The PCF Theorem Revisited

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Dedicated to Paul Erdős

Summary. *The* pcf *theorem (of the possible cofinability theory) was proved for reduced products* $\prod_{i<\kappa} \lambda_i/I$, *where* $\kappa < \min_{i<\kappa} \lambda_i$. *Here we prove this theorem under weaker assumptions such as* $\mathrm{wsat}(I) < \min_{i<\kappa} \lambda_i$, *where* $\mathrm{wsat}(I)$ *is the minimal* θ *such that* κ *cannot be divided to* θ *sets* $\notin I$ *(or even slightly weaker condition). We also look at the existence of exact upper bounds relative to* $<_I$ *(*$<_I$ *−eub) as well as cardinalities of reduced products and the cardinals* $T_D(\lambda)$. *Finally we apply this to the problem of the depth of ultraproducts (and reduced products) of Boolean algebras.*

0. Introduction

An aim of the pcf theory is to answer the question, what are the possible cofinalities (pcf) of the partial orders $\prod_{i<\kappa} \lambda_i/I$, where $\mathrm{cf}(\lambda_i) = \lambda_i$, for different ideals I on κ. For a quick introduction to the pcf theory see [Sh400a], and for a detailed exposition, see [Sh-g] and more history. In §1 and §2 we generalize the basic theorem of this theory by weakening the assumption $\kappa < \min_{i<\kappa} \lambda_i$ to the assumption that I extends a fixed ideal I^* with $\mathrm{wsat}(I^*) < \min_{i<\kappa} \lambda_i$, where $\mathrm{wsat}(I^*)$ is the minimal θ such that κ cannot be divided to θ sets $\notin I^*$ (not just that the Boolean algebra $\mathcal{P}(\kappa)/I^*$ has no θ pairwise disjoint non zero elements). So §1, §2 follow closely [Sh-g, Ch. I=Sh345a], [Sh-g, II 3.1], [Sh-g, VIII §1]. It is interesting to note that some of (as presented in courses and see a forthcoming survey of Kojman) those proofs which look to be superseded when by [Sh420, §1] we know that for regular $\theta < \lambda$, $\theta^+ < \lambda \Rightarrow \exists$ stationary $S \in I[\lambda]$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$, give rise to proofs here which seem necessary. Note $\mathrm{wsat}(I^*) \leq |\mathrm{Dom}(I^*)|^+$ (and $\mathrm{reg}_*(I^*) \leq |\mathrm{Dom}(I^*)|^+$ so [Sh-g, I §1, §2, II §1, VII 2.1, 2.2, 2.6] are really a special case of the proofs here.

During the sixties the cardinalities of ultraproducts and reduced products were much investigated (see Chang and Keisler [CK]). For this the notion "regular filter" (and (λ,μ)-regular filter) were introduced, as: if $\lambda_i \geq \aleph_0$, D a regular ultrafilter (or filter) on κ then $\prod_{i<\kappa} \lambda_i/D = (\limsup_D \lambda_i)^\kappa$. We reconsider these problems in §3 (again continuing [Sh-g]). We also draw a conclusion on the depth of the reduced product of Boolean algebras partially answering a problem of Monk; and make it clear that the truth of the full expected result is translated to a problem on pcf. On those problems on Boolean algebras see Monk [M]. In this section we include known proofs for completeness (mainly 3.6).

Let us review the paper in more details. In 1.1, 1.2 we give basic definition of cofinality, true cofinality, $\mathrm{pcf}(\bar\lambda)$ and $J_{<\lambda}[\bar\lambda]$ where usually $\bar\lambda = \langle \lambda_i : i < \kappa \rangle$ a sequence of regular cardinals, I^* a fixed ideal on κ such that we consider only

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ideals extending it (and filter disjoint to it). Let wsat(I^*) be the first θ such that we cannot partition κ to θ I^*-positive set (so they are pairwise disjoint, not just disjoint modulo I^*). In 1.3, 1.4 we give the basic properties. In lemma 1.5 we phrase the basic property enabling us to do anything: (1.5(*)): $\liminf_{I^*}(\bar\lambda) \geq \theta \geq \mathrm{wsat}(I^*)$, $\prod \bar\lambda/I^*$ is θ^+-directed; we prove that $\prod \bar\lambda/J_{<\lambda}[\bar\lambda]$ is λ-directed. In 1.6, 1.8 we deduce more properties of $\langle J_{<\lambda}[\bar\lambda] : \lambda \in \mathrm{pcf}(\bar\lambda)\rangle$ and in 1.7 deal with $<_{J_{<\lambda}[\bar\lambda]}$-increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$ with no $<_{J_{<\lambda}[\bar\lambda]}$-bound in $\prod \bar\lambda$. In 1.9 we prove $\mathrm{pcf}(\bar\lambda)$ has a last element and in 1.10, 1.11 deal with the connection between the true cofinality of $\prod\limits_{i<\kappa} \lambda_i/D^*$ and $\prod\limits_{i<\sigma} \mu_i/E$ when $\mu_i =: \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i/D_i)$ and D^* is the E-limit of the D_i's.

In 2.1 we define normality of λ for $\bar\lambda$: $J_{\leq\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B_\lambda$ and we define semi-normality: $J_{\leq}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$ where $B_\alpha/J_{<\lambda}[\bar\lambda]$ is increasing. We then (in 2.2) characterize semi normality (there is a $<_{J_{<\lambda}[\bar\lambda]}$-increasing $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ cofinal in $\prod \bar\lambda/D$ for every ultrafilter D (disjoint to I^* of course) such that $\mathrm{tcf}(\prod \bar\lambda/D) = \lambda$) and when semi normality implies normality (if some such $\bar f$ has a $<_{J_{<\lambda}[\bar\lambda]}$ $-$eub).

We then deal with continuity system $\bar a$ and $<_{J_{<\lambda}[\bar\lambda]}$- increasing sequence obeying $\bar a$, in a way adapted to the basic assumption (*) of 1.5.

Here as elsewhere if $\min(\bar\lambda) \geq \theta^+$ our life is easier than when we just assume $\lim sup_{I^*}(\bar\lambda) \geq \theta$, $\prod \bar\lambda/I^*$ is θ^+-directed (where $\theta \geq \mathrm{wsat}(I^*)$ of course). In 2.3 we give the definitions, in 2.4 we quote existence theorem, show existence of obedient sequences (in 2.5), essential uniqueness (in 2.7) and better consequence to 1.7 (in the direction to normality). We define (2.9) generating sequence and draw a conclusion (2.10(1)). Now we get some desirable properties: in 2.8 we prove semi normality, in 2.10(2) we compute $\mathrm{cf}(\prod \bar\lambda/I^*)$ as $\max \mathrm{pcf}(\bar\lambda)$. Next we relook at the whole thing: define several variants of the pcf-th (Definition 2.11). Then (in 2.12) we show that e.g. if $\min(\bar\lambda) > \theta^+$, we get the strongest version (including normality using 2.6, i.e. obedience). Lastly we try to map the implications between the various properties when we do not use the basic assumption 1.5 (*) (in fact there are considerable dependence, see 2.13, 2.14).

In 3.1, 3.3 we present measures of regularity of filters, in 3.2 we present measures of hereditary cofinality of $\prod \bar\lambda/D$: allowing to decrease $\bar\lambda$ and/or increase the filter. In 3.4 - 3.8 we try to estimate reduced products of cardinalities $\prod\limits_{i<\kappa} \lambda_i/D$ and in 3.9 we give a reasonable upper bound by hereditary cofinality $(\leq (\theta^\kappa/D + \mathrm{hcf}_{D,\theta}(\prod\limits_{i<\kappa} \lambda_I))^{<\theta}$ when $\theta \geq \mathrm{reg}_\otimes(D))$.

In 3.10-3.11 we return to existence of eub's and obedience and in 3.12 draw conclusion on "downward closure". On $T_D(f)$, starting with Galvin and Hajnal [GH] see [Sh-g].

In 3.13 - 3.14 we estimate $T_D(\bar\lambda)$ and in 3.15 try to translate it more fully to pcf problem (countable cofinality is somewhat problematic (so we restrict ourselves to $T_D(\bar\lambda) > \mu = \mu^{\aleph_0}$). We also mention $\aleph_1$-complete filters; (3.16, 3.17) and see what can be done without relaying on pcf (3.20)).

Now we deal with depth: define it (3.18, see 3.19), give lower bound (3.22), compute it for ultraproducts of interval Boolean algebras of ordinals (3.24). Lastly we connect the problem "does $\lambda_i < \mathrm{Depth}^+(B_i)$ for $i < \kappa$ implies $\mu < \mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D)$" at least when $\mu > 2^\kappa$ and $(\forall \alpha < \mu)[|\alpha|^{\aleph_0} < \mu]$, to a pcf problem (in 3.26). This is continued in [Sh589].

In the last section we phrase a reason why 1.5(∗) works (see 4.1), analyze the case we weaken 1.5(∗) to $\liminf_{I^*}(\bar\lambda) \ge \theta \ge \mathrm{wsat}(I^*)$ proving the pseudo pcf-th (4.3).

1. Basic pcf

Notation 1.0. I, J denote ideals on a set $\mathrm{Dom}(I)$, $\mathrm{Dom}(J)$ resp., called its domain (possibly $\bigcup_{A\in I} A \subset \mathrm{Dom} I$). If not said otherwise the domain is an infinite cardinal denoted by κ and also the ideal is proper i.e. $\mathrm{Dom}(I) \notin I$. Similarly D denotes a filter on a set $\mathrm{Dom} D$; we do not always distinguish strictly between an ideal on κ and the dual filter on κ. Let $\bar\lambda$ denote a sequence of the form $\langle \lambda_i : i < \kappa\rangle$. We say $\bar\lambda$ is regular if every λ_i is regular, $\min\bar\lambda = \min\{\lambda_i : i < \kappa\}$ (of course also in $\bar\lambda$ we can replace κ by another set), and let $\prod\bar\lambda = \prod\limits_{i<\kappa} \lambda_i$; usually we are assuming $\bar\lambda$ is regular. Let I^* denote a fixed ideal on κ. Let $I^+ = \mathcal{P}(\kappa) \setminus I$ (similarly $D^+ = \{A \subseteq \kappa : \kappa \setminus A \notin D\}$), let

$$\liminf_I \bar\lambda = \min\{\mu : \{i < \kappa : \lambda_i \le \mu\} \in I^+\} \quad \text{and}$$

$$\limsup_I \bar\lambda = \min\{\mu : \{i < \kappa : \lambda_i > \mu\} \in I\} \quad \text{and}$$

$$\mathrm{atom}_I \bar\lambda = \{\mu : \{i : \lambda_i = \mu\} \in I^+\}.$$

For a set A of ordinals with no last element, $J_A^{\mathrm{bd}} = \{B \subseteq A : \sup(B) < \sup(A)\}$, i.e. the ideal of bounded subsets. Generally, if $\mathrm{inv}(X) = \sup\{|y| : \vDash \varphi[X,y]\}$ then $\mathrm{inv}^+(X) = \sup\{|y|^+ : \vDash \varphi[X,y]\}$, and any y such that $\vDash \varphi[X,y]$ is a witness for $|y| \le \mathrm{inv}(X)$ (and $|y| < \mathrm{inv}^+(X)$), and it exemplifies this. Let $\bar A^*_\theta[\bar\lambda] = \langle A^*_\alpha : \alpha < \theta\rangle = \langle A^*_{\theta,\alpha}[\bar\lambda] : \alpha < \theta\rangle$ be defined by: $A^*_\alpha = \{i < \kappa : \lambda_i > \alpha\}$. Let Ord be the class of ordinals.

Definition 1.1.

(1) For a partial order* P:
  - (a) P is λ-directed if: for every $A \subseteq P$, $|A| < \lambda$ there is $q \in P$ such that $\bigwedge_{p\in A} p \le q$, and we say: q is an upper bound of A;
  - (b) P has true cofinality λ if there is $\langle p_\alpha : \alpha < \lambda\rangle$ cofinal in P, i.e.: $\bigwedge_{\alpha<\beta} p_\alpha < p_\beta$ and $\forall q \in P[\bigvee_{\alpha<\lambda} q \le p_\alpha]$ [and one writes $\mathrm{tcf}(P) = \lambda$ for the minimal such λ] (note: if P is linearly ordered it always has a true cofinality but e.g. $(\omega, <) \times (\omega_1, <)$ does not).
  - (c) P is called endless if $\forall p \in P \exists q \in P[q > p]$ (so if P is endless, in clauses (a), (b), (d) above we can replace $\le$ by $<$).
  - (d) $A \subseteq P$ is a cover if: $\forall p \in P \exists q \in A[p \le q]$; we also say "$A$ is cofinal in P".
  - (e) $\mathrm{cf}(P) = \min\{|A| : A \subseteq P$ is a cover$\}$.
  - (f) We say that, in P, p is a lub (least upper bound) of $A \subseteq P$ if:
    - (α) p is an upper bound of A (see (a))
    - (β) if p' is an upper bound of A then $p \le p'$.

(2) If D is a filter on S, α_s (for $s \in S$) are ordinals, $f, g \in \prod\limits_{s\in S} \alpha_s$, then: $f/D < g/D$, $f <_D g$ and $f < g \bmod D$ all mean $\{s \in S : f(s) < g(s)\} \in D$. Also if f, g are partial functions from S to ordinals, D a filter on S <u>then</u> $f < g$ mod D means

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* actually we do not require $p \le q \le p \Rightarrow p = q$ so we should say quasi partial order

$\{i \in \mathrm{Dom}(D) : i \notin \mathrm{Dom}(f)$ <u>or</u> $f(i) < g(i)$ (so both are defined)$\}$ belongs to D. We write $X = A \bmod D$ if $\mathrm{Dom}(D) \setminus [(X \setminus A) \cup (A \setminus X)]$ belongs to D. Similarly for $\leq$, and we do not distinguish between a filter and the dual ideal in such notions. So if J is an ideal on κ and $f, g \in \prod \bar{\lambda}$, then $f < g \bmod J$ iff $\{i < \kappa : \neg f(i) < g(i)\} \in J$. Similarly if we replace the α_s's by partial orders.

(3) For $f, g: S \to$ Ordinals, $f < g$ means $\bigwedge_{s \in S} f(s) < g(s)$; similarly $f \leq g$. So $(\prod \bar{\lambda}, \leq)$ is a partial order, we denote it usually by $\prod \bar{\lambda}$; similarly $\prod f$ or $\prod\limits_{i<\kappa} f(i)$.

(4) If I is an ideal on κ, $F \subseteq {}^{\kappa}\mathrm{Ord}$, we call $g \in {}^{\kappa}\mathrm{Ord}$ an $\leq_I$-eub (exact upper bound) of F if:
  - (α) g is an $\leq_I$-upper bound of F (in ${}^{\kappa}\mathrm{Ord}$)
  - (β) if $h \in {}^{\kappa}\mathrm{Ord}$, $h <_I \mathrm{Max}\{g, 1\}$ <u>then</u> for some $f \in F$, $h < \max\{f, 1\} \bmod I$.
  - (γ) if $A \subseteq \kappa$, $A \neq \emptyset \bmod I$ and $[f \in F \Rightarrow f \restriction A =_I 0_A$, i.e. $\{i \in A : f(i) \neq 0\} \in I]$ then $g \restriction A =_J 0_A$.

(5) (a) We say the ideal I (on κ) is θ-weakly saturated if κ cannot be divided to θ pairwise disjoint sets from I^+ (which is $\mathcal{P}(\kappa) \setminus I$)
  (b) $\mathrm{wsat}(I) = \min\{\theta : I$ is θ-weakly saturated$\}$

Remark 1.1A.

(1) Concerning 1.1(4), note: $g' = \mathrm{Max}\{g, 1\}$ means $g'(i) = \mathrm{Max}\{g(i), 1\}$ for each $i < \kappa$; if there is $f \in F$, $\{i < \kappa : f(i) = 0\} \in I$ we can replace $\mathrm{Max}\{g, 1\}$, $\mathrm{Max}\{f, 1\}$ by g, f respectively in clause (β) and omit clause (γ).

(2) Considering $\prod\limits_{i<\kappa} f(i)$, $<_I$ formally if $(\exists i) f(i) = 0$ then $\prod\limits_{i<\kappa} f(i) = \emptyset$; but we usually ignore this, particularly when $\{i : f(i) = 0\} \in I$.

Definition 1.2. Below if Γ is "a filter disjoint to I", we write I instead Γ.

(1) For a property Γ of ultrafilters:

$$\mathrm{pcf}_{\Gamma}(\bar{\lambda}) = \mathrm{pcf}(\bar{\lambda}, \Gamma) = \{\mathrm{tcf}(\prod \bar{\lambda}/D) : D \text{ is an ultrafilter on } \kappa \text{ satisfying } \Gamma\}$$

(so $\bar{\lambda}$ is a sequence of ordinals, usually of regular cardinals, note: as D is an ultrafilter, $\prod \bar{\lambda}/D$ is linearly ordered hence has true cofinality).

(1A) More generally, for a property Γ of ideals on κ we let $\mathrm{pcf}_{\Gamma}(\bar{\lambda}) = \{\mathrm{tcf}(\prod \bar{\lambda}/J) : J$ is an ideal on κ satisfying Γ such that $\prod \bar{\lambda}/J$ has true cofinality$\}$. Similarly below.

(2) $J_{<\lambda}[\bar{\lambda}, \Gamma] = \{B \subseteq \kappa$: for no ultrafilter D on κ satisfying Γ to which B belongs, is $\mathrm{tcf}(\prod \bar{\lambda}/D) \geq \lambda\}$.

(3) $J_{\leq\lambda}[\bar{\lambda}, \Gamma] = J_{<\lambda^+}[\bar{\lambda}, \Gamma]$.

(4) $\mathrm{pcf}_{\Gamma}(\bar{\lambda}, I) = \{\mathrm{tcf}(\prod \bar{\lambda}/D) : D$ a filter on κ disjoint to I satisfying $\Gamma\}$.

(5) If $B \in I^+$, $\mathrm{pcf}_I(\bar{\lambda} \restriction B) = \mathrm{pcf}_{I + (\kappa \setminus B)}(\bar{\lambda})$ (so if $B \in I$ it is $\emptyset$), also $J_{<\lambda}(\bar{\lambda} \restriction B, I) \subseteq \mathcal{P}(B)$ is defined similarly.

(6) If $I = I^*$ we may omit it, similarly in (2), (4).
If $\Gamma = \Gamma_{I^*} = \{D : D$ a filter on κ disjoint to $I^*\}$ we may omit it.

Remark. We mostly use $\mathrm{pcf}(\bar{\lambda})$, $J_{<\lambda}[\bar{\lambda}]$.

Claim 1.3.

(0) $(\prod\bar\lambda, <_J)$ and $(\prod\bar\lambda, \le_J)$ are endless (even when each λ_i is just a limit ordinal);

(1) $\min(\mathrm{pcf}_I(\bar\lambda)) \ge \liminf_I(\bar\lambda)$ for $\bar\lambda$ regular;

(2) (i) If $B_1 \subseteq B_2$ are from I^+ then $\mathrm{pcf}_I(\bar\lambda \restriction B_1) \subseteq \mathrm{pcf}_I(\bar\lambda \restriction B_2)$;

(ii) if $I \subseteq J$ then $\mathrm{pcf}_J(\bar\lambda) \subseteq \mathrm{pcf}_I(\bar\lambda)$; and

(iii) for $B_1, B_2 \subseteq \kappa$ we have $\mathrm{pcf}_I(\bar\lambda \restriction (B_1 \cup B_2)) = \mathrm{pcf}_I(\bar\lambda \restriction B_1) \bigcup \mathrm{pcf}_I(\bar\lambda \restriction B_2)$. Also

(iv) $A \in J_{<\lambda}[\bar\lambda \restriction (B_1 \cup B_2)] \Leftrightarrow A \cap B_1 \in J_{<\lambda}[\bar\lambda \restriction B_1]$ & $A \cap B_2 \in J_{<\lambda}[\bar\lambda \restriction B_2]$

(v) If $A_1, A_2 \in I^+$, $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = \kappa$, and $\mathrm{tcf}(\prod\bar\lambda \restriction A_\ell, <_I) = \lambda$ for $\ell = 1, 2$ then $\mathrm{tcf}(\prod\bar\lambda, <_I) = \lambda$; and if the sequence $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ witness both assumptions then it witness the conclusion.

(3) (i) if $B_1 \subseteq B_2 \subseteq \kappa$, B_1 finite and $\bar\lambda$ regular then

$$\mathrm{pcf}_I(\bar\lambda \restriction B_2) \setminus \mathrm{Rang}(\bar\lambda \restriction B_1) \subseteq \mathrm{pcf}_I(\bar\lambda \restriction (B_2 \setminus B_1)) \subseteq \mathrm{pcf}_I(\bar\lambda \restriction B_2)$$

(ii) if in addition $i \in B_1 \Rightarrow \lambda_i < \min(\mathrm{Rang}[\bar\lambda \restriction (B_2 \setminus B_1)])$, then $\mathrm{pcf}_I(\bar\lambda \restriction B_2) \setminus \mathrm{Rang}(\bar\lambda \restriction B_1) = \mathrm{pcf}_I(\bar\lambda \restriction (B_2 \setminus B_1))$.

(4) Let $\bar\lambda$ be regular (i.e. each λ_i is regular);

(i) If $\theta = \liminf_I \bar\lambda$ then $\prod\bar\lambda/I$ is θ-directed

(ii) If $\theta = \liminf_I \bar\lambda$ is singular then $\prod\bar\lambda/I$ is θ^+-directed

(iii) If $\theta = \liminf_I \bar\lambda$ is inaccessible (i.e. a limit regular cardinal), the set $\{i < \kappa : \lambda_i = \theta\}$ is in the ideal I and for some club E of θ, $\{i < \kappa \colon \lambda_i \in E\} \in I$ then $\prod\bar\lambda/I$ is θ^+-directed. We can weaken the assumption to "I is not weakly normal for $\bar\lambda$" (defined in the next sentence). Let "I is not medium normal for $(\theta, \bar\lambda)$" mean: for some $h \in \prod\bar\lambda$, for no $j < \theta$ is $\{i < \kappa \colon \lambda_i \le \theta \Rightarrow h(i) < j\} = \kappa \bmod I$; and let "$I$ is not weakly normal for $(\theta, \bar\lambda)$" mean: for some $h \in \prod\bar\lambda$, for no $\zeta < \liminf_I(\bar\lambda) = \theta$, is $\{i < \kappa \colon \lambda_i \le \theta \Rightarrow h(i) < \zeta\} \in I^+$.

(iv) If $\{i : \lambda_i = \theta\} = \kappa \mod I$ and I is medium normal for $\bar\lambda$ then $(\prod\bar\lambda, <_I)$ has true cofinality θ.

(v) If $\prod\bar\lambda/I$ is θ-directed then $\mathrm{cf}(\prod\bar\lambda/I) \ge \theta$ and $\min \mathrm{pcf}_I(\prod\bar\lambda) \ge \theta$.

(vi) $\mathrm{pcf}_I(\bar\lambda)$ is non empty set of regular cardinals. [see part (7)]

(5) Assume $\bar\lambda$ is regular and: if $\theta' =: \limsup_I(\bar\lambda)$ is regular then I is not medium normal for $(\theta', \bar\lambda)$. Then $\mathrm{pcf}_I(\bar\lambda) \not\subseteq (\limsup_I(\bar\lambda))^+$; in fact for some ideal J extending I, $\prod\bar\lambda/J$ is $(\limsup_I(\bar\lambda))^+$-directed.

(6) If D is a filter on a set S and for $s \in S$, α_s is a limit ordinal then:

(i) $\mathrm{cf}(\prod_{s\in S}\alpha_s, <_D) = \mathrm{cf}(\prod_{s\in S}\mathrm{cf}(\alpha_s), <_D) = \mathrm{cf}(\prod_{s\in S}(\alpha_s, <)/D)$, and

(ii) $\mathrm{tcf}(\prod_{s\in S}\alpha_s, <_D) = \mathrm{tcf}(\prod_{s\in S}(\mathrm{cf}(\alpha_s), <_D)) = \mathrm{tcf}(\prod_{s\in S}(\alpha_s, <)/D)$.

In particular, if one of them is well defined, then so are the others. This is true even if we replace α_s by linear orders or even partial orders with true cofinality.

(7) If D is an ultrafilter on a set S, λ_s a regular cardinal, then $\theta =: \mathrm{tcf}(\prod_{s\in S}\lambda_s, <_D)$ is well defined and $\theta \in \mathrm{pcf}(\{\lambda_s \colon s \in S\})$.

(8) If D is a filter on a set S, for $s \in S$, λ_s is a regular cardinal, $S^* = \{\lambda_s \colon s \in S\}$ and

$$E =: \{B\colon \quad B \subseteq S^* \text{ and } \{s \colon \lambda_s \in B\} \in D\}$$

and $\lambda_s > |S|$ or at least $\lambda_s > |\{t : \lambda_t = \lambda_s\}|$ for any $s \in S$ then:

(i) E is a filter on S^*, and if D is an ultrafilter on S then E is an ultrafilter on S^*.

(ii) S^* is a set of regular cardinals and if $s \in S \Rightarrow \lambda_s > |S|$ then $(\forall\lambda \in S^*)\lambda > |S^*|$,

(iii) $F = \{f \in \prod_{s\in S}\lambda_s \colon s = t \Rightarrow f(s) = f(t)\}$ is a cover of $\prod_{s\in S}\lambda_s$,

(iv) $\mathrm{cf}(\prod_{s\in S}\lambda_s/D) = \mathrm{cf}(\prod S^*/E)$ and $\mathrm{tcf}(\prod_{s\in S}\lambda_s/D) = \mathrm{tcf}(\prod S^*/E)$.

(9) Assume I is an ideal on κ, $F \subseteq {}^{\kappa}\mathrm{Ord}$ and $g \in {}^{\kappa}\mathrm{Ord}$. If g is a $\leq_I$-eub of F then g is a $\leq_I$-lub of F.
(10) $\sup \mathrm{pcf}_I(\bar{\lambda}) \leq |\prod \bar{\lambda}/I|$
(11) If I is an ideal on S and $(\prod_{s \in S} \alpha_s, <_I)$ has true cofinality λ as exemplified by $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ then the function $\langle \alpha_s : s \in S \rangle$ is a $<_I$ $-$eub (hence $<_I$ $-$lub) of $\bar{f}$.
(12) The inverse of (11) holds: if I is an ideal on S and $f_\alpha \in {}^S\mathrm{Ord}$ for $\alpha < \lambda = \mathrm{cf}(\lambda)$, $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_I$-increasing with $<_I$ $-$eub f then $\mathrm{tcf}(\prod_i f(i), <_I) = \mathrm{tcf}(\prod \mathrm{cf}[f(i)], <_I) = \lambda$.
(13) If $I \subseteq J$ are ideals on κ then
(a) $\mathrm{wsat}(I) \geq \mathrm{wsat}(J)$
(b) $\liminf_I(\bar{\lambda}) \leq \liminf_J(\bar{\lambda})$
(c) if $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_I)$ then $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_J)$
(14) If f_1, f_2 are $<_I$ $-$lub of F then $f_1 =_I f_2$.

Proof. They are all very easy, e.g.

(0) We shall show $(\prod \bar{\lambda}, <_J)$ is endless (assuming, of course, that J is a proper ideal on κ). Let $f \in \prod \bar{\lambda}$, then $g =: f+1$ (defined $(f+1)(\gamma) = f(\gamma)+1$) is in $\prod \bar{\lambda}$ too as each λ_α being an infinite cardinal is a limit ordinal and $f < g \bmod J$.

(5) Let $\theta' =: \limsup_I(\bar{\lambda})$ and define

$$J =: \{A \subseteq \kappa : \text{ for some } \theta < \theta', \{i < \kappa : \lambda_i > \theta \text{ and } i \in A\} \text{ belongs to } I\}.$$

Clearly J is an ideal on κ extending I (and $\kappa \notin J$) and $\limsup_J(\bar{\lambda}) = \liminf_J(\bar{\lambda}) = \theta'$.

Case 1: θ' is singular

By part (4) clause (ii), $\prod \bar{\lambda}/J$ is $(\theta')^+$-directed and we get the desired conclusion.

Case 2: θ' is regular.

Let $h \in \prod \bar{\lambda}$ witness that "I is not medium normal for $(\theta', \bar{\lambda})$" and let

$$J^* = \{A \subseteq \kappa : \text{ for some } j < \theta' \text{ we have } \{i \in A : h(i) < j\} = A \mod I\}.$$

Note that if $A \in J$ then for some $\theta < \theta'$, $A' =: \{i \in A : \theta_i > \theta\} \in I$ hence the choice $j =: \theta$ witness $A \in J^*$. So $J \subseteq J^*$. Also $J^* \subseteq \mathcal{P}(\kappa)$ by its definition. J^* is closed under subsets (trivial) and under union [why? assume A_0, $A_1 \in J^*$, $A = A_0 \cup A_1$; choose j_0, $j_1 < \theta'$ such that $A'_\ell =: \{i \in A_\ell : h(i) < j_\ell\} = A_\ell \mod I$, so $j =: \max\{j_0, j_1\} < \theta$ and $A' = \{i \in A : h(i) < j\} = A \mod I$; so $A \in J^*$]. Also $\kappa \notin J^*$ [why? as h witness that I is not medium normal for $(\theta', \bar{\lambda})$]. So together J^* is an ideal on κ extending I. Now J^* is not weakly normal for $(\theta, \bar{\lambda})$, as witnessed by h. Lastly $\prod \bar{\lambda}/J^*$ is $(\theta')^+$-directed (by part (4) clause (iii)), and so $\mathrm{pcf}_J(\bar{\lambda})$ is disjoint to $(\theta')^+$.

(9) Let us prove g is a $\leq_I$ $-$lub of F in $({}^{\kappa}\mathrm{Ord}, \leq_I)$. As we can deal separately with $I + A$, $I + (\kappa \setminus A)$ where $A =: \{i : g(i) = 0\}$, and the later case is trivial we can assume $A = \emptyset$. So assume g is not a $\leq_I$ $-$lub, so there is an upper bound g' of F, but not $g \leq_I g'$. Define $g'' \in {}^{\kappa}\mathrm{Ord}$:

$$g''(i) = \begin{cases} 0 & \text{if } g(i) \leq g'(i) \\ g'(i) & \text{if } g'(i) < g(i) \end{cases}.$$

Clearly $g'' <_I g$. So, as g in an $\leq_I$ $-$eub of F for I, there is $f \in F$ such that $g'' <_I \max\{f, 1\}$, but $B =: \{i : g'(i) < g(i)\} \neq \emptyset \bmod I$ (as not $g \leq_I g'$) so $g' \restriction B =$

$g'' \restriction B <_I \max\{f, 1\} \restriction B$. But we know that $f \le_I g'$ (as g' is an upper bound of F) hence $f \restriction B \le_I g' \restriction B$, so by the previous sentence necessarily $f \restriction B =_I 0_B$ hence $g' \restriction B =_I 0_B$; as g' is a $\le_I$-upper bound of F we know $[f' \in F \Rightarrow f' \restriction B =_I 0_B]$, hence by (γ) of Definition 1.1(4) we have $g \restriction B =_I 0_B$, a contradiction to $B \notin I$ (see above). $\blacksquare_{1.3}$

Remark 1.3A. In 1.3 we can also have the straight monotonicity properties of

$$\mathrm{pcf}_I(\prod \bar\lambda, \Gamma) \text{ in } \Gamma \text{ and in } I.$$

CLAIM 1.4:

(1) $J_{<\lambda}[\bar\lambda]$ is an ideal (of $\mathcal{P}(\kappa)$ i.e. on κ, but the ideal may not be proper).
(2) if $\lambda \le \mu$, then $J_{<\lambda}[\bar\lambda] \subseteq J_{<\mu}[\bar\lambda]$
(3) if λ is singular, $J_{<\lambda}[\bar\lambda] = J_{<\lambda^+}[\bar\lambda] = J_{\le\lambda}[\bar\lambda]$
(4) if $\lambda \notin \mathrm{pcf}(\bar\lambda)$, <u>then</u> we have $J_{<\lambda}[\bar\lambda] = J_{\le\lambda}[\bar\lambda]$.
(5) If $A \subseteq \kappa$, $A \notin J_{<\lambda}[\bar\lambda]$, and $f_\alpha \in \prod \bar\lambda \restriction A$, $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing cofinal in $(\prod \bar\lambda \restriction A)/J_{<\lambda}[\bar\lambda]$ <u>then</u> $A \in J_{\le\lambda}[\bar\lambda]$. Also this holds when we replace $J_{<\lambda}[\bar\lambda]$ by any ideal J on κ, $I^* \subseteq J \subseteq J_{\le\lambda}[\bar\lambda]$.
(6) The earlier parts hold for $J_{<\lambda}[\bar\lambda, \Gamma]$ too.

Proof. Straight.

LEMMA 1.5: Assume

(*) $\bar\lambda$ is regular and
  (α) $\min\bar\lambda > \theta \ge \mathrm{wsat}(I^*)$ (see 1.1(5)(b)) or at least
  (β) $\liminf_{I^*}(\bar\lambda) \ge \theta \ge \mathrm{wsat}(I^*)$, and $\prod \bar\lambda/I^*$ is θ^+-directed**.

If λ is a cardinal $\ge \theta$, and $\kappa \notin J_{<\lambda}[\bar\lambda]$ <u>then</u> $(\prod\bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is λ-directed (remember: $J_{<\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda, I^*]$).

Proof. Note: if $f \in \prod\bar\lambda$ then $f < f+1 \in \prod\bar\lambda$, (i.e. $(\prod\bar\lambda, <_{J_\lambda[\bar\lambda]})$ is endless) where $f+1$ is defined by $(f+1)(i) = f(i)+1$. Let $F \subseteq \prod\bar\lambda$, $|F| < \lambda$, and we shall prove that for some $g \in \prod\bar\lambda$ we have $(\forall f \in F)(f \le g \bmod J_{<\lambda}[\bar\lambda])$, this suffices. The proof is by induction on $|F|$. If $|F|$ is finite, this is trivial. Also if $|F| \le \theta$, when (α) of $(*)$ holds it is easy: let $g \in \prod\bar\lambda$ be $g(i) = \sup\{f(i) : f \in F\} < \lambda_i$; when (β) of $(*)$ holds use second clause of (β). So assume $|F| = \mu$, $\theta < \mu < \lambda$ so let $F = \{f^0_\alpha : \alpha < \mu\}$. By the induction hypothesis we can choose by induction on $\alpha < \mu$, $f^1_\alpha \in \prod\bar\lambda$ such that:

(a) $f^0_\alpha \le f^1_\alpha \bmod J_{<\lambda}[\bar\lambda]$
(b) for $\beta < \alpha$ we have $f^1_\beta < f^1_\alpha \bmod J_{<\lambda}[\bar\lambda]$.

If μ is singular, there is $C \subseteq \mu$ unbounded, $|C| = \mathrm{cf}(\mu) < \mu$, and by the induction hypothesis there is $g \in \prod\bar\lambda$ such that for $\alpha \in C$, $f^1_\alpha \le g \bmod J_{<\lambda}[\bar\lambda]$. Now g is as required: $f^0_\alpha \le f^1_\alpha \le f^1_{\min(C\setminus\alpha)} \le g \bmod J_{<\lambda}[\bar\lambda]$. So without loss of generality μ is regular. Let us define $A^*_\varepsilon =: \{i < \kappa : \lambda_i > |\varepsilon|\}$ for $\varepsilon < \theta$, so $\varepsilon < \zeta < \theta \Rightarrow A^*_\zeta \subseteq A^*_\varepsilon$ and $\varepsilon < \theta \Rightarrow A^*_\varepsilon = \kappa \bmod I^*$. Now we try to define by induction on $\varepsilon < \theta$, g_ε, $\alpha_\varepsilon = \alpha(\varepsilon) < \mu$ and $\langle B^\varepsilon_\alpha : \alpha < \mu\rangle$ such that:

** note if $\mathrm{cf}(\theta) < \theta$ then "θ^+-directed" follows from "θ-directed" which follows from "$\liminf_{I^*}(\bar\lambda) \ge \theta$, i.e. first part of clause (β) implies clause (β). Note also that if clause (α) holds then $\prod\bar\lambda/I^*$ is θ^+-directed (even $(\prod\bar\lambda, <)$ is θ^+-directed), so clause (α) implies clause (β).

(i) $g_\varepsilon \in \prod \bar\lambda$
(ii) for $\varepsilon < \zeta$ we have $g_\varepsilon \restriction A^*_\zeta \le g_\zeta \restriction A^*_\zeta$
(iii) for $\alpha < \mu$ let $B^\varepsilon_\alpha =: \{i < \kappa: f^1_\alpha(i) > g_\varepsilon(i)\}$
(iv) for each $\varepsilon < \theta$, for every $\alpha \in [\alpha_{\varepsilon+1}, \mu)$, $B^\varepsilon_\alpha \ne B^{\varepsilon+1}_\alpha \bmod J_{<\lambda}[\bar\lambda]$.

We cannot carry this definition: as letting $\alpha(*) = \sup\{\alpha_\varepsilon: \varepsilon < \theta\}$, then $\alpha(*) < \mu$ since $\mu = \mathrm{cf}(\mu) > \theta$. We know that $B^\varepsilon_{\alpha(*)} \cap A^*_{\varepsilon+1} \ne B^{\varepsilon+1}_{\alpha(*)} \cap A^*_{\varepsilon+1}$ for $\alpha < \theta$ (by (iv) and as $A^*_{\varepsilon+1} = \kappa \bmod I^*$ and $I^* \subseteq J_{<\lambda}[\bar\lambda]$) and $B^\varepsilon_{\alpha(*)} \subseteq \kappa$ (by (iii)) and $[\varepsilon < \zeta \Rightarrow B^\zeta_{\alpha(*)} \cap A^*_\zeta \subseteq B^\varepsilon_{\alpha(*)}]$ (by (ii)), together $\langle A^*_{\varepsilon+1} \cap (B^\varepsilon_{\alpha(*)} \setminus B^{\varepsilon+1}_{\alpha(*)}): \varepsilon < \theta\rangle$ is a sequence of θ pairwise disjoint members of $(I^*)^+$, a contradiction*** to the definition of $\theta = \mathrm{wsat}(I^*)$.
Now for $\varepsilon = 0$ let g_i be f^1_0 and $\alpha_\varepsilon = 0$.
For ε limit let $g_\varepsilon(i) = \bigcup_{\zeta<\varepsilon} g_\zeta(i)$ for $i \in A^*_\varepsilon$ and zero otherwise (note: $g_\varepsilon \in \prod \bar\lambda$ as $\varepsilon < \theta$, $\lambda_i > \varepsilon$ for $i \in A^*_\varepsilon$ and $\bar\lambda$ is a sequence of regular cardinals) and let $\alpha_\varepsilon = 0$.
For $\varepsilon = \zeta + 1$, suppose that g_ζ hence $\langle B^\zeta_\alpha: \alpha < \mu\rangle$ are defined. If $B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]$ for unboundedly many $\alpha < \mu$ (hence for every $\alpha < \mu$) then g_ζ is an upper bound for F mod $J_{<\lambda}[\bar\lambda]$ and the proof is complete. So assume this fails, then there is a minimal $\alpha(\varepsilon) < \mu$ such that $B^\zeta_{\alpha(\varepsilon)} \notin J_{<\lambda}[\bar\lambda]$. As $B^\zeta_{\alpha(\varepsilon)} \notin J_{<\lambda}[\bar\lambda]$, by Definition 1.2(2) for some ultrafilter D on κ disjoint to $J_{<\lambda}[\bar\lambda]$ we have $B^\zeta_{\alpha(\varepsilon)} \in D$ and $\mathrm{cf}(\prod \bar\lambda / D) \ge \lambda$. Hence $\{f^1_\alpha / D: \alpha < \mu\}$ has an upper bound h_ε / D where $h_\varepsilon \in \prod \bar\lambda$. Let us define $g_\varepsilon \in \prod \bar\lambda$:

$$g_\varepsilon(i) = \mathrm{Max}\{g_\zeta(i), h_\varepsilon(i)\}.$$

Now (i), (ii) hold trivially and B^ε_α is defined by (iii). Why does (iv) hold (for ζ) with $\alpha_{\zeta+1} = \alpha_\varepsilon =: \alpha(\varepsilon)$? Suppose $\alpha(\varepsilon) \le \alpha < \mu$. As $f^1_{\alpha(\varepsilon)} \le f^1_\alpha \bmod J_{<\lambda}[\bar\lambda]$ clearly $B^\zeta_{\alpha(\varepsilon)} \subseteq B^\zeta_\alpha \bmod J_{<\lambda}[\bar\lambda]$. Moreover $J_{<\lambda}[\bar\lambda]$ is disjoint to D (by its choice) so $B^\zeta_{\alpha(\varepsilon)} \in D$ implies $B^\zeta_\alpha \in D$.
On the other hand B^ε_α is $\{i < \kappa: f^1_\alpha(i) > g_\varepsilon(i)\}$ which is equal to $\{i \in \bar\lambda: f^1_\alpha(i) > g_\zeta(i), h_\varepsilon(i)\}$ which does not belong to D (h_ε was chosen such that $f^1_\alpha \le h_\varepsilon \bmod D$). We can conclude $B^\varepsilon_\alpha \notin D$, whereas $B^\zeta_\alpha \in D$; so they are distinct $\bmod J_{<\lambda}[\bar\lambda]$ as required in clause (iv).

Now we have said that we cannot carry the definition for all $\varepsilon < \theta$, so we are stuck at some ε; by the above ε is successor, say $\varepsilon = \zeta + 1$, and g_ζ is as required: an upper bound for F modulo $J_{<\lambda}[\bar\lambda]$. $\blacksquare_{1.5}$

LEMMA 1.6: If (*) of 1.5, D is an ultrafilter on κ disjoint to I^* and $\lambda = \mathrm{tcf}(\prod \bar\lambda, <_D)$, <u>then</u> for some $B \in D$, $(\prod \bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality λ. (So $B \in J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$ by 1.4(5).)

Proof. By the definition of $J_{<\lambda}[\bar\lambda]$ clearly we have $D \cap J_{<\lambda}[\bar\lambda] = \emptyset$.
Let $\langle f_\alpha / D: \alpha < \lambda\rangle$ be increasing unbounded in $\prod \bar\lambda / D$ (so $f_\alpha \in \prod \bar\lambda$). By 1.5 without loss of generality $(\forall \beta < \alpha)(f_\beta < f_\alpha \bmod J_{<\lambda}[\bar\lambda])$.
Now 1.6 follows from 1.7 below: its hypothesis clearly holds. If $\bigwedge_{\alpha<\lambda} B_\alpha = \emptyset \bmod D$, (see (A) of 1.7) then (see (D) of 1.7) $J \cap D = \emptyset$ hence (see (D) of 1.7) g/D contradicts the choice of $\langle f_\alpha / D: \alpha < \lambda\rangle$. So for some $\alpha < \lambda$, $B_\alpha \in D$; by (C) of 1.7 and 1.4(5) we get the desired conclusion. $\blacksquare_{1.6}$

*** in fact note that for no $B_\varepsilon \subseteq \kappa$ $(\varepsilon < \theta)$ do we have: $B_\varepsilon \ne B_{\varepsilon+1}$ mod I^* and $\varepsilon < \zeta < \theta \Rightarrow B_\varepsilon \cap A_\zeta \subseteq B_\zeta$ where $A_\zeta = \kappa$ mod I^* (e.g. $A_\zeta = A^*_\zeta$)

LEMMA 1.7: Suppose (*) of 1.5, $\mathrm{cf}(\lambda) > \theta$, $f_\alpha \in \prod\bar{\lambda}$, $f_\alpha < f_\beta \bmod J_{<\lambda}[\bar{\lambda}]$ for $\alpha < \beta < \lambda$, and there is no $g \in \prod\bar{\lambda}$ such that for every $\alpha < \lambda$, $f_\alpha < g \bmod J_{<\lambda}[\bar{\lambda}]$. Then there are B_α (for $\alpha < \lambda$) such that:

(A) $B_\alpha \subseteq \kappa$ and for some $\alpha(*) < \lambda$: $\alpha(*) \le \alpha < \lambda \Rightarrow B_\alpha \notin J_{<\lambda}[\bar{\lambda}]$
(B) $\alpha < \beta \Rightarrow B_\alpha \subseteq B_\beta \bmod J_{<\lambda}[\bar{\lambda}]$ (i.e. $B_\alpha \setminus B_\beta \in J_{<\lambda}[\bar{\lambda}]$)
(C) for each α, $\langle f_\beta \restriction B_\alpha : \beta < \lambda\rangle$ is cofinal in $(\prod\bar{\lambda} \restriction B_\alpha, <_{J_{<\lambda}[\bar{\lambda}]})$ (better restrict yourselves to $\alpha \ge \alpha(*)$ (see (A)) so that necessarily $B_\alpha \notin J_{<\lambda}[\bar{\lambda}]$);.
(D) for some $g \in \prod\bar{\lambda}$, $\bigwedge_{\alpha<\lambda} f_\alpha \le g \bmod J$ where† $J = J_{<\lambda}[\bar{\lambda}] + \{B_\alpha : \alpha < \lambda\}$;

in fact

(D)$^+$ for some $g \in \prod\bar{\lambda}$ for every $\alpha < \lambda$, we have† $f_\alpha \le g \bmod (J_{<\lambda}[\bar{\lambda}] + B_\alpha)$, in fact $B_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$
(E) if $g \le g' \in \prod\bar{\lambda}$, then for arbitrarily large $\alpha < \lambda$:

$$\{i < \kappa : [g(i) \ge f_\alpha(i) \Leftrightarrow g'(i) \ge f_\alpha(i)]\} = \kappa \bmod J_{<\lambda}[\bar{\lambda}]$$

(hence for every large enough $\alpha < \lambda$ this holds)
(F) if δ is a limit ordinal $< \lambda$, f_δ is a $\le_{J_{<\lambda}[\bar{\lambda}]}$ $-$lub of $\{f_\alpha : \alpha < \delta\}$ then B_δ is a lub of $\{B_\alpha : \alpha < \delta\}$ in $\mathcal{P}(\kappa)/J_{<\lambda}[\bar{\lambda}]$.

Proof of 1.7. Remember that for $\varepsilon < \theta$, $A^*_\varepsilon = \{i < \kappa : \lambda_i > |\varepsilon|\}$ so $A^*_\varepsilon = \kappa \bmod I^*$ and $\varepsilon < \zeta \Rightarrow A^*_\zeta \subseteq A^*_\varepsilon$. We now define by induction on $\varepsilon < \theta$, g_ε, $\alpha(\varepsilon) < \lambda$, $\langle B^\varepsilon_\alpha : \alpha < \lambda\rangle$ such that:

(i) $g_\varepsilon \in \prod\bar{\lambda}$
(ii) for $\zeta < \varepsilon$, $g_\zeta \restriction A^*_\varepsilon \le g_\varepsilon \restriction A^*_\varepsilon$
(iii) $B^\varepsilon_\alpha =: \{i \in \kappa : f_\alpha(i) > g_\varepsilon(i)\}$
(iv) if $\alpha(\varepsilon) \le \alpha < \lambda$ then $B^\varepsilon_\alpha \ne B^{\varepsilon+1}_\alpha \bmod J_{<\lambda}[\bar{\lambda}]$

For $\varepsilon = 0$ let $g_\varepsilon = f_0$, and $\alpha(\varepsilon) = 0$.
For ε limit let $g_\varepsilon(i) = \bigcup_{\zeta<\varepsilon} g_\zeta(i)$ if $i \in A^*_\varepsilon$ and zero otherwise; now

$$[\zeta < \varepsilon \Rightarrow g_\zeta \restriction A^*_\varepsilon \le g_\varepsilon \restriction A^*_\varepsilon]$$

holds trivially and $g_\varepsilon \in \prod\bar{\lambda}$ as each λ_i is regular and $[i \in A^*_\varepsilon \Leftrightarrow \lambda_i > \varepsilon]$), and let $\alpha(\varepsilon) = 0$.
For $\varepsilon = \zeta + 1$, if $\{\alpha < \lambda : B^\zeta_\alpha \in J_{<\lambda}[\bar{\lambda}]\}$ is unbounded in λ, then g_ζ is a bound for $\langle f_\alpha : \alpha < \lambda\rangle \bmod J_{<\lambda}[\bar{\lambda}]$, contradicting an assumption. Clearly

$$\alpha < \beta < \lambda \quad \Rightarrow B^\zeta_\alpha \subseteq B^\zeta_\beta \bmod J_{<\lambda}[\bar{\lambda}]$$

hence $\{\alpha < \lambda : B^\zeta_\alpha \in J_{<\lambda}[\bar{\lambda}]\}$ is an initial segment of λ. So by the previous sentence there is $\alpha(\varepsilon) < \lambda$ such that for every $\alpha \in [\alpha(\varepsilon), \lambda)$, we have $B^\zeta_\alpha \notin J_{<\lambda}[\bar{\lambda}]$ (of course, we may increase $\alpha(\varepsilon)$ later). If $\langle B^\zeta_\alpha : \alpha < \lambda\rangle$ satisfies the desired conclusion, with $\alpha(\varepsilon)$ for $\alpha(*)$ in (A) and g_ζ for g in (D), (D)$^+$ and (E), we are done. Now among the conditions in the conclusion of 1.7, (A) holds by the definition of B^ζ_α and of $\alpha(\varepsilon)$, (B) holds by B^ζ_α's definition as $\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J_{<\lambda}[\bar{\lambda}]$, (D)$^+$ holds with $g = g_\zeta$ by the choice of B^ζ_α hence also clause (D) follows. Lastly if (E) fails, say for g', then it can serve as g_ε. Now condition (F) follows immediately from (iii) (if (F) fails for δ, then there is $B \subseteq B^\zeta_\delta$ such that $\bigwedge_{\alpha<\delta} B^\zeta_\alpha \subseteq B \bmod J_{<\lambda}[\bar{\lambda}]$ and

† Of course, if $B_\alpha = \kappa \bmod J_{<\lambda}[\bar{\lambda}]$, this becomes trivial.

$B^\zeta_\delta \setminus B \notin J_{<\lambda}[\bar\lambda]$; now the function $g^* =: (g_\zeta \restriction (\kappa \setminus B)) \cup (f_\delta \restriction B)$ contradicts "f_δ is a $\le_{J_{<\lambda}[\bar\lambda]}$ –lub of $\{f_\alpha : \alpha < \delta\}$", because: $g^* \in \prod \bar\lambda$ (obvious), $\neg(f_\delta \le g^* \mod J_{<\lambda}[\bar\lambda])$ [why? as $B^\zeta_\delta \setminus B \notin J_{<\lambda}[\bar\lambda]$ and $g^* \restriction (B^\zeta_\delta \setminus B) = g_\zeta \restriction (B^\zeta_\delta \setminus B) < f_\delta \restriction (B^\zeta_\delta \setminus B)$ by the choice of B^ζ_δ], and for $\alpha < \delta$ we have:

$$f_\alpha \restriction B \le_{J_{<\lambda}[\bar\lambda]} f_\delta \restriction B = g^* \restriction B \quad \text{and}$$

$$f_\alpha \restriction (\kappa \setminus B) \le_{J_{<\lambda}[\bar\lambda]} g_\zeta \restriction (\kappa \setminus B) = g^* \restriction (\kappa \setminus B)$$

(the $\le_{J_{<\lambda}[\bar\lambda]}$ holds as $(\kappa \setminus B) \cap B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]$ and the definition of B^ζ_α). So only clause (C) (of 1.7) may fail, without loss of generality for $\alpha = \alpha(\varepsilon)$. I.e. $\langle f_\beta \restriction B^\zeta_{\alpha(\varepsilon)} : \beta < \lambda \rangle$ is not cofinal in $(\prod \bar\lambda \restriction B^\zeta_{\alpha(\varepsilon)}, <_{J_{<\lambda}[\bar\lambda]})$. As this sequence of functions is increasing w.r.t. $<_{J_{<\lambda}[\bar\lambda]}$, there is $h_\alpha \in \prod(\bar\lambda \restriction B^\zeta_{\alpha(\varepsilon)})$ such that for no $\beta < \lambda$ do we have $h_\alpha \le f_\beta \restriction B^j_{\alpha(\varepsilon)} \mod J_{<\lambda}[\bar\lambda]$. Let $h'_\varepsilon = h_\varepsilon \cup 0_{(\kappa \setminus B^\zeta_{\alpha(\varepsilon)})}$ and $g_\varepsilon \in \prod \bar\lambda$ be defined by $g_\varepsilon(i) = \mathrm{Max}\{g_\zeta(i), h'_\varepsilon(i)\}$. Now define B^ε_α by (iii) so (i), (ii), (iii) hold trivially, and we can check (iv).

So we can define g_ε, $\alpha(\varepsilon)$ for $\varepsilon < \theta$, satisfying (i)–(iv). As in the proof of 1.5, this is impossible: because (remembering $\mathrm{cf}(\lambda) = \lambda > \theta$) letting $\alpha(*) =: \bigcup_{\varepsilon<\theta} \alpha(\varepsilon) < \lambda$ we have: $\langle B^\varepsilon_{\alpha(*)} \cap A^*_\zeta : \varepsilon < \zeta \rangle$ is $\subseteq$-decreasing, for each $\zeta < \theta$, and $A^*_\varepsilon = \kappa \mod I^*$ and $B^{\varepsilon+1}_{\alpha(*)} \ne B^\varepsilon_{\alpha(*)} \mod J_{<\lambda}[\bar\lambda]$ so $\langle B^\varepsilon_{\alpha(*)} \cap A^*_{\varepsilon+1} \setminus B^{\varepsilon+1}_{\alpha(*)} : \varepsilon < \theta \rangle$ is a sequence of θ pairwise disjoint members of $(J_{<\lambda}[\bar\lambda])^+$ hence of $(I^*)^+$ which give the contradiction to $(*)$ of 1.5; so the lemma cannot fail. $\blacksquare_{1.7}$

LEMMA 1.8: Suppose $(*)$ of 1.5.

(1) For every $B \in J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$, we have:

$$(\prod \bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]}) \text{ has true cofinality } \lambda \text{ (hence } \lambda \text{ is regular).}$$

(2) If D is an ultrafilter on κ, disjoint to I^*, <u>then</u> $\mathrm{cf}(\prod \bar\lambda / D)$ is $\min\{\lambda : D \cap J_{\le\lambda}[\bar\lambda] \ne \emptyset\}$.

(3) (i) For λ limit $J_{<\lambda}[\bar\lambda] = \bigcup_{\mu<\lambda} J_{<\mu}[\bar\lambda]$ hence
(ii) for every λ, $J_{<\lambda}[\bar\lambda] = \bigcup_{\mu<\lambda} J_{\le\mu}[\bar\lambda]$.

(4) $J_{\le\lambda}[\bar\lambda] \ne J_{<\lambda}[\bar\lambda]$ <u>iff</u> $J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda] \ne \emptyset$ <u>iff</u> $\lambda \in \mathrm{pcf}(\bar\lambda)$.

(5) $J_{\le\lambda}[\bar\lambda]/J_{<\lambda}[\bar\lambda]$ is λ-directed (i.e. if $B_\gamma \in J_{\le\lambda}[\bar\lambda]$ for $\gamma < \gamma^*$, $\gamma^* < \lambda$ <u>then</u> for some $B \in J_{\le\lambda}[\bar\lambda]$ we have $B_\gamma \subseteq B \mod J_{<\lambda}[\bar\lambda]$ for every $\gamma < \gamma^*$.)

Proof. (1) Let

$$J = \{B \subseteq \kappa : B \in J_{<\lambda}[\bar\lambda] \text{ or } B \in J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda] \text{ and}$$
$$(\prod \bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]}) \text{ has true cofinality } \lambda\}.$$

By its definition clearly $J \subseteq J_{\le\lambda}[\bar\lambda]$; it is quite easy to check it is an ideal (use 1.3(2)(v)). Assume $J \ne J_{\le\lambda}[\bar\lambda]$ and we shall get a contradiction. Choose $B \in J_{\le\lambda}[\bar\lambda] \setminus J$; as J is an ideal, there is an ultrafilter D on κ such that: $D \cap J = \emptyset$ and $B \in D$. Now if $\mathrm{tcf}(\prod \bar\lambda / D) \ge \lambda^+$, then $B \notin J_{\le\lambda}[\bar\lambda]$ (by the definition of $J_{\le\lambda}[\bar\lambda]$); contradiction. On the other hand if $F \subseteq \prod \bar\lambda$, $|F| < \lambda$ then there is $g \in \prod \bar\lambda$ such that $(\forall f \in F)(f < g \mod J_{<\lambda}[\bar\lambda])$ (by 1.5), so $(\forall f \in F)[f < g \mod D]$ (as $J_{<\lambda}[\bar\lambda] \subseteq J$, $D \cap J = \emptyset$), and this implies $\mathrm{cf}(\prod \bar\lambda / D) \ge \lambda$. By the last two sentences we know that $\mathrm{tcf}(\prod \bar\lambda / D)$ is λ. Now by 1.6 for some $C \in D$, $\left(\prod(\bar\lambda \restriction C), <_{J_{<\lambda}[\bar\lambda]}\right)$ has true cofinality λ, of course $C \cap B \subseteq C$ and $C \cap B \in D$ hence $C \cap B \notin J_{<\lambda}[\bar{\lambda}]$. Clearly if $C' \subseteq C$, $C' \notin J_{<\lambda}[\bar{\lambda}]$ then also $(\prod \bar{\lambda} \restriction C', <_{J_{<\lambda}[\bar{\lambda}]})$ has true cofinality λ, hence by the last sentence without loss of generality $C \subseteq B$; hence by 1.4(5) we know that $C \in J_{\leq\lambda}[\bar{\lambda}]$ hence by the definition of J we have $C \in J$. But this contradicts the choice of D as disjoint from J.

We have to conclude that $J = J_{\leq\lambda}[\bar{\lambda}]$ so we have proved 1.8(1).

(2) Let λ be minimal such that $D \cap J_{\leq\lambda}[\bar{\lambda}] \neq \emptyset$ (it exists as by 1.3(10) $J_{<(\prod \bar{\lambda})^+}[\bar{\lambda}] = \mathcal{P}(\kappa)$) and choose $B \in D \cap J_{\leq\lambda}[\bar{\lambda}]$. So $[\mu < \lambda \Rightarrow B \notin J_{\leq\mu}[\bar{\lambda}]]$ (by the choice of λ) hence by 1.8(3)(ii) below, we have $B \notin J_{<\lambda}[\bar{\lambda}]$. It similarly follows that $D \cap J_{<\lambda}[\bar{\lambda}] = \emptyset$. Now $(\prod \bar{\lambda} \restriction B, <_{J_{<\lambda}[\bar{\lambda}]})$ has true cofinality λ by 1.8(1). As we know that $B \in D \cap J_{\leq\lambda}[\bar{\lambda}]$, and $J_{<\lambda}[\bar{\lambda}] \cap D = \emptyset$; clearly we have finished the proof.

(3) (i) Let $J =: \bigcup_{\mu<\lambda} J_{<\mu}[\bar{\lambda}]$. Now J is an ideal by 1.4(1)+(2) and $(\prod \bar{\lambda}, <_J)$ is λ-directed; i.e. if $\alpha^* < \lambda$ and $\{f_\alpha : \alpha < \alpha^*\} \subseteq \prod \bar{\lambda}$, then there exists $f \in \prod \bar{\lambda}$ such that

$$(\forall \alpha < \alpha^*)(f_\alpha < f \bmod J).$$

[Why? if $\alpha^* < \theta^+$ as $(*)$ of 1.5 holds, this is obvious, suppose not; λ is a limit cardinal, hence there is μ^* such that $\alpha^* < \mu^* < \lambda$. Without loss of generality $|\alpha^*|^+ < \mu^*$. By 1.5, there is $f \in \prod \bar{\lambda}$ such that $(\forall \alpha < \alpha^*)(f_\alpha < f \bmod J_{<\mu^*}[\bar{\lambda}])$. Since $J_{<\mu^*}[\bar{\lambda}] \subseteq J$, it is immediate that

$$(\forall \alpha < \alpha^*)(f_\alpha < f \bmod J).]$$

Clearly $\bigcup_{\mu<\lambda} J_{<\mu}[\bar{\lambda}] \subseteq J_{<\lambda}[\bar{\lambda}]$ by 1.4(2). On the other hand, let us suppose that there is $B \in (J_{<\lambda}[\bar{\lambda}] \setminus \bigcup_{\mu<\lambda} J_{<\mu}[\bar{\lambda}])$. Choose an ultrafilter D on κ such that $B \in D$ and $D \cap J = \emptyset$. Since $(\prod \bar{\lambda}, <_J)$ is λ-directed and $D \cap J = \emptyset$, one has $\mathrm{tcf}(\prod \bar{\lambda}/D) \geq \lambda$, but $B \in D \cap J_{<\lambda}[\bar{\lambda}]$, in contradiction to Definition 1.2(2).

(3)(ii) If λ limit — by part (i) and 1.4(2); if λ successor — by 1.4(2) and Definition 1.2(3).

(4) Easy.

(5) Let $\langle f^\gamma_\alpha : \alpha < \lambda \rangle$ be $<_{J_{<\lambda}[\bar{\lambda}]+(\kappa \setminus B_\gamma)}$-increasing and cofinal in $\prod \bar{\lambda}$ (for $\gamma < \gamma^*$). Let us choose by induction on $\alpha < \lambda$ a function $f_\alpha \in \prod \bar{\lambda}$, as a $<_{J_{<\lambda}[\bar{\lambda}]}$-bound to $\{f_\beta : \beta < \alpha\} \cup \{f^\gamma_\alpha : \gamma < \gamma^*\}$, such f_α exists by 1.5 and apply 1.7 to $\langle f_\alpha : \alpha < \lambda \rangle$, getting $\langle B'_\alpha : \alpha < \lambda \rangle$, now B'_α for α large enough is as required. $\blacksquare_{1.8}$

Conclusion 1.9: If $(*)$ of 1.5, then $\mathrm{pcf}(\bar{\lambda})$ has a last element.

Proof. This is the minimal λ such that $\kappa \in J_{\leq\lambda}[\bar{\lambda}]$. λ exists, since $\lambda^* =: |\prod \bar{\lambda}| \in \{\lambda : \kappa \in J_{\leq\lambda}[\bar{\lambda}]\} \neq \emptyset$ and by 1.4(2); and $\lambda \in pcf(\bar{\lambda})$ by 1.8(4) and $\lambda = \max pcf(\bar{\lambda})$ by 1.4(7)+1.8(4). $\blacksquare_{1.9}$

Claim 1.10: Suppose (*) of 1.5 holds. Assume for $j < \sigma$, D_j is a filter on κ extending $\{\kappa \setminus A : A \in I^*\}$, E a filter on σ and $D^* = \{B \subseteq \kappa : \{j < \sigma : B \in D_j\} \in E\}$ (a filter on κ). Let $\mu_j =: \mathrm{tcf}(\prod \bar{\lambda}, <_{D_j})$ be well defined for $j < \sigma$, and assume further $\mu_j > \sigma + \theta$ (where θ is from $(*)$ of 1.5).
Let

$$\lambda = \mathrm{tcf}(\prod \bar{\lambda}, <_{D^*}), \mu = \mathrm{tcf}(\prod_{j<\sigma} \mu_j, <_E).$$

<u>Then</u> $\lambda = \mu$ (in particular, if one is well defined, than so is the other).

Proof. Wlog $\sigma \geq \theta$ (otherwise we can add $\mu_j =: \mu_0$, $D_j =: D_0$ for $j \in \theta \setminus \sigma$, and replace σ by θ and E by $E' = \{A \subseteq \theta: A \cap \sigma \in E\}$). Let $\langle f^j_\alpha: \alpha < \mu_j \rangle$ be an $<_{D_j}$-increasing cofinal sequence in $(\prod \bar\lambda, <_{D_j})$.

Now $\ell = 0, 1$, for each $f \in \prod \bar\lambda$, define $G_\ell(f) \in \prod_{j<\sigma} \mu_j$ by $G_\ell(f)(j) = \min\{\alpha < \mu_j$: if $\ell = 1$ then $f \leq f^j_\alpha \bmod D_j$ <u>and</u> if $\ell = 0$ then: not $f^j_\alpha < f \bmod D_j\}$ (it is well defined for $f \in \prod \bar\lambda$ by the choice of $\langle f^j_\alpha: \alpha < \mu_j \rangle$).
Note that for f^1, $f^2 \in \prod \bar\lambda$ and $\ell < 2$ we have:

$$\begin{aligned} f^1 &\leq f^2 \bmod D^* \Leftrightarrow B(f^1, f^2) =: \{i < \kappa: f^1(i) \leq f^2(i)\} \in D^* \\ &\Leftrightarrow A(f^1, f^2) =: \{j < \sigma: B(f^1, f^2) \in D_j\} \in E \\ &\Leftrightarrow \text{ for some } A \in E, \text{ for every } i \in A \text{ we have } f^1 \leq_{D_i} f^2 \\ &\Rightarrow \text{ for some } A \in E \text{ for every } i \in A \text{ we have} \\ &\qquad\qquad G_\ell(f^1)(i) \leq G_\ell(f^2)(i) \\ &\Leftrightarrow G_\ell(f^1) \leq G_\ell(f^2) \bmod E. \end{aligned}$$

So

$\otimes_1$ G_ℓ is a mapping from $(\prod \bar\lambda, \leq_{D^*})$ into $(\prod_{j<\sigma} \mu_j, \leq_E)$ preserving order.

Next we prove that

$\otimes_2$ for every $g \in \prod_{j<\sigma} \mu_j$ for some $f \in \prod \bar\lambda$, we have $g \leq G_0(f) \bmod E$.

[Why? Note that $\min\{\mu_j: j < \sigma\} \geq \sigma^+ \geq \theta^+$ and $J_{\leq\theta}[\bar\lambda] \subseteq J_{\leq\sigma}[\bar\lambda]$. By 1.5 we know $(\prod \bar\lambda, <_{J_{\leq\sigma}[\bar\lambda]})$ is σ^+-directed, hence for some function $f \in \prod \bar\lambda$:

$(*)_1$ for $j < \sigma$ we have $f^j_{g(j)} < f \bmod J_{\leq\sigma}[\bar\lambda]$.

We here assumed $\sigma < \mu_j$, hence $J_{\leq\sigma}[\bar\lambda] \subseteq J_{<\mu_j}[\bar\lambda]$ (by 1.4(2)) but $J_{<\mu_j}[\bar\lambda]$ is disjoint to D_j by the definition of $J_{<\mu_j}[\bar\lambda]$ (by 1.8(2) + 1.3(13)(c)) so together with $(*)_1$:

$(*)_2$ for $j < \sigma$, $f^j_{g(j)} < f \bmod D_j$.

So by the definition of G_0 for every $j < \sigma$ we have $g(j) < G_0(f)(j)$ hence clearly $g < G_0(f)$.]

$\otimes_3$ for $f \in \prod \bar\lambda$ we have $G_0(f) \leq G_1(f)$ [Why? read the definitions]

$\otimes_4$ if f_1, $f_2 \in \prod \bar\lambda$ and $G_1(f_1) <_E G_0(f_2)$ then $f_1 <_{D^*} f_2$

[Why? as $G_1(f_1) <_E G_0(f_2)$ there is $B \in E$ such that: $j \in B \Rightarrow G_1(f_1)(j) < G_0(f_2)(j)$ so for each $j \in B$ we have $f_1 \leq_{D_j} f^j_{G_1(f_1)(j)}$ (by the definition of $G_1(f_1)$) and $f^j_{G_1(f_1)(j)} <_{D_j} f_2$ (as $G_1(f_1)(j) < G_0(f_2)(j)$ and the definition of $G_0(f_2)(j)$) so together $f_1 <_{D_j} f_2$. So $A(f_1, f_2) = \{i < \kappa : f_1(i) < f_2(i)\}$ satisfies: $A(f_1, f_2) \in D_j$ for every $j \in B$ but B was chosen in E, hence $A(f_1, f_2) \in D^*$ (by the definition of D^*) hence $f_1 <_{D^*} f_2$ as required]

Now first assume $\lambda = \mathrm{tcf}(\prod \bar\lambda, <_{D^*})$ is well defined, so there is a sequence $\bar f = \langle f_\alpha : \alpha < \lambda \rangle$ of members of $\prod \bar\lambda$, $<_{D^*}$-increasing and cofinal. So $\langle G_0(f_\alpha) : \alpha < \lambda \rangle$ is $\leq_E$-increasing in $\prod_{j<\sigma} \mu_j$ (by $\otimes_1$), for every $g \in \prod_{j<\sigma} \mu_j$ for some $f \in \prod \bar\lambda$ we have $g \leq_E G_0(f)$ (why? by $\otimes_2$), but by the choice of $\bar f$ for some $\beta < \lambda$ we have $f <_{D^*} f_\beta$ hence by $\otimes_1$ we have $g \leq_E G_0(f) \leq_E G_0(f_\beta)$, so $\langle G_0(f_\alpha) : \alpha < \lambda \rangle$ is cofinal in $(\prod_{j<\sigma} \mu_j, <_E)$. Also for every $\alpha < \lambda$, applying the previous sentence to $G(f_\alpha) + 1$ $(\in \prod_{j<\sigma} \mu_j)$ we can find $\beta < \lambda$ such that $G(f_\alpha) + 1 \leq_E G(f_\beta)$, so $G(f_\alpha) <_E G(f_\alpha)$, so for some club C of λ, $\langle G_0(f_\alpha) : \alpha \in C\rangle$ is $<_E$-increasing cofinal in $(\prod\limits_{j<\sigma} \mu_j, <_E)$. So if λ is well defined then $\mu = \mathrm{tcf}(\prod\limits_{j<\sigma} \mu_j, <_E)$ is well defined and equal to λ.

Lastly assume that μ is well defined i.e. $\prod\limits_{j<\sigma} \mu_j/E$ has true cofinality μ, let $\bar g = \langle g_\alpha : \alpha < \mu\rangle$ exemplifies it. Choose by induction on $\alpha < \mu$, a function f_α and ordinals β_α, γ_α such that

(i) $f_\alpha \in \prod \bar\lambda$ and $\beta_\alpha < \mu$ and $\gamma_\alpha < \mu$
(ii) $g_{\beta_\alpha} <_E G_0(f_\alpha) \le_E G_1(f_\alpha) <_E g_{\gamma_\alpha}$ (so $\beta_\alpha < \gamma_\alpha$)
(iii) $\alpha_1 < \alpha_2 < \mu \Rightarrow \gamma_{\alpha_1} < \beta_{\alpha_2}$ (so $\beta_\alpha \ge \alpha$)

In stage α, first choose $\beta_\alpha = \bigcup\{\gamma_{\alpha_1} + 1 : \alpha_1 < \alpha\}$, then choose $f_\alpha \in \prod\bar\lambda$ such that $g_{\beta_\alpha} + 1 <_E G_0(f_\alpha)$ (possible by $\otimes_2$) then choose γ_α such that $G_1(f_\alpha) <_E g_{\gamma_\alpha}$. Now $G_0(f_\alpha) \le_E G_1(f_\alpha)$ by $\otimes_3$. By $\otimes_4$ we have $\alpha_1 < \alpha_2 \Rightarrow f_{\alpha_1} <_{D^*} f_{\alpha_2}$. Also if $f \in \prod\bar\lambda$ then $G_1(f) \in \prod\limits_{j<\sigma} \mu_j$ hence by the choice of $\bar g$, for some $\alpha < \mu$ we have $G_1(f) <_E g_\alpha$ but $\alpha \le \beta_\alpha$ so $G_1(f) <_E g_\alpha \le_E G_0(f_\alpha)$ hence by $\otimes_4$, $f <_{D^*} f_\alpha$. Altogether, $\langle f_\alpha : \alpha < \mu\rangle$ exemplifies that $(\prod\bar\lambda, <_{D^*})$ has true cofinality μ, so λ is well defined and equal to μ. $\blacksquare_{1.11}$

CONCLUSION 1.11: If $(*)$ of 1.5 holds, and σ, $\bar\mu = \langle \mu_j : j < \sigma\rangle$, $\langle D_j : j < \sigma\rangle$ are as in 1.10 and $\sigma + \theta < \min(\bar\mu)$, and J is an ideal on σ and I an ideal on κ such that $I^* \subseteq I \subseteq \{A \subseteq \kappa :$ for some $B \in J$ for every $j \in \sigma \setminus A$ we have $B \notin D_j\}$ (e.g. $I = I^*$) <u>then</u> $\mathrm{pcf}_J(\{\mu_j : j < \sigma\}) \subseteq \mathrm{pcf}_I(\bar\lambda)$.

Proof. Let E be an ultrafilter on σ disjoint to J then we can define an ultrafilter D^* on κ as in 1.10, so clearly D^* is disjoint to I and we apply 1.10. $\blacksquare_{1.11}$

2. Normality of $\lambda \in \mathrm{pcf}(\bar\lambda)$ for $\bar\lambda$

Having found those ideals $J_{<\lambda}[\bar\lambda]$, we would like to know more. As $J_{<\lambda}[\bar\lambda]$ is increasing continuous in λ, the question is how $J_<[\bar\lambda]$, $J_{<\lambda^+}[\bar\lambda]$ are related.

The simplest relation is $J_{<\lambda^+}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$ for some $B \subseteq \kappa$, and then we call λ normal (for $\bar\lambda$) and denote $B = B_\lambda[\bar\lambda]$ though it is unique only modulo $J_{<\lambda}[\bar\lambda]$. We give a sufficient condition for existence of such B, using this in 2.8; giving the necessary definition in 2.3 and needed information in 2.4, 2.5, 2.6; lastly 2.7 is the essential uniqueness of cofinal sequences in appropriate $\prod\bar\lambda/I$.

Definition 2.1.

(1) We say $\lambda \in \mathrm{pcf}(\bar\lambda)$ is normal (for $\bar\lambda$) if for some $B \subseteq \kappa$, $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$.
(2) We say $\lambda \in \mathrm{pcf}(\bar\lambda)$ is semi-normal (for $\bar\lambda$) if there are B_α for $\alpha < \lambda$ such that:
  (i) $\alpha < \beta \Rightarrow B_\alpha \subseteq B_\beta \bmod J_{<\lambda}[\bar\lambda]$
  and
  (ii) $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$.
(3) We say $\bar\lambda$ is normal if every $\lambda \in \mathrm{pcf}(\bar\lambda)$ is normal for $\bar\lambda$. Similarly for semi normal.
(4) In (1), (2), (3) instead $\bar\lambda$ we can say $(\bar\lambda, I)$ or $\prod\bar\lambda/I$ or $(\prod\bar\lambda, <_I)$ if we replace I^* by I (an ideal on $\mathrm{Dom}(\bar\lambda)$).

Fact 2.2. Suppose (*) of 1.5 and $\lambda \in \mathrm{pcf}(\bar\lambda)$. Now:

(1) λ is semi-normal for $\bar\lambda$ iff for some $F = \{f_\alpha : \alpha < \lambda\} \subseteq \prod\bar\lambda$ we have: $[\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J_{<\lambda}[\bar\lambda]]$ and for every ultrafilter D over κ disjoint to $J_{<\lambda}[\bar\lambda]$, F is unbounded in $(\prod\bar\lambda, <_D)$ whenever $\mathrm{tcf}(\prod\bar\lambda, <_D) = \lambda$.

(2) In 2.1(2), without loss of generality, we may assume that
either: $B_\alpha = B_0 \bmod J_{\leq\lambda}[\bar\lambda]$ (so λ is normal)
or: $B_\alpha \neq B_\beta \bmod J_{\leq\lambda}[\bar\lambda]$ for $\alpha < \beta < \lambda$ so λ is not normal.

(3) Assume λ is semi normal for $\bar\lambda$. Then λ is normal for $\bar\lambda$ iff for some F as in part (1) (of 2.2), F has a $<_{J_{<\lambda}[\bar\lambda]}$-exact upper bound $g \in \prod_{i<\kappa}(\lambda_i + 1)$ and then $B =: \{i < \kappa : g(i) = \lambda_i\}$ generates $J_{\leq\lambda}[\bar\lambda]$ over $J_{<\lambda}[\bar\lambda]$.

(4) If λ is semi normal for $\bar\lambda$ then for some $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$, $\bar B = \langle B_\alpha : \alpha < \lambda\rangle$ we have: $\bar B$ is increasing modulo $J_{<\lambda}[\bar\lambda]$, $J_{\leq\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$, and the sequences $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing and $\bar f$, $\bar B$ are as in 1.7.

Proof. 1) For the direction $\Rightarrow$, given $\langle B_\alpha : \alpha < \lambda\rangle$ as in Definition 2.1(2), for each $\alpha < \lambda$, by 1.8(1) we have $(\prod\bar\lambda \restriction B_\alpha, <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality λ, and let it be exemplified by $\langle f^\alpha_\beta : \beta < \lambda\rangle$. By 1.5 we can choose by induction on $\gamma < \lambda$ a function $f_\gamma \in \prod\bar\lambda$ such that: $\beta, \gamma \leq \alpha \Rightarrow f^\alpha_\beta \leq_{J_{<\lambda}[\bar\lambda]} f_\gamma$ and $\beta < \gamma \Rightarrow f_\beta <_{J_{<\lambda}[\bar\lambda]} f_\gamma$.

Now $F =: \{f_\alpha : \alpha < \lambda\}$ is as required. [Why? First, obviously $\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J_{<\lambda}[\bar\lambda]$. Second, if D is an ultrafilter on κ disjoint to I^* and $(\prod\bar\lambda, <_D)$ has true cofinality λ, then by 1.6 for some $B \in J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$ we have $B \in D$, so for some $\alpha < \lambda$, $B \subseteq B_\alpha \bmod J_{<\lambda}[\bar\lambda]$ hence $B_\alpha \in D$. As $f^\alpha_\beta \leq_{J_{<\lambda}[\bar\lambda]} f_\beta$ for $\beta \in [\alpha, \lambda)$ clearly F is cofinal in $(\prod\bar\lambda, <_D)$.]

The other direction, $\Leftarrow$ follows from 1.7 applied to $F = \{f_\alpha : \alpha < \lambda\}$. [Why? we get there $\langle B_\alpha : \alpha < \lambda\rangle$, $B_\alpha \in J_{\leq\lambda}[\bar\lambda]$ increasing modulo $J_{<\lambda}[\bar\lambda]$ so $J =: J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\} \subseteq J_{\leq\lambda}[\bar\lambda]$.

If equality does not hold then for some ultrafilter D over κ, $D \cap J = \emptyset$ but $D \cap J_{\leq\lambda}[\bar\lambda] \neq \emptyset$ so by clause (D) of 1.7, F is bounded in $\prod\lambda/D$ whereas by 1.8(1),(2), $\mathrm{tcf}(\prod\bar\lambda, <_D) = \lambda$ contradicting the assumption on F.]

2) Because we can replace $\langle B_\alpha : \alpha < \lambda\rangle$ by $\langle B_{\alpha_i} : i < \lambda\rangle$ whenever $\langle \alpha_i : i < \lambda\rangle$ is non decreasing, non eventually constant.

3) If λ is normal for $\bar\lambda$, let $B \subseteq \kappa$ be such that $J_{\leq\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$. By 1.8(1) we know that $(\prod(\bar\lambda \restriction B), <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality λ, so let it be exemplified by $\langle f^0_\alpha : \alpha < \lambda\rangle$. Let $f_\alpha = f^0_\alpha \cup 0_{(\kappa\setminus B)}$ for $\alpha < \lambda$ and let $g \in {}^\kappa\mathrm{Ord}$ be defined by $g(i) = \lambda_i$ if $i \in B$ and $g(i) = 0$ if $i \in \kappa \setminus B$. Now $\langle f_\alpha : \alpha < \lambda\rangle$, g are as required by 1.3(11).

Now suppose $\langle f_\alpha : \alpha < \lambda\rangle$ is as in part (1) of 2.2 and g is a $<_{J_{<\lambda}[\bar\lambda]}$ $-$eub of F, $g \in \prod\limits_{i<\kappa}(\lambda_i + 1)$ and $B = \{i : g(i) = \lambda_i\}$. Let D be an ultrafilter on κ disjoint to $J_{<\lambda}[\bar\lambda]$. If $B \in D$ then for every $f \in \prod\bar\lambda$, let $f' = (f \restriction B) \cup 0_{(\kappa\setminus B)}$, now necessarily $f' < \max\{g, 1\}$ (as $[i \in B \Rightarrow f'(i) < \lambda_i = g(i)]$ and $[i \in \kappa\setminus B \Rightarrow f'(i) = 0 \leq g < 1]$), hence (see Definition 1.2(4)) for some $\alpha < \lambda$ we have $f' < \max\{f_\alpha, 1\} \bmod J_{<\lambda}[\bar\lambda]$ hence for some $\alpha < \lambda$, $f' \leq f_\alpha \bmod J_{<\lambda}[\bar\lambda]$ hence $f \leq f' \leq f_\alpha \bmod D$; also $\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod D$, hence together $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifies $\mathrm{tcf}(\prod\bar\lambda, <_D) = \lambda$. If $B \notin D$ then $\kappa \setminus B \in D$ so $g' = g \restriction (\kappa \setminus B) \cup 0_B = g \bmod D$ and $\alpha < \lambda \Rightarrow f_\alpha <_D f_{\alpha+1} \leq_D g =_D g'$, so $g' \in \prod\bar\lambda$ exemplifies F is bounded in $(\prod\bar\lambda, <_D)$ so as F is as in 2.2(1), $\mathrm{tcf}(\prod\bar\lambda, <_D) = \lambda$ is impossible. As D is disjoint to $J_{<\lambda}[\bar\lambda]$, necessarily $\mathrm{tcf}(\prod\bar\lambda, <_D) > \lambda$. The last two arguments together give, by 1.8(2) that $J_{\leq\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$ as required in the definition of normality.

4) Should be clear. $\blacksquare_{2.2}$

We shall give some sufficient conditions for normality.

Remark. In the following definitions we slightly deviate from [Sh-g, Ch I =Sh345a]. The ones here are perhaps somewhat artificial but enable us to deal also with case (β) of 1.5(*). I.e. in Definition 2.3 below we concentrate on the first θ elements of an a_α and for "obey" we also have $\bar{A}^* = \langle A_\alpha : \alpha < \theta\rangle$ and we want to cover also the case θ is singular.

Definition 2.3. Let there be given regular λ, $\theta < \mu < \lambda$, μ possibly an ordinal, $S \subseteq \lambda$, $\sup(S) = \lambda$ and for simplicity S is a set of limit ordinals or at least have no two successive members.

(1) We call $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ a continuity condition for (S, μ, θ) (or is an (S, μ, θ)-continuity condition) if: S is an unbounded subset of λ, $a_\alpha \subseteq \alpha$, $\mathrm{otp}(a_\alpha) < \mu$, and $[\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta]$ and, for every club E of λ, for some‡ $\delta \in S$ we have $\theta = \mathrm{otp}\{\alpha \in a_\delta : \mathrm{otp}(a_\alpha) < \theta$ and for no $\beta \in a_\delta \cap \alpha$ is $(\beta, \alpha) \cap E = \emptyset\}$. We say $\bar{a}$ is continuous in S^* if $\alpha \in S^* \Rightarrow \alpha = \sup(a_\alpha)$.

(2) Assume $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \lambda$ and $\bar{A}^* = \langle A^*_\alpha : \alpha < \theta\rangle$ be a decreasing sequence of subsets of κ such that $\kappa \setminus A^*_\alpha \in I^*$. We say $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ obeys $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ for $\bar{A}^*$ if:
   (i) for $\beta \in a_\alpha$, if $\varepsilon =: \mathrm{otp}(a_\alpha) < \theta$ then we have $f_\beta \restriction A^*_\varepsilon \le f_\alpha \restriction A^*_\varepsilon$ (note: $\bar{A}^*$ determine θ).

(2A) Let κ, $\bar{\lambda}$, I^* be as usual. We say $\bar{f}$ obeys $\bar{a}$ for $\bar{A}^*$ continuously on S^* if: $\bar{a}$ is continuous in S^* and $\bar{f}$ obeys $\bar{a}$ for $\bar{A}^*$ and in addition $S^* \subseteq S$ and for $\alpha \in S^*$ (a limit ordinal) we have $f_\alpha = f_{a_\alpha}$ from (2B), i.e. for every $i < \kappa$ we have $f_\alpha(i) = \sup\{f_\beta(i) : \beta \in a_\alpha\}$ when $|a_\alpha| < \lambda_i$.

(2B) For given $\bar{\lambda} = \langle \lambda_i : i < \kappa\rangle$, $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ where $f_\alpha \in \prod \bar{\lambda}$ and $a \subseteq \lambda$, and θ let $f_a \in \prod \bar{\lambda}$ be defined by: $f_a(i)$ is 0 if $|a| \ge \lambda_i$ and $\cup\{f_\alpha(i) : \alpha \in a\}$ if $|a| < \lambda_i$.

(3) Let (S, θ) stands for $(S, \theta + 1, \theta)$; (λ, μ, θ) stands for "(S, μ, θ) for some unbounded subset S of λ" and (λ, θ) stands for $(\lambda, \theta + 1, \theta)$.
   If each A^*_α is κ then we omit "for $\bar{A}^*$" (but θ should be fixed or said).

(4) We add to "continuity condition" (in part (1)) the adjective "weak" ["θ-weak"] if "$\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$" is replaced by "$\alpha \in S \& \beta \in a_\alpha \Rightarrow (\exists \gamma < \alpha)[a_\alpha \cap \beta \subseteq a_\gamma \;\&\; \gamma < \min(a_\alpha \setminus (\beta + 1)) \;\&\; [|a_\alpha \cap \beta| < \theta \Rightarrow |a_\gamma| < \theta]]$" [but we demand that γ exists only if $\mathrm{otp}(a_\alpha \cap \beta) < \theta$]. (Of course a continuity condition is a weak continuity condition which is a θ-weak continuity condition).

Remark 2.3A. There are some obvious monotonicity implications, we state below only 2.4(3).

Fact 2.4.

(1) Let $\theta_r = \begin{cases} \theta & \mathrm{cf}(\theta) = \theta \\ \theta^+ & \mathrm{cf}(\theta) < \theta \end{cases}$ and assume $\lambda = \mathrm{cf}(\lambda) > \theta_r^+$. Then for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta_r\}$, there is a continuity condition $\bar{a}$ for (S, θ_r); moreover, it is continuous in S and $\delta \in S \Rightarrow \mathrm{otp}(a_\delta) = \theta_r$; so for every club E of λ for some $\delta \in S$, $\forall \alpha, \beta[\alpha < \beta \;\&\; \alpha \in a_\delta \;\&\; \beta \in a_\delta \rightarrow (\alpha, \beta) \cap E \neq \emptyset\}]$.

(2) Assume $\lambda = \theta^{++}$, then for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\theta)\}$ there is a continuity condition for $(S, \theta + 1, \theta)$. (In fact continuous in S and $[\delta \in S \Rightarrow a_\delta$ closed in $\delta]$ and $[\alpha \in a_\delta$ and $\delta \in S \Rightarrow a_\alpha = a_\delta \cap \alpha]$.)

(3) If $\bar{a}$ is a (λ, μ, θ_1)-continuity condition and $\theta_1 \ge \theta$ then there is a $(\lambda, \theta + 1, \theta)$-continuity condition.

‡ Note: if $\mathrm{otp}(a_\delta) = \theta$ and $\delta = \sup(a_\delta)$ (holds if $\delta \in S$, $\mu = \theta + 1$ and $\bar{a}$ continuous in S (see below)) and $\delta \in acc(E)$ then δ is as required.

Proof. 1) By [Sh420, §1].

2) By [Sh351, 4.4(2)] and§ [Sh-g, III 2.14(2), clause (c), p.135-7].

3) Check. $\blacksquare_{2.4}$

Remark 2.4A. Of course also if $\lambda = \theta^+$ the conclusion of 2.4(2) may well hold. We suspect but do not know that the negation is consistent with ZFC.

Fact 2.5. Suppose (*) of 1.5, $f_\alpha \in \prod \bar\lambda$ for $\alpha < \lambda$, $\lambda = \mathrm{cf}(\lambda) > \theta$ (of course $\kappa = \mathrm{dom}(\bar\lambda)$) and $\bar A^* = \bar A^*[\bar\lambda]$ is as in the proof of 1.5, i.e. $A^*_\alpha = \{i < \kappa : \lambda_i > \alpha\}$). Then

(1) Assume $\bar a$ is a θ-weak continuity condition for (S, θ), $\lambda = \sup(S)$, then we can find $\bar f' = \langle f'_\alpha : \alpha < \lambda \rangle$ such that:
 (i) $f'_\alpha \in \prod \bar\lambda$,
 (ii) for $\alpha < \lambda$ we have $f_\alpha \le f'_\alpha$
 (iii) for $\alpha < \beta < \lambda$ we have $f'_\alpha <_{J_{<\lambda}[\bar\lambda]} f'_\beta$
 (iv) $\bar f'$ obeys $\bar a$ for $\bar A^*$

(2) If in addition $\min(\bar\lambda) \ge \mu$, $S^* \subseteq S$ are stationary subsets of λ but $\bar a$ is a continuity condition for (S, μ, θ) and $\bar a$ is continuous on S^* then we can find $\bar f' = \langle f'_\alpha : \alpha < \lambda \rangle$ such that
 (i) $f'_\alpha \in \prod \bar\lambda$
 (ii) for $\alpha \in \lambda \setminus S^*$ we have $f_\alpha \le f'_\alpha$ and $\alpha = \beta + 1 \in \lambda \setminus S^*$ & $\beta \in S^* \Rightarrow f_\beta \le f'_\alpha$
 (iii) for $\alpha < \beta < \lambda$ we have $f'_\alpha <_{J_{<\lambda}[\bar\lambda]} f'_\beta$
 (iv) $\bar f'$ obeys $\bar a$ for $\bar A^*$ continuously on S^*; moreover 2.3(2)(i) can be strengthened to $\beta \in a_\alpha \Rightarrow f_\beta < f_\alpha$.

(3) Suppose $\langle f'_\alpha : \alpha < \lambda \rangle$ obeys $\bar a$ continuously on S^* and satisfies 2.5(2)(ii) (and 2.5(2)'s assumption holds). If $g_\alpha \in \prod \bar\lambda$ and $\langle g_\alpha : \alpha < \lambda \rangle$ obeys $\bar a$ continuously on S^* and $[\alpha \in \lambda \setminus S^* \Rightarrow g_\alpha \le f_\alpha]$ then $\bigwedge_\alpha g_\alpha \le f'_\alpha$.

(4) If $\zeta < \theta$, for $\varepsilon < \zeta$ we have $\bar f^\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \lambda \rangle$, where $f^\varepsilon_\alpha \in \prod \bar\lambda$, then in 2.5(1) (and 2.5(2)) we can find $\bar f'$ as there for all $\bar f^\varepsilon$ simultaneously. Only in clause (ii) we replace $f_\alpha \le f'_\alpha$ by $f_\alpha \restriction A^*_\zeta \le f'_\alpha \restriction A^*_\zeta$ (and $f_\beta \le f'_\alpha$ by $f_\beta \restriction A^*_\zeta \le f'_\alpha \restriction A^*_\zeta$.

Proof. Easy (using 1.5 of course).

Claim 2.5A: In 2.5 we can replace "(*) from 1.5" by "$\prod \bar\lambda / J_{<\lambda}[\bar\lambda]$ is λ-directed and $\liminf_{I^*}(\bar\lambda) \ge \theta$".

Claim 2.6: Assume (*) of 1.5 and let $\bar A^*$ be as there,

(1) in 1.7, if $\langle f_\alpha : \alpha < \lambda \rangle$ obeys some (S, θ)-continuity condition or just a θ-weak one for $\bar A^*$ (where $S \subseteq \lambda$ is unbounded) then we can deduce also:
 (G) the sequence $\langle B_\alpha / J_{<\lambda}[\bar\lambda] : \alpha < \lambda \rangle$ is eventually constant.

(2) If $\theta^+ < \lambda$ then $J_{\le\lambda}[\bar\lambda] / J_{<\lambda}[\bar\lambda]$ is λ^+-directed (hence if λ is semi normal for $\bar\lambda$ then it is normal to $\bar\lambda$).

Proof. 1) Assume not, so for some club E of λ we have

(*) $\alpha < \delta < \lambda \,\&\, \delta \in E \Rightarrow B_\alpha \ne B_\delta \mod J_{<\lambda}[\bar\lambda]$.

As $\bar a$ is a θ-weak (S, θ)-continuity condition, there is $\delta \in S$ such that $b =: \{\alpha \in a_\delta : \mathrm{otp}(a_\delta \cap \alpha) < \theta$ and for no $\beta \in a_\delta \cap \alpha$ is $(\beta, \alpha) \cap E = \emptyset$ and for some $\gamma < \alpha$, $a_\alpha \cap \beta \subseteq a_\gamma$ and $\gamma < \min(a_\alpha \setminus (\beta + 1))$ and $|a_\gamma| < \theta\}$ has order type θ. Let $\{\alpha_\varepsilon : \varepsilon < \theta\}$ list b (increasing with ε). So for every $\varepsilon < \theta$ there is $\gamma_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap E$, and let $\beta_\varepsilon < \alpha_{\varepsilon+1}$ be such that $a_\delta \cap \alpha_\varepsilon \subseteq a_{\beta_\varepsilon}$ and $\mathrm{otp}(a_{\beta_\varepsilon} \cap \alpha_\varepsilon) < \theta$; by shrinking b and

§ the definition of B^α_i in the proof of [Sh-g, III 2.14(2)] should be changed as in [Sh351, 4.4(2)] renaming wlog $\beta_\varepsilon < \gamma_\varepsilon$ and $\alpha_\varepsilon \in a_{\beta_\varepsilon}$. Let $\xi(\varepsilon) =: \mathrm{otp}(a_{\beta_\varepsilon})$. Lastly let $B^0_\varepsilon =: \{i < \kappa: f_{\alpha_\varepsilon}(i) < f_{\beta_\varepsilon}(i) < f_{\gamma_\varepsilon}(i) < f_{\alpha_{\varepsilon+1}}(i)\}$, clearly it is $= \kappa \bmod I^*$ and let (remember $(*)$ above) $B^*_\varepsilon =: A^*_{\xi(\varepsilon)+1} \cap (B_{\gamma_\varepsilon} \setminus B_{\beta_\varepsilon}) \cap B^0_\varepsilon$, now $B_{\alpha_\varepsilon} \subseteq B_{\beta_\varepsilon} \subseteq B_{\gamma_\varepsilon} \bmod J_{<\lambda}[\bar\lambda]$ by clause (B) of 1.7, and $B_{\gamma_\varepsilon} \neq B_{\beta_\varepsilon} \bmod J_{<\lambda}[\bar\lambda]$ by $(*)$ above hence $B_{\gamma_\varepsilon} \setminus B_{\beta_\varepsilon} \neq \emptyset \bmod J_{<\lambda}[\bar\lambda]$. Now B^0_ε, $A^*_{\xi(\varepsilon)+1} = \kappa \bmod I^*$ by the previous sentence and by 1.5$(*)$ which we are assuming respectively and $I^* \subseteq J_{<\lambda}[\bar\lambda]$ by the later's definition; so we have gotten $B^*_\varepsilon \neq \emptyset \bmod J_{<\lambda}[\bar\lambda]$. But for $\varepsilon < \zeta < \theta$ we have $B^*_\varepsilon \cap B^*_\zeta = \emptyset$, for suppose $i \in B^*_\varepsilon \cap B^*_\zeta$, so $i \in A^*_{\xi(\varepsilon)+1}$ and also $f_{\gamma_\varepsilon}(i) < f_{\alpha_{\varepsilon+1}}(i) \leq f_{\beta_\zeta}(i)$ (as $i \in B^0_\varepsilon$ and as $\alpha_{\varepsilon+1} \in a_{\beta_\zeta}$ & $i \in A^*_{\xi(\zeta)+1}$ respectively); now $i \in B^*_\varepsilon$ hence $i \in B_{\gamma_\varepsilon}$ i.e. (where g is from 1.7 clause $(\mathrm{D})^+$) $f_{\gamma_\varepsilon}(i) > g(i)$ hence (by the above) $f_{\beta_\zeta}(i) > g(i)$ hence $i \in B_{\beta_\zeta}$ hence $i \notin B^*_\zeta$, contradiction. So $\langle B^*_\varepsilon : \varepsilon < \theta\rangle$ is a sequence of θ pairwise disjoint members of $(J_{<\lambda}[\bar\lambda])^+$, contradiction.

2) The proof is similar to the proof of 1.8(5), using 2.6(1) instead 1.7 (and $\bar a$ from 2.4(1) if $\lambda > \theta^+_r$ or 2.4(2) if $\lambda = \theta^{++}$). $\blacksquare_{2.6}$

We note also (but shall not use):

Claim 2.7: Suppose $(*)$ of 1.5 and

(a) $f_\alpha \in \prod\bar\lambda$ for $\alpha < \lambda$, $\lambda \in \mathrm{pcf}(\bar\lambda)$ and $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing
(b) $\bar f$ obeys $\bar a$ continuously on S^*, where $\bar a$ is a continuity condition for (S, θ) and $\lambda = \sup(S)$ (hence $\lambda > \theta$ by the last phrase of 2.3(1))
(c) J is an ideal on κ extending $J_{<\lambda}[\bar\lambda]$, and $\langle f_\alpha/J : \alpha < \lambda\rangle$ is cofinal in $(\prod\bar\lambda, <_J)$ (e.g. $J = J_{<\lambda}[\bar\lambda] + (\kappa \setminus B)$, $B \in J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$).
(d) $\langle f'_\alpha : \alpha < \lambda\rangle$ satisfies (a), (b) above.
(e) $f_\alpha \leq f'_\alpha$ for $\alpha \in \lambda \setminus S^*$ (alternatively: $\langle f'_\alpha : \alpha < \lambda\rangle$ satisfies (c)).
(f) if $\delta \in S^*$ <u>then</u> J is $\mathrm{cf}(\delta)$-indecomposable (i.e. if $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\rangle$ is a $\subseteq$-increasing sequence of members, of J then $\bigcup_{\varepsilon < \mathrm{cf}(\delta)} A_\varepsilon \in J$).

<u>Then</u>:

(A) the set

$$\{\delta < \lambda : \text{ if } \delta \in S^* \text{ and } \mathrm{otp}(a_\delta) = \theta \text{ then } f'_\delta = f_\delta \bmod J\}$$

contains a club of λ.

(B) the set

$$\{\delta < \lambda : \text{if } \alpha \in S \text{ and } \delta = \sup(\delta \cap a_\alpha) \text{ and } \mathrm{otp}(\alpha \cap a_\delta) = \theta \\ \text{then } f'_{\alpha \cap a_\delta} = f_{\alpha \cap a_\delta} \bmod J\}$$

contains a club of λ.

Proof. We concentrate on proving (A). Suppose $\delta \in S^*$, and $f_\delta \neq f'_\delta \bmod J$. Let

$$\begin{aligned} A_{1,\delta} &= \{i < \kappa : f_\delta(i) < f'_\delta(i)\} \\ A_{2,\delta} &= \{i < \kappa : f_\delta(i) > f'_\delta(i)\}, \end{aligned}$$

So $A_{1,\delta} \cup A_{2,\delta} \in J^+$, suppose first $A_{1,\delta} \in J^+$. By Definition 2.3(2A), for every $i \in A_{1,\delta}$ for every large enough $\alpha \in a_\delta$, $f_\delta(i) < f'_\alpha(i)$, say for $\alpha \in a_\delta \setminus \beta_i$. As J is $\mathrm{cf}(\delta)$-indecomposable for some $\beta < \alpha$ we have $\{i < \kappa : \beta_i < \beta\} \in J^+$ so $f_\delta \restriction A_{1,\delta} < f'_\beta \restriction A_{1,\delta}$ (and $\beta < \delta$). Now by clause (c), $E =: \{\delta < \lambda$: for every $\beta < \delta$ we have $f'_\beta < f_\delta \bmod J\}$ is a club of λ, and so we have proved

$$\delta \in E \Rightarrow A_{1,\delta} \in J.$$

If $\bigwedge_{\alpha<\lambda} f_\alpha \leq f'_\alpha$ (first possibility in clause (e) implies it) also $A_{2,\delta} \in J$ hence for no $\delta \in S^* \cap E$ do we have $f_\delta \neq f'_\delta \bmod J$. If the second possibility of clause (e) holds, we can interchange $\bar{f}$, $\bar{f}'$ hence $[\delta \in E \Rightarrow A_{2,\delta} \in J]$ and we are done. $\blacksquare_{2.7}$

We now return to investigating the $J_{<\lambda}[\bar{\lambda}]$, first without using continuity conditions.

LEMMA 2.8: Suppose $(*)$ of 1.5 and $\lambda = \mathrm{cf}(\lambda) \in \mathrm{pcf}(\bar{\lambda})$. Then λ is semi normal for $\bar{\lambda}$.

Proof. We assume λ is not semi normal for $\bar{\lambda}$ and eventually get a contradiction. Note that by our assumption $(\prod \bar{\lambda}, <_I)$ is θ^+-directed hence $\min \mathrm{pcf}_I(\bar{\lambda}) \geq \theta^+$ (by 1.3(4)(v)) hence let us define by induction on $\xi \leq \theta$, $\bar{f}^\xi = \langle f^\xi_\alpha : \alpha < \lambda \rangle$, B_ξ and D_ξ such that:

(I) (i) $f^\xi_\alpha \in \prod \bar{\lambda}$
 (ii) $\alpha < \beta < \lambda \Rightarrow f^\xi_\alpha \leq f^\xi_\beta \bmod J_{<\lambda}[\bar{\lambda}]$
 (iii) $\alpha < \lambda \,\&\, \xi < \theta \Rightarrow f^\xi_\alpha \leq f^\theta_\alpha \bmod J_{<\lambda}[\bar{\lambda}]$
 (iv) for $\zeta < \xi < \theta$ and $\alpha < \lambda$: $f^\zeta_\alpha \restriction A^*_\xi \leq f^\xi_\alpha \restriction A^*_\xi$
(II) (i) D_ξ is an ultrafilter on κ such that: $\mathrm{cf}(\prod \bar{\lambda}/D_\xi) = \lambda$
 (ii) $\langle f^\xi_\alpha / D_\xi : \alpha < \lambda \rangle$ is not cofinal in $\prod \bar{\lambda}/D_\xi$
 (iii) $\langle f^{\xi+1}_\alpha / D_\xi : \alpha < \lambda \rangle$ is increasing and cofinal in $\prod \bar{\lambda}/D_\xi$; moreover
 (iii)$^+$ $B_\xi \in D_\xi$ and $\langle f^{\xi+1}_\alpha : \alpha < \lambda \rangle$ is increasing and cofinal in $\prod \bar{\lambda}/(J_{<\lambda}[\bar{\lambda}] + (\kappa \setminus B_\xi))$
 (iv) $f^{\xi+1}_0 / D_\xi$ is above $\{f^\xi_\alpha / D_\xi : \alpha < \lambda\}$.

For $\xi = 0$. No problem. [Use 1.8(1)+(4)].

For ξ limit $< \theta$. Let $g^\xi_\alpha \in \prod \bar{\lambda}$ be defined by $g^\xi_\alpha(i) = \sup\{f^\zeta_\alpha(i) : \zeta < \xi\}$ for $i \in A^*_\xi$ and $f^\xi_\alpha(i) = 0$ else, (remember that $\kappa \setminus A^*_\xi \in I^*$). Then choose by induction on $\alpha < \lambda$, $f^\xi_\alpha \in \prod \bar{\lambda}$ such that $g^\xi_\alpha \leq f^\xi_\alpha$ and $\beta < \alpha \Rightarrow f_\beta < f_\alpha \bmod J_{<\lambda}[\bar{\lambda}]$. This is possible by 1.5 and clearly the requirements (I)(i),(ii),(iv) are satisfied. Use 2.2(1) to find an appropriate D_ξ (i.e. satisfying II(i)+(ii)). Now $\langle f^\xi_\alpha : \alpha < \lambda \rangle$ and D_ξ are as required. (The other clauses are irrelevant.)

For $\xi = \theta$. Choose f^θ_α by induction of α satisfying I(i), (ii), (iii) (possible by 1.5).

For $\xi = \zeta + 1$. Use 1.6 to choose $B_\zeta \in D_\zeta \cap J_{\leq\lambda}[\bar{\lambda}] \setminus J_{<\lambda}[\bar{\lambda}]$. Let $\langle g^\xi_\alpha : \alpha < \lambda \rangle$ be cofinal in $(\prod \bar{\lambda}, <_{D_\xi})$ and even in $(\prod \bar{\lambda}, <_{J_<[\bar{\lambda}]+(\kappa \setminus B_\xi)})$ and without loss of generality $\bigwedge_{\alpha<\lambda} f^\zeta_\alpha / D_\zeta < g^\xi_0 / D_\zeta$ and $\bigwedge_{\alpha<\lambda} f^\zeta_\alpha \restriction A^*_\xi \leq g^\xi_\alpha \restriction A^*_\xi$. We get $\langle f^\xi_\alpha : \alpha < \lambda \rangle$ increasing and cofinal $\bmod (J_{<\lambda}[\bar{\lambda}] + (\kappa \setminus B_\xi))$ such that $g^\xi_\alpha \leq f^\xi_\alpha$ by 1.5 from $\langle g^\xi_\alpha : \alpha < \lambda \rangle$. Then get D_ξ as in the case "ξ limit".

So we have defined the f^ξ_α's and D_ξ's. Now for each $\xi < \theta$ we apply (II) (iii)$^+$ for $\langle f^{\xi+1}_\alpha : \alpha < \lambda \rangle$, $\langle f^\theta_\alpha : \alpha < \lambda \rangle$. We get a club C_ξ of λ such that:

$$\alpha < \beta \in C_\xi \Rightarrow f^\theta_\alpha \restriction B_\xi < f^{\xi+1}_\beta \restriction B_\xi \bmod J_{<\lambda}[\bar{\lambda}] \tag{$*$}$$

So $C =: \bigcap_{\xi<\theta} C_\xi$ is a club of λ. By 2.2(1) applied to $\langle f^\theta_\alpha : \alpha < \lambda \rangle$ (and the assumption "λ is not semi-normal for $\bar{\lambda}$") there is $g \in \prod \bar{\lambda}$ such that

$$\neg g \leq f^\theta_\alpha \bmod J_{<\lambda}[\bar{\lambda}] \text{ for } \alpha < \lambda \tag{$*$)$_1$}$$

(not used) and by 1.5 wlog

$$f_0^\xi < g \bmod J_{<\lambda}[\bar\lambda] \quad \text{for } \xi < \theta \tag{$*$)$_2$}$$

For each $\xi < \theta$, by II (iii), (iii)$^+$ for some $\alpha_\xi < \lambda$ we have

$$g \restriction B_\xi < f_{\alpha_\xi}^{\xi+1} \restriction B_\xi \bmod J_{<\lambda}[\bar\lambda] \tag{$*$)$_3$}$$

Let $\alpha(*) = \sup_{\xi<\theta} \alpha_\xi$, so $\alpha(*) < \lambda$ and so

$$g \restriction B_\xi < f_{\alpha(*)}^{\xi+1} \restriction B_\xi \bmod J_{<\lambda}[\bar\lambda] \tag{$*$)$_4$}$$

For $\zeta < \theta$, let $B_\zeta^* = \{i \in A_\zeta^* : g(i) < f_{\alpha(*)}^\zeta(i)\}$. By $(*)_4$, $B_{\xi+1}^* \in D_\xi$; by (II)(iv)+$(*)_2$ we know $B_\xi^* \notin D_\xi$, hence $B_\xi^* \neq B_{\xi+1}^* \bmod D_\xi$ hence $B_\xi^* \neq B_{\xi+1}^* \bmod J_{<\lambda}[\bar\lambda]$.
On the other hand by (I)(iv) for each $\zeta < \theta$ we have $\langle B_\xi^* \cap A_\zeta^* : \xi \le \zeta\rangle$ is $\subseteq$-increasing and (as $A_\zeta^* = \kappa \bmod J_{<\lambda}[\bar\lambda]$ for each $\zeta < \theta$) hence by I(iv) we have $\langle B_\xi^*/I^* : \xi < \theta\rangle$ is $\subseteq$-increasing, and by the previous sentence $B_\xi^* \neq B_{\xi+1}^* \bmod J_{<\lambda}[\bar\lambda]$ hence $\langle B_\xi^*/I^* : \xi < \theta\rangle$ is strictly $\subseteq$-increasing. Together clearly $\langle B_{\xi+1}^* \cap A_{\xi+1}^* \backslash B_\xi^* : \xi < \theta\rangle$ is a sequence of θ pairwise disjoint members of $(J_{<\lambda}[\bar\lambda])^+$, hence of $(I^*)^+$; contradiction to $\theta \ge \mathrm{wsat}(I^*)$. $\blacksquare_{2.8}$

Definition 2.9.

(1) We say $\langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ is a <u>generating sequence</u> for $\bar\lambda$ if:
  (i) $B_\lambda \subseteq \kappa$ and $\mathfrak{c} \subseteq \mathrm{pcf}(\bar\lambda)$
  (ii) $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B_\lambda$ for each $\lambda \in \mathfrak{c}$
(2) We call $\bar B = \langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ <u>smooth</u> if:

$$i \in B_\lambda \,\&\, \lambda_i \in \mathfrak{c} \Rightarrow B_{\lambda_i} \subseteq B_\lambda.$$

(3) We call $\bar B = \langle B_\lambda : \lambda \in \mathrm{Rang}(\bar\lambda)\rangle$ <u>closed</u> if for each λ

$$B_\lambda \supseteq \{i < \kappa : \lambda_i \in \mathrm{pcf}(\bar\lambda \restriction B_\lambda)\}$$

Fact 2.10. Assume $(*)$ of 1.5.

(1) Suppose $\mathfrak{c} \subseteq \mathrm{pcf}(\bar\lambda)$, $\bar B = \langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\bar\lambda$, and $B \subseteq \kappa$, $\mathrm{pcf}(\bar\lambda \restriction B) \subseteq \mathfrak{c}$ <u>then</u> for some finite $\mathfrak{d} \subseteq \mathfrak{c}$, $B \subseteq \bigcup_{\mu\in\mathfrak{d}} B_\mu \bmod I^*$.
(2) $\mathrm{cf}(\prod \bar\lambda / I^*) = \max \mathrm{pcf}(\bar\lambda)$

Remark 2.10A. For another proof of 2.10(2) see 2.12(2)+ 2.12(4) and for another use of the proof of 2.10(2) see 2.14(1).

Proof. (1) If not, then $I = I^* + \{B \cap \bigcup_{\mu\in\mathfrak{d}} B_\mu : \mathfrak{d} \subseteq \mathfrak{c},\ \mathfrak{d} \text{ finite}\}$ is a family of subsets of κ, closed under union, $B \notin I$, hence there is an ultrafilter D on κ disjoint from I to which B belongs. Let $\mu =: \mathrm{cf}(\prod_{i<\kappa} \lambda_i / D)$; necessarily $\mu \in \mathrm{pcf}(\bar\lambda \restriction B)$, hence by the last assumption of 2.10(1) we have $\mu \in \mathfrak{c}$. By 1.8(2) we know $B_\mu \in D$ hence $B \cap B_\mu \in D$, contradicting the choice of D.

(2) The case $\theta = \aleph_0$ is trivial (as $\mathrm{wsat}(I^*) \le \aleph_0$ implies $\mathcal{P}(\kappa)/I^*$ is a Boolean algebra satisfying the $\aleph_0$-c.c. (as here we can subtract) hence this Boolean algebra is finite hence also $\mathrm{pcf}(\bar\lambda)$ is finite) so we assume $\theta > \aleph_0$. For $B \in (I^*)^+$ let $\lambda(B) = \max \mathrm{pcf}_{I^* \restriction B}(\bar\lambda \restriction B)$.

We prove by induction on λ that for every $B \in (I^*)^+$, $\mathrm{cf}(\prod \bar\lambda, <_{I^* + (\kappa\backslash B)}) = \lambda(B)$ when $\lambda(B) \le \lambda$; this will suffice (use $B = \kappa$ and $\lambda = |\prod_{i<\kappa} \lambda_i|^+$). Given B let $\lambda = \lambda(B)$, by notational change wlog $B = \kappa$. By 1.9, $\mathrm{pcf}(\prod \bar\lambda)$ has a last element, necessarily it is $\lambda =: \lambda(B)$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ be $<_{J_{<\lambda}[\bar\lambda]}$ increasing cofinal in

$\prod \bar{\lambda}/J_{<\lambda}[\bar{\lambda}]$, it clearly exemplifies $\max \mathrm{pcf}(\bar{\lambda}) \le \mathrm{cf}(\prod \bar{\lambda}/I^*)$. Let us prove the other inequality. For $A \in J_{<\lambda}[\bar{\lambda}] \setminus I^*$ choose $F_A \subseteq \prod \bar{\lambda}$ which is cofinal in $\prod \bar{\lambda}/(I^* + (\kappa \setminus A))$, $|F_A| = \lambda(A) < \lambda$ (exists by the induction hypothesis). Let χ be a large enough regular, and we now choose by induction on $\varepsilon < \theta$, N_ε, g_ε such that:

(A) (i) $N_\varepsilon \prec (H(\chi), \in, <^*_\chi)$
  (ii) $\|N_\varepsilon\| = \lambda$
  (iii) $\langle N_\xi : \xi \le \varepsilon \rangle \in N_{\varepsilon+1}$
  (iv) $\langle N_\varepsilon : \varepsilon < \theta \rangle$ is increasing continuous
  (v) $\{\varepsilon : \varepsilon \le \lambda + 1\} \subseteq N_0$, $\{\bar{\lambda}, I^*\} \in N_0$, $\langle f_\alpha : \alpha < \lambda \rangle \in N_0$ and the function $A \mapsto F_A$ belongs to N_0.

(B) (i) $g_\varepsilon \in \prod \bar{\lambda}$ and $g_\varepsilon \in N_{\varepsilon+1}$
  (ii) for no $f \in N_\varepsilon \cap \prod \bar{\lambda}$ does $g_\varepsilon <_{I^*} f$
  (iii) $\zeta < \varepsilon \,\&\, \lambda_i > |\varepsilon| \Rightarrow g_\zeta(i) < g_\varepsilon(i)$.

There is no problem to define N_ε, and if we cannot choose g_ε this means that $N_\varepsilon \cap \prod \bar{\lambda}$ exemplifies $\mathrm{cf}(\prod \bar{\lambda}, <) \le \lambda$ as required. So assume $\langle N_\varepsilon, g_\varepsilon : \varepsilon < \theta \rangle$ is defined. For each $\varepsilon < \theta$ for some $\alpha(\varepsilon) < \lambda$, $g_\varepsilon < f_{\alpha(\varepsilon)} \bmod J_{<\lambda}[\bar{\lambda}]$ hence $\alpha(\varepsilon) \le \alpha < \lambda \Rightarrow g_\varepsilon <_{J_{<\lambda}[\bar{\lambda}]} f_\alpha$. As $\lambda = \mathrm{cf}(\lambda) > \theta$, we can choose $\alpha < \lambda$ such that $\alpha > \bigcup_{\varepsilon<\theta} \alpha(\varepsilon)$. Let $B_\varepsilon = \{i < \kappa : g_\varepsilon(i) \ge f_\alpha(i)\}$; so for each $\xi < \theta$ we have $\langle B_\varepsilon \cap A^*_\xi : \varepsilon \le \xi \rangle$ is increasing with ε, (by clause (B)(iii)), hence as usual as $\theta \ge \mathrm{wsat}(I^*)$ (and $\theta > \aleph_0$) we can find $\varepsilon(*) < \theta$ such that $\bigwedge_n B_{\varepsilon(*)+n} = B_{\varepsilon(*)} \bmod I^*$ [why do we not demand $\varepsilon \in (\varepsilon(*), \theta) \Rightarrow B_\varepsilon = B_{\varepsilon(*)} \bmod I^*$? as θ may be singular]. Now as $g_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ and $f_\alpha \in N_0 \prec N_{\varepsilon(*)+1}$ clearly, by its definition, $B_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ hence $F_{B_{\varepsilon(*)}} \in N_{\varepsilon(*)+1}$. Now:

$$\begin{aligned} g_{\varepsilon(*)+1} \restriction (\kappa \setminus B_{\varepsilon(*)}) =_{I^*} g_{\varepsilon(*)+1} \restriction (\kappa \setminus B_{\varepsilon(*)+1}) &< f_\alpha \restriction (\kappa \setminus B_{\varepsilon(*)+1}) \\ &=_{I^*} f_\alpha \restriction (\kappa \setminus B_{\varepsilon(*)}) \end{aligned}$$

[why first equality and last equality? as $B_{\varepsilon(*)+1} = B_{\varepsilon(*)} \bmod I^*$, why the $<$ in the middle? by the definition of $B_{\varepsilon(*)+1}$].

But $g_{\varepsilon(*)+1} \restriction B_{\varepsilon(*)} \in \prod_{i \in B_{\varepsilon(*)}} \lambda_i$, and $B_{\varepsilon(*)} \in J_{<\lambda}[\bar{\lambda}]$ as $g_\varepsilon < f_{\alpha(\varepsilon)} \le f_\alpha \bmod J_{<\lambda}[\bar{\lambda}]$ so for some $f \in F_{B_{\varepsilon(*)}} \subseteq \prod \bar{\lambda}$ we have $g_{\varepsilon(*)+1} \restriction B_{\varepsilon(*)} < f \restriction B_{\varepsilon(*)} \bmod I^*$. By the last two sentences

$$g_{\varepsilon(*)+1} < \max\{f, f_\alpha\} \bmod I^* \tag{$*$}$$

Now $f_\alpha \in N_{\varepsilon(*)+1}$ and $f \in N_{\varepsilon(*)+1}$ (as $f \in F_{B_{\varepsilon(*)}}$, $|F_{B_{\varepsilon(*)}}| \le \lambda$, $\lambda + 1 \subseteq N_{\varepsilon(*)+1}$ the function $B \mapsto F_B$ belongs to $N_0 \prec N_{\varepsilon(*)+1}$ and $B_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ as $\{g_{\varepsilon(*)}, f_\alpha\} \in N_{\varepsilon(*)+1}$) so together

$$\max\{f, f_\alpha\} \in N_{\varepsilon(*)+1}; \tag{$**$}$$

But $(*)$, $(**)$ together contradict the choice of $g_{\varepsilon(*)+1}$ (i.e. clause (B)(ii)). $\blacksquare_{2.10}$

Definition 2.11.

(1) We say that I^* satisfies the pcf-th for (the regular) $(\bar{\lambda}, \theta)$ if $\prod \bar{\lambda}/I^*$ is θ-directed and $(\prod \bar{\lambda}, <_{J_{<\lambda}[\bar{\lambda}]})$ is λ-directed for each λ and we can find $\langle B_\lambda : \lambda \in \mathrm{pcf}_{I^*}(\bar{\lambda}) \rangle$, such that:

$B_\lambda \subseteq \kappa$, $J_{<\lambda}[\bar{\lambda}, I^*] = I^* + \{B_\mu : \mu \in \lambda \cap \mathrm{pcf}_{I^*}(\bar{\lambda})\}$, $B_\lambda \notin J_{<\lambda}[\bar{\lambda}, I^*]$ and $\prod(\bar{\lambda} \restriction B_\lambda)/J_{<\lambda}[\bar{\lambda}, I^*]$ has true cofinality λ (so $B_\lambda \in J_{\le\lambda}[\bar{\lambda}] \setminus J_{<\lambda}[\bar{\lambda}]$ and $J_{\le\lambda}[\bar{\lambda}] = J_{<\lambda}[\bar{\lambda}] + B_\lambda$).

(1A) We say that I^* satisfies the weak pcf-th for $(\bar{\lambda}, \theta)$ if

$(\prod \bar\lambda, <_{I^*})$ is θ-directed
each $(\prod \bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is λ-directed and
there are $B_{\lambda,\alpha} \subseteq \kappa$ for $\alpha < \lambda \in \mathrm{pcf}_{I^*}(\bar\lambda)$ such that

$$\alpha < \beta < \mu \in \mathrm{pcf}_{I^*}(\bar\lambda) \Rightarrow B_{\mu,\alpha} \subseteq B_{\mu,\beta} \mod J_{<\mu}[\bar\lambda, I^*]$$

$$J_{<\lambda}[\bar\lambda] = I^* + \{B_{\mu,\alpha}: \alpha < \mu < \lambda, \mu \in \mathrm{pcf}_{I^*}(\bar\lambda)\}$$

and

$$(\prod(\bar\lambda \restriction B_{\mu,\alpha}), <_{J_{<\lambda}[\bar\lambda]}) \text{ has true cofinality } \lambda$$

(1B) We say that I^* satisfies the weaker pcf-th for $(\bar\lambda, \theta)$ <u>if</u> $(\prod \bar\lambda, <_{I^*})$ is θ-directed and each $(\prod \bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is λ-directed and for any ultrafilter D on κ disjoint to $J_{<\theta}[\bar\lambda]$ letting $\lambda = \mathrm{tcf}(\prod \bar\lambda, <_D)$ we have: $\lambda \geq \theta$ and for some $B \in D \cap J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$, the partial order $(\prod(\bar\lambda \restriction B), <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality λ.

(1C) We say that I^* satisfies the weakest pcf-th for $(\bar\lambda, \theta)$ <u>if</u> $(\prod \bar\lambda, <_{I^*})$ is θ-directed and $(\prod \bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is λ-directed for any $\lambda \geq \theta$

(1D) Above we write $\bar\lambda$ instead $(\bar\lambda, \theta)$ when we mean

$$\theta = \sup\{\theta : (\prod \bar\lambda, <_{I^*}) \text{ is } \theta^+\text{-directed}\}.$$

(2) We say that I^* satisfies the pcf-th for θ if for any regular $\bar\lambda$ such that $\liminf_{I^*}(\bar\lambda) \geq \theta$, we have: I^* satisfies the pcf-th for $\bar\lambda$. We say that I^* satisfies the pcf-th above μ (above μ^-) if it satisfies the pcf-th for $\bar\lambda$ with $\liminf_{I^*}(\bar\lambda) > \mu$ (with $\{i : \lambda_i \geq \mu\} = \kappa \bmod I^*$). Similarly (in both cases) for the weak pcf-th and the weaker pcf-th.

(3) Given I^*, θ let $J_\theta^{\mathrm{pcf}} = \{A \subseteq \kappa: A \in I^*$ or $A \notin I^*$ and $I^* + (\kappa \setminus A)$ satisfies the pcf-theorem for $\theta\}$.
$J_\theta^{\mathrm{wsat}} =: \{A \subseteq \kappa: \mathrm{wsat}(I^* \restriction A) \leq \theta$ or $A \in I^*\}$;
similarly J_θ^{wpcf}; we may write $J_\theta^x[I^*]$.

(4) We say that I^* satisfies the pseudo pcf-th for $\bar\lambda$ <u>if</u> for every ideal I on κ extending I^*, for some $A \in I^+$ we have $(\prod(\bar\lambda \restriction A), <_I)$ has a true cofinality.

CLAIM 2.12:

(1) If $(*)$ of 1.5 then I^* satisfies the weak pcf-th for $(\bar\lambda, \theta^+)$.

(2) If $(*)$ of 1.5 holds, and $\prod \bar\lambda/I^*$ is θ^{++}-directed (i.e. $\theta^+ < \min \bar\lambda$) or just there is a continuity condition for (θ^+, θ)) <u>then</u> I^* satisfies the pcf-th for $(\bar\lambda, \theta^+)$.

(3) If I^* satisfy the pcf-th for $(\bar\lambda, \theta)$ <u>then</u> I^* satisfy the weak pcf-th for $(\bar\lambda, \theta)$ which implies that I^* satisfies the weaker pcf-th for $(\bar\lambda, \theta)$, which implies that I^* satisfies the weakest pcf-th for $(\bar\lambda, \theta)$.

Proof. (1) Let appropriate $\bar\lambda$ be given. By 1.5, 1.8 most demands holds, but we are left with seminormality. By 2.8, if $\lambda \in \mathrm{pcf}(\bar\lambda)$, then $\bar\lambda$ is semi normal for λ. This finishing the proof of (1).

(2) Let $\lambda \in \mathrm{pcf}(\bar\lambda)$ and let $\bar f$, $\bar B$ be as in 2.2(4). By 2.4(1)+(2) there is $\bar a$, a (λ, θ)-continuity condition; by 2.5(1) wlog $\bar f$ obeys $\bar a$, by 2.6(1) the relevant B_α/I^* are eventually constant which suffices by 2.2(2).

(3) Should be clear. $\blacksquare_{2.12}$

CLAIM 2.13: Assume $(\prod \bar\lambda, <_{I^*})$ is given (but possibly $(*)$ of 1.5 fails).

(1) If I^*, $\bar\lambda$ satisfies (the conclusion of) 1.6, <u>then</u> I^*, $\bar\lambda$ satisfy (the conclusions of) 1.8(1), 1.8(2), 1.8(3), 1.8(4), 1.9.

(1A) If I^* satisfies the weaker pcf-th for $\bar\lambda$ <u>then</u> they satisfy the conclusions of 1.6 and 1.5.
(2) If I^*, $\bar\lambda$ satisfies (the conclusion of) 1.5 <u>then</u> I^*, $\bar\lambda$ satisfies (the conclusion of) 1.10.
(2A) If I^* satisfies the weakest pcf-th for $\bar\lambda$ <u>then</u> I^*, $\bar\lambda$ satisfy the conclusion of 1.5.
(3) If I^*, $\bar\lambda$ satisfies 1.5, 1.6 <u>then</u> I^*, $\bar\lambda$ satisfies 2.2(1) (for 2.2(2) - no assumptions).
(4) If I^*, $\bar\lambda$ satisfies 1.8(1), 1.8(2) <u>then</u> I^*, $\bar\lambda$ satisfies 2.2(3) when we interpret "seminormal" by the second phrase of 2.2(1)
(5) If I^*, $\bar\lambda$ satisfies 1.8(2) <u>then</u> I^*, $\bar\lambda$ satisfies 2.10(1).
(6) If I^* $\bar\lambda$ satisfy 1.8(1) + 1.8(3)(i) <u>then</u> I^*, $\bar\lambda$ satisfies 1.8(2)
(7) If I^*, $\bar\lambda$ satisfies 1.8(1) + 1.8(2) and is semi normal <u>then</u> 2.10(2) holds i.e.

$$\mathrm{cf}(\prod\bar\lambda, <_{I^*}) \leq \sup \mathrm{pcf}_{I^*}(\lambda).$$

(8) If I^*, $\bar\lambda$ satisfies 1.5+1.6 <u>then</u> they satisfy 2.10(2).

Proof. (1) We prove by parts.

Proof of 1.8(2). Let $\lambda = \mathrm{tcf}(\prod\bar\lambda/D)$; by the definition of $J_{<\lambda}[\bar\lambda]$, clearly $D \cap J_{<\lambda}[\bar\lambda] = \emptyset$. Also by 1.6 for some $B \in D$ we have $\lambda = \mathrm{tcf}(\prod(\bar\lambda \restriction B), <_{J_{<\lambda}[\bar\lambda]})$, so by the previous sentence $B \notin J_{<\lambda}[\bar\lambda]$, and by 1.4(5) we have $B \in J_{\leq\lambda}[\bar\lambda]$, together we finish.

Proof of 1.8(1). Repeat the proof of 1.8(1) replacing the use of 1.5 by 1.8(2).

Proof of 1.8(3)(i). Let $J =: \bigcup\limits_{\mu<\lambda} J_{<\mu}[\bar\lambda]$, so $J \subseteq J_{<\lambda}[\bar\lambda]$ is an ideal because $\langle J_{<\mu}[\bar\lambda] : \mu < \lambda\rangle$ is $\subseteq$-increasing (by 1.4(2)), if equality fail choose $B \in J_{<\lambda}[\bar\lambda] \setminus J$ and choose D an ultrafilter on κ disjoint to J to which B belongs. Now if $\mu = \mathrm{cf}(\mu) < \lambda$ then $\mu^+ < \lambda$ (as λ is a limit cardinal) and $\mu = \mathrm{cf}(\mu)$ & $\mu^+ < \lambda \Rightarrow D \cap J_{\leq\mu}[\bar\lambda] = D \cap J_{<\mu^+}[\bar\lambda] = \emptyset$ hence by 1.8(2) we have $\mu \neq \mathrm{cf}(\prod\bar\lambda/D)$. Also if $\mu = \mathrm{cf}(\mu) \geq \lambda$ then $D \cap J_{<\mu}[\bar\lambda] \subseteq D \cap J_{<\lambda}[\bar\lambda] \neq \emptyset$ hence by 1.8(2) we have $\mu \neq \mathrm{cf}(\prod\bar\lambda/D)$. Together contradiction by 1.3(7).

Proof of 1.8(3)(ii). Follows.

Proof of 1.8(4). Follows.

Proof of 1.9. As in 1.9.
(1A) Check.
(2) Read the proof of 1.10.
(2A) Check.
(3) The direction $\Rightarrow$ is proved directly as in the proof of 2.2(1) (where the use of 1.8(1) is justified by 2.13(1)).

So let us deal with the direction $\Leftarrow$. So assume $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ is a sequence of members of $\prod\bar\lambda$ which is $<_{J_{<\lambda}[\bar\lambda]}$-increasing such that for every ultrafilter D on κ disjoint to $J_{<\lambda}[\bar\lambda]$ we have: $\lambda = \mathrm{tcf}(\prod\bar\lambda, <_D)$ iff $\bar f$ is unbounded (equivalently cofinal) in $(\prod\bar\lambda, <_D)$. By (the conclusion of) 1.5 wlog $\bar f$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing.

By 1.5 there is $g \in \prod\bar\lambda$ such that $f_\alpha < g \mod J_{\leq\lambda}[\bar\lambda]$ for each $\alpha < \lambda$, and let $B_\alpha =: \{i < \kappa : g(i) \leq f_\alpha(i)\}$. Hence $B_\alpha \in J_{\leq\lambda}[\bar\lambda]$ (by the previous sentence) and $\langle B_\alpha/J_{<\lambda}[\bar\lambda] : \alpha < \lambda\rangle$ is $\subseteq$-increasing (as $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing). Lastly if $B \in J_{\leq\lambda}[\bar\lambda]$, but $B \setminus B_\alpha \notin J_{<\lambda}[\bar\lambda]$ for each $\alpha < \lambda$, let D be an ultrafilter on κ disjoint to $J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$ but to which B belongs, so $\mathrm{tcf}(\prod\bar\lambda, <_D) = \lambda$ (by 1.8(2) which holds by 2.13(1)) but $\{f_\alpha/D : \alpha < \lambda\}$ is bounded by g/D (as

$f_\alpha/D \le g/D$ by the definition of B_α), contradiction. So the sequence $\langle B_\alpha : \alpha < \lambda\rangle$ is as required.

4) – 6) Left to the reader.

7) Let for $\lambda \in \mathrm{pcf}(\bar\lambda)$, $\langle B_i^\lambda : i < \lambda\rangle$ be such that $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + \{B_i^\lambda : i < \lambda\}$ (exists by seminormality; we use only this equality). Let $\langle f_\alpha^{\lambda,i} : \alpha < \lambda\rangle$ be cofinal in $(\prod(\bar\lambda \restriction B_i^\lambda), <_{J_{<\lambda}[\bar\lambda]})$, it exists by 1.8(1). Let F be the closure of $\{f_\alpha^{\lambda,i} : \alpha < \lambda, i < \lambda, \lambda \in \mathrm{pcf}(\bar\lambda)\}$, under the operation $\max\{g,h\}$. Clearly $|F| \le \sup\mathrm{pcf}(\bar\lambda)$, so it suffice to prove that F is a cover of $(\prod\bar\lambda, <_{I^*})$. Let $g \in \prod\bar\lambda$, if $(\exists f \in F)(g \le f)$ we are done, if not

$$I = \{A \cup \{i < \kappa : f(i) > g(i)\} : f \in F, A \in I^*\}$$

is $\aleph_0$-directed, $\kappa \notin I$, so there is an ultrafilter D on κ disjoint to I, (so $f \in F \Rightarrow f <_D g$) and let $\lambda = \mathrm{tcf}(\prod\bar\lambda/D)$, so by 1.8(2) we have $D \cap J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda] \ne \emptyset$, hence for some $i < \lambda$, $B_i^\lambda \in D$, and we get contradiction to the choice of the $\{f_\alpha^{\lambda,\alpha} : \alpha < \lambda\}$ $(\subseteq F)$.

8) Repeat the proof of 2.10(2) (only using $J = \{A \subseteq \kappa$: if $A \notin J_{<\lambda}[\bar\lambda]$ then $\mathrm{cf}(\prod\bar\lambda/I^*) \le \lambda\}$; if $\kappa \notin J$ let D be an ultrafilter on κ disjoint to J, and use 1.6). $\blacksquare_{2.13}$

Claim 2.14: If I^* satisfies pseudo pcf-th <u>then</u>

(1) We can find $\langle(J_\zeta, \theta_\zeta) : \zeta < \zeta^*\rangle$, ζ^* a successor ordinal such that $J_0 = I^*$, $J_{\zeta+1} = \{A \subseteq \kappa :$ if $A \notin J_\zeta$ then $\mathrm{tcf}(\prod(\bar\lambda \restriction A), <_{J_\zeta}) = \theta_\zeta\}$ and for no $A \in (J_\zeta)^+$ does $(\prod(\bar\lambda \restriction A), <_{J_\zeta})$ has true cofinality which is $< \theta_\zeta$.

(2) If I^* satisfies the weaker pcf-th for $\bar\lambda$ <u>then</u> I^* satisfies the pseudo pcf-th for $\bar\lambda$.

Proof. 1) Check (we can also present those ideals in other ways).
2) Check. $\blacksquare_{2.14}$

3. Reduced products of cardinals

We characterize here the cardinalities $\prod\limits_{i<\kappa}\lambda_i/D$ and $T_D(\langle\lambda_i : i < \kappa\rangle)$ using pcf's and the amount of regularity of D (in 3.1-3.4). Later we give sufficient conditions for the existence of $<_D$-lub or $<_D$-eub. Remember the old result of Kanamori [Kn] and Ketonen [Kt]: for D an ultrafilter the sequence $\langle\alpha/D : \alpha < \kappa\rangle$ (i.e. the constant functions) has a $<_D$-lub if $\mathrm{reg}(D) < \kappa$; and see [Sh-g, III 3.3] (for filters). Then we turn to depth of ultraproducts of Boolean algebras.

The questions we would like to answer are (restricting ourselves to "$\lambda_i \ge 2^\kappa$" or "$\lambda_i \ge 2^{2^\kappa}$" and D an ultrafilter on κ will be good enough).

Question A: What can be $\mathrm{Car}_D =: \{\prod\limits_{i<\kappa}\lambda_i/D : \lambda_i$ a cardinal for $i < \kappa\}$ i.e. characterize it by properties of D; (or at least $\mathrm{Card}_D \setminus 2^\kappa$) (for D a filter also $T_D(\prod\lambda_i)$ is natural).

Question B: What can be $\mathrm{DEPTH}_D^+ = \{\mathrm{Depth}^+(\prod\limits_{i<\kappa}\lambda_i/D) : \lambda_i$ a regular cardinal$\}$ (at least $\mathrm{DEPTH}_D^+ \setminus 2^\kappa$, see Definition 3.18).

If D is an $\aleph_1$-complete ultrafilter, the answer is clear. For D a regular ultrafilter on κ, $\lambda_i \ge \aleph_0$ the answer to question A is known ([CK]) in fact it was the reason for defining "regularity of filters" (for $\lambda_i < \aleph_0$ see [Sh7], [Sh-a, VI §3 Th 3.12 and pp 357-370] better [Sh-c VI§3] and Koppleberg [Ko].) For D a regular ultrafilter on κ, the answer to the question is essentially completed in 3.22(1), the remaining problem can be answered by *pp* (see [Sh-g]) except the restriction $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$, which can be removed if the cov $=$ *pp* problem is completed (see [Sh-g, AG]). So the problem is for the other ultrafilters D, on which we give a reasonable amount on information translating to a pcf problem, sometimes depending on the pcf theorem.

Definition 3.1.

(1) For a filter D let $\mathrm{reg}(D) = \min\{\theta\colon D \text{ is not } \theta\text{-regular}\}$ (see below).
(2) A filter D is θ-regular if there are $A_\varepsilon \in D$ for $\varepsilon < \theta$ such that the intersection of any infinitely many A_ε-s' is empty.
(3) For a filter D let

$$\mathrm{reg}_*(D) = \min\{\theta\colon \text{ there are no } A_\varepsilon \in D^+ \text{ for } \varepsilon < \theta \text{ such that no } i < \kappa \text{ belongs to infinitely many } A_\varepsilon\text{'s}\}$$

and

$$\mathrm{reg}_\otimes(D) =: \{\theta : \text{ there are no } A_\varepsilon \in D^+ \text{ for } \varepsilon < \theta \text{ such that : } \varepsilon < \zeta \Rightarrow A_\zeta \subseteq A_\varepsilon \mod D \text{ and no } i < \kappa \text{ belongs to infinitely many } A_\varepsilon\text{'s}\}.$$

(4) $\mathrm{reg}^\sigma(D) = \min\{\theta : D \text{ is not } (\theta,\sigma)\text{-regular}\}$ where "D is (θ,σ)-regular" means that there are $A_\varepsilon \in D$ for $\alpha < \theta$ such that the intersection of any σ of them is empty. Lastly $\mathrm{reg}^\sigma_*(D)$, $\mathrm{reg}^\sigma_\otimes(D)$ are defined similarly using $A_\varepsilon \in D^+$. Of course $\mathrm{reg}(I)$ etc. means $\mathrm{reg}(D)$ where D is the dual filter.

Definition 3.2.

(1) Let

$$\mathrm{htcf}_{D,\mu}(\prod \gamma_i) = \sup\{\mathrm{tcf}(\textstyle\prod_{i<\kappa}\lambda_i/D)\colon \mu \le \lambda_i = \mathrm{cf}\,\lambda_i \le \gamma_i \text{ for } i < \kappa \text{ and } \mathrm{tcf}(\prod \lambda_i/D) \text{ is well defined}\} \text{ and}$$

$$\mathrm{hcf}_{D,\mu}(\prod_{i<\kappa}\gamma_i) = \sup\{\mathrm{cf}(\prod_{i<\kappa}\lambda_i/D)\colon \mu \le \lambda_i = \mathrm{cf}\,\lambda_i \le \gamma_i\};$$

if $\mu = \aleph_0$ we may omit it.
(2) For E a family of filters on κ let $\mathrm{htcf}_{E,\mu}(\prod_{i<\kappa}\alpha_i)$ be

$$\sup\{\mathrm{tcf}(\textstyle\prod_{i<\kappa}\lambda_i/D)\colon D \in E \text{ and } \mu \le \lambda_i = \mathrm{cf}\,\lambda_i \le \alpha_i \text{ for } i < \kappa \text{ and } \mathrm{tcf}(\prod_{i<\kappa}\lambda_i/D) \text{ is well defined}\}.$$

Similarly for $\mathrm{hcf}_{E,\mu}$ (using cf instead tcf).
(3) $\mathrm{hcf}^*_{D,\mu}(\prod_{i<\kappa}\alpha_i)$ is $\mathrm{hcf}_{E,\mu}(\prod_{i<\kappa}\alpha_i)$ for $E = \{D'\colon D' \text{ a filter on } \kappa \text{ extending } D\}$. Similarly for $\mathrm{htcf}^*_{D,\mu}$.
(4) When we write I e.g. in $\mathrm{hcf}_{I,\mu}$ we mean $\mathrm{hcf}_{D,\mu}$ where D is the dual filter.

CLAIM 3.3:

(1) $\mathrm{reg}(D)$ is always regular
(2) If $\theta < \mathrm{reg}_*(D)$ then some filter extending D is θ-regular.

(3) $\mathrm{wsat}(D) \le \mathrm{reg}_*(D)$
(4) $\mathrm{reg}(D) \le \mathrm{reg}_\otimes(D) \le \mathrm{reg}_*(D)$
(5) $\mathrm{reg}_*(D) = \min\{\theta :$ no ultrafilter D_1 on κ extending D is θ-regular$\}$
(6) If $D \subseteq E$ are filters on κ then:
 (a) $\mathrm{reg}(D) \le \mathrm{reg}(E)$
 (b) $\mathrm{reg}_*(D) \ge \mathrm{reg}_*(E)$

Proof. Should be clear. E.g (2) let $\langle u_\varepsilon : \varepsilon < \theta\rangle$ list the finite subsets of θ, and let $\{A_\varepsilon : \varepsilon < \theta\} \subseteq D^+$ exemplify "$\theta < \mathrm{reg}_*(D)$". Now let $D^* =: \{A \subseteq \kappa$: for some finite $u \subseteq \theta$, for every $\varepsilon < \theta$ we have: $u \subseteq u_\varepsilon \Rightarrow A_\varepsilon \subseteq A \bmod D\}$, and let $A^*_\varepsilon = \bigcup\{A_\zeta : \varepsilon \in u_\zeta\}$. Now D^* is a filter on κ extending D and for $\varepsilon < \theta$ we have $A^*_\varepsilon \in D$. Finally the intersection of $A^*_{\varepsilon_0} \cap A^*_{\varepsilon_1} \cap \dots$ for distinct $\varepsilon_n < \theta$ is empty, because for any member j of it we can find $\zeta_n < \theta$ such that $j \in A_{\zeta_n}$ and $\varepsilon_n \in u_{\zeta_n}$. Now <u>if</u> $\{\zeta_n : n < \omega\}$ is infinite then there is no such j by the choice of $\langle A_\varepsilon : \varepsilon < \theta\rangle$, and <u>if</u> $\{\zeta_n : n < \omega\}$ is finite then wlog $\bigwedge_{n,\omega} \zeta_n = \zeta_0$ contradicting "u_{ζ_0} is finite" as $\bigwedge_{n<\omega} \varepsilon_n \in u_{\zeta_n}$. Lastly $\emptyset \notin D^*$ because $A^*_\varepsilon \ne \emptyset \bmod D$. $\blacksquare_{3.3}$

Observation 3.4. $|\prod_{i<\kappa}\lambda_i/I| \ge |\aleph_0^\kappa/I|$ holds when $\bigwedge_{i<\kappa}\lambda_i \ge \aleph_0$.

Observation 3.5.

(1) $|\prod_{i<\kappa}\lambda_i/I| \ge \mathrm{htcf}^*_I(\prod_{i<\kappa}\lambda_i)$.
(2) If I^* satisfies the pcf-th for $\bar\lambda$ or even the weaker pcf-th for $\bar\lambda$ (see Definition 2.11) <u>then</u>: $\mathrm{cf}(\prod\bar\lambda/I^*) = \max\mathrm{pcf}_{I^*}(\bar\lambda)$.
(3) If I^* satisfies the pcf-th for μ for and $\min(\bar\lambda) \ge \mu$ <u>then</u>

$$\mathrm{hcf}_{D,\mu}(\prod\bar\lambda) = \mathrm{hcf}^*_{D,\mu}(\prod\bar\lambda) = \mathrm{htcf}^*_{D,\mu}(\prod\bar\lambda)$$

 whenever D is disjoint to I^*.
(4) $\mathrm{hcf}_{E,\mu}(\prod_{i<\kappa}\lambda_i) = \mathrm{hcf}^*_{E,\mu}(\prod_{i<\kappa}\lambda_i)$.
(5) $\prod_{i<\kappa}\lambda_i/I \ge \mathrm{hcf}_{I,\mu}(\prod_{i<\kappa}\lambda_i) = \mathrm{hcf}^*_{I,\mu}(\prod_{i<\kappa}\lambda_i) \ge \mathrm{htcf}^*_{I,\mu}(\prod_{i<\kappa}\lambda_i)$ and $\mathrm{hcf}_{I,\mu}(\prod_{i<\kappa}\lambda_i) \ge \mathrm{htcf}_{I,\mu}(\prod_{i<\kappa}\lambda_i)$.

Remark 3.5A. In 3.5(3) concerning $\mathrm{htcf}_{D,\mu}$ see 3.10.

Proof. 1) By the definition of htcf^*_I it suffices to show $|\prod_{i<\kappa}\lambda_i/I| \ge \mathrm{tcf}(\prod\lambda'_i/I')$, when I' is an ideal on κ extending I, $\lambda'_i = \mathrm{cf}\,\lambda'_i \le \lambda_i$ for $i < \kappa$ and $\mathrm{tcf}(\prod_{i<\kappa}\lambda'_i/I')$ is well defined. Now $|\prod_{i<\kappa}\lambda_i/I| \ge |\prod_{i<\kappa}\lambda'_i/I| \ge |\prod_{i<\kappa}\lambda'_i/I'| \ge \mathrm{cf}(\prod\lambda'_i/I')$, so we have finished.

2) By 2.13(1A)clearly I^*, $\bar\lambda$ satisfies 1.5, 1.6 hence by 2.13(1), (2) also 1.8(1), (2), (3), (4) and 1.9 and 1.10. Now by 2.13(8) also (the conclusion of) 2.10(2) holds which is what we need.

3) Left to the reader (see Definition 2.11(2) and part (2)).

4), 5) Check. $\blacksquare_{3.5}$

Claim 3.6: If $\lambda = |\prod_{i<\kappa}\lambda_i/I|$ (and $\lambda_i \ge \aleph_0$ and, of course, I an ideal on κ) and $\theta < \mathrm{reg}(I)$ <u>then</u> $\lambda = \lambda^\theta$.

Proof. For each $i < \kappa$, let $\langle \eta^i_\alpha : \alpha < \lambda_i \rangle$ list the finite sequences from λ_i. Let $M_i = (\lambda_i, F_i, G_i)$ where $F_i(\alpha) = \ell g(\eta^i_\alpha)$, $G_i(\alpha, \beta)$ is $\eta^i_\alpha(\beta)$ if $\beta < \ell g(\eta^i_\alpha)$ $(= F_i(\alpha))$, and $F(\alpha, \beta) = 0$ otherwise; let $M = \prod_{i<\kappa} M_i / I$ so $\|M\| = |\prod \lambda_i / I|$ and let $M = (\prod \lambda_i / I, F, G)$. Let $\langle A_i : i < \theta \rangle$ exemplifies I is θ-regular. Now

- $(*)_1$ We can find $f \in {}^\kappa\omega$ and $f_\varepsilon \in \prod_{i<\kappa} f(i)$ for $\varepsilon < \theta$ such that: $\varepsilon < \zeta < \theta \Rightarrow f_\varepsilon <_I f_\zeta$ [just for $i < \kappa$ let $w_i = \{\varepsilon < \theta : i \in A_\varepsilon\}$, it is finite and let $f(i) = |w_i| + 1$ and $f_\varepsilon(i) = |\varepsilon \cap w_i| \le f(i)$, and note $\varepsilon < \zeta \& i \in A_\varepsilon \cap A_\zeta \Rightarrow f_\varepsilon(i) < f_\zeta(i)$].
- $(*)_2$ For every sequence $\bar g = \langle g_\varepsilon : \varepsilon < \theta \rangle$ of members of $\prod_{i<\kappa} \lambda_i$, there is $h \in \prod_{i<\kappa} \lambda_i$ such that $\varepsilon < \theta \Rightarrow M \models F(h/I, f_\varepsilon/I) = g_\varepsilon/I$ [why? let, in the notation of $(*)_1$, $h(i)$ be such that $\eta^i_{h(i)} = \langle g_\varepsilon(i) : \varepsilon \in w_i \rangle$ (in the natural order)].

So in M, every θ-sequence of members is coded using f/I, f_ε/I (for $\varepsilon < \theta$) by at least one member so $\|M\|^\theta = \|M\|$, but $\|M\| = |\prod_{i<\kappa} \lambda_i / I|$ hence we have proved 3.6. $\blacksquare_{3.6}$

Fact 3.7.

(1) For D a filter on κ, $\langle A_1, A_2 \rangle$ a partition of κ and (non zero) cardinals λ_i for $i < \kappa$ we have

$$|\prod_{i<\kappa} \lambda_i / D| = |\prod_{i<\kappa} \lambda_i / (D + A_1)| \times |\prod_{i<\kappa} \lambda_i / (D + A_2)|$$

(note: $|\prod_{i<\kappa} \lambda_i / \mathcal{P}(\kappa)| = 1$).

(2) $D^{[\mu]} =: \{A \subseteq \kappa : |\prod_{i<\kappa} \lambda_i / (D + (\kappa \setminus A))| < \mu\}$ is a filter on κ (μ an infinite cardinal of course) and if $\aleph_0 \le \mu \le \prod_{i<\kappa} \lambda_i / D$ <u>then</u> $D^{[\mu]}$ is a proper filter.

(3) If $\lambda \le |\prod\limits_{i<\kappa} \lambda_i / I|$, ($\lambda_i$ infinite, of course, I an ideal on κ) and $A \in I^+ \Rightarrow |\prod\limits_{i \in A} \lambda_i / I| \ge \lambda$ and $\sigma < \mathrm{reg}_*(I)$ <u>then</u> $|\prod \lambda_i / I| \ge \lambda^\sigma$

Proof. Check (part (3): by the proof of 3.3(2) we can find $A_\varepsilon \in I^+$ for $\varepsilon < \sigma$ such that for finite $u \subseteq \sigma$, $\cap_{\varepsilon \in u} A_\varepsilon \in I^+$ and continue as in the proof of 3.6).

Claim 3.8: If $D \subseteq E$ are filters on κ then

$$|\prod_{i<\kappa} \lambda_i / D| \le |\prod_{i<\kappa} \lambda_i / E| + \sup_{A \in E \setminus D} |\prod_{i<\kappa} \lambda_i / (D + (\kappa \setminus A))| + (2^\kappa / D) + \aleph_0.$$

We can replace $2^\kappa / D$ by $|\mathcal{P}|$ if $\mathcal{P}$ is a maximal subset of E such that $A \ne B \in \mathcal{P} \Rightarrow (A \setminus B) \cup (B \setminus A) \ne \emptyset \bmod D$.

Proof. Think.

Lemma 3.9: $|\prod_{i<\kappa} \lambda_i / D| \le (\theta^\kappa / D + \mathrm{hcf}_{D,\theta}(\prod_{i<\kappa} \lambda_i))^{<\theta}$ (see Definition 3.2(1)) provided that:

$$\theta \ge \mathrm{reg}_\otimes(D) \tag{$*$}$$

Remark 3.9A. 1) If $\theta = \theta_1^+$, we can replace θ^κ / D by θ_1^κ / D. In general we can replace θ^κ / D by $\sup\{\prod\limits_{i<\kappa} f(i)/D : f \in \theta^\kappa\}$.

2) If D satisfies the pcf-th above θ (see 2.11(1A), 2.12(2)) then by 3.5(3) we can use htcf* (sometime even htcf, see 3.10). But by 3.7(1) we can ignore the $\lambda_i \le \theta$, and when $i < \kappa \Rightarrow \lambda_i > \theta$ we know that 1.5(*)(α) holds by 3.3(3).

Proof. Let $\lambda = \theta^\kappa/D + \mathrm{hcf}_{D,\theta}(\prod_{i<\kappa}\lambda_i)$. Let for $\zeta < \theta$, $\mu_\zeta =: \lambda^{|\zeta|}$ i.e. $\mu_\zeta =: (\theta^\kappa/D + \mathrm{hcf}_{D,\theta}\prod_{i<\kappa}\lambda_i)^{|\zeta|}$, clearly $\mu_\zeta = \mu_\zeta^{|\zeta|}$. Let $\chi = \beth_8(\sup_{i<\kappa}\lambda_i)^+$ and $N_\zeta \prec (H(\chi), \in, <^*_\chi)$ be such that $\|N_\zeta\| = \mu_\zeta$, $N^{\leq|\zeta|} \subseteq N_\zeta$, $\lambda + 1 \subseteq N_\zeta$ and $\{D, \langle\lambda_i : i < \kappa\rangle\} \in N_\zeta$ and $[\varepsilon < \zeta \Rightarrow N_\varepsilon \prec N_\zeta]$. Let $N = \cup\{N_\zeta : \zeta < \theta\}$. Let $g^* \in \prod_{i<\kappa}\lambda_i$ and we shall find $f \in N$ such that $g^* = f \bmod D$, this will suffice. We shall choose by induction on $\zeta < \theta$, $f^e_\zeta (e < 3)$ and $\bar{A}^\zeta$ such that:

(a) $f^e_\zeta \in \prod_{i<\kappa}(\lambda_i + 1)$
(b) $f^1_\zeta \in N_\zeta$ and $f^2_\zeta \in N_\zeta$.
(c) $\bar{A}^\zeta = \langle A^\zeta_i : i < \kappa\rangle \in N_\zeta$.
(d) $\lambda_i \in A^\zeta_i \subseteq \lambda_i + 1$, $|A^\zeta_i| \leq |\zeta| + 1$, and $\langle A^\zeta_i : \zeta < \theta\rangle$ is increasing continuous (in ζ).
(e) $f^0_\zeta(i) = \min(A^\zeta_i \setminus g^*(i))$; note: it is well defined as $g^*(i) < \lambda_i \in A^\zeta_i$
(f) $f^1_\zeta = f^0_\zeta \bmod D$
(g) $g^* < f^2_\zeta < f^1_\zeta \bmod (D + \{i < \kappa : g^*(i) \neq f^1_\zeta(i)\})$.
(h) $f^2_\zeta(i) \in A^{\zeta+1}_i$

So assume everything is defined for every $\varepsilon < \zeta$. If $\zeta = 0$, let $A^\zeta_i = \{\lambda_i\}$, if ζ limit $A^\zeta_i = \bigcup_{\varepsilon<\zeta} A^\varepsilon_i$, for $\zeta = \varepsilon + 1$, A^ζ_i will be defined in stage ε. So arriving to ζ, $\bar{A}^\zeta$ is well defined and it belongs to N_ζ: for $\zeta = 0$ check, for $\zeta = \varepsilon + 1$, done in stage ε, for ζ limit it belongs to N_ζ as we have $N_\zeta^{\leq|\zeta|} \subseteq N_\zeta$ and: $\xi < \zeta \Rightarrow N_\xi \prec N_\zeta$ and $\bar{A}^\xi \in N_\xi$. Now use clause (e) to define f^0_ζ/D. As $\langle A^\zeta_i : i < \kappa\rangle \in N_\zeta$, $|A^\zeta_i| \leq |\zeta| + 1 < \theta$ and $\theta^\kappa/D \leq \lambda < \lambda + 1 \subseteq N_\zeta$, clearly $|\prod_{i<\kappa}|A^\zeta_i|/D| \leq \lambda$ hence $\{f/D : f \in \prod_{i<\kappa} A^\zeta_i\} \subseteq N_\zeta$ hence $f^0_\zeta/D \in N_\zeta$ hence there is $f^1_\zeta \in N_\zeta$ such that $f^1_\zeta \in f^0_\zeta/D$ i.e. clause (f) holds. As $g^* \leq f^0_\zeta$ clearly $g^* \leq f^1_\zeta \bmod D$, let $y^\zeta_0 =: \{i < \kappa : g^*(i) \geq f^1_\zeta(i)\}$, $y^\zeta_1 =: \{i < \kappa : i \notin y^\zeta_0$ and $\mathrm{cf}(f^1_\zeta(i)) < \theta\}$ and $y^\zeta_2 =: \kappa \setminus y^\zeta_0 \setminus y^\zeta_1$. So $\langle y^\zeta_e : e < 3\rangle$ is a partition of κ and $g^* < f^1_\zeta \bmod (D + y^\zeta_e)$ for $e = 1, 2$.

Let $y^\zeta_4 = \{i < \kappa : \mathrm{cf}(f^1_\zeta(i)) \geq \theta\}$ so $f^1_\zeta \in N_\zeta$, and $\theta \in N_\zeta$ hence $y^\zeta_4 \in N_\zeta$, so $(\prod_{i<\kappa} f^1_\zeta(i), <_{D+y^\zeta_4}) \in N_\zeta$. Clearly $y^\zeta_2 \subseteq y^\zeta_4 \subseteq y^\zeta_0 \cup y^\zeta_2$. Now

$$\mathrm{cf}(\prod_{i<\kappa} f^1_\zeta(i), <_{D+y^\zeta_4}) \leq \mathrm{hcf}_{D+y^\zeta_4,\theta}(\prod_{i<\kappa}\lambda_i) \leq \mathrm{hcf}_{D,\theta}(\prod_{i<\kappa}\lambda_i) \subseteq \lambda < \lambda+1 \subseteq N_\zeta$$

hence there is $F \in N_\zeta$, $|F| \leq \lambda$, $F \subseteq \prod_{i\in y^\zeta_4} f^1_\zeta(i)$ such that:

$$(\forall g)[g \in \prod_{i\in y^\zeta_4} f^1_\zeta(i) \Rightarrow (\exists f \in F)(g < f \bmod (D + y^\zeta_4)))].$$

As $\lambda + 1 \subseteq N$ necessarily $F \subseteq N_\zeta$. Apply the property of F to $(g^* \restriction y^\zeta_2) \cup 0_{(\kappa \setminus y^\zeta_2)}$ and get $f^\zeta_4 \in F \subseteq N_\zeta$ such that $g^* < f^\zeta_4 \bmod (D + y^\zeta_2)$. Now use similarly $\prod_{i<\kappa}\mathrm{cf}(f^1_\zeta(i))/(D + (\kappa \setminus y^\zeta_4)) \leq |\theta^\kappa/D| \leq \lambda$; by the proof of 3.7(1) there is a function $f^2_\zeta \in N_\zeta \cap \prod_{i<\kappa} f^1_\zeta(i)$ such that $g^* \restriction (y^\zeta_1 + y^\zeta_2) < f^2_\zeta \bmod D$. Let $A^{\zeta+1}_i$ be: $A^\zeta_i \cup \{f^2_\zeta(i)\}$.

It is easy to check clauses (g), (h). So we have carried the definition. Let

$$X_\zeta =: \{i < \kappa : f^0_{\zeta+1}(i) < f^0_\zeta(i)\}.$$

Note that by the choice of f^1_ζ, $f^1_{\zeta+1}$ we know $X_\zeta = y^\zeta_1 \cup y^\zeta_2 \mod D$, if this last set is not D-positive then $g^* \geq f^1_\zeta \operatorname{mod} D$, hence $g^*/D = f^1_\zeta/D \in N_\zeta$, contradiction, so $y^\zeta_1 \cup y^\zeta_2 \neq \emptyset \operatorname{mod} D$ hence $X_\zeta \in D^+$. Also $\langle (y^\zeta_1 \cup y^\zeta_2)/D : \zeta < \theta\rangle$ is $\subseteq$-decreasing hence $\langle X_\zeta/D : \zeta < \theta\rangle$ is $\subseteq$-decreasing.

Also if $i \in X_{\zeta_1} \cap X_{\zeta_2}$ and $\zeta_1 < \zeta_2$ then $f^0_{\zeta_2}(i) \leq f^0_{\zeta_1+1}(i) < f^0_{\zeta_1}(i)$ (first inequality: as $A^{\zeta_1+1}_i \subseteq A^{\zeta_2}_i$ and clause (e) above, second inequality by the definition of X_{ζ_1}), hence for each ordinal i the set $\{\zeta < \theta : i \in X_\zeta\}$ is finite. So $\theta < \operatorname{reg}_\otimes(D)$, contradiction to the assumption $(*)$. $\blacksquare_{3.9}$

Note we can conclude

CLAIM 3.9B:

$$\prod_{i<\kappa} \lambda_i/D = \sup\{(\prod_{i<\kappa} f(i))^{<\operatorname{reg}_\otimes(D_1)} + \operatorname{hcf}_{D_1}(\prod_{i<\kappa}\lambda_i)^{<\operatorname{reg}_\otimes(D_1)} : D_1 \text{ is a filter on } \kappa \text{ extending } D \text{ such that}$$
$$A \in D^+_1 \Rightarrow \prod_{i<\kappa}\lambda_i/(D_1 + A) = \prod_{i<\kappa}\lambda_i/D_1 \text{ and } f \in \theta^\kappa, f(i) \leq \lambda_i\}$$

Proof. The inequality $\geq$ should be clear by 3.7(3). For the other direction let μ be the right side cardinality and let $D_0 = \{\kappa \setminus A : A \subseteq \kappa$ and if $A \in D^+$ then $\prod_{i<\kappa}\lambda_i/(D+A) \leq \mu\}$, so we know by 3.7(2) that D_0 is a filter on κ extending D. If $\emptyset \in D_0$ we are done so assume not. Now $\mu \geq 2^\kappa/D$ (by the term $(\prod_i f(i)/D_0)^{<\operatorname{reg}_\otimes(D_1)}$) so by 3.8 we have $\prod_{i<\kappa}\lambda_i/D_0 > \mu$ (use 3.8 with D, D_0 here corresponding to D, E there). Now the same holds for $D_0 + A$ for every $A \in D^+_0$. Also $A \subseteq B \subseteq \kappa$ and $A \in D^+_0 \Rightarrow \prod_{i<\kappa}\lambda_i/(D_0 + A) \leq \prod_{i<\kappa}\lambda_i/(D_1 + B)$ so for some $B \in D^+_0$, $D_1 =: D_0 + B$ satisfies the requirement inside the definition of μ, so $\mu \geq \operatorname{hcf}_{D_1}(\prod_{i<\kappa}\lambda_1)^{<\operatorname{reg}_\otimes(D_1)}$. By 3.9 (see 3.9A(1)) we get a contradiction. $\blacksquare_{3.9B}$

Next we deal with existence of $<_D$ –eub.

CLAIM 3.10: 1) Assume D a filter on κ, $g^*_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \delta$, $\bar{g}^* = \langle g^*_\alpha : \alpha < \delta\rangle$ is $\leq_D$-increasing, and

$$\operatorname{cf}(\delta) \geq \theta \geq \operatorname{reg}_*(D). \tag{$*$}$$

<u>Then</u> at least one of the following holds:

(A) $\langle g^*_\alpha : \alpha < \delta\rangle$ has a $<_D$-eub $g \in {}^\kappa\mathrm{Ord}$; moreover $\theta \leq \liminf_D\langle \operatorname{cf}[g(i)] : i < \kappa\rangle$

(B) $\operatorname{cf}(\delta) = \operatorname{reg}_*(D)$

(C) for some club C of δ and some $\theta_1 < \theta$ and $\gamma_i < \theta^+_1$ and $w_i \subseteq \mathrm{Ord}$ of order type γ_i for $i < \kappa$, there are $f_\alpha \in \prod_{i<\kappa} w_i$ (for $\alpha \in C$) such that $f_\alpha(i) = \min(w_i \setminus g^*_\alpha(i))$ and $\alpha \in C$ & $\beta \in C$ & $\alpha < \beta \Rightarrow f_\alpha \leq_D f_\beta$ & $\neg f_\alpha =_D f_\beta$ & $\neg f_\alpha \leq_D g^*_\beta$ & $g^*_\alpha \leq f_\alpha$.

2) In (C) above if for simplicity D is an ultrafilter we can find $w_i \subseteq \mathrm{Ord}$, $\operatorname{otp}(w_i) = \gamma_i$, $\langle \alpha_\xi : \xi < \operatorname{cf}(\delta)\rangle$ increasing continuous with limit δ, and $h_\varepsilon \in \prod_{i<\kappa} w_i$ such that $f_{\alpha_\varepsilon} <_D h_\varepsilon <_D f_{\alpha_{\varepsilon+1}}$, moreover, $\bigwedge_{i<\kappa} \gamma_i < \omega$.

Proof. 1) Let $\sigma = \mathrm{reg}_*(D)$. We try to choose by induction on $\zeta < \sigma$, g_ζ, $f_{\alpha,\zeta}$ (for $\alpha < \delta$), $\bar{A}^\zeta$, α_ζ such that

(a) $\bar{A}^\zeta = \langle A_i^\zeta : i < \kappa \rangle$.
(b) $A_i^\zeta = \{f_{\alpha_\varepsilon,\varepsilon}(i), g_\varepsilon(i) : \varepsilon < \zeta\} \cup \{[\sup_{\alpha<\delta} g_\alpha^*(i)] + 1\}$.
(c) $f_{\alpha,\zeta}(i) = \min(A_i^\zeta \setminus g_\alpha^*(i))$ (and $f_{\alpha,\zeta} \in {}^\kappa\mathrm{Ord}$, of course).
(d) α_ζ is the first α, $\bigcup_{\varepsilon<\zeta} \alpha_\varepsilon < \alpha < \delta$ such that $[\beta \in [\alpha, \delta) \Rightarrow f_{\beta,\zeta} = f_{\alpha,\zeta} \bmod D]$ if there is one.
(e) $g_\zeta \leq f_{\alpha_\zeta,\zeta}$ moreover $g_\zeta < \max\{f_{\alpha_\zeta,\zeta}, 1_\kappa\}$ but for no $\alpha < \delta$ do we have $g_\zeta < \max\{g_\alpha^*, 1\} \bmod D$.

Let ζ^* be the first for which they are not defined (so $\zeta^* \leq \sigma$). Note

$$\varepsilon < \xi < \zeta^* \& \alpha_\xi \leq \alpha < \delta \Rightarrow f_{\alpha_\varepsilon,\varepsilon} =_D f_{\alpha,\varepsilon} \& f_{\alpha,\xi} \leq f_{\alpha,\varepsilon} \& f_{\alpha,\xi} \neq_D f_{\alpha,\varepsilon}. \qquad (*)$$

[Why last phrase? applying clause (e) above, second phrase with α, ε here standing for α, ζ there we get $A_0 =: \{i < \kappa : \max\{g_\alpha^*(i), 1\} \leq g_\varepsilon(i)\} \in D^+$ and applying clause (e) above first phrase with ε here standing for ζ there we get $A_1 = \{i < \kappa : g_\varepsilon(i) < f_{\alpha,\varepsilon}(i)$ or $g_\varepsilon(i) = 0 = f_{\alpha,\varepsilon}(i)\} \in D$, hence $A_0 \cap A_1 \in D^+$, and $g_\varepsilon(i) > 0$ for $i \in A_0 \cap A_1$ (even for $i \in A_0$). Also by clause (c) above $g_\alpha^*(i) \leq g_\varepsilon(i) \Rightarrow f_{\alpha,\xi}(i) \leq g_\varepsilon(i)$. Now by the last two sentences $i \in A_0 \cap A_1 \Rightarrow g_\alpha^*(i) \leq g_\varepsilon(i) < f_{\alpha,\varepsilon}(i) \Rightarrow f_{\alpha,\xi}(i) \leq g_\varepsilon(i) < f_{\alpha,\varepsilon}(i)$, together $f_{\alpha,\xi} \neq_D f_{\alpha,\varepsilon}$ as required]

Case A. $\zeta^* = \sigma$ and $\bigcup_{\zeta<\sigma} \alpha_\zeta < \delta$. Let $\alpha(*) = \bigcup_{\zeta<\sigma} \alpha_\zeta$, for $\zeta < \sigma$ let $y_\zeta = \{i < \kappa : f_{\alpha(*),\zeta}(i) \neq f_{\alpha(*),\zeta+1}(i)\} \neq \emptyset \bmod D$. Now for $i < \kappa$, $\langle f_{\alpha(*),\zeta}(i) : \zeta < \sigma \rangle$ is non increasing so i belongs to finitely many y_ζ's only, so $\langle y_\zeta : \zeta < \sigma \rangle$ contradict $\sigma \geq \mathrm{reg}_*(D)$.

Case B. $\zeta^* = \sigma$ and $\bigcup_{\zeta<\sigma} \alpha_\zeta = \delta$. So possibility (B) of Claim 3.10 holds.

Case C. $\zeta^* < \sigma$.

Still $A_i^{\zeta^*}$ $(i < \kappa)$, f_{α,ζ^*} $(\alpha < \delta)$ are well defined.

Subcase C1. α_{ζ^*} cannot be defined.

Then possibility C of 3.10 holds (use $w_i =: A_i^{\zeta^*}$, $f_\beta = f_{\alpha_{\zeta^*}+\beta,\zeta^*}$).

Subcase C2. α_{ζ^*} can be defined.

Then $f_{\alpha_{\zeta^*},\zeta^*}$ is a $<_D$-eub of $\langle g_\alpha^* : \alpha < \delta \rangle$ as otherwise there is g_{ζ^*} as required in clause (e). Now $f_{\alpha_\zeta^*,\zeta^*}$ is almost as required in possibility (A) of Claim 3.10 only the second phrase is missing. If for no $\theta_1 < \theta$, $\{i < \kappa : \mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)] \leq \theta_1\} \in D^+$, then possibility (A) holds.

So assume $\theta_1 < \theta$ and $B =: \{i < \kappa : \aleph_0 \leq \mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)] \leq \theta_1\}$ belongs to D^+, we shall try to prove that possibility (C) holds, thus finishing. Now we choose w_i for $i < \kappa$: for $i \in \kappa$ we let $w_i^0 =: \{f_{\alpha_{\zeta^*},\zeta^*}(i), [\sup_{\alpha<\delta} g_\alpha^*(i)] + 1\}$, for $i \in B$ let w_i^1 be an unbounded subset of $f_{\alpha_{\zeta^*},\zeta^*}(i)$ of order type $\mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)]$ and for $i \in \kappa \setminus B$ let $w_i^1 = \emptyset$, lastly let $w_i = w_i^0 \cup w_i^1$, so $|w_i| \leq \theta_1$ as required in possibility (C). Define $f_\alpha \in {}^\kappa\mathrm{Ord}$ by $f_\alpha(i) = \min(w_i \setminus g_\alpha^*(i))$ (by the choice of w_i^0 it is well defined). So $\langle f_\alpha : \alpha < \delta \rangle$ is $\leq_D$-increasing; if for some $\alpha^* < \delta$, for every $\alpha \in [\alpha^*, \delta)$ we have $f_\alpha/D = f_{\alpha^*}/D$, we could define $g_{\zeta^*} \in {}^\kappa\mathrm{Ord}$ by:

$g_{\zeta^*} \restriction B = f_{\alpha^*}$ (which is $< f_{\alpha_{\zeta^*},\zeta^*}$),

$g_{\zeta^*} \restriction (\kappa \setminus B) = 0_{\kappa\setminus B}$.

Now g_{ζ^*} is as required in clause (e) so we get contradiction to the choice of ζ^*. So there is no $\alpha^* < \delta$ as above so for some club C of δ we have $\alpha < \beta \in C \Rightarrow f_\alpha \neq_D f_\beta$, so we have actually proved possibility (C).

2) Easy (for $\bigwedge_i \gamma_i < \omega$, wlog $\theta = \mathrm{reg}_*(D)$ but $\mathrm{reg}_*(D) = \mathrm{reg}(D)$ so $\theta_1 < \mathrm{reg}(D)$). $\blacksquare_{3.10}$

CLAIM 3.11:

(1) In 3.10(1), if $\lambda = \delta = \mathrm{cf}(\lambda)$, $\bar{g}^*$ obeys $\bar{a}$ ($\bar{a}$ as in 2.1), $\bar{a}$ a θ-weak (S,θ)-continuity condition, $S \subseteq \lambda$ unbounded, then clause (C) of 3.10 implies:
(C)′ there are $\theta_1 < \mathrm{reg}_*(D)$ and $A_\varepsilon \in D^+$ for $\varepsilon < \theta$ such that the intersection of any θ_1^+ of the sets A_ε is empty (equivalently $i < \kappa \Rightarrow (\exists^{\leq \theta_1} \varepsilon)[i \in A_\varepsilon]$ (reminds (σ, θ_1^+)-regularity of ultrafilters).

(2) We can in 3.10(1) weaken the assumption $(*)$ to $(*)'$ below if in the conclusion we weaken clause (A) to (A)′ where
$(*)'$ $\mathrm{cf}(\delta) \geq \theta \geq \mathrm{reg}(D)$
(A)′ there is a $\leq_D$-upper bound f of $\{g^*_\alpha : \alpha < \delta\}$ such that
no $f' <_D f$ (of course $f' \in {}^\kappa\mathrm{Ord}$) is a $\leq_D$-upper bound of $\{g^*_\alpha : \alpha < \delta\}$
and $\theta \leq \liminf_D \langle \mathrm{cf}[f(i)] : i < \kappa \rangle$

(3) If $g^*_\alpha \in {}^\kappa\mathrm{Ord}$, $\langle g^*_\alpha : \alpha < \delta \rangle$ is $<_D$-increasing and $f \in {}^\kappa\mathrm{Ord}$ satisfies (A)′ above and
$(*)''$ $\mathrm{cf}(\delta) \geq \mathrm{wsat}(D)$ and for some $A \in D$ for every $i < \kappa$, $\mathrm{cf}(f(i)) \geq \mathrm{wsat}(D)$

<u>then</u> for some $B \in D^+$ we have $\prod_{i<\kappa} \mathrm{cf}[f(i)]/(D+B)$ has true cofinality $\mathrm{cf}(\delta)$.

Remark. Compare with 2.6.

Proof. 1) By the choice of $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ as C (in clause (c) of 3.11(1)) is a club of λ, we can find $\beta < \lambda$ such that letting $\langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ list $\{\alpha \in a_\beta : \mathrm{otp}(\alpha \cap a_\beta) < \theta\}$ (or just a subset of it) we have $(\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap C \neq \emptyset$.

Let $\gamma_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap C$, and $\xi_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1})$ be such that $\{\alpha_\zeta : \zeta \leq \varepsilon\} \subseteq a_{\xi_\varepsilon}$, and as we can use $\langle \alpha_{2\varepsilon} : \varepsilon < \theta \rangle$, wlog $\xi_\varepsilon < \gamma_\varepsilon$. For $\zeta < \theta$ let $B_\zeta = \{i < \kappa : f_{\alpha_\zeta}(i) < f_{\beta_\zeta}(i) < f_{\gamma_\zeta}(i) < f_{\alpha_{\zeta+1}}(i)$ and $\sup\{f_{\alpha_\xi}(i)+1 : \xi < \zeta\} < \sup\{f_{\alpha_\xi}(i)+1 : \xi < \zeta+1\}$.
2) In the proof of 3.10 we replace clause (e) by
(e′) $g_\zeta \leq f_{\alpha_\zeta,\zeta}$ and for $\alpha < \delta$ we have $f_\alpha \leq g_\zeta \bmod D$
3) By 1.8(1) $\blacksquare_{3.11}$

CLAIM 3.12:

(1) Assume $\lambda = \mathrm{tcf}(\prod \bar{\lambda}/D)$ and $\mu = \mathrm{cf}(\mu) < \lambda$ <u>then</u> there is $\bar{\lambda}' <_D \bar{\lambda}$, $\bar{\lambda}'$ a sequence of regular cardinals and $\mu = \mathrm{tcf}(\prod \bar{\lambda}'/D)$ provided that

$$\mu > \mathrm{reg}_*(D),\ \min(\bar{\lambda}) > \mathrm{reg}_*^{\sigma^+}(D) \text{ whenever } \sigma < \mathrm{reg}_*(D) \qquad (*)$$

(2) Let I^* be the ideal dual to D, and assume $(*)$ above. If $(*)(\alpha)$ of 1.5 holds and μ is semi-normal (for $(\bar{\lambda}, I^*)$) *then* it is normal.

Proof. Part (2) follows from part (1) by 2.2(3). Let us prove (1).

Case 1. $\mu < \liminf_D(\bar{\lambda})$
We let

$$\lambda' = \begin{cases} \mu & \text{if } \mu < \lambda_i \\ 1 & \text{if } \mu \geq \lambda_i \end{cases}$$

and we are done.

Case 2. $\liminf_D(\bar\lambda) \geq \theta \geq \mathrm{reg}_*(D)$, $\mu > \theta$, and $(\forall \sigma < \mathrm{reg}_*(D))[\mathrm{reg}_*^\sigma(D) < \theta]$.

Let $\theta =: \mathrm{reg}_*(D)$. There is an unbounded $S \subseteq \mu$ and an (S,θ)-continuity system $\bar a$ (see 2.4). As $\prod \bar\lambda/D$ has true cofinality λ, $\lambda > \mu$ clearly there are $g^*_\alpha \in \prod \bar\lambda$ for $\alpha < \mu$ such that $\bar g^* = \langle g^*_\alpha : \alpha < \mu\rangle$ obeys $\bar a$ for $\bar A^*[\bar\lambda]$ (exists as $\theta \leq \liminf_D(\bar\lambda)$).

Now if in claim 3.10(1) for $\bar g^*$ possibility (A) holds, we are done. By 3.11(1) we get that for some $\sigma < \mathrm{reg}_*(D)$ we have $\mathrm{reg}_*^\sigma(I) \geq \mu$, contradiction.

Case 3. $\liminf_D(\bar\lambda) \geq \theta \geq \mathrm{reg}_*(D)$, $\mu \geq \theta$, and $(\forall \sigma < \mathrm{reg}_*(D))[\mathrm{reg}_*^\sigma(D) < \theta]$.

Like the proof of [Sh-g, Ch II 1.5B] using the silly square. $\blacksquare_{3.12}$

* * *

We turn to other measures of $\prod \bar\lambda/D$.

Definition 3.13.

(a) $T^0_D(\bar\lambda) = \sup\{|F| : F \subseteq \prod \bar\lambda \text{ and } f_1 \neq f_2 \in F \Rightarrow f_1 \neq_D f_2\}$.

(b) $T^1_D(\bar\lambda) = \min\{|F| :$ (i) $F \subseteq \prod\bar\lambda$
(ii) $f_1 \neq f_2 \in F \Rightarrow f_1 \neq_D f_2$
(iii) F maximal under (i)+(ii)$\}$

(c) $T^2_D(\bar\lambda) = \min\{|F| : F \subseteq \prod\bar\lambda$ and for every $f_1 \in \prod\bar\lambda$, for some $f_2 \in F$ we have $\neg f_1 \neq_D f_2\}$.

(d) If $T^0_D(\bar\lambda) = T^1_D(\bar\lambda) = T^2_D(\bar\lambda)$ then let $T_D(\bar\lambda) = T^l_D(\bar\lambda)$ for $l < 3$.

(e) for $f \in {}^\kappa\mathrm{Ord}$ and $\ell < 3$ let $T^l_D(f)$ means $T^l_D(\langle f(\alpha) : \alpha < \kappa\rangle)$.

THEOREM 3.14:

(0) If $D_0 \subseteq D_1$ are filters on κ then $T^\ell_{D_0}(\bar\lambda) \leq T^\ell_{D_1}(\bar\lambda)$ for $\ell = 0,2$. Also if $\kappa = A_0 \cup A_1$, $A_0 \in D^+$, and $A_1 \in D^+$ then $T^\ell_D(\bar\lambda) = \min\{T^\ell_{D+A_0}(\bar\lambda), T^\ell_{D+A_1}(\bar\lambda)\}$ for $\ell = 0,2$.

(1) $\mathrm{htcf}_D(\prod\bar\lambda) \leq T^2_D(\bar\lambda) \leq T^1_D(\bar\lambda) \leq T^0_D(\bar\lambda)$

(2) If $T^0_D(\bar\lambda) > |\mathcal{P}(\kappa)/D|$ or just $T^0_D(\lambda) > \mu$, and $\mathcal{P}(\kappa)/D$ satisfies the μ^+-c.c. <u>then</u> $T^0_D(\bar\lambda) = T^1_D(\bar\lambda) = T^2_D(\bar\lambda)$ so the supremum in 3.13(a) is obtained (so e.g. $T^0_D(\bar\lambda) > 2^\kappa$ suffice)

(3) $T^0_D(\bar\lambda)^{<\mathrm{reg} D} = T^0_D(\bar\lambda)$ (each λ_i infinite of course).

(4) $[\mathrm{htcf}_D \prod_{i<\kappa} f(i)] \leq T^2_D(f) \leq [\mathrm{htcf}_D \prod_{i<\kappa} f(i)]^{<\theta} + \mathrm{reg}(D)^\kappa/D$ where $\theta = \mathrm{reg}_*(D)$ in fact $\theta = \mathrm{reg}(D) + \mathrm{wsat}(D)$ suffice

(5) If D is an ultrafilter $|\prod\bar\lambda/D| = T^e_D(\bar\lambda)$ for $e \leq 2$.

(6) In (4), if $\bigwedge_{i<\kappa} f(i) \geq 2^\kappa$ (or just $(\mathrm{reg}(D) + 2)^\kappa/D \leq \min_{i<\kappa} f(i)$), <u>then</u>
$[\mathrm{htcf}_D \prod_{i<\kappa} f(i)]^{<\mathrm{reg}\ D} \leq T^0_D(f)$

(7) If the sup in the definition of $T^0_D(\bar\lambda)$ is not obtained then it has cofinality $\geq \mathrm{reg}(D)$ and even is regular.

Proof. (0) Check.

(1) First assume $\mu =: T^2_D(\bar\lambda) < \mathrm{htcf}_D(\prod\bar\lambda)$; then we can find $\mu^* = \mathrm{cf}(\mu^*) \in (\mu, \mathrm{htcf}_D(\prod\bar\lambda)]$ and $\bar\mu = \langle \mu_i : i < \kappa\rangle$, a sequence of regular cardinals, $\bigwedge_{i<\kappa} \mu_i \leq \lambda_i$ such that $\mu^* = \mathrm{tcf}(\prod\bar\mu/D)$ and let $\langle f_\alpha : \alpha < \mu^*\rangle$ exemplify this. Now let F exemplify $\mu = T^2_D(\bar\lambda)$, for each $g \in F$ let

$$g' \in \prod_{i<\kappa} \mu_i \text{ be} : g'(i) = \begin{cases} g(i) & \text{if } g(i) < \mu_i \\ 0 & \text{otherwise.} \end{cases}$$

So there is $\alpha(g) < \mu^*$ such that $g' <_D f_{\alpha(g)}$. Let $\alpha^* = \sup\{\alpha(g): g \in F\}$, now $\alpha^* < \mu^*$ (as $\mu^* = \mathrm{cf}\,\mu^* > \mu = |F|$). So $g \in F \Rightarrow g \neq_D f_{\alpha^*}$, contradiction. So really $T^2_D(\bar\lambda) \leq \mathrm{htcf}_D(\prod \bar\lambda)$ as required.

If F exemplifies the value of $T^1_D(\bar\lambda)$, it also exemplifies $T^2_D(\bar\lambda) \leq |F|$ hence $T^2_D(\bar\lambda) \leq T^1_D(\bar\lambda)$.

Lastly if F exemplifies the value of $T^1_D(f)$ it also exemplifies $T^0_D(\bar\lambda) \geq |F|$, so $T^1_D(\bar\lambda) \leq T^0_D(\bar\lambda)$.

(2) Let μ be $|\mathcal{P}(\kappa)/D|$ or at least μ is such that the Boolean algebra $\mathcal{P}(\kappa)/D$ satisfies the μ^+-c.c. Assume that the desired conclusion fails so $T^2_D(\bar\lambda) < T^0_D(\bar\lambda)$, so there is $F_0 \subseteq \prod \bar\lambda$, such that $[f_1 \neq f_2 \in F_0 \Rightarrow f_1 \neq_D f_2]$, and $|F_0| > T^2_D(\bar\lambda) + \mu$ (by the definition of $T^0_D(\bar\lambda)$). Also there is $F_2 \subseteq \prod \bar\lambda$ exemplifying the value of $T^2_D(\bar\lambda)$. For every $f \in F_0$ there is $g_f \in F_2$ such that $\neg f \neq_D g_f$ (by the choice of F_2). As $|F_0| > T^2_D(\bar\lambda) + \mu$ for some $g \in F_2$, $F^* =: \{f \in F_0 : g_f = g\}$ has cardinality $> T^2_D(f) + \mu$. Now for each $f \in F^*$ let $A_f = \{i < \kappa : f(i) = g(i)\}$, clearly $A_f \in D^+$. Now $f \mapsto A_f/D$ is a function from F^* into $\mathcal{P}(\kappa)/D$, hence, if $\mu = |\mathcal{P}(\kappa)/D|$, it is not one to one (by cardinality consideration) so for some $f' \neq f''$ from F^* (hence form F_0) we have $A_{f'}/D = A_{f''}/D$; but so

$$\{i < \kappa : f'(i) = f''(i)\} \supseteq \{i < \kappa : f'(i) = g(i)\} \cap \{i < \kappa : f''(i) = g(i)\} = A_{f'}/D$$

hence is $\neq \emptyset \bmod D$, so $\neg f' \neq_D f''$, contradiction the choice of F_0. If $\mu \neq |\mathcal{P}(\kappa)/D|$ (as $F^* \subseteq F_0$ by the choice of F_0) we have:

$$f_1 \neq f_2 \in F^* \Rightarrow A_{f_1} \cap A_{f_2} = \emptyset \mod D$$

so $\{A_f : f \in F^*\}$ contradicts "the Boolean algebra $\mathcal{P}(\kappa)/D$ satisfies the μ^+-c.c.".

(3) Assume that $\theta < \mathrm{reg}(D)$ and¶ $\mu \leq^+ T^0_D(\bar\lambda)$. As $\mu \leq^+ T^0_D(\bar\lambda)$ we can find $f_\alpha \in \prod \bar\lambda$ for $\alpha < \mu$ such that $[\alpha < \beta \Rightarrow f_\alpha \neq_D f_\beta]$. Also (as $\theta < \mathrm{reg}(D)$) we can find $\{A_\varepsilon : \varepsilon < \theta\} \subseteq D$ such that for every $i < \kappa$ the set $w_i =: \{\varepsilon < \theta : i \in A_\varepsilon\}$ is finite. Now for every function $h: \theta \to \mu$ we define g_h, a function with domain κ:

$$g_h(i) = \{(\varepsilon, f_{h(\varepsilon)}(i)) : \varepsilon \in w_i\}$$

So $|\{g_h(i) : h \in {}^\theta\mu\}| \leq (\lambda_i)^{|w_i|} = \lambda_i$, and if $h_1 \neq h_2$ are from ${}^\theta\mu$ then for some $\varepsilon < \theta$, $h_1(\varepsilon) \neq h_2(\varepsilon)$ so $B_{h_1,h_2} = \{i : f_{h_1(\varepsilon)}(i) \neq f_{h_2(\varepsilon)}(i)\} \in D$ that is $B_{h_1,h_2} \cap A_\varepsilon \in D$ so

$\otimes_1$ if $i \in B_{h_1,h_2} \cap A_\varepsilon$ then $\varepsilon \in w_i$, so $g_{h_1}(i) \neq g_{h_2}(i)$.

$\otimes_2$ $B_{h_1,h_2} \cap A_\varepsilon \in D$

So $\langle g_h : h \in {}^\theta\mu\rangle$ exemplifies $T^0_D(\bar\lambda) \geq \mu^\theta$. If the supremum in the definition of $T^0_D(\bar\lambda)$ is obtained we are done. If not then $T^0_D(\bar\lambda)$ is a limit cardinal, and by the proof above:

$$[\mu < T^0_D(\bar\lambda) \quad \& \quad \theta < \mathrm{reg}(D) \quad \Rightarrow \quad \mu^\theta < T^0_D(\bar\lambda)].$$

So if $T^0_D(\bar\lambda)$ has cofinality $\geq \mathrm{reg}(D)$ we are done; otherwise let it be $\sum_{\varepsilon<\theta} \mu_\varepsilon$ with $\mu_\varepsilon < T^0_D(\bar\lambda)$ and $\theta < \mathrm{reg} D$. Note that by the previous sentence $T^0_D(\bar\lambda)^\theta = T^0_D(\bar\lambda)^{<\mathrm{reg}(D)} = \prod_{\varepsilon<\theta} \mu_\varepsilon$, and let $\{f^\varepsilon_\alpha : \alpha < \mu_\varepsilon\} \subseteq \prod \bar\lambda$ be such that $[\alpha < \beta \Rightarrow f^\varepsilon_\alpha \neq_D f^\varepsilon_D]$ and repeat the previous proof with $f^\varepsilon_{h(\varepsilon)}$ replacing $f_{h(\varepsilon)}$.

¶ $\leq^+$ means here that the right side is a supremum, right bigger than the left or equal but the supremum is obtained

(4) For the first inequality. assume it fails so $\mu =: T^2_D(f) < \mathrm{htcf}_D(\prod_{i<\kappa} f(i))$ hence for some $g \in \prod_{i<f(i)} (f(i)+1)$, $\mathrm{tcf}(\prod_{i<\kappa} g(i), <_D)$ is λ with $\lambda = \mathrm{cf}(\lambda) > \mu$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifies this. Let F be as in the definition of $T^2_D(f)$, now for each $h \in F$, there is $\alpha(h) < \lambda$ such that

$$\{i < \kappa : \text{ if } h(i) < g(i) \text{ then } h(i) < f_{\alpha(g)}(i)\} \in D.$$

Let $\alpha^* = \sup\{\alpha(h) + 1 : h \in F\}$, now $f_{\alpha^*} \in \prod_{i<\kappa} f(i)$ and $h \in F \Rightarrow h \neq_D f_{\alpha^*}$ contradicting the choice of F.

For the second inequality. Repeat the proof of 3.9 except that here we prove $F =: \bigcup_{\zeta<\theta} (N_\zeta \cap \prod_{i<\kappa} f(i))$ exemplifies $T^2_D(f) \leq \lambda$. So let $g^* \in \prod_{i<\kappa} \lambda_i$, and we should find $f \in N$ such that $(g^* \neq_D f)$; we replace clause (g) in the proof by
(g)′ $g^* < f^2_{\zeta+1} < f^1_\zeta \bmod D$
the construction is for $\zeta < \mathrm{reg}(D)$ and if we are stuck in ζ then $\neg f^1_\zeta \neq_D g^*$ and so we are done.

(5) Straightforward.

(6) Note that all those cardinals are $\geq 2^\kappa$ and $2^\kappa \geq \mathrm{reg}(D)^\kappa/D$. Now write successively inequalities from (2), (4), (1) and (3):

$$T^0_D(f) = T^2_D(f) \leq [\mathrm{htcf}_D \prod_{i<\kappa} f(i)]^{<\mathrm{reg}(D)} \leq [T^0_D(f)]^{<\mathrm{reg}(D)} = T^0_D(f).$$

(7) See proof of part (3). Moreover, let $\mu = \sum_{\varepsilon<\tau} \mu_\varepsilon$, $\tau < T^0_D(\bar\lambda)$, $\mu_\varepsilon < T^0_D(\bar\lambda)$ as exemplified by $\{f_\varepsilon : \varepsilon < \tau\}$, $\{f^\varepsilon_\alpha : \alpha < \mu_\varepsilon\}$ respectively. Let g_α be: if $\sum_{\varepsilon<\zeta} \mu_\varepsilon < \alpha < \sum_{\varepsilon\leq\zeta} \mu_\varepsilon$ then $g_\alpha(i) = (f_\varepsilon(i), f^\varepsilon_\alpha(i))$. So $\{g_\alpha : \alpha < \mu\}$ show: if $T^0_D(\bar\lambda)$ is singular then the supremum is obtained. $\blacksquare_{3.14}$

CLAIM 3.15: Assume D is a filter on κ, $f \in {}^\kappa\mathrm{Ord}$, $\mu^{\aleph_0} = \mu$ and $2^\kappa < \mu$, $T_D(f)$, (see Definition 3.13(d) and Theorem 3.14(2)) and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$. If $\mu < T_D(f)$ <u>then</u> for some sequence $\bar\lambda \leq f$ of regulars, $\mu^+ = \mathrm{tcf}(\prod \bar\lambda/D)$, or at least

(∗) there are $\langle\langle\lambda_{i,n} : n < n_i\rangle : i < \kappa\rangle$, $\lambda_{i,n} = \mathrm{cf}(\lambda_{i,n}) < f(i)$ and a filter D^* on $\bigcup_{i<\kappa}\{i\} \times n_i$ such that: $\mu^+ = \mathrm{tcf}(\prod_{(i,n)} \lambda_{i,n}/D^*)$ and $D = \{A \subseteq \kappa : \bigcup_{i\in A}\{i\} \times n_i \in D^*\}$.

Also the inverse is true.

Remark 3.15A. (1) It is not clear whether the first possibility may fail. We have explained earlier the doubtful role of $\mu^{\aleph_0} = \mu$.

(2) We can replace μ^+ by any regular μ such that $\bigwedge_{\alpha<\mu} |\alpha|^{\aleph_0} < \mu$ and then we use 3.14(4) to get $\mu \leq^+ T_D(f)$.

(3) The assumption $2^\kappa < \mu$ can be omitted.

Proof. The inverse should be clear (as in the proof of 3.6, by 3.14(3)).

Wlog $f(i) > 2^\kappa$ for $i < \kappa$, and trivially $(\mathrm{reg}(D))^\kappa/D \le 2^\kappa$, so by 3.14(4)

$$T_D(f) \le [\mathrm{htcf}_D(\prod_{i<\kappa} f(i)]^{<\mathrm{reg}_*(D)}.$$

If $\mu < \mathrm{htcf}_D(\prod_{i<\kappa} f(i))$ we are done (by 3.12(1)), so assume $\mathrm{htcf}_D(\prod_{i<\kappa} f(i)) \le \mu$, but we have assumed $\mu < T_D(f)$ so by 3.14(4) as $\mathrm{reg}_*(D) = \mathrm{reg}(D)$ we have $\mu^{<\mathrm{reg}(D)} \ge \mu^+$. Let $\chi \le \mu$ be minimal such that $\bigvee_{\theta<\mathrm{reg}(D)} \chi^\theta \ge \mu$, and let $\theta =: \mathrm{cf}(\chi)$ so, as $\mu > 2^\kappa$ we know $\chi^{\mathrm{cf}\,\chi} = \chi^{<\mathrm{reg}(D)} = \mu^{<\mathrm{reg}(D)} \ge \mu^+$, $\chi > 2^\kappa$, $\bigwedge_{\alpha<\chi} |\alpha|^{<\mathrm{reg}(D)} < \chi$. By the assumption $\mu = \mu^{\aleph_0}$ we know $\theta > \aleph_0$ (of course θ is regular). By [Sh-g, VIII 1.6(2), IX 3.5] and [Sh513, 6.12] there is a strictly increasing sequence $\langle \mu_\varepsilon : \varepsilon < \theta\rangle$ of regular cardinals with limit χ such that $\mu^+ = \mathrm{tcf}(\prod_{\varepsilon<\theta} \mu_\varepsilon/J^{\mathrm{bd}}_\theta)$.

As clearly $\chi \le \mathrm{htcf}_D(\prod_{i<\kappa} f(i))$, by 2.12(1) there is for for each each $\varepsilon < \theta$, a sequence $\bar\lambda^\varepsilon = \langle \lambda^\varepsilon_i : i < \kappa\rangle$ such that $\lambda^\varepsilon_i = \mathrm{cf}(\lambda^\varepsilon_i) \le f(i)$, and $\mathrm{tcf}(\prod_{i<\kappa} \lambda^\varepsilon_i/D) = \mu_\varepsilon$, also wlog $\lambda^\varepsilon_i > 2^\kappa$. Let $\langle A_\varepsilon : \varepsilon < \theta\rangle$ exemplify $\theta < \mathrm{reg}(D)$ and $n_i = |\{\varepsilon < \theta : i \in A_\varepsilon\}|$ and $\{\lambda_{i,n} : n < \omega\}$ enumerate $\{\lambda^\varepsilon_i : \varepsilon \text{ satisfies } i \in A_\varepsilon\}$, so we have gotten $(*)$. $\blacksquare_{3.15}$

Conclusion 3.16. Suppose D is an $\aleph_1$-complete filter on κ and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$. If $\lambda_i \ge 2^\kappa$ for $i < \kappa$ and $\sup_{A\in D^+} T_{D+A}(\bar\lambda) > \mu^{\aleph_0}$ <u>then</u> for some $\lambda'_i = \mathrm{cf}(\lambda'_i) \le \lambda_i$ we have

$$\sup_{A\in D^+} \mathrm{htcf}_{D+A}(\prod_{i<\kappa} \lambda'_i) > \mu.$$

Conclusion 3.17. Let D be an $\aleph_1$-complete filter on κ and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$. If for $i < \kappa$, B_i is a Boolean algebra and $\lambda_i < \mathrm{Depth}^+(B_i)$ (see below) and

$$2^\kappa < \mu^{\aleph_0} < \sup_{A\in D^+} T_{D+A}(\bar\lambda)$$

<u>then</u> $\mu^+ < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$.

Proof. Use 3.25 below and 3.16 above.

Definition 3.18. For a partial order P (e.g. a Boolean algebra) let $\mathrm{Depth}^+(P) = \min\{\lambda :$ we cannot find $a_\alpha \in P$ for $\alpha < \lambda$ such that $\alpha < \beta \Rightarrow a_\alpha <_P a_\beta\}$.

Discussion 3.19.

(1) We conjecture that in 3.16 (and 3.17) the assumption "D is $\aleph_1$-complete" can be omitted. See [Sh589].

(2) Note that our results are for $\mu = \mu^{\aleph_0}$ only; to remove this we need first to improve the theorem on $pp = cov$ (i.e. to prove $\mathrm{cf}(\lambda) = \aleph_0 < \lambda \Rightarrow pp(\lambda) = cov(\lambda, \lambda, \aleph_1, 2)$ (or $\sup\{pp(\mu) : \mathrm{cf}\,\mu = \aleph_0 < \mu < \lambda\} = \mathrm{cf}(S_{\le\aleph_0}(\lambda), \subseteq)$ (see [Sh-g], [Sh430, §1]), which seems to me a very serious open problem (see [Sh-g, Analitic guide, 14]).

(3) In 3.17, if we can find $f_\alpha \in \prod_{i<\kappa} \lambda_i$ for $\alpha < \lambda$: $[\alpha < \beta < \lambda \Rightarrow f_\alpha \le f_\beta \bmod D]$ and $\neg f_\alpha =_D f_{\alpha+1}$ then $\lambda < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$. But this does not help for λ regular $> 2^\kappa$.

(4) We can approach 3.15 differently, by 3.20–3.23 below.

Claim 3.20: If $2^{2^\kappa} \le \mu < T_D(\bar\lambda)$, (or at least $2^{|D|+\kappa} \le \mu < T_D(\bar\lambda)$) and $\mu^{<\theta} = \mu$, <u>then</u> for some θ-complete filter $E \subseteq D$ we have $T_E(\bar\lambda) > \mu$.

Proof. Wlog θ is regular (as $\mu^{<\theta} = \mu$ &, $\mathrm{cf}(\theta) < \theta \Rightarrow \mu^{<\theta^+} = \mu$). Let $\{f_\alpha : \alpha < \mu^+\} \subseteq \prod \bar\lambda$, be such that $[\alpha < \beta \Rightarrow f_\alpha \ne_D f_\beta]$. We choose by induction on ζ, $\alpha_\zeta < \mu^+$ as follows: α_ζ is the minimal ordinal $\alpha < \mu^+$ such that $E_{\zeta,\alpha} \subseteq D$ where $E_{\zeta,\alpha}$ = the θ-complete filter generated by

$$\big\{\{i < \kappa : f_{\alpha_\varepsilon}(i) \ne f_\alpha(i)\} : \varepsilon < \zeta\big\}$$

(note: each generator of $E_{\zeta,\alpha}$ is in D but not necessarily $E_{\zeta,\alpha} \subseteq D$!).
Let α_ζ be well defined if $\zeta < \zeta^*$, clearly $\varepsilon < \zeta \Rightarrow \alpha_\varepsilon < \alpha_\zeta$. Now if $\zeta^* < \mu^+$, then clearly $\alpha^* = \bigcup_{\zeta<\zeta^*} \alpha_\zeta < \mu^+$ and for every $\alpha \in (\alpha^*, \mu^+)$, $E_{\zeta^*,\alpha} \not\subseteq D$, so for every such α there are $A_\alpha \in D^+$ and $a_\alpha \in [\zeta^*]^{<\theta}$ such that $A_\alpha = \bigcup_{\varepsilon\in a_\alpha} \{i < \kappa : f_{\alpha_\varepsilon}(i) = f_\alpha(i)\}$. But for every $A \in D^+$, $a \in [\zeta^*]^{<\theta}$ we have

$$\{\alpha : \alpha \in (\alpha^*, \mu^+), A_\alpha = A, a_\alpha = a\} \subseteq \{\alpha : f_\alpha \restriction A \in \prod\nolimits_{i<\kappa} \{f_{\alpha_\varepsilon}(i) : \varepsilon \in a_\alpha\}\},$$

hence has cardinality $\le \theta^\kappa \le 2^\kappa < \mu$. Also $|[\zeta^*]^{<\theta}| \le \mu^{<\theta} < \mu^+$, $|D^+| \le 2^\kappa < \mu^\kappa$ so we get easy contradiction.

So $\zeta^* = \mu^+$, but the number of possible E's is $\le 2^{2^\kappa}$, hence for some E we have $|\{\varepsilon < \mu^+ : E_{\varepsilon,\alpha_\varepsilon} = E\}| = \mu^+$. Necessarily $E \subseteq D$ and E is θ-complete, and $\{f_{\alpha_\varepsilon} : \varepsilon < \mu^+$, and $E_{\alpha_\varepsilon} = E\}$ exemplifies $T_E(\bar\lambda) > \mu$, so E is as required. $\blacksquare_{3.20}$

Fact 3.21. 1. In 3.20 we can replace μ^+ by μ^* if $2^{2^\kappa} < \mathrm{cf}(\mu^*) \le \mu^* \le T^0_D(\bar\lambda)$ and $\bigwedge_{\alpha<\mu^*} |\alpha|^{<\theta} < \mu^*$.

Proof. The same proof as 3.20.

Claim 3.22:

(1) If $2^\kappa < |\prod \bar\lambda/D|$, D an ultrafilter on κ, $\mu = \mathrm{cf}(\mu) \le |\prod \bar\lambda/D|$, $\bigwedge_{i<\kappa} |i|^{\aleph_0} < \mu$, and D is regular <u>then</u> $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} \lambda_i/D)$

(2) Similarly for D just a filter but $A \in D^+ \Rightarrow \prod \bar\lambda/(D + A) = \prod \bar\lambda/D$.

Proof. 1) Wlog $\lambda =: \lim_D \bar\lambda = \sup(\bar\lambda)$, so $|\prod \bar\lambda/D| = \lambda^\kappa$ (see 3.6, by [CK]). If $\mu \le \lambda$ we are done; otherwise let $\chi = \min\{\chi : \chi^\kappa = \lambda^\kappa\}$, so $\chi^{\mathrm{cf}(\chi)} = \lambda^\kappa$, $\mathrm{cf}(\chi) \le \kappa$ but $\lambda < \mu \le \lambda^\kappa$ hence $\lambda^{\aleph_0} < \mu$ hence $\mathrm{cf}(\chi) > \aleph_0$, also by χ's minimality $\bigwedge_{i<\chi} |i|^{\mathrm{cf}\,\chi} \le |i|^\kappa < \chi$, and remember $\chi < \mu = \mathrm{cf}\,\mu \le \chi^{\mathrm{cf}\,\chi}$ so by [Sh-g. VIII 1.6(2)] there is $\langle \mu_\varepsilon : \varepsilon < \mathrm{cf}(\chi)\rangle$ strictly increasing sequence of regular cardinals with limit χ, $\prod_{\varepsilon<\mathrm{cf}(\chi)} \mu_\varepsilon/J^{bd}_{\mathrm{cf}\,\chi}$ has true cofinality μ. Let $\chi_\varepsilon = \sup\{\mu_\zeta : \zeta < \varepsilon\} + 2^\kappa$, let $\mathrm{i} : \kappa \to \mathrm{cf}(\chi)$ be $\mathrm{i}(i) = \sup\{\varepsilon + 1 : \lambda_i \ge \chi_\varepsilon\}$. If there is a function $h \in \prod_{i<\kappa} \mathrm{i}(i)$ such that $\bigwedge_{j<\mathrm{cf}(\chi)} \{i < \kappa : h(i) < j\} = \emptyset \bmod D$ then $\prod_{i<\kappa} \mu_{h(i)}/D$ has true cofinality μ as required; if not (D, i) is weakly normal (i.e. there is no such h - see [Sh420]). But for D regular, D is $\mathrm{cf}(\chi)$-regular, some $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\kappa)\rangle$ exemplifies it and $h(i) = \max\{\varepsilon : \varepsilon < \mathrm{i}(i)$ and $i \in A_\varepsilon\}$ (maximum over a finite set) is as required.

2) Similarly using $\lambda =: \liminf_D(\bar\lambda)$. $\blacksquare_{3.22}$

Discussion 3.23.

1. In 3.20 (or 3.21) we can apply [Sh 410, §6] so $\mu = \mathrm{tcf}(\prod \bigcup_{i<\mu} \mathfrak{a}_i/D^*)$, where $D = \{A \subseteq \kappa \colon \bigcup_{i \in A} \mathfrak{a}_i \in D^*\}$ and each $\mathfrak{a}_i$ is finite.

   See also in 3.15.

CLAIM 3.24: If D is a filter on κ, B_i is the interval Boolean algebra on the ordinal α_i, and $|\prod\limits_{i<\kappa} \alpha_i/D| > 2^\kappa$ <u>then</u> for regular μ we have: $\mu < \mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D)$ iff for some $\mu_i \leq \alpha_i$ (for $i < \kappa$) and $A \in D^+$, the true cofinality of $\prod\limits_{i<\kappa} \mu_i/(D+A)$ is well defined and equal to μ.

Proof. The $\Leftarrow$ (i.e. if direction) is clear. For the $\Rightarrow$ direction assume μ is regular $< \mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D)$ so there are $f_\alpha \in \prod\limits_{i<\kappa} B_i$ such that $\prod\limits_{i<\kappa} B_i/D \vDash f_\alpha/D < f_\beta/D$ for $\alpha < \beta$.

Wlog $\mu > 2^\kappa$. Let $f_\alpha(i) = \bigcup\limits_{\ell < n(\alpha,i)} [j_{\alpha,i,2\ell}, j_{\alpha,i.2\ell+1})$ where $j_{\alpha,i,\ell} < j_{\alpha,i,\ell+1} < \alpha_i$ for $\ell < 2n(\alpha,i)$. As $\mu = \mathrm{cf}(\mu) > 2^\kappa$ wlog $n_{\alpha,i} = n_i$. By [Sh430, 6.6D] (see more [Sh513, 6.1]) we can find $A \subseteq A^* =: \{(i,\ell) : i < \kappa, \ell < 2n_\alpha\}$ and $\langle \gamma^*_{i,\ell} : i < \kappa, \ell < 2n_i\rangle$ such that $(i,\ell) \in A \Rightarrow \gamma^*_{i,\ell}$ is a limit ordinal and

$(*)$ for every $f \in \prod\limits_{(i,\ell)\in A} \gamma^*_{i,\ell}$ and $\alpha < \mu$ there is $\beta \in (\alpha,\mu)$ such that

$(i,\ell) \in A^* \setminus A \Rightarrow j_{\alpha,i,\ell} = \gamma^*_{i,\ell}$
$(i,\ell) \in A \Rightarrow f(i,\ell) < j_{\beta,i,\ell} < \gamma^*_{i,\ell}$
$(i,\ell) \in A \Rightarrow \mathrm{cf}(\gamma^*_{i,\ell}) > 2^\kappa$

Let $\ell(i) = \max\{\ell < 2n(i) : (i,\ell) \in A\}$ and let $B = \{i : \ell(i) \text{ well defined}\}$. Clearly $B \in D^+$ (otherwise we can find $\alpha < \beta < \mu$ such that $f_\alpha/D = f_\beta/D$, contradiction). For $(i,\ell) \in A$ define $\beta^*_{i,\ell}$ by $\beta^*_{i,\ell} = \sup\{\gamma^*_{j,m} + 1 : (j,m) \in A^*$ and $\gamma^*_{j,m} < \gamma^*_{i,\ell}\}$. Now $\beta^*_{i,\ell} < \gamma^*_{i,\ell}$ as $\mathrm{cf}(\gamma^*_{i,\ell}) > 2^\kappa$. Let

$$Y = \{\alpha < \mu : \text{ if } (i,\ell) \in A^* \setminus A \text{ then } j_{\alpha,i,\ell} = \gamma^*_{i,\ell}$$
$$\text{and if } (i,\ell) \in A \text{ then } \beta^*_{i,\ell} < j_{\alpha,\ell,i} < \gamma^*_{\ell,i}\}$$

Let $B_1 = \{i \in B : \ell(i) \text{ is odd}\}$. Clearly $B_1 \subseteq B$ and $B \setminus B_1 = \emptyset \bmod D$ (otherwise as in $(*)_1$, $(*)_2$ below get contradiction) hence $B_1 \in D^+$. Now

$(*)_1$ for $\alpha < \beta$ from Y we have

$$\langle j_{\alpha,i,\ell(i)} : i \in B_1\rangle \leq \langle j_{\beta,i,\ell(i)} : i \in B_1\rangle \bmod (D \restriction B_1)$$

[Why? as f_α/D was non decreasing in $\prod\limits_{i<\kappa} B_i/D$]

$(*)_2$ for every $\alpha \in Y$ for some β, $\alpha < \beta \in Y$ we have

$$\langle j_{\alpha,i,\ell(i)} : i \in B_1\rangle < \langle j_{\beta,i,\ell(i)} : i \in B_1\rangle \bmod (D \restriction B_1)$$

[Why? by $(*)$ above]

Together for some unbounded $Z \subseteq Y$, $\big\langle \langle j_{\alpha,\ell,\ell(i)} : i \in B_1\rangle/(D \restriction B_1) : \alpha \in Z\big\rangle$ is $<_{D\restriction B_1}$-increasing, so it has a $<_{(D\restriction B_1)}$ $-$eub (as $\mu > 2^\kappa$, see 3.10, and more in [Sh-g, II §1]), say $\langle j^*_i : i \in B_1\rangle$ hence $\prod\limits_{i\in B_1} j^*_i/(D \restriction B_1)$ has true cofinality μ by 1.3(12) and clearly $j^*_i \leq \gamma^*_{i,\ell(i)} \leq \alpha_i$, so we have finished. $\blacksquare_{3.24}$

CLAIM 3.25: If D is a filter on κ, B_i a Boolean algebra, $\lambda_i < \mathrm{Depth}^+(B_i)$ <u>then</u>

(a) $\mathrm{Depth}(\prod_{i<\kappa} B_i/D) \geq \sup_{A\in D^+} \mathrm{tcf}(\prod_{i<\kappa} \lambda_i/(D+A))$ (i.e. on the cases tcf is well defined).
(b) $\mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$ is $\geq \mathrm{Depth}^+(\mathcal{P}(\kappa)/D)$ and is at least

$$\sup\{[\mathrm{tcf}(\prod_{i<\kappa} \lambda'_i/(D+A))]^+ : \lambda'_i < \mathrm{Depth}^+(B_i), A \in D^+\}.$$

Proof. Check.

CLAIM 3.26: Let D be a filter on κ, $\langle \lambda_i : i < \kappa\rangle$ a sequence of cardinals and $2^\kappa < \mu = \mathrm{cf}(\mu)$. Then $(\alpha) \Leftrightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta)$, and if $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$ and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$ we also have $(\gamma) \Leftrightarrow (\delta)$ where
(α) if B_i is a Boolean algebra, $\lambda_i < \mathrm{Depth}^+(B_i)$ <u>then</u> $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$
(β) there are $\mu_i = \mathrm{cf}(\mu_i) \leq \lambda_i$ for $i < \kappa$ and $A \in D^+$ such that $\mu = \mathrm{tcf}(\prod \mu_i/(D+A))$
(γ) there are $\langle\langle \lambda_{i,n} : n < n_i\rangle : i < \kappa\rangle$, $\lambda_{i,n} = \mathrm{cf}(\lambda_{i,n}) < \lambda_i$, $A^* \in D^+$ and a filter D^* on $\bigcup_{i<\kappa} \{i\} \times n_i$ such that:

$$\mu = \mathrm{tcf}(\prod_{(i,n)} \lambda_{i,n}/D^*) \text{ and } D+A^* = \{A \subseteq \kappa : \text{the set } \bigcup_{i\in A}\{i\}\times n_i \text{ belongs to } D^*\}.$$

(δ) for some $A \in D^+$, $\mu \leq T_{D+A}(\langle \lambda_i : i < \kappa\rangle)$

Remark. So the question whether $(\alpha) \Leftrightarrow (\delta)$ assuming $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$ is equivalent to $(\beta) \leftrightarrow (\gamma)$ which is a "pure" pcf problem.

Proof. Note $(\gamma) \Rightarrow (\delta)$ is easy (as in 3.15, i.e. as in the proof of 3.6, only easier). Now $(\beta) \Rightarrow (\gamma)$ is trivial and $(\beta) \Rightarrow (\alpha)$ by 3.25. Next $(\alpha) \Rightarrow (\beta)$ holds as we can use (α) for $B_i =:$ the interval Boolean algebra of the order λ_i and use 3.24. Lastly assume $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$ and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$, now $(\gamma) \Leftrightarrow (\delta)$ by 3.15. $\blacksquare_{3.26}$

Discussion. We would like to have (letting B_i denote Boolean algebra)

$$\mathrm{Depth}^{(+)}(\prod_{i<\kappa} B_i/D) \geq \prod_{i<\kappa} \mathrm{Depth}^{(+)}(B_i)/D$$

if D is just filter we should use T_D and so by the problem of attainment (serious by Magidor Shelah [MgSh433]), we ask

⊗ for D an ultrafilter on κ, does $\lambda_i < \mathrm{Depth}^+(B_i)$ for $i < \kappa$ implies

$$\prod_{i<\kappa} \lambda_i/D < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$$

at least when $\lambda_i > 2^\kappa$;

⊗′ for D a filter on κ, does $\lambda_i < \mathrm{Depth}^+(B_i)$ for $i < \kappa$ implies, assuming $\lambda_i > 2^\kappa$ for simplicity,

$$T_D(\langle \lambda_i : i < \kappa\rangle) < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$$

As explained in 3.26 this is a pcf problem.

In [Sh589] we deal with this under reasonable assumption (e.g. $\mu = \chi^+$ and $\chi = \chi^{\aleph_0}$). We also deal with a variant, changing the invariant (closing under homomorphisms, see [M]).

4. Remarks on the conditions for the pcf analysis

We consider a generalization whose interest is not so clear.

CLAIM 4.1: Suppose $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ is a sequence of regular cardinals, and θ is a cardinal and I^* is an ideal on κ; and H is a function with domain κ. We consider the following statements:

$(**)_H$ $\liminf_{I^*}(\bar{\lambda}) \geq \theta \geq \mathrm{wsat}(I^*)$ and H is a function from κ to $\mathcal{P}(\theta)$ such that:

(a) for every $\varepsilon < \theta$ we have $\{i < \kappa : \varepsilon \in H(i)\} = \kappa \bmod I^*$

(b) for $i < \kappa$ we have $\mathrm{otp}(H(i)) \leq \lambda_i$ or at least $\{i < \kappa : |H(i)| \geq \lambda_i\} \in I^*$

$(**)^+$ similarly but

(b)$^+$ for $i < \kappa$ we have $\mathrm{otp}(H(i)) < \lambda_i$

(1) In 1.5 we can replace the assumption $(*)$ by $(**)_H$ above.
(2) Also in 1.6, 1.7, 1.8, 1.9, 1.10, 1.11 we can replace 1.5$(*)$ by $(**)_H$.
(3) Suppose in Definition 2.3(2) we say $\bar{f}$ obeys $\bar{a}$ for H (instead of for $\bar{A}^*$) if
(i) for $\beta \in a_\alpha$ such that $\varepsilon =: \mathrm{otp}(a_\alpha) < \theta$ we have

$$\mathrm{otp}(a_\beta), \mathrm{otp}(a_\alpha) \in H(i) \Rightarrow f_\beta(i) \leq f_\alpha(i)$$

and in 2.3(2A), $f_\alpha(i) = \sup\{f_\beta(i) : \beta \in a_\alpha \text{ and } \mathrm{otp}(a_\beta), \mathrm{otp}(a_\alpha) \in H(i)\}$.

Then we can replace 1.5$(*)$ by $(**)_H$ in 2.5, 2.5A, 2,6; and replace 1.5$(*)$ by $(**)^+_H$ in 2.7 (with the natural changes).

Proof. (1) Like the proof of 1.5, but defining the g_ε's by induction on ε we change requirement (ii) to

(ii)$'$ if $\zeta < \varepsilon$, and $\{\zeta, \varepsilon\} \subseteq H(i)$ then $g_\zeta(i) < g_\varepsilon(i)$.

We can not succeed as

$$\langle (B^\varepsilon_{\alpha(*)} \setminus B^{\varepsilon+1}_{\alpha(*)}) \cap \{i < \kappa : \varepsilon, \varepsilon+1 \in H(i)\} : \varepsilon < \theta \rangle$$

is a sequence of θ pairwise disjoint member of $(I^*)^+$.

In the induction, for ε limit let $g_\varepsilon(i) < \cup\{g_\zeta(i) : \zeta \in H(i) \text{ and } \varepsilon \in H(i)\}$ (so this is a union at most $\mathrm{otp}(H(i) \cap \varepsilon)$ but only when $\varepsilon \in H(i)$ hence is $< \mathrm{otp}(H(i)) \leq \lambda_i$).

(2) The proof of 1.6 is the same, in the proof of 1.7 we again replace (ii) by (ii)$'$. Also the proof of the rest is the same.

(3) Left to the reader. $\blacksquare_{4.1}$

We want to see how much weakening $(*)$ of 1.5 to "$\liminf_{I^*}(\bar{\lambda}) \geq \theta \geq \mathrm{wsat}(I^*)$ suffices. If θ singular or $\liminf_{I^*}(\bar{\lambda}) > \theta$ or just $(\prod \bar{\lambda}, <_{I^*})$ is θ^+-directed then case (β) of 1.5 applies. This explains $(*)$ of 4.2 below.

CLAIM 4.2: Suppose $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$, $\lambda_i = \mathrm{cf}(\lambda_i)$, I^* an ideal on κ, and

$$\liminf_I(\bar{\lambda}) = \theta \geq \mathrm{wsat}(I^*), \quad \theta \text{ regular} \tag{$*$}$$

Then we can define a sequence $\bar{J} = \langle J_\zeta : \zeta < \zeta(*) \rangle$ and an ordinal $\zeta(*) \leq \theta^+$ such that

(a) $\bar{J}$ is an increasing continuous sequence of ideals on κ.
(b) $J_0 = I^*$, $J_{\zeta+1} =: \{A : A \subseteq \kappa$, and: $A \in J_\zeta$ or we can find $h: A \to \theta$ such that $\lambda_i > h(i)$ and $\varepsilon < \theta \Rightarrow \{i : h(i) < \varepsilon\} \in J_\zeta\}$.
(c) for $\zeta < \zeta(*)$ and $A \in J_{\zeta+1} \setminus J_\zeta$, the pair $(\prod \bar{\lambda}, J_\zeta + (\kappa \setminus A))$ (equivalently $(\prod \bar{\lambda} \restriction A, J_\zeta \restriction A)$) satisfies condition 1.5$(*)$ (case (β)) hence its consequences, (in particular it satisfies the weak pcf-th for θ).
(d) if $\kappa \notin \cup_{\zeta < \zeta(*)} J_\zeta$ then $(\prod \bar{\lambda}, \cup_{\zeta < \zeta(*)} J_\zeta)$ has true cofinality θ.

Proof. Straight. (We define J_ζ for $\zeta \leq \theta^+$ by clause (b) for $\zeta = 0$, ζ successor and as $\bigcup_{\varepsilon<\zeta} J_\varepsilon$ for ζ limit. Clause (c) holds by claim 4.4 below. It should be clear that $J_{\theta^++1} = J_{\theta^+}$, and let $\zeta(*) = \min\{\zeta: J_{\zeta+1} = \bigcup_{\varepsilon<\zeta} J_\varepsilon\}$ so we are left with checking clause (d). If $A \in J^+_{\zeta(*)}$, $h \in \prod_{i\in A} \lambda_i$, choose by induction on $\zeta < \theta$, $\varepsilon(\zeta) < \theta$ increasing with ζ such that $\{i < \kappa: h(i) \in (\varepsilon(\zeta), \varepsilon(\zeta+1)) \in J^+_{\zeta(*)}$. If we succeed we contradict $\theta \geq \mathrm{wsat}(I^*)$ as θ is regular. So for some $\zeta < \theta$, $\varepsilon(\zeta)$ is well defined but not $\varepsilon(\zeta+1)$. As $J_{\zeta(*)} = J_{\zeta(*)+1}$, clearly $\{i < \kappa: h(i) \leq \varepsilon(\zeta)\} = \kappa \mod J_{\zeta(*)}$. So $g_\varepsilon(i) = \begin{cases} \varepsilon & \text{if } \varepsilon < \lambda_i \\ 0 & \text{if } \varepsilon \geq \lambda_i \end{cases}$ exemplifies $\mathrm{tcf}(\prod \bar\lambda / J_{\zeta(*)}) = \theta$. $\blacksquare_{4.2}$

Now:

Conclusion 4.3: Under the assumptions of 4.2, I^* satisfies the pseudo pcf-th (see Definition 2.11(4)).

Claim 4.4: Under the assumption of 4.2, if J is an ideal on κ extending I^* the following conditions are equivalent

(a) for some $h \in \prod \bar\lambda$, for every $\varepsilon < \theta$ we have $\{i \in A : h(i) < \varepsilon\} \in J$
(b) $(\prod \bar\lambda, <_{J+(\kappa\setminus A)})$ is θ^+-directed.

Proof. $\underline{(a) \Rightarrow (b)}$

Let $f_\zeta \in \prod \bar\lambda$ for $\zeta < \theta$, we define $f^* \in \prod \bar\lambda$ by

$$f^*(i) = \sup\{f_\zeta(i) + 1 : \zeta < h(i)\}.$$

Now $f^*(i) < \lambda_i$ as $h(i) < \lambda_i = \mathrm{cf}(\lambda_i)$ and $f_\zeta \restriction A <_J f^* \restriction A$ as $\{i \in A : h(i) < \zeta\} \in J$.

$\underline{(b) \Rightarrow (a)}$

Let f_ζ be the following function with domain κ:

$$f_\zeta(i) = \begin{cases} \zeta & \text{if } \zeta < \lambda_i \\ 0 & \text{if } \zeta \geq \lambda_i \end{cases}$$

As $\liminf_{I^*} \geq \theta$, clearly $\varepsilon < \zeta \Rightarrow f_\varepsilon <_{I^*} f_\zeta$ and of course $f_\zeta \in \prod \bar\lambda$. By our assumption (b) there is $h \in \prod \bar\lambda$ such that $\zeta < \theta \Rightarrow f_\zeta \restriction A < h \restriction A \mod J$. Clearly h is as required. $\blacksquare_{4.4}$

References

[CK] C. C. Chang and H. J. Keisler, *Model Theory*, North Holland Publishing Company (1973).

[GH] F. Galvin and A. Hajnal, *Inequalities for cardinal power*, Annals of Math., **10** (1975) 491–498.

[Kn] A. Kanamori, *Weakly normal filters and irregular ultra-filter*, Trans of A.M.S., **220** (1976) 393–396.

[Ko] S. Koppelberg, *Cardinalities of ultraproducts of finite sets*, The Journal of Symbolic Logic, **45** (1980) 574–584.

[Kt] J. Ketonen, *Some combinatorial properties of ultra-filters,* Fund Math. VII (1980) 225–235.

[M] J. D. Monk, *Cardinal Function on Boolean Algebras,* Lectures in Mathematics, ETH Zürich, Bikhäuser, Verlag, Baser, Boston, Berlin, 1990.

[Sh-b] S. Shelah, *Proper forcing* Springer Lecture Notes, 940 (1982) 496+xxix.

[Sh-g] S. Shelah, *Cardinal Arithmetic,* volume 29 of Oxford Logic Guides, General Editors: Dov M. Gabbai, Angus Macintyre and Dana Scott, Oxford University Press, 1994.

[Sh7] S. Shelah, *On the cardinality of ultraproduct of finite sets*, Journal of Symbolic Logic, **35** (1970) 83–84.

[Sh345] S. Shelah, *Products of regular cardinals and cardinal invariants of Boolean Algebra,* Israel Journal of Mathematics, **70** (1990) 129–187.

[Sh400a] S. Shelah, *Cardinal arithmetic for skeptics*, American Mathematical Society. Bulletin. New Series, **26** (1992) 197–210.

[Sh410] S. Shelah, *More on cardinal arithmetic*, Archive of Math Logic, **32** (1993) 399–428.

[Sh420] S. Shelah, *Advances in Cardinal Arithmetic,* Proceedings of the Conference in Banff, Alberta, April 1991, ed. N. W. Sauer et al., Finite and Infinite Combinatorics, Kluwer Academic Publ., (1993) 355–383.

[Sh430] S. Shelah, *Further cardinal arithmetic*, Israel Journal of Mathematics, **accepted**.

[MgSh433] M. Magidor and S. Shelah, *λ_i inaccessible $> \kappa$, $\prod\limits_{i<\kappa} \lambda_i/D$ of order type μ^+*, preprint.

[Sh589] S. Shelah, *PCF theory: Application*, in preparation.