CODING AND RESHAPING WHEN THERE ARE NO SHARPS

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ABSTRACT. Assuming $0^\sharp$ does not exist, κ is an uncountable cardinal and for all cardinals λ with $\kappa \leq \lambda < \kappa^{+\omega}$, $2^\lambda = \lambda^+$, we present a "mini-coding" between κ and $\kappa^{+\omega}$. This allows us to prove that any subset of $\kappa^{+\omega}$ can be coded into a subset, W of κ^+ which, further, "reshapes" the interval $[\kappa, \kappa^+)$, i.e., for all $\kappa < \delta < \kappa^+$, $\kappa = (\text{card } \delta)^{L[W \cap \delta]}$. We sketch two applications of this result, assuming $0^\sharp$ does not exist. First, we point out that this shows that any set can be coded by a real, via a **set** forcing. The second application involves a notion of abstract condensation, due to Woodin. Our methods can be used to show that for any cardinal μ, condensation for μ holds in a generic extension by a **set** forcing.

1. INTRODUCTION

Theorem. *Assume that $V \models ZFC +$ "$0^\sharp$ does not exist", and, in V, $\kappa \geq \aleph_2$, $Z \subseteq \kappa^{+\omega}$ and for cardinals λ with $\kappa \leq \lambda < \kappa^{+\omega}$, $2^\lambda = \lambda^+$.* **THEN** there is a cofinality preserving forcing $\mathbf{S}(\kappa) = \mathbf{S}(\kappa, Z)$ of cardinality $\kappa^{+(\omega+1)}$ such that if G is V-generic for $\mathbf{S}(\kappa)$, there is $W \subseteq \kappa^+$ such that $V[G] = V[W]$, $Z \in L[W, Z \cap \kappa]$, for all cardinals λ with $\kappa \leq \lambda < \kappa^{+\omega}$, and for all limit ordinals δ with $\kappa < \delta < \kappa^+$, $\kappa = (\text{card } \delta)^{L[W \cap \delta]}$.

Our forcing $\mathbf{S}(\kappa)$ can be thought of as a kind of Easton product between κ and $\kappa^{+\omega}$ of partial orderings which *simultaneously* perform the tasks of coding (§1.2 of [1]) and reshaping (§1.3 of [1]). Our new idea is to introduce an additional coding area used for "marking" certain ordinals. This "marking" technique is the crucial addition to the arguments of §1 of [1]. We appeal to the Covering Lemma twice: in (3.1), and again in the proof of the Proposition in (3.3). The referee has informed us that the hypothesis that $0^\sharp$ does not cannot be eliminated. Jensen first used this hypothesis in [1] to facilitate certain arguments, and then realized that *his* uses *were* eliminatable. It is not the purpose of this paper to discuss the nature of Jensen's appeals to the Covering Lemma; the interested reader may consult pp. 62, 96 and the Introduction to Chapter 8 of [1] for insight

H. Judah et al. (eds.), *Set Theory of the Continuum*
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into Jensen's uses of the Covering Lemma, and how he was able to eliminate them. In [2], S. Friedman presents a rather different, more streamlined approach to avoiding such uses of Covering. It should be clear from the preceding that Jensen's appeals to the Covering Lemma are of a rather different character than ours.

To better understand the role of this "marking" technique, let us briefly recall some material from [1]. Let us first consider the possibility of coding $R \subseteq \kappa^+$ into a subset of κ, when κ is regular. In order to use almost disjoint set coding, we seem to need extra properties of the ground model, or of the set R, since, in order to carry out the *decoding* recursion across $[\kappa,\ \kappa^+)$ we need, e.g., an almost disjoint sequence $\vec{b} = (b_\alpha : \alpha \in [\kappa,\ \kappa^+))$ of cofinal subsets of κ satisfying:

$$(*) \qquad \text{for all } \theta \in [\kappa,\ \kappa^+),\ (b_\alpha : \alpha \leq \theta) \in L[R \cap \theta], \text{ and is "canonically definable" there.}$$

Such a $\vec{b}$ is called *decodable*, and it is easy to obtain a decodable $\vec{b}$ if R satisfies:

$$(**) \qquad \text{for all } \theta \in [\kappa,\ \kappa^+),\ (\text{card } \theta)^{L[R\cap\theta]} = \kappa.$$

If $(**)$ holds, we say that R *promptly collapses fake cardinals.*

Of course, typically $(**)$ fails, and the "reshaping" conditions of §1.3 of [1] are introduced to obtain $(**)$ in a generic extension. Our κ and R, from the previous paragraph are called γ and B in §1.3 of [1]. Unfortunately, the distributivity argument for the reshaping partial ordering given there *seems* to really require not merely that $H_{\gamma^+} = L_{\gamma^+}[B]$, but that $H_{\gamma^{++}} = L_{\gamma^{++}}[B]$, where $B \subseteq \gamma^+$. This will be the case if B is the result of coding as far as γ^+, but that is another story, which leads to Jensen's original approach to the Coding Theorem. Our appeals to the Covering Lemma focus on this point: essentially, to prove a distributivity property of the reshaping conditions. As already indicated, in Jensen's treatment, the appeals to the Covering Lemma were designed to overcome different sorts of obstacles and proved to be eliminatable.

Our approach to guaranteeing that the unions of certain increasing chains of reshaping conditions collapse the suprema of their domains is to have "marked" a cofinal sequence of small order type. Because of the need to meet certain dense sets in the course of the construction, it is too much to expect that the ordinals we **intentionally** marked are the only marked ordinals. However, what we will be able to guarantee is that they are the

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only members of a certain club subset which have been marked. The club will exist in a small enough inner model, thanks to the Covering Lemma. This argument is given in (3.3). We are grateful to the referee for suggesting the use of "fast clubs" in the argument of (3.3). This allowed us to streamline a more complicated argument (which also suffered from some [probably reparable] inaccuracies) in an earlier version of this paper. We use "1" to mark ordinals. To guarantee that this does not collide with requirements imposed by the "coding" part of the conditions, we set aside the limit ordinals as the only potentially marked ordinals and do not use them for coding.

1.1. Summary and organization

We now give a brief overview of the contents of this paper. In §2, we build to the definition, in (2.5), of the $\mathbf{S}(\kappa)$, along with auxiliary forcings, $\mathbf{S}_k(\kappa)$. In §3, we prove that the $\mathbf{S}(\kappa)$ are as required. The heart of the matter is (3.3), where we prove the distributivity properties of the $\mathbf{S}_k(\kappa)$. Preliminary observations are given in (3.1) and (3.2). The former shows that only increasing sequences of certain lengths are problematical. The latter is a rather routine observation about how the coding works. In the argument of (3.3), we use this in the context of forcing over $\hat{\mathcal{N}}$, a transitive set model of enough ZFC, introduced in the proof of (3.3), below. In (3.4) we put together the material of (3.1)–(3.3) to prove the Theorem. In (3.5) we make a few remarks and briefly sketch the applications mentioned in the abstract.

The partial ordering $\mathbf{S}(T,\ \lambda)$, introduced in (2.2), below, is the analogue of the reshaping partial ordering of §(1.3) of [1]. It adds a subset of λ^+, which, together with T, promptly collapses fake cardinals in $(\lambda,\ \lambda^+)$. The partial ordering $\mathbf{P}_{\kappa,\ T,\ g}$, introduced in (2.4), is a version of the coding partial ordering of §(1.2) of [1], **relative to** g. We require that $T \subseteq \kappa^+$, $g \in S(T,\ \kappa^+)$. If $p \in P_{\kappa,\ T,\ g}$, then p will have the form $(\ell(p),\ r(p))$; $\ell(p)$ is the "function part" of p and $r(p)$ is the "promise part" of p. We require that $\ell(p)$ starts to code not only T, but also g **and** that $\ell(p) \in S(T \cap \kappa,\ \kappa)$. If g were not merely a condition but generic for $\mathbf{S}(T,\ \kappa^+)$, then $\mathbf{P}_{\kappa,\ T,\ g}$ would just be the usual forcing for the almost-disjoint set coding of the "join" of T and g, with the extra requirement above, that for conditions, p, $\ell(p)$, together with $T \cap \kappa$, collapses sup dom $\ell(p)$.

Finally, the $\mathbf{S}(\kappa)$, introduced in (2.5), is the forcing which accomplishes the task of coding and reshaping, between κ and κ^+. It is defined relative to the choice of a fixed $Z \subseteq \kappa^{+\omega}$ such that $H_{\kappa^{+n}} = L_{\kappa^{+n}}[Z \cap \kappa^{+n}]$, for all $n \leq \omega$. The elements of $S(\kappa)$, are ω-sequences, $(p(n) :\ n < \omega)$, where

for all $n < \omega$, $p(n) = (\ell(p(n)),\ r(p(n)))$, $\ell(p(n)) \in S(Z \cap \kappa_n,\ \kappa_n)$ and $p(n) \in P_{\kappa_n,\ Z\cap\kappa_{n+1},\ \ell(p(n+1))}$. Thus, letting $\dot{G}$ be the canonical name for the generic of $\mathbf{S}(\kappa)$, letting $\dot{G}(n)$ be the canonical name for $\{\ell(p(n)) : p \in \dot{G}\}$, and letting $\dot{W}(n)$ be the canonical name for $\bigcup \dot{G}(n)$, $\mathbf{S}(\kappa)$ is a sort of Easton product of the $P_{\kappa_n,\ Z\cap\kappa_{n+1},\ \dot{W}(n+1)}$.

1.2. Notation and terminology

Our notation and terminology is intended to be standard, or have a clear meaning, e.g., *o.t.* for order type, card for cardinality. A catalogue of possible exceptions follows. When forcing, $p \leq q$ means q gives more information. Closed unbounded sets are *clubs* . The set of limit points of a set X of ordinals is denoted by X'. $A\Delta B$ is the symmetric difference of A and B, and $A\setminus B$ is the relative complement of B in A. For ordinals, $\alpha \leq \beta$, $[\alpha,\ \beta)$ is the half-open interval $\{\gamma : \alpha \leq \gamma < \beta\}$. The notation for the three other intervals are clear. It should be clear from context whether the open interval or the ordered pair is meant. OR is the class of all ordinals. For infinite cardinals, κ, H_κ is the set of all sets hereditarily of cardinality $< \kappa$, i.e. those sets x such that if t is the transitive closure of x, then card $t < \kappa$. For ordinals α, β, we write $\alpha >> \beta$ to mean that α is MUCH greater than β; the precise sense of how much greater we must take it to be is supposed to be clear from context. For models, $\mathcal{M}$, $Sk_{\mathcal{M}}$ denotes the Skolem operation in $\mathcal{M}$, where the Skolem functions are obtained in some reasonable fixed fashion. In this paper, we often suppress mention of the membership relation as a relation of a model, but it is always intended that it be one. Thus, $(M,\ A,\ \cdots)$ denotes the same model as $(M,\ \in,\ A,\ \cdots)$. All other notation is introduced as needed (we hope).

2. The Forcings

2.1 Definition. If g is a function, $\overline{g} = \{x \in \text{dom } g : g(x) = 1\}$.

2.2 Definition. If λ is a infinite cardinal, $T \subseteq \lambda$, then $g \in S(T,\ \lambda)$ iff there's $\delta = \delta(g) \in (\lambda,\ \lambda^+)$ such that $g : (\lambda,\ \delta) \to \{0,\ 1\}$ and for all $\alpha \in (\lambda,\ \delta]$:

$(*)_{\alpha,\ g}$ (card $\alpha)^{L[T,\ g|\alpha]} = \lambda$ (we say: g promptly collapses α).

$\mathbf{S}(T,\ \lambda) = (S(T,\ \lambda),\ \subseteq)$.

2.3 Definition. Let κ be an infinite cardinal, $T \subseteq \kappa^+$, $g \in S(T, \kappa^+)$. $\vec{b}^g = (b^g_\alpha : \alpha \in (\kappa^+, \delta(g)])$ is a sequence of almost disjoint cofinal subsets of successor ordinals $\beta \in (\kappa, \kappa^+)$ which are multiples of 3, such that for all $\alpha \in (\kappa, \delta(g)]$, $(b^g_\xi : \xi \in (\kappa^+, \alpha])$ is canonically defined in $L[T, g|\alpha]$.

2.4 Definition. With κ, T, g as in (1.5), $p = (\ell(p), r(p)) \in P_{\kappa, T, g}$ iff

(1) $\ell(p) \in S^+(T \cap \kappa, \kappa)$,
(2) if $\alpha \in (\kappa, \delta(\ell(p)))$, $\alpha = 3\alpha' + 1$, then $\ell(p)(\alpha) = 1$ iff $\alpha' \in T$. (we say: $\ell(p)$ codes T),
(3) $r(p) : \text{dom } r(p) \to \kappa^+$, $\text{dom } r(p) \in [\text{dom } g]^{<\kappa^+}$, and whenever $\alpha \in \text{dom } r(p)$, $r(p)(\alpha) \leq \xi \in b^g_\alpha \cap \delta(\ell(p))$, $\ell(p)(\xi) = g(\alpha)$,
(4) if $\alpha_1, \alpha_2 \in \text{dom } r(p)$ and $g(\alpha_1) \neq g(\alpha_2)$, then $b^g_{\alpha_1} \setminus r(p)(\alpha_1) \cap b^g_{\alpha_2} \setminus r(p)(\alpha_2) = \emptyset$.

For $p, q \in P_{\kappa, T, g}$, $p \leq q$ iff $\ell(p) \subseteq \ell(q)$, $r(p) \subseteq r(q)$; $\mathbf{P}_{\kappa, T, g} = (P_{\kappa, T, g}, \leq)$.

2.5 Definition. Let κ be an infinite cardinal. For $n \leq \omega$, let κ_n be κ^{+n}. Let $Z \subseteq \kappa_\omega$ be such that for all $n \leq \omega$, $H_{\kappa_n} = L_{\kappa_n}[Z \cap \kappa_n]$. $p \in S(\kappa, Z) = S(\kappa)$ iff dom $p = \omega$, for all $n < \omega$, $p(n) = (\ell(p(n)), r(p(n)))$, $\ell(p(n)) \in S(Z \cap \kappa_n, \kappa_n)$ and $p(n) \in P_{\kappa_n, Z \cap \kappa_{n+1}, \ell(p(n+1))}$. For $p, q \in S(\kappa)$, $p \leq q$ iff for all $n < \omega$, $\ell(p(n)) \subseteq \ell(q(n))$, $r(p(n)) \subseteq r(q(n))$. $\mathbf{S}(\kappa) = \mathbf{S}(\kappa, Z) = (S(\kappa), \leq)$.

If $k < \omega$, $S_k(\kappa) = S_k(\kappa, Z) = \{p|[k, \omega) : p \in S(\kappa)\}$; $\leq_k$ is the obvious projection of $\leq$ onto $S_k(\kappa)$. $\mathbf{S}_k(\kappa) = \mathbf{S}_k(\kappa, Z) = (S_k(\kappa), \leq_k)$.

3. The Results

Our ultimate goal in this section will be to prove that for cardinals κ with $\kappa \geq \aleph_2$, for all $k < \omega$, $\mathbf{S}_k(\kappa)$ is (κ_k, ∞)- distributive. As will be clear from what follows, by this we mean that the intersection κ_k open dense sets is dense, and not the weaker notion involving fewer than κ_k open dense sets. We denote the latter notion by $(< \kappa_k, \infty$-distributive. A useful first step will be to establish something stronger than this latter notion.

3.1 Proposition. *For all $k < \omega$, $\mathbf{S}_k(\kappa)$ is $< \kappa_k$- complete.*

Proof. Let $\theta < \kappa_k$, $(p_i : i < \theta)$ be a $\leq_k$-increasing sequence from $S_k(\kappa)$. For $i < \theta$, $k \leq n < \omega$, let $\delta_i(n) = \delta(\ell(p_i(n)))$, so, for such n, $(\delta_i(n) : i < \theta)$ is non-decreasing. Let $\delta(n) = \sup\{\delta_i(n) : i < \theta\}$. Let $\ell(p(n)) = \bigcup\{\ell(p_i(n)) : i < \theta\}$, $r(p(n)) = \bigcup\{r(p_i(n)) : i < \theta\}$, and let $p(n) = (\ell(p(n)), r(p(n)))$, for

$k \leq n < \omega$. We shall prove that $p \in S_k(\kappa)$. The only difficulty is to prove that for $k \leq n < \omega$, (card $\delta(n))^{L[Z \cap \kappa_n,\ \ell(p(n))]} = \kappa_n$. If θ is a successor ordinal or $\delta(n) = \delta_i(n)$ for some $i < \theta$, this is clear. Otherwise, $\delta(n)$ is a limit ordinal of cofinality $\leq cf\ \theta < \kappa_n$, so, by the Covering Lemma, already $(\text{cf } \delta(n))^L < \kappa_n$. But then, since $(\forall \alpha < \delta(n))(\text{card } \alpha)^{L[Z \cap \kappa_n,\ \ell(p(n))]} \leq \kappa_n$, the conclusion is clear. $\square$

(3.2) Before proving the main lemma of the section, in (3.3), it will be helpful to simply remark (the proofs are easy, and the reader may consult [2] for an outline) that letting $\dot{G}$ be the canonical name for the generic, letting $\dot{G}(n)$ be the canonical name for $\{\ell(p(n)) : p \in \dot{G}\}$, and letting $\dot{W}(n)$ be the canonical name for $\bigcup \dot{G}(n)$, then for all $k < \omega$,

$$\Vdash_{\mathbf{S}_k(\kappa)} \text{“}(\forall k \leq n < \omega)\dot{W}(n),\ Z \cap \kappa_n \in L[Z \cap \kappa_k,\ \dot{W}(k)]\text{”}.$$

We shall use a variant of this fact with no further comment below, in the proof of the main lemma. We note only that by an easy density argument, it can be shown that for $k \leq n < \omega$ and $\alpha \in [\kappa_{n+1},\ \kappa_{n+2})$, there is $\eta < \kappa_{n+1}$ such that whenever $\xi \in b_\alpha^{\dot{W}(n+1)|\alpha} \setminus \eta$, $\dot{W}(n)(\xi) = 0 \Rightarrow \dot{W}(n)(\xi + 1) = \dot{W}(n+1)(\alpha)$, and that $\{\xi \in b_\alpha^{\dot{W}(n+1)|\alpha} : \dot{W}(n)(\xi) = 0\}$ is cofinal in κ_{n+1}. Thus, $\dot{W}(n+1)(\alpha)$ is read by: $\dot{W}(n+1)(\alpha) = i$ iff there is a final segment $x \subseteq b_\alpha^{\dot{W}(n+1)|\alpha}$ such that for all $\xi \in x$, $\dot{W}(n)(\xi) = i$.

(3.3) We are now ready for the main Lemma.

Lemma. *For all $k < \omega$, $\mathbf{S}_k(\kappa)$ is $(\kappa_k,\ \infty)$-distributive.*

Proof. We first note that it suffices to prove that for all $k < \omega$

> $(*)_k$: Let $p_0 \in S_k(\kappa)$, let χ be regular $\chi >> 2^{2^{\kappa_\omega}}$; let $<(*)$ be a well-ordering of H_χ in type χ; let $\mathcal{M} = (H_\chi,\ <(*),\ \{\mathbf{S}_k(\kappa)\},\ \{Z\},\ \{p_0\})$; let $\mathcal{N} \prec \mathcal{M}$, $\kappa_k + 1 \subseteq |\mathcal{N}|$, card $|\mathcal{N}| = \kappa_k$. Then there is $p_0 \leq_k p^*$ which is $(\mathcal{N},\ \mathbf{S}_k(\kappa) \cap |\mathcal{N}|)$-generic.

The argument that $(*)_k$ suffices is well-known, so fix the above data. Without loss of generality, we may assume that $[|\mathcal{N}|]^{<\kappa_k} \subseteq |\mathcal{N}|$. It will often be convenient to work with the transitive collapse of $\mathcal{N}$, so let $\pi : \hat{\mathcal{N}} \to \mathcal{N}$ be the inverse of the transitive collapse map; thus, $[|\hat{\mathcal{N}}|]^{<\kappa_k} \subseteq |\hat{\mathcal{N}}|$. Let $\sigma = \pi^{-1} =$ the transitive collapse map. If $X \subseteq |\hat{\mathcal{N}}|$ and $(\hat{\mathcal{N}},\ X)$ is amenable, then we let $\pi(X) = \bigcup\{\pi(a \cap X) : a \in |\hat{\mathcal{N}}|\}$, and similarly for $\sigma(Y)$ if $(\mathcal{N},\ Y)$ is amenable. We let $\hat{\kappa}_n = \sigma(\kappa_n)$. We also let $\theta_n = \sup\ (|\mathcal{N}| \cap \kappa_n)$.

For $k < n < \omega$, note that $\hat{\kappa}_n = (\kappa_n)^{\hat{\mathcal{N}}}$, and that $\hat{\kappa}_{k+1} = \theta_{k+1}$. Note that by applying Proposition 3.1 to forcing over $\hat{\mathcal{N}}$ with $\sigma(\mathbf{S}_{k+1}(\kappa))$, we

easily construct $\hat{p} \in \sigma(\mathbf{S}_{k+1}(\kappa))$ which is $(\hat{\mathcal{N}},\ \sigma(\mathbf{S}_{k+1}(\kappa)))$-generic, such that $\sigma(p_0)|[k+1,\ \omega)$ is extended by $\hat{p}$, in $\sigma(\leq_{k+1})$, such that for $k+1 \leq n < \omega$, $\hat{p}(n) \subseteq |\hat{\mathcal{N}}|$ and all proper initial segments of $\hat{p}(n)$ lie in $|\hat{\mathcal{N}}|$. In view of the discussion in (2.2), for forcing over $\hat{\mathcal{N}}$,

$$\hat{\mathcal{N}}[\hat{p}] \models \text{“}(\forall n)(k+1 \leq n < \omega \Rightarrow \hat{p}(n) \in L[\sigma(Z \cap \kappa_{k+1}),\ \hat{p}(k+1)]\text{”}.$$

Thus, $\hat{\mathcal{N}}[\hat{p}] \models$ “$\sigma(Z \cap \kappa_\omega) \in L[\sigma(Z \cap \kappa_{k+1}),\ \hat{p}(k+1)]$”.

A crucial observation is:

Proposition. $OR \cap |\hat{\mathcal{N}}| < ((\hat{\kappa}_{k+1})^+)^L$.

Proof. Let $\hat{\theta} = OR \cap |\hat{\mathcal{N}}|$, $\theta = \sup\ (OR \cap |\mathcal{N}|)$. Note that $\pi|L^{\hat{\mathcal{N}}} : L_{\hat{\theta}} \to_{\Sigma_1} L_\theta$, with critical point $\hat{\kappa}_{k+1}$. If $\hat{\theta} \geq ((\hat{\kappa}_{k+1})^+)^L$, then $0^\sharp$ exists, which proves the Proposition. $\square$

Thus, $(cf\ \hat{\kappa}_n)^L \leq (cf\ \hat{\kappa}_{k+1})^L$, for all $k+1 \leq n < \omega$. Typically, of course, $\hat{\kappa}_{k+1}$ is a (regular cardinal)^L. Let $x_{k+1} = Z \cap \hat{\kappa}_{k+1}$, $h_{k+1} = \ell(\hat{p}(k+1))$.

We shall construct in V, $\hat{p}(k)$ which is $|\hat{\mathcal{N}}|$-generic for $\mathbf{P}_{\kappa_k,\ x_{k+1},\ h_{k+1}}$, as defined in $\hat{\mathcal{N}}$. Among other properties, letting $h_k = \ell(\hat{p}(k))$, h_k will code h_{k+1}. This will be **clear** from the construction; we shall use this fact **before** showing that $(cf\ \hat{\kappa}_{k+1})^{L[Z \cap \kappa_k,\ h_k]} = \kappa_k$. This is exactly what is required to show that if we define p by letting $p(n) = (\pi(\ell(\hat{p}(n))),\ \pi(r(\hat{p}(n))))$ (recall our convention about $\pi(X)$ for $(\hat{\mathcal{N}},\ X)$ amenable), then $p \in S_k(\kappa)$ (and p is $|\mathcal{N}|$-generic for $\mathbf{S}_k(\kappa) \cap |\mathcal{N}|$).

We shall have $h_k = \ell(q_{\kappa_k})$, $r(\hat{p}(k)) = r(q_{\kappa_k})$, where $q_i = (\ell(q_i),\ r(q_i))$ and $(q_i : i \leq \kappa_k)$ is defined recursively in V, with $q_0 = \sigma(p_0(k))$. For this, in V, we let $(D_i : i < \kappa_k)$ enumerate the dense subsets, in $|\hat{\mathcal{N}}|$, of $\mathbf{P}_{\kappa_k,\ x_{k+1},\ h_{k+1}}$, as defined in $\hat{\mathcal{N}}$. For all $\theta < \kappa_k$, $(D_i : i < \theta) \in |\hat{\mathcal{N}}|$, in virtue of the closure property we have assumed for $|\hat{\mathcal{N}}|$. For all $i < \theta$, we'll have $q_i \in |\hat{\mathcal{N}}|$, so, by the same observation, for $\theta < \kappa_k$, $(q_i : i < \theta) \in |\hat{\mathcal{N}}|$.

Also, for $j < \hat{\kappa}_{k+1}$, letting $D(j)$ be the subset of $\mathbf{P}_{\kappa_k,\ x_{k+1},\ h_{k+1}}$ consisting of those r with $\delta(\ell(r)) \geq j$, as defined in $\hat{\mathcal{N}}$, clearly $D(j)$ is dense and so is among the D_i. This will guarantee that $\sup\ \{\delta(\ell(q_i)) : i < \kappa_k\} = \hat{\kappa}_{k+1}$, provided that we know that $q_{i+1} \in D_i$. This will be part of the construction and will also guarantee the genericity of p.

For $i < \kappa_k$, we'll set $\alpha_i = \delta(\ell(q_i))$. For limit $\theta \leq \kappa_k$, we let $\ell(q_\theta) = \bigcup\{\ell(q_i) : i < \theta\}$, $r(q_\theta) = \bigcup\{r(q_i) : i < \theta\}$. If $\theta < \kappa_k$, by the covering argument of the proposition of (2.1), these are always conditions, and, if $\theta < \kappa_k$, as noted above, $(q_i : i < \theta) \in |\hat{\mathcal{N}}|$, so also $q_\theta \in |\hat{\mathcal{N}}|$. So, we must define q_{i+1}, where our crucial work is done.

For each $\alpha_i \leq \alpha < \gamma < \hat{\kappa}_{k+1}$, α a limit ordinal, we define $p^{\gamma,\ \alpha,\ 1} \geq q_i$ as follows: $r(p^{\gamma,\ \alpha,\ 1}) = r(q_i)$; if $\alpha_i \leq \beta < \gamma$ and $\beta \equiv 1$ $(mod\ 3)$ then $\ell(p^{\gamma,\ \alpha,\ 1})(\beta) = 0$ if $\beta' \notin Z$ & $= 1$, if $\beta' \in Z$, where β' is such that $\beta = 3\beta'+1$. If $\gamma \geq \alpha_i+\kappa_k$, we fix a subset $b \in |\hat{\mathcal{N}}| \cap L$, $b \subseteq \kappa_k$ which codes a well-ordering of κ_k in type γ, and for $\beta < \kappa_k$, we set $\ell(p^{\gamma,\ \alpha,\ 1})(\alpha_i + 3\beta + 2) = 0$ if $\beta \notin b$ & $= 1$ if $\beta \in b$. If $\alpha_i + \kappa_k \leq \beta < \gamma$ and $\beta \equiv 2$ $(mod\ 3)$, we set $\ell(p^{\gamma,\ \alpha,\ 1})(\beta) = 0$. Similarly, if $\gamma < \alpha_i + \kappa_k$, we set $\ell(p^{\gamma,\ \alpha,\ 1})(\beta) = 0$ for all $\alpha_i \leq \beta < \gamma$ such that $\beta \equiv 2$ $(mod\ 3)$.

If $\alpha_i \leq \beta < \gamma$ and for some $\tau \in$ dom $r(q_i)$, $\beta \in b_\tau^{h_{k+1}} \setminus r(q_i)(\tau)$, then $\ell(p^{\gamma,\ \alpha,\ 1})(\beta) = h_{k+1}(\tau)$. Note that in virtue of (4) of (1.4), this is well-defined. For all other successor ordinals, $\alpha_i \leq \beta\gamma$ which are multiples of 3, we set $\ell(p^{\gamma,\ \alpha,\ 1})(\beta) = 0$.

Now, suppose β is a limit ordinal, $\alpha_i \leq \beta < \gamma$. We set $\ell(p^{\gamma,\ \alpha,\ 1})(\beta) = 0$, **unless** $\beta = \alpha$ & $= 1$), if $\beta = \alpha$ (in this case, we mark α).

Then, let $p^{\gamma,\ \alpha,\ 2} \geq p^{\gamma,\ \alpha,\ 1}$ be chosen canonically in D_i. Now $(\gamma,\ \alpha) \mapsto p^{\gamma,\ \alpha,\ 2}$ is definable in $\hat{\mathcal{N}}$, and so, for each γ, in $\hat{\mathcal{N}}$, we can compute a bound, $\eta(\gamma) < \hat{\kappa}_{k+1}$, for sup $\{$dom $\ell(p^{\gamma,\ \alpha,\ 2}) : \alpha_i \leq \alpha < \gamma,\ \alpha$ a limit ordinal $\}$, as a function of γ. Iterating η in $\hat{\mathcal{N}}$ gives us a club, E_i, of $\hat{\kappa}_{k+1}$, $E_i \in |\hat{\mathcal{N}}|$. Now, $(H_{\hat{\kappa}_{k+2}})^{\hat{\mathcal{N}}} = L_{\hat{\kappa}_{k+2}}[\sigma(Z) \cap \hat{\kappa}_{k+2}]$, so all clubs of $\hat{\kappa}_{k+1}$ which lie in $|\hat{\mathcal{N}}|$, and, in particular, E_i, lie in $L[\sigma(Z) \cap \hat{\kappa}_{k+2}]$. Already in L, card $\hat{\kappa}_{k+2} =$ card $\hat{\kappa}_{k+1}$. So, in $L[\sigma(Z) \cap \hat{\kappa}_{k+2}]$ there is $\theta < (\hat{\kappa}_{k+1})^+$ such that all clubs of $\hat{\kappa}_{k+1}$ which lie in $|\hat{\mathcal{N}}|$, in fact, lie in $L_\theta[\sigma(Z) \cap \hat{\kappa}_{k+2}]$. This, however, readily gives us that unless (card $\hat{\kappa}_{k+1})^{L[\sigma(Z)\cap\hat{\kappa}_{k+2}]} = \kappa_k$ (and in this case, there is no problem in proving that q_{κ_k} is a condition), there is a club C of $\hat{\kappa}_{k+1}$, $C \in L[\sigma(Z) \cap \hat{\kappa}_{k+2}]$, such that C grows faster than any club of $\hat{\kappa}_{k+1}$ which lies in $|\hat{\mathcal{N}}|$. In particular, C grows faster than E_i, so that for sufficiently large $\gamma < \hat{\kappa}_{k+1}$, all E_i-intervals above γ miss C. In V, fix $C^* \subseteq C$, *o.t.* $C^* = \kappa_k$, C^* a club of $\hat{\kappa}_{k+1}$.

The idea of the above is that in constructing $p^{\gamma,\ \alpha,\ 1}$, we have "marked" α and our hope is that in passing from $p^{\gamma,\ \alpha,\ 1}$ to $p^{\gamma,\ \alpha,\ 2}$, we have not inadvertently "marked" anything else. While this is too much to hope for, in general, we shall be able to get that we have not marked anything else **in C**, provided we choose γ sufficiently large so that every interval of E_i, above γ, misses C. So, GOOD's winning strategy, finally, to go from q_i to q_{i+1}, is to take γ to be the least ordinal $> \alpha_i$, $\gamma \in C$ which, as above, is sufficiently large that the interval $[\gamma,\ \eta(\gamma)) \cap C = \emptyset$, **and such that there is** $\alpha^* \in [\alpha_i,\ \gamma) \cap \mathbf{C}^*$ and then to take $q_{i+1} = p^{\gamma,\ \alpha^*,\ 2}$. Thus, GOOD has "marked" a member of C^* and nothing else in C, while obtaining $q_{i+1} \in D_i$.

Now, since, as remarked above, we know from the construction that h_k

codes h_{k+1}, in $L[Z \cap \kappa_k,\ h_k]$, we can recover $\sigma(Z) \cap \hat{\kappa}_{k+2}$, and therefore C. But then, by the construction, we have that $\{\alpha \in C : (h_k(\alpha),\ h_k(\alpha+1)) = (1,\ 1)\}$ is a cofinal subset of C^*. Thus, as required, $(cf\ \hat{\kappa}_{k+1})^{L[Z\cap\kappa_k,\ h_k]} = \kappa_k$. This completes the proof. □

(3.4) Taken together, (3.1)–(3.3) give us the following Lemma, which, in turn, gives us the Theorem of the Introduction:

Lemma. *Forcing with* $\mathbf{S}(\kappa)$ *preserves cofinalities, GCH, and if* G *is* V*-generic for* $\mathbf{S}(\kappa)$*, then, in* $V[G]$ *there is* $W \subseteq \kappa^+$ *such that* $V[G] = V[W]$, $Z \in L[W,\ Z \cap \kappa]$ *and for all* $n \leq \omega$, $H_{\kappa_n} = L_{\kappa_n}[W]$ *and for* $\kappa < \alpha < \kappa^+$, $(\text{card } \alpha)^{L[W\cap\alpha]} = \kappa$.

Proof. Of course $W = \bigcup\{\overline{\ell(p(0))} : p \in \dot{G}\}$. It is a routine generalization of arguments from Chapter 1 of [1] to see that for all k, there is $\mathbf{Q}_k \in V^{\mathbf{S}_k(\kappa)}$ such that $\mathbf{S}(\kappa) \cong \mathbf{S}_k(\kappa) * \mathbf{Q}_k$, and $\Vdash_{\mathbf{S}_k(\kappa)}$ "$\mathbf{Q}_k$ is $\kappa_{k+1}-$ c.c. and card $Q_k = \kappa_{k+1}$". Further, for $k = 0$, (2.3) gives us that $\mathbf{S}(\kappa)$ is $(\kappa,\ \infty)$-distributive and clearly card $S(\kappa) = \kappa_\omega^+$. Thus, preservation of GCH is clear, as is the preservation of all cardinals except possibly κ_ω^+. The argument here is routine: if this failed, then letting $\gamma = (cf\ \kappa_\omega^+)^{V^{\mathbf{S}(\kappa)}}$, for some $0 < k < \omega$, $\gamma = \kappa_k$. But then, since $(cf\ \kappa_\omega^+)^{V^{\mathbf{S}_k(\kappa)}} > \kappa_k$, forcing with $\mathbf{Q}_k$ over $V^{\mathbf{S}_k(\kappa)}$ would have to collapse a cardinal $\geq \kappa_{k+1}$ which is impossible. □

3.5 Remarks and applications

(1) If we start from an arbitrary $Z' \subseteq \kappa_\omega$, we can, of course, code Z' by first coding Z' into a Z, as above (e.g., by coding Z' into Z on odd ordinals), and then proceeding as above.

(2) In work in progress, we are attempting to develop a combinatorial approach to coding the universe by a real (when $0^\sharp$ does not exist). Part of our approach is to use the Easton product of the $\mathbf{S}(\kappa)$, for $\kappa = \aleph_2$, or κ a limit cardinal, as a preliminary forcing, to simplify the universe before doing the main coding.

(3) Several people have observed that the $\mathbf{S}(\kappa)$ afford a method of coding any **set** of ordinals using a **set** forcing over models of GCH where $0^\sharp$ does not exist. This can be done as follows. Let $X \subseteq \lambda$, and assume, without loss of generality, that $\lambda \geq \aleph_2$. Code X into a $Z \subseteq \lambda^{+\omega}$, where Z has the properties assumed above. Then, force

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with $\mathbf{S}(\lambda)$ to get W, as above. Finally, since W reshapes the interval $(\lambda,\ \lambda^+)$, we can continue to code W down to a real, using one of the usual methods of coding by a real.

(4) Woodin has introduced the following abstract notion of condensation. $A \subseteq \delta$ **has condensation** iff there's an algebra, $\mathcal{A} \in V$ with underlying set δ, such that for any generic extension V' of V:

(*) if $X \subseteq \delta$ and X is the underlying set of a subalgebra of $\mathcal{A}$, and $\pi:\ (\mathcal{A}^*,\ A^*) \to (\mathcal{A}|X,\ A \cap X)$, where π is the inverse of the transitive collapse map, then $A^* \in V$.

δ has condensation iff for all $A \subseteq \delta$, A has condensation. This notion has been investigated by Woodin's student, D. Law, in his dissertation [3], and by Woodin himself.

S. Friedman has observed that using (3), above, it can be shown that for any cardinal μ, we can force condensation for μ via a set forcing. We omit the proof, except to say that according to Friedman, this is not a routine consequence of the usual sort of condensation for $L[r]$, but rather involves a closer look at the coding apparatus.

Acknowledgments. The research of the first author was partially supported by the NSF, the Basic Research Fund of the Israel Academy of Science and the Mathematical Sciences Research Institute. The research of the second author was partially supported by NSF grant DMS-8806536 and the Mathematical Sciences Research Institute, and, for the preparation of the final version of this paper, by a grant from the Reidler Foundation.

This paper is Number 294 in the first author's publication list.

We are both grateful to the administration and staff of the Mathematical Sciences Research Institute for their hospitality during portions of 1989–90.

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