpha}$ is a limit point of C_α^J, then $\bar{\alpha} \in R_4$.

3.21. We put α in E_5 iff $\alpha \in R_4$ and $\varrho(\alpha) < \delta(\alpha)$. It is clear from Lemma 3.13 that E_5 is disjoint from A. Note that for any $\alpha \in R_4$, we have $\varrho(\alpha) \leqslant \delta(\alpha)$ since $\delta(\alpha)$ is the largest limit ordinal in $J_{\beta(\alpha)}$. If follows that for $\alpha \in R_5$, we have $\varrho(\alpha) = \delta(\alpha)$.

Before defining C_α for $\alpha \in E_5$, it will be necessary to develop a certain amount of preliminary material. For $\alpha \in R_4$, we let $E(\alpha)$ be the set of ordinals ζ less than $\delta(\alpha)$ such that $\varkappa < \zeta$ and J_ζ is an elementary submodel of J_δ.

Lemma. *Let* $\alpha \in R_4$. *Then* $\varrho(\alpha) = \delta(\alpha)$ *iff* $E(\alpha)$ *is cofinal in* $\delta(\alpha)$.

Proof. Since $\alpha \in R_4$, we know that $\beta(\alpha) = \delta(\alpha)+1$ and that $p^*(\alpha) = \{\delta(\alpha)\}$. It follows that $\varrho(\alpha)$ can be characterized as the least ordinal ϱ such that the Σ_1 elementary submodel of $J_{\delta(\alpha)+1}$ generated by $\varrho \cup \{\delta(\alpha)\}$ is all of $J_{\delta(\alpha)+1}$. In view of Lemma 3.7, $\varrho(\alpha)$ can be characterized as the least ordinal ϱ such the Σ_ω elementary submodel of J_δ generated by the ordinals less than ϱ is all of J_δ. If $E(\alpha) = \emptyset$, then $\varrho(\alpha) \leqslant \varkappa+1$. If $E(\alpha)$ has sup, say γ, strictly less than $\delta(\alpha)$, then $\varrho(\alpha) \leqslant \gamma+1$. We leave it to the reader to verify that if $E(\alpha)$ is cofinal in $\delta(\alpha)$, then $\varrho(\alpha) = \delta(\alpha)$.

Let $\alpha \in R_4$, and let $\bar{\alpha}$ be a limit point of C_α^J. Then we have mentioned previously (in § 1.20) the canonical Σ_1 map $\tilde{\pi}$ of $J_{\delta(\bar{\alpha})+1}$ into $J_{\delta(\alpha)+1}$. Since $\tilde{\pi}(\delta(\bar{\alpha})) = \delta(\alpha)$, it induces a map π_* of $J_{\delta(\bar{\alpha})}$ into $\sigma_{\delta(\alpha)}$. It follows from Lemma 1.7 that π_* is elementary. If greater precision is needed in specifying the map π_*, we shall refer to is as $\pi_*(\bar{\alpha})$ (if alpha can be supplied by the context) or as $\pi_*(\bar{\alpha}, \alpha)$ if absolute precision is required.

3.22. Lemma. *Let* $\alpha \in R_4$. *Let* $\bar{\alpha}$ *be a limit point of* C_α^J.

(a) *If* $\varrho(\alpha) = \delta(\alpha)$, *then the range of* $\pi_*(\bar{\alpha})$ *is not cofinal in* $\delta(\alpha)$.

(b) *If* $\varrho(\alpha) < \delta(\alpha)$, *and the limit points of* C_α^J *are cofinal in* C_α^J, *then there is a limit point* α^* *of* C_α^J *such that the range of* $\pi_*(\alpha^*)$ *is cofinal in* $\delta(\alpha)$.

Proof. We first prove (a). Let $\bar{\alpha}$ be the λ^{th} member of C_α^J. We use the notation of the proof of Theorem 5.2 of [J1]. Then the range of $\pi_*(\bar{\alpha})$ is a subset of the Σ_ω submodel of $J_{\delta(\alpha)}$ generated by $m(\lambda)$. (This uses Lemma 1.7.) By Lemma 3.21, $E(\alpha)$ is cofinal in $\delta(\alpha)$. But clearly, any member of $E(\alpha)$ which is greater than $m(\lambda)$ will be greater than any member of the range of $\pi_*(\bar{\alpha})$.

We turn to the proof of (b). If $\varrho(\alpha) < \delta(\alpha)$,then by Lemma 3.21, we can find $\zeta < \delta(\alpha)$ such that ζ is greater than every member of $E(\alpha)$. If α^* is a sufficiently large limit point of C_α^J, then $\zeta \in$ range $\tilde{\pi}(\alpha^*)$. Moreover the range of $\tilde{\pi}(\alpha^*)$ is an elementary submodel of $J_\delta(\alpha)$. But then Lemma 1.9 says that the range is cofinal in $\delta(\alpha)$.

3.23. We now define C_α for $\alpha \in E_5$.

Case 1. The set of limit points of C_α^J is not cofinal in C_α^J.

In this case, α has cofinality ω. We take C_α to be some ω-sequence cofinal in α.

Case 2. Otherwise.

In this case, we take C_α to consist of those limit points, $\bar{\alpha}$, of C_α^J such that $\pi_*(\bar{\alpha})$ maps cofinally into $\delta(\alpha)$.

The check that the requirements (i) through (iv) are met is somewhat less trivial than before. In the first place, it follows from Lemma 3.13 that if $\alpha \in A$, then $\varrho(\alpha) = \delta(\alpha)$. Thus E_5 is disjoint from A, and requirement (iv) is met. If $\alpha \in E_5$ falls under Case 1, then requirements (i) through (iii) are trivial, since C_α has no limit points.

If α_1 and α_2 are limit points of C_α^J (with $\alpha_1 < \alpha_2$), then it is easy to see that range $\pi_*(\alpha_1)$ is a subset of range $\pi_*(\alpha_2)$. It follows that, in Case 2, C_α consists of a tail of the limit points of C_α^J. Moreover, by part (b) of Lemma 3.22, C_α is non-empty. Requirement (iii) is now clear for α falling under Case 2 of the definition.

We next check requirement (i). Let $\alpha \in E_5$ fall under Case 2 of the definition, and let $\bar{\alpha}$ be a limit point of C_α. Let α_0 be the least point of C_α. It is not hard to see that $\pi_*(\alpha_0, \alpha)$ is the composition of the maps $\pi_*(\bar{\alpha}, \alpha)$ and $\pi_*(\alpha_0, \bar{\alpha})$. Since $\pi_*(\alpha_0, \alpha)$ maps cofinally into $\delta(\alpha)$, it follows that $\pi_*(\alpha_0, \bar{\alpha})$ maps cofinally into $\delta(\bar{\alpha})$. It now follows from part (a) of Lemma 3.22, that $\bar{\alpha} \in E_5$.

Finally, we check requirement (ii). Let $\bar{\alpha}$ and α be as in the preceding paragraph. Then $\bar{\alpha}$ is a limit point of C_α^J. By a previous remark, we know that $\bar{\alpha} \in R_4$. (Cf. the end of § 3.20.) Hence $C_{\bar{\alpha}}^J = C_\alpha^J \cap \bar{\alpha}$. Since $\bar{\alpha}$ is a limit point of C_α, $\bar{\alpha}$ is a limit of limit points of C_α^J. It follows that $\bar{\alpha}$ comes under Case 2 of the definition of C_α. Now let α_1 be a limit point of $C_{\bar{\alpha}}^J$. We have previously remarked that $\pi_*(\alpha_1, \alpha)$ is the composition of the maps $\pi_*(\alpha_1, \bar{\alpha})$ and $\pi_*(\bar{\alpha}, \alpha)$. Since $\pi_*(\bar{\alpha}, \alpha)$ maps $\delta(\bar{\alpha})$ into $\delta(\alpha)$ in an order-preserving cofinal fashion, it is clear that $\pi_*(\alpha_1, \bar{\alpha})$ maps cofinally into $\delta(\bar{\alpha})$ iff $\pi_*(\alpha_1, \alpha)$ maps cofinally into $\delta(\alpha)$. It is now quite easy to verify that requirement (ii) is satisfied.

Remark. Note that this is the first case where we did not simply set $C_\alpha = C_\alpha^J$.

3.24. Lemma. *Let $\alpha \in A$. The set $E(\alpha)$ has order-type $\leqslant \varkappa$.*

Proof. Recall from § 3.3 the sequence $\langle M_\xi;\ \xi \leqslant \lambda\rangle$ of elementary submodels of J_δ. For each $\xi < \lambda$, let γ_ξ be the sup of the ordinals in M_ξ. By Lemma 3.9, each of the γ_ξ's lies in $E(\alpha)$. We shall show that these are all the ordinals in $E(\alpha)$. It will follow that the order-type of $E(\alpha)$ is equal to $\lambda \leqslant \varkappa$.

Let E be $E(\alpha)$, and let E_1 be the set of γ_ξ's (for $\xi < \lambda$). It is easy to see that E_1 is a closed subset of δ. Towards a contradiction, assume that $E - E_1$ is not empty. Hence, if η is least in $E - E_1$, there must be a largest element γ of E_1 which is less than η. (It could happen that there are no elements of E_1 less than η. In that case, we set $\gamma = \varkappa$.) Suppose first that $\gamma = \gamma_\xi$. We have remarked previously (in § 3.11) that M_ξ is definable in J_δ from the parameters ξ and γ_ξ. Since ξ and γ_ξ lie in the model J_η, and J_η is an elementary submodel of J_δ, we have $M_\xi \in J_\eta$. Hence, $M_{\xi+1} \subseteq J_\eta$, and therefore $\gamma_{\xi+1} \leqslant \eta$. This contradicts the choice of γ. The case when $\gamma = \varkappa$ is quite similar and is left to the reader.

3.25. We are now in a position to define the set E_6. An ordinal α will be put in E_6 iff $\alpha \in R_5$ and either (a) the parameter $p(\alpha)$ is not $\varnothing$ or (b) the set $E(\alpha)$ has order-type greater than $\varkappa$. It follows readily from Lemmas 3.14 and 3.24 that E_6 is disjoint from A. Hence requirement (iv) of § 3.15 is satisfied.

The definition of C_α for $\alpha \in E_6$ will be somewhat involved. Since $\alpha \in R_5$, we have $\delta(\alpha) = \varrho(\alpha)$. Recall also from the definition of C_α^J, the functions k, l and m, which are all defined on some limit ordinal $\theta \leqslant \varkappa$. There is also an ordinal $\tilde{\theta} < \theta$ and a normal map $t\colon \tilde{\theta} \to \theta$. We let $\tilde{l}\colon \tilde{\theta} \to \alpha$ be defined by $\tilde{l}(\eta) = l(t(\eta))$. Similarly, we define $\tilde{m}\colon \tilde{\theta} \to \varrho$ by $\tilde{m}(\eta) = m(t(\eta))$. The set C_α^J is precisely the set of $\tilde{l}(\eta)$ for $\eta < \tilde{\theta}$.

The function $\langle \tilde{m}(\eta);\ \eta < \tilde{\theta}\rangle$ enumerates in increasing order a club subset of $\varrho(\alpha) = \delta(\alpha)$. Also, the set $E(\alpha)$ is a club subset of $\delta(\alpha)$ by Lemma 3.21. Thus, we can define a closed subset K of $\tilde{\theta}$ by stipulating that $\eta \in K$ iff $\tilde{m}(\eta) \in E(\alpha)$.

Case 1. K is not cofinal in $\tilde{\theta}$.

In this case, the intersection of the two club subsets, $E(\alpha)$ and range $\tilde{m}$ is not cofinal in $\delta(\alpha)$. Hence $\delta(\alpha)$ is cofinal with ω. It follows that $\tilde{\theta}$ and α are also cofinal with ω. We take C_α to be some ω-sequence cofinal in α. It is clear that requirements (i) through (iii) are met in this case.

Case 2. K is cofinal in $\tilde{\theta}$ but $p(\alpha) \neq 0$.

In this case, we take C_α to be $\{\tilde{l}(\eta)\colon \eta \in K\}$. It is clear that C_α is a closed subset of C_α^J. So requirement (iii) is clearly met.

Let $\bar{\alpha}$ be a limit point of C_α. We shall show that $\bar{\alpha} \in E_6$ and indeed that $\bar{\alpha}$ comes under the current Case 2. It is clear from the definition of C_α that $\bar{\alpha}$ is a limit point of C_α^J. Hence, by the remark at the end of § 3.20, $\bar{\alpha} \in R_4$. Also by the analogue (for our current version of C_α^J) of Lemma 5.9 (a) of [J1], we have $p(\bar{\alpha}) \neq 0$.

Let $\lambda < \tilde{\theta}$ be such that $\bar{\alpha} = \tilde{l}(\lambda)$. Let $A(\alpha)$ be as in the next-to-last paragraph of § 3.20. Similarly, let π be the canonical Σ_0 map of $\langle J_{\varrho(\bar{\alpha})}, A(\bar{\alpha})\rangle$ into $\langle J_{\varrho(\alpha)}, A(\alpha)\rangle$. Let M_0 be the range of π. As discussed in § 3.21, π has a canonical prolongation to a Σ_ω map, π_*, of $J_{\delta(\bar{\alpha})}$ into $J_{\delta(\alpha)}$. Let M_1 be the range of π_*.

M_0 can be characterized as the Σ_1-elementary submodel of $\langle J_{\tilde{m}(\lambda)};\ A(\alpha) \cap J_{\tilde{m}(\lambda)}\rangle$ generated by $\varkappa \cup \{\langle \varkappa, p(\alpha)\rangle\}$. Moreover, it is clear from the construction in [J1], that $\{m(\eta)\colon \eta < t(\lambda)\}$ is a subset of M_0. M_1 is just the Σ_ω elementary submodel of $J_{\delta(\alpha)}$ generated by M_0. Since M_0 is a subset of $J_{m(\lambda)}$, and $m(\lambda) \in E(\alpha)$, it follows that $M_1 \subseteq J_{m(\lambda)}$.

$J_{\delta(\bar{\alpha})}$ is the transitive collapse of M_1. Let ψ be the transitive collapse map for M_1. Let E_* be the set of ordinals of the form $\psi(\tilde{m}(\eta))$ (for $\eta \in (K \cap \lambda)$). It is clear from the remarks above that E_* is cofinal in $\delta(\bar{\alpha})$. Since ψ is a Σ_ω-elementary map, it is clear that if $\eta_1 < \eta_2$ are members of E_*, then J_{η_1} is an elementary submodel of J_{η_2}. It follows that $E_* \subseteq E(\bar{\alpha})$. So $E(\bar{\alpha})$ is cofinal in $\delta(\bar{\alpha})$. By Lemma 3.21, $\bar{\alpha} \in R_5$. Moreover, if $\bar{k}$, $\bar{l}$, $\bar{m}$, and $\bar{t}$ play for $\bar{\alpha}$ the same roles that k, l, m, and t play for α, then it follows from what we have just shown that if $\eta \in (K \cap \lambda)$, then $\bar{m}(\bar{t}(\eta)) \in E(\bar{\alpha})$. It follows from this and the fact that $p(\bar{\alpha}) \neq 0$, that $\bar{\alpha} \in E_6$, and falls under the current Case 2. The verification of requirement (i) for Case 2 is complete.

It remains to verify requirement (ii). In view of what we have done so far, (ii) amounts to the following claim. Let λ_1 be an ordinal less than λ. Then $\bar{m}(\bar{t}(\lambda_1)) \in E(\bar{\alpha})$ iff $m(t(\lambda_1)) \in E(\alpha)$. The "if" part of this claim has already been established. For the other direction, choose λ_2 such that $\lambda_1 < \lambda_2 < \lambda$ and $\lambda_2 \in K$. (This is possible since λ is a limit point of K.) If $\lambda_1 \in E(\bar{\alpha})$, then $J_{\bar{m}(\bar{t}(\lambda_1))}$ is an elementary submodel of $J_{\bar{m}(\bar{t}(\lambda_2))}$. Since π_* is Σ_ω, it follows that $J_{m(t(\lambda_1))}$ is an elementary submodel of $J_{m(t(\lambda_2))}$. But $m(t(\lambda_2)) \in E(\alpha)$. It follows that $m(t(\lambda_1)) \in E(\alpha)$. This completes the proof of the claim and thus shows that requirement (ii) is met in Case 2.

Case 3. Otherwise.

In this case, K is cofinal in $\bar{\theta}$, $p(\alpha) = \emptyset$, and the order-type of $E(\alpha)$ is greater than $\varkappa$. Let $\alpha_1 \in C^J_\alpha$. We put α_1 into C_α iff (1) $\alpha_1 \in E(\alpha)$ and (2) there are $\varkappa$ members of $E(\alpha)$ less than α_1. It is easy to see that C_α is a closed cofinal subset of C^J_α. Hence requirement (iii) is clearly satisfied in this case.

We now consider requirement (i). The argument is quite similar to that used in Case 2, and we use the notation established in that case. Let $\bar{\alpha}$ be a limit point of C_α. As before, we see that $\bar{\alpha} \in R_4$, that $\varrho(\bar{\alpha}) = \delta(\bar{\alpha})$ (so that $\bar{\alpha} \in R_5$), and that $K(\bar{\alpha})$ is cofinal in λ, (so that if $\bar{\alpha} \in E_6$, then $\bar{\alpha}$ does not come under Case 1). Moreover, since $p(\alpha) = \emptyset$, and π maps $p(\bar{\alpha})$ to $p(\alpha)$, it is clear that $p(\bar{\alpha}) = \emptyset$. (Thus, if $\bar{\alpha} \in E_6$, $\bar{\alpha}$ does not come under Case 2.)

To complete the proof that requirement (i) is satisfied, it suffices to show that the order-type of $E(\bar{\alpha})$ is greater than $\varkappa$. Let α_0 be the least element of C_α. Then $\alpha_0 \in E(\alpha)$, and there are $\varkappa$ members of $E(\alpha)$ less than α_0. An argument similar to that of § 3.11 shows that (for $\xi < \varkappa$), the ξth element of $E(\alpha)$ is definable in $J_{\delta(\alpha)}$ from α_0 and ξ. But M_1 is an elementary submodel of $J_{\delta(\alpha)}$ containing α_0 and all the ordinals $\leqslant \varkappa$. Since we have previously seen that the transitive collapse map for M_1 maps members of $E(\alpha)$ onto members of $E(\bar{\alpha})$, it is clear that the order type of $E(\bar{\alpha})$ is greater than $\varkappa$. The proof that requirement (i) is satisfied is complete.

The argument of the preceding paragraph shows that if $\xi < \varkappa$, the ξth member of $E(\alpha)$ is collapsed onto the ξth member of $E(\bar{\alpha})$. It is now quite easy to see that requirement (ii) is met along the lines used in Case 2. We leave the details to the reader.

3.26. We are now near the end of the road. To complete the proof, we have only to show that $R_6 = A$. The inclusion $A \subseteq R_6$ is clear. We fix $\alpha \in R_6$, and show that $\alpha \in A$.

Let λ be the order-type of $E(\alpha)$. The proof of Lemma 3.24 can be easily adapted to show that M_ξ is defined by the procedure of § 3.3 for $\xi \leqslant \lambda$, and that the ordinals of M_λ are cofinal in δ. To show that $\alpha \in A$ amounts to showing that $M_\lambda = J_\delta$. Fix $x \in J_\delta$. We shall show that $x \in M_\lambda$.

Since $p(\alpha) = \emptyset$, x is $\Sigma_1(\langle J_\delta, A_*\rangle)$ definable from $\varkappa$ and some ordinal ζ less than $\varkappa$. Therefore, if η is a sufficiently big member of $E(\alpha)$, x is $\Sigma_1(J_\delta)$ definable from $\varkappa$, ζ and $A_* \cap J_\eta$. Let $\xi < \lambda$ be such that η is the sup of the ordinals in M_ξ. We have to show that $x \in M_\lambda$. Since $\varkappa \cup \{\varkappa\} \subseteq M_\lambda$, and M_λ is an elementary submodel of J_δ it suffices to show that $A_* \cap J_\eta$ is a member of M_λ.

The set A_* is essentially the set of pairs $\langle i, x\rangle$ such that $i \in \omega$, $x \in J_\delta$, and $\varphi_i(x, \delta)$ holds in $J_{\delta+1}$. (Here $\langle \varphi_i\rangle$ is some recursive enumeration of the Σ_1 formulas with two free variables.) Let B_* be the set of triples $\langle i, n, x\rangle$ such that $i \in \omega$, $n \in \omega$, $x \in J_\delta$ and $\varphi_i(x, \delta)$ holds in $S_{\omega\delta+n}$. It is clear that $A_* \cap J_\eta$ is definable in J_δ from $B_* \cap J_\eta$.

Let $\langle \psi_i, i \in \omega\rangle$ be a recursive enumeration of the Σ_ω formulas with one free variable. Let C_* be the set of all pairs $\langle i, x\rangle$ such that $i \in \omega$, $x \in J_\delta$, and $\psi_i(x)$ holds in J_δ. The proof of Lemma 3.7 shows that $B_* \cap J_\eta$ is definable in J_δ from $C_* \cap J_\eta$.

Since $\eta \in E(\alpha)$, $C_* \cap J_\eta$ is essentially the satisfaction relation for J_η, and is definable from η in J_δ. So it suffices to see that $\eta \in M_\lambda$ to conclude that $A_* \cap J_\eta \in M_\lambda$. But it is clear from the construction of the M's that $M_\xi \in M_\lambda$, and hence, since η is the sup of the ordinals in M_ξ, that $\eta \in M_\lambda$.

This completes (1) the proof that $A_* \cap J_\eta \in M_\lambda$; (2) the proof that $x \in M_\lambda$; (3) the proof that $R_6 = A$; (4) the proof of Theorem 3.3.

4. Construction of a tree.

4.1. CONVENTIONS. (1) In what follows $\varkappa$ is an infinite uncountable cardinal, possibly singular. (2) The letter T is reserved for trees, the trees are of height $\leqslant \varkappa^+$, every level T_α is of cardinality $\varkappa$, and any point has $\varkappa$ many successors at any higher level (unless it is in a last level). For a set of ordinals C, $T|C = \bigcup_{\alpha \in C} T_\alpha$. (3) $[a, b]$ is the unordered pair of $a \neq b$. We define the square of T somewhat differently from usual in that we look at unordered pairs and the diagonal is omitted

$$[T]^2 = \{[a, b]|\ a \neq b \text{ are both in some } T_\alpha\}.$$

$[T]^2$ is partially ordered by

$$[a, b] < [c, d] \quad \text{iff } a < c \text{ and } b < d \text{ (or } a < d \text{ and } b < c)$$

(4) We define $Q = {}^{\omega}_{\circ}\varkappa - \{\emptyset\}$. So Q is the set of all nonempty finite sequences from $\varkappa$. Q is ordered lexicographically:

$h < g$ iff $g \subset h$ or, for some n, $g \restriction n = h \restriction n$ and $g(n) > h(n)$. (So the shorter sequence is bigger.)

Q has no maximal element since $\emptyset$ is omitted

(5) A *special map* is a function f: $[T]^2 \to Q$ which is order preserving.

$$[a, b] < [c, d] \to f(a, b) < f(c, d).$$

($f(a, b)$ is $f([a, b])$.) We say the tree T has a *special square* iff there is a special map defined on $[T]^2$.

4.2. THEOREM. *Assume* $\boxed{\Diamond}$, *then there is a Souslin tree of height* $\varkappa^+$ *with a special square.*

Before going into the proof proper we need some preliminaries.

4.3. DEFINITIONS. Let T be a tree and f a special map on T, fixed throughout the definition.

(1) The *imposition* on T, derived from f, is the function i: $[T]^2 \to Q$ defined by:

$$\text{If } f(a, b) = l^\frown\langle\zeta\rangle \text{ (where } l \neq \emptyset\text{), then } i(a, b) = l.$$

$$\text{If } f(a, b) = \langle\zeta\rangle, \text{ then } i(a, b) = \langle\zeta+1\rangle.$$

we clearly have $f(a, b) < i(a, b)$.

(2) Let i be the imposition on T derived from f, as defined above. Another function i^+: $[T]^2 \to Q$, called the imposition$^+$ on T derived from f, is defined thus:

Sh:221

$i^+(a,b) = i(a,b)^\frown\langle\xi_0\rangle$, where $\xi_0 \in \varkappa$ is the least ξ such that

$$f(a,b) < i(a,b)^\frown\langle\xi\rangle.$$

Equivalently, we can say:

If $f(a,b) = l^\frown\langle\zeta\rangle$, then $i^+(a,b) = l^\frown\langle\zeta+1\rangle$.

If $f(a,b) = \langle\zeta\rangle$, then $i^+(a,b) = \langle\zeta+1, 0\rangle$.

Of course, $f(a,b) < i^+(a,b) < i(a,b)$.

(3) Let $\alpha < \alpha' <$ height T. A map $e\colon T_\alpha \to T_{\alpha'}$ is called an orderer iff:

(a) $x < e(x)$ for every $x \in T_\alpha$.

(b) $i(a,b) = i(e(a), e(b))$ for every $a \neq b \in T_\alpha$.

(4) Similarly, $e^+\colon T_\alpha \to T_{\alpha'}$ is called an orderer$^+$ iff:

(a) $x < e^+(x)$ for $x \in T_\alpha$.

(b) $i^+(a,b) = i(e^+(a), e^+(b))$ for $a \neq b$ in T_α.

(5) Again, $e^*\colon T_\alpha \to T_{\alpha'}$ is called an orderer* iff

(a) $x < e^*(x)$ for $x \in T_\alpha$, and

(b) $i(a,b) = f(e^*(a), e^*(b))$ for $a \neq b$ in T_α.

(6) We say that α and α' are *in order* ($\alpha < \alpha' <$ height T) iff for any $x \in T_\alpha$ and $y \in T_{\alpha'}$ with $x < y$:

(a) There is an orderer $e\colon T_\alpha \to T_{\alpha'}$, such that $e(x) = y$.

(b) There is also an orderer$^+$, $e^+\colon T_\alpha \to T_{\alpha'}$, such that $e^+(x) = y$.

(c) If α' is a successor ordinal, then there is also an orderer*, $e^*\colon T_\alpha \to T_{\alpha'}$, with $e^*(x) = y$.

4.4. Lemma. *Let T be a tree and f a special map on $[T]^2$. Suppose $\alpha < \alpha' < \alpha''$ $<$ height T, and α, α' as well as α', α'' are in order, then α, α'' are in order too. In fact*

(1) *If $e\colon T_\alpha \to T_{\alpha'}$, $e'\colon T_{\alpha'} \to T_{\alpha''}$ are orderers, then $e' \circ e\colon T_\alpha \to T_{\alpha''}$ is an orderer.*

(2) *If $e^+\colon T_\alpha \to T_{\alpha'}$ is an orderer$^+$ and $e'\colon T_{\alpha'} \to T_{\alpha''}$ is an orderer, then $e' \cdot e^+\colon T_\alpha \to T_{\alpha''}$ is an orderer$^+$.*

(3) *If $e\colon T_\alpha \to T_{\alpha'}$ is an orderer and $e^*\colon T_{\alpha'} \to T_{\alpha''}$ is an orderer*, then $e^* \circ e\colon T_\alpha \to T_{\alpha''}$ is an orderer*.*

(4) *For e^+ and e^* as above, $h = e^* \circ e^+\colon T_\alpha \to T_{\alpha''}$ is an orderer satisfying $f(h(a), h(b)) = i^+(a,b)$.*

Proof is done by checking the definitions.

4.5. Lemma. *Suppose height of T is $\alpha+1$ (so T_α is the last level of T), f is a special map on $[T]^2$. Let T' be the extension of T formed by adding $\varkappa$ many successors to any point in T_α. Then it is possible to define f on $[T'_{\alpha+1}]^2$ so that $\alpha, \alpha+1$ are in order.*

Proof. We can find for each $y \in T'_{\alpha+1}$ three sets $A_{y,i}$ (i is a, b, or c) so that $y \in A_{y,i}$ and for any $x \in T_\alpha$ there is exactly one $x' \in A_{y,i}$ above x, and such that, for $i \neq j$ or $y \neq y'$, $A_{y,i}$ and $A_{y',j}$ contain at most one common point. Now we define f on $[A_{y,i}]^2$ so that condition (6) (i) is satisfied. Then, on the remaining pairs, f is defined so as to satisfy the requirement that f is order-preserving on pairs — 4.1 (5).

4.6. Definition of Systems. Let T be of height $\mu < \varkappa^+$. f a special map on $[T]^2$. *A system* $\mathscr{S}$ for T and f consists of:

(1) A closed unbounded $C \subseteq \mu$ of order type $\leqslant \varkappa$. (If $\mu = \beta+1$ then $\beta \in C$.)

(2) A collection $\{e(\alpha,\alpha')\colon \alpha, \alpha' \in C$ and $\alpha < \alpha'\}$ where $e(\alpha,\alpha')\colon T_\alpha \to T_{\alpha'}$ is an orderer and the maps are associative: for $\alpha < \alpha' < \alpha''$ in C, $e(\alpha,\alpha'') = e(\alpha',\alpha'') \circ e(\alpha,\alpha')$.

The *restriction* of the system $\mathscr{S}$ to $\mu' < \mu$ is defined naturally from $C \cap \mu'$ and the maps $e(\alpha,\alpha')$ for $\alpha' < \mu'$. $\mathscr{S}$ is said to be an *extension* of its restriction.

4.7. Definition of a good system. Let $\mathscr{S}$ be a system for T and f. Let $C = \{\gamma_i\colon i < \lambda\}$ be an increasing and continuous enumeration of the club of $\mathscr{S}$. We say that $\mathscr{S}$ is a *good system* iff:

(*) $\forall j \in \lambda$ $\forall a \neq b \in T_{\gamma_j}$ let $i^+(a,b) = l^\frown\langle\zeta\rangle$, and let ϱ be any ordinal with $i = j+1+\varrho < \lambda$, put $a' = e(\gamma_j,\gamma_i)(a)$ and $b' = e(\gamma_j,\gamma_i)(b)$, then $f(a',b') = l^\frown\langle\zeta+\varrho\rangle$.

4.8. Definition. Let $\mathscr{S}$ be a system for T and f. The equivalence relation $\equiv$ on $T|C$ (induced by $\mathscr{S}$) is defined by:

$x \equiv y$ iff $x = y$ or $x \in T_\alpha$, $y \in T_{\alpha'}$, $\alpha \neq \alpha'$ are in C (say $\alpha < \alpha'$) and $e(\alpha,\alpha')(x) = y$.

The equivalence class to which x belongs is denoted by $[x]$. We let $[x](\gamma)$ be the unique $y \in T_\gamma$ with $y \in [x]$; if there is no such y then $[x](\gamma)$ is undefined. Then $[x](\gamma)$ is defined for a cofinal segment of C.

4.9. Let $\mathscr{S}$ be a good system for T and f as above. Assume the height of T is a limit ordinal μ. We are going to define $T_\mu = T(\mathscr{S})$ (so that T with T_μ is a tree of height $\mu+1$), and extend f on $[T_\mu]^2$ in some definable way.

First, $T_\mu = \{[x]\colon x \in T|C\}$. We define

$$a < [x] \text{ iff } a < y \text{ for some } y \in [x].$$

It is clear that any $a \in T$ has some $[x] \in T_\mu$ above it. To define $f = f(\mathscr{S})$ on $[T_\mu]^2$, let be given $[x] \neq [y] \in T_\mu$ and let γ_j be the first γ in C such that $[x](\gamma)$ and $[y](\gamma)$ are defined. Put $a = [x](\gamma_j)$ and $b = [y](\gamma_j)$. $a \neq b$ for otherwise $[x] = [y]$. Let $l = i(a,b)$, then $l = i([x](\gamma), [y](\gamma))$ for any $\gamma \in C - \gamma_j$ (this follows from the properties of orderers). Let $i^+(a,b) = l^\frown\langle\zeta\rangle$. Since $\mathscr{S}$ is a good system, for any ϱ with $i = j+1+\varrho < \lambda$,

4.10. $f([x](\gamma_i), [y](\gamma_i)) = l^\frown\langle\zeta+\varrho\rangle$.

We now let ϱ_0 be such that $j+1+\varrho_0 = \lambda$, and define

4.11. $f([x],[y]) = l^\frown\langle\zeta+\varrho_0\rangle$.

We have

4.12. Lemma. (1) *The extended f is a special map on $[T \cup T_\mu]^2$*

(2) *The map $e(\gamma_i,\mu)\colon T_{\gamma_i} \to T_\mu$, defined by $x \to [x]$, is an orderer.*

(3) *By adding the maps $e(\gamma_i,\mu)$ we extend the good system $\mathscr{S}$ and get a good system.*

Proof. (1) Let $[a,b] \in [T]^2$ be below $[[x],[y]] \in [T_\mu]^2$, we have to show $f(a,b) < f([x],[y])$. Since the order relation defined on Q is transitive, we can change $[a, b]$ by any higher pair which is below $[[x], [y]]$; for example by some $[[x](\gamma_i), [y](\gamma_i)]$. Now 4.10 and 4.11 clearly give the proof. (2) and (3) are left to the reader.

4.13. Let us turn to the proof of Theorem 4.2. Let $\varkappa$ be a singular cardinal and assume $\boxed{\diamondsuit}(\varkappa^+)$. We are going to define a Souslin tree T on $\varkappa^+$ with a special square.

The construction of T_μ is done by induction on $\mu < \varkappa^+$. The underlying set of T_α will be the ordinal interval $[\varkappa\cdot\mu, \varkappa\cdot(\mu+1))$. At stage μ, we will also define the restriction of the special map f to $[T_\mu]^2$, and if μ is limit, a good system $\mathscr{S}_\mu$.

Any two levels of T are in order. In fact, for any $\alpha < \alpha'$ the family $\mathcal{O}(\alpha, \alpha')$, of all orderers, orderers$^+$ and orderers* and composition of such functions from T_α into $T_{\alpha'}$ that are defined and used in the proof, is a family of cardinality $\varkappa$. If $A \subseteq T_\alpha$ consists of the first τ members of T_α, $\tau < \varkappa$ a limit, and if $e\colon T_\alpha \to T_{\alpha'}$ is in $\mathcal{O}(\alpha, \alpha')$, then $e[A]$ is called a *small set*. There are only $\varkappa$ many small subsets to T_α.

Successor stages. If $\mu = \alpha+1$ we use Lemma 3.5 to define $T_{\alpha+1}$ and to extend the special map f.

Limit stages. Assume $\mu < \varkappa^+$ is a limit ordinal and $T|\mu$ etc. is given. We are going to define a good system $\mathscr{S}_\mu$ and then define $T_\mu = T_\mu(\mathscr{S}_\mu)$ and extend f and define the orderers as in 4.9–4.12. The description of $\mathscr{S}_\mu$ depends uniformly on the squared diamond sequence of μ. Let $\langle S_\alpha |\ \alpha \in \lim \varkappa^+\rangle$ be a fixed $\boxed{\diamondsuit}$ diamond sequence.

Any S_γ, $\gamma < \mu$, can be interpreted as giving us two subsets of $T|\gamma$, so we write $S_\gamma = (A_\gamma, B_\gamma)$. (The interpretation is obtained using Gödel's correspondence $\gamma \leftrightarrow 2\times\gamma$.)

4.14. Without loss of generality we can assume B_γ is a dense subset of $T|\gamma$, and A_γ is either (1) or (2).

(1) A small subset of some T_β, $\beta < \gamma$. (So A_γ as a set of ordinals has a well-ordering of length $< \varkappa$.)

(2) All $T|\gamma$.

In case (1), S_γ is understood as part of a mission to define an orderer which sends any element of A_γ above a member of the dense set B_μ. In case (2), S_γ is understood as part of a mission which, when successful, makes any member of T_μ extends some member of B_μ. Let us give the details of the definition of $\mathscr{S}_\mu$.

Let $C_\mu \subseteq \mu$ be the closed unbounded subset of μ given by the square. We shall modify C_μ a little and replace any α in C_μ which is not a limit of members of C_μ by $\alpha+1$. Let C be the resulting closed unbounded subset of μ and let $\{\gamma_i |\ i < \lambda\}$ be the increasing and continuous enumeration of C. C is the closed unbounded set of $\mathscr{S}_\mu$. It remains to define the orderers $e(\alpha, \alpha')\colon T_\alpha \to T_{\alpha'}$ for $\alpha < \alpha'$ in C. This is done by induction on α'. $e(\alpha, \alpha')$ is picked from $\mathcal{O}(\alpha, \alpha')$. We have to be careful so that (*) of 4.7 finally holds.

Case α' is a limit. If α' is a limit ordinal and if the system $\mathscr{S}_{\alpha'}$ consists of the maps $e(\gamma_i, \gamma_j)$, $\gamma_j < \alpha'$, defined so far and of the club $C \cap \alpha'$, then we know $T_\alpha = T_\alpha(\mathscr{S}_{\alpha'})$ and we define the orderer $e(\gamma_i, \alpha')$ by the formula $x \to [x]$. Lemma 4.12 (3) promises that (*) of 4.7 still holds. If the premises above do not hold, then the definition breaks down.

Case α' is a successor. If $\alpha' = \gamma_{i+1}$ is a successor, it is enough to define first $e(\gamma_i, \gamma_{i+1})$ and then to complete by composition the remaining maps.

Put $\gamma = \gamma_{i+\omega}$ and look at $S_\gamma = (A_\gamma, B_\gamma)$. Two cases were mentioned in 4.14:

4.15. Assume case 3.14 (1) holds. Then $A_\gamma \subseteq T_\beta$ is a small set with an order-type $< \varkappa$. We assume $\beta \leqslant \gamma_i$, otherwise put $A_\gamma = \emptyset$ and $\beta = \gamma_i$ in the following. Let a be the ith member of A_γ (if there is no ith member to A_γ then put a to be the first member of T_{γ_i}). Put $e_0 = e(\beta, \gamma_i)$, an orderer which is assumed to be defined by the induction hypothesis. Let $a' = e_0(a)$. Now let $a'' \in T_{\alpha'}$ be the first extension of a' which is above some member of B_γ (if there is no such extension let a'' be the first extension of a' in $T_{\alpha'}$).

4.16. PROPOSITION. *There is in $\mathcal{O}(\gamma_i, \gamma_{i+1})$ an orderer $e\colon T_{\gamma_i} \to T_{\gamma_{i+1}}$ such that $e(a') = a''$ and*

$$f(e(x), e(y)) = i^+(x, y) \text{ for } [x, y] \in [T_{\gamma_i}]^2. \tag{4.17}$$

Proof. We know $\gamma_{i+1} \in C$ is a successor ordinal. Put $\gamma_{i+1} = \alpha+1$, so $\gamma_i < \alpha < \alpha+1$ (we can assume no two points in C are adjacent). Since any two levels of T are in order, there is an orderer$^+$, $e^+\colon T_{\gamma_i} \to T_\alpha$, and an orderer*, $e^*\colon T_\alpha \to T_{\alpha+1}$ such that $a'' = e^* \circ e^+(a')$. Now let $e = e^* \circ e^+$ (Lemma 4.4 (4)).

Now we define $e(\gamma_i, \gamma_{i+1})$ to be the first orderer as in this proposition.

4.18. Assume now case 3.14 (2) holds. So A_γ is $T|\gamma$ and B_γ is a dense subset of $T|\gamma$. If $i \geqslant \mathrm{cf}(\varkappa)$ we let $e = e(\gamma_i, \gamma_{i+1})$ be the first orderer satisfying (4.17). If $i < \mathrm{cf}(\varkappa)$ we proceed as follows.

We fix a bijection $\mathrm{cf}(\varkappa)\times\mathrm{cf}(\varkappa) \cong \mathrm{cf}(\varkappa)$, denoted by $\langle a_0, a_1\rangle = a$ (with $a_0, a_1 \leqslant a$). Also, $\langle \varkappa_j |\ j < \mathrm{cf}(\varkappa)\rangle$ is a continuous sequence cofinal in $\varkappa$ fixed in advance.

Now let $i = \langle i_0, i_1\rangle$, and define E to be the first $\varkappa_{i_0}$ members of $T_{\gamma_{i_1}}$. Put $E' = e(\gamma_{i_1}, \gamma_i)[E]$, E' is a small set. Let us ask: is there an orderer $d\colon T_{\gamma_i} \to T_{\gamma_{i+1}}$ (in $\mathcal{O}(\gamma_i, \gamma_{i+1})$) such that

(4.19) (i) $x \in d[E'] \to x$ is above some member of B_γ. and

(ii) $f(d(x), d(y)) = i^+(x, y)$ for any $[x, y] \in [T_{\gamma_i}]^2$?

If there is one, then we define $e(\gamma_i, \gamma_{i+1})$ to be the first such d. If there is not, then we let $e(\gamma_i, \gamma_{i+1})$ be the first orderer d which satisfies (ii) above.

This defines $\mathscr{S}_\mu$ and hence T_μ, and the inductive definition of T is presented. Before proving T is Souslin we need a lemma.

4.20. LEMMA. *If $A \subseteq T_\beta$ is a small set and if $B \subseteq T$ is dense, then there is $\mu > \beta$ and an orderer $T_\beta \to T_\mu$ which sends any member of A above a member of B.*

Proof. Let τ be the order type of A. Look at (A, B). There is a closed unbounded $D \subseteq \varkappa^+ - (\beta+1)$ consisting of ordinals greater than $\varkappa$ closed under Gödel's pairing function, such that if $\alpha < \zeta \in D$ and $x \in T$ then there is $y \in T_\zeta$ extending both x and an element of the sense set B. There is some $\mu < \varkappa^+$ such that $\mathrm{otp}(C_\mu) = \tau$ and $C_\mu \subseteq D$ and, for $\gamma \in C'_\mu$ $(A, B|\gamma) = S_\gamma$. Now, in the construction of S_μ, case 3.14 (1) always hold. Hence any $[x]$ for $x \in A$ is above a member of B and we have our orderer.

4.21. Claim. *T is a Souslin tree.*

Proof. Let $B \subseteq T$ be dense. Let $D \subseteq \varkappa^+$ be a closed unbounded set of ordinals greater than $\varkappa$ closed under Gödel's pairing function such that for $\alpha < \alpha' \in D$ any small set $A \subseteq T$ has an orderer $T_\alpha \to T_\mu$ as in the previous Lemma for $\mu < \alpha'$. Observe that if $\gamma = \alpha' + 1$ then an orderer d: $T_\alpha \to T_\gamma$ can be found which sends A above B and moreover satisfies 4.19 (ii) for $[x, y] \in [T_\alpha]^2$. To see this pick first any orderer$^+$ $T_\alpha \to T_{\alpha+1}$; look at the image of A; apply an orderer as in Lemma 4.20 and then an orderer*. (See Lemma 4.4 (4).)

Now we can find $\mu < \varkappa^+$ of cofinality $\mathrm{cf}(\varkappa)$ such that $C_\mu \subseteq D$ and for $\gamma \in C'_\mu$, $S_\gamma = (T|\gamma, B \cap \gamma)$. It follows now that any $[x] \in T_\mu$ extends some member of B.

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BEN GURION UNIVERSITY OF THE NEGEV
Beer Sheva
Israel
THE HEBREW UNIVERSITY
Jerusalem
Israel
UNIVERSITY OF CALIFORNIA
Berkeley, CA
U.S.A.

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A note on the Mac Dowell-Specker theorem

by

Peter G. Clote * (Boston)

Dedicated to G. Takeuti on his 60th birthday

Abstract. By using formalized recursion theoretic arguments, here reminiscent of a finite-injury priority argument, one can remove the countability assumption in the Kirby–Paris refinement of the Mac Dowell–Specker theorem on end extensions of models of arithmetic.

The Mac Dowell–Specker Theorem states that any model of Peano arithmetic has a proper elementary end extension. In [KiPa], L. Kirby and J. Paris refined this result obtaining a correlation between subsystems of Peano arithmetic and the existence of proper end extensions which are elementary with respect to Σ_n and Π_n formulas. Their result is.

Theorem 1. *For any countable model M of $I\Sigma_0$ and $n \geqslant 2$ $M \models B\Sigma_n$ iff M admits a proper n-elementary end extension K which satisfies $I\Sigma_0$.*

The Kirby–Paris construction used very strongly the countability of the model. In view of the cardinality-free statement of the Mac Dowell–Specker Theorem, we might expect the conclusion of Theorem 1 to hold for models of any cardinality. Such a possibility was first suggested by A. Wilkie. By using formalized recursion theoretic arguments (in a manner reminiscent of a simple priority argument mixed with G. Kreisel's proof of Gödel's second independence theorem), we obtain the desired result, thus answering Question 2 of [C1]. Since the early work of L. Kirby and J. Paris, many results in models of arithmetic have been obtained for *countable* models (consider also the notion of recursive saturation in the case of countable vs. uncountable models). G. Müller has mentioned the desire to extend results in models of arithmetic into the uncountable, so as to make precise those concepts and theorems which rely on cardinality considerations and those which do not. R. Kossak has established several results in this direction and the present note should be seen as a very minor contribution to this program.

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