CICHOŃ'S MAXIMUM WITHOUT LARGE CARDINALS

MARTIN GOLDSTERN, JAKOB KELLNER, DIEGO A. MEJÍA, AND SAHARON SHELAH

Abstract. Cichoń's diagram lists twelve cardinal characteristics (and the provable inequalities between them) associated with the ideals of null sets, meager sets, countable sets, and σ-compact subsets of the irrationals.

It is consistent that all entries of Cichoń's diagram are pairwise different (apart from $\mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M})$, which are provably equal to other entries). However, the consistency proofs so far required large cardinal assumptions.

In this work, we show the consistency without such assumptions.

Introduction

How many Lebesgue null sets do we need to cover the real line? Countably many are not enough, as the countable union of null sets is null; and continuum many are enough, as $\bigcup_{r\in\mathbb{R}}\{r\} = \mathbb{R}$.

The answer to this question (and similar ones) is called a *cardinal characteristic* (sometimes also called cardinal invariant); in our case the characteristic is called "$\mathrm{cov}(\mathcal{N})$".

As we have argued, $\aleph_0 < \mathrm{cov}(\mathcal{N}) \leq 2^{\aleph_0}$. So if the Continuum Hypothesis (CH) holds, then $\mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$. It has been shown by Gödel [Göd40] and Cohen [Coh63] that CH is independent of ZFC. I.e., one can prove: If ZFC is consistent, then so is ZFC+CH as well as ZFC+¬CH.

Under ¬CH, $\mathrm{cov}(\mathcal{N})$ could be some cardinal less than $2^{\aleph_0}$, and one can indeed show that $\aleph_1 = \mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$, $\aleph_1 < \mathrm{cov}(\mathcal{N}) = 2^{\aleph_0}$ and $\aleph_1 = \mathrm{cov}(\mathcal{N}) < 2^{\aleph_0}$ are all consistent.

Some more characteristics associated with the σ-ideal $\mathcal{N}$ of null sets are defined:

- $\mathrm{add}(\mathcal{N})$ is the smallest number of null sets whose union is not null.
- $\mathrm{non}(\mathcal{N})$ is the smallest cardinality of a non-null set.
- $\mathrm{cof}(\mathcal{N})$ is the smallest size of a cofinal family of null sets, i.e., a family that contains for each null set N a superset of N.

Replacing $\mathcal{N}$ with another σ-ideal I gives us the analogously defined characteristics for I. In particular, for the meager ideal $\mathcal{M}$ we get $\mathrm{add}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, $\mathrm{cov}(\mathcal{M})$, $\mathrm{cof}(\mathcal{M})$.

For the σ-ideal $\mathtt{ctbl}$ of countable sets, it is easy to see that $\mathrm{add}(\mathtt{ctbl}) = \mathrm{non}(\mathtt{ctbl}) = \aleph_1$ and $\mathrm{cov}(\mathtt{ctbl}) = \mathrm{cof}(\mathtt{ctbl}) = 2^{\aleph_0}$, which is also called $\mathfrak{c}$ (for "continuum").

Date: April 16, 2020.

2010 *Mathematics Subject Classification.* 03E17, 03E35, 03E40.

This work was supported by the following grants: Austrian Science Fund (FWF): project number I3081, P29575 (first author) and P26737, P30666 (second author); Grant-in-Aid for Early Career Scientists 18K13448, Japan Society for the Promotion of Science (third author); European Research Council grant 338821 (fourth author). This is publication number 1177 of the fourth author.

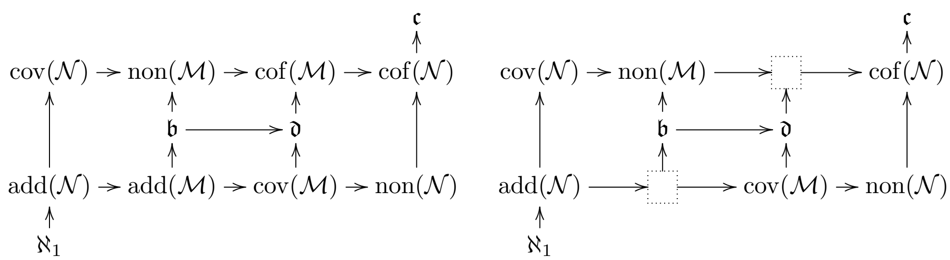

FIGURE 1. Cichoń's diagram (left). In the version on the right, the two "dependent" values are removed; the "independent" ones remain (nine entries excluding $\aleph_1$, or ten including it). It is consistent that these ten entries are pairwise different.

For $\mathcal{K}$, the σ-ideal generated by the compact subsets of the irrationals, it turns out that $\text{add}(\mathcal{K}) = \text{non}(\mathcal{K})$. This characteristic is more commonly called $\mathfrak{b}$. We also have $\text{cov}(\mathcal{K}) = \text{cof}(\mathcal{K})$, called $\mathfrak{d}$.

These characteristics are customarily displayed in Cichoń's diagram, see Figure 1. An arrow from $\mathfrak{x}$ to $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, one can show that $\max\{\mathfrak{d}, \text{non}(\mathcal{M})\} = \text{cof}(\mathcal{M})$ and $\min\{\mathfrak{b}, \text{cov}(\mathcal{M})\} = \text{add}(\mathcal{M})$. A series of results [Bar84, CKP85, BJS93, JS90, Kam89, Mil81, Mil84, RS83, RS85], summarized in [BJ95, Ch. 7], proves these (in)equalities in ZFC and shows that they are the only ones provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the nine "independent" characteristics in Cichoń's diagram (excluding $\aleph_1$ and including $\mathfrak{c}$) are consistent with ZFC, provided they honor the inequalities given by the arrows.

This leaves the question on how to separate more than two entries simultaneously. There was a lot of progress in recent years, giving four and up to seven values [Mej13, FGKS17, GMS16, FFMM18, Mej19a]. Finally, it was shown [GKS19] that the following statement, which we call **"Cichoń's maximum"**, is consistent:

> The maximal possible number of entries of Cichoń's diagram, i.e., all ten "independent" entries (including $\aleph_1$ and $\mathfrak{c}$), are pairwise different.

However, the proof required four Boolean ultrapower embeddings, constructed from four strongly compact cardinals.[1] A strongly compact cardinal is an example of a so-called "large cardinal" (LC). Such cardinals turned out to be an important scale for measuring consistency strengths of mathematical (and in particular set theoretic) statements: There are many examples of statements φ where one cannot prove

> The consistency of ZFC implies the consistency of (ZFC plus φ),

but only:

> The consistency of (ZFC plus LC) implies the consistency of (ZFC plus φ)

[1] A simpler example of this Boolean ultrapower construction, giving only eight different values and using three compacts, can be found in [KTT18]; and later a construction for Cichoń's maximum requiring only three compacts was given in [BCM18]. [Git19] notes that superstrongs are sufficient for the constructions. However until now all proofs showing the consistency of eight or more different values needed some large cardinals assumptions.

for some specific large cardinal axiom LC. In many cases, one can even show that φ is equiconsistent to LC (i.e., one can also prove that the consistency of (ZFC plus φ) implies the consistency of (ZFC plus LC)). For example, "there is an extension of Lebesgue measure to a σ-complete measure which measures *all* sets of reals" is equiconsistent with a so-called measurable cardinal (a notion much weaker than a strongly compact).

In case of φ being Cichoń's maximum, we previously could only prove an upper bound for the consistency strength, but conjectured that Cichoń's maximum is actually equiconsistent with ZFC. This turns out to be correct.

In this work, we introduce a new method to control cardinal characteristics when modifying a finite support ccc iteration (by taking intersections with σ-closed elementary submodels). This method can replace the Boolean ultrapower embeddings in previous constructions, so in particular we can get Cichoń's maximum without assuming large cardinals. Furthermore, we can get arbitrary regular cardinals as the values of the entries in Cichoń's diagram. As the method is quite general, we expect that it can be applied to control the values of other characteristics, in other constructions, as well.

This paper should be reasonably self-contained (modulo an understanding of forcing, such as presented in [Kun11]). However, in Section 2 we just quote the result (from [GKS19] or alternatively from [BCM18]) that a suitable preparatory forcing P^{pre} for the left hand side exists, without proofs or much explanation.

Annotated contents:

- S. 1 We define the properties LCU and COB for a forcing P, which give us the "strong witnesses" that will guarantee the desired equalities (or rather: both sides of the required inequalities) for the respective cardinal characteristics. We show how these properties are preserved when intersecting P with a σ-complete elementary submodel.
- S. 2 We just quote (without proof) the result from [GKS19] (or [BCM18]) that a suitable forcing P^{pre} for the left hand side with suitable LCU and COB properties exists.
- S. 3 We prove the main result: There is a complete subforcing P^{fin} of P^{pre} which forces ten different values to Cichoń's diagram (we can actually choose any desired regular values).
- S. 4 We remark that the same argument can be applied to alternative "initial forcings" for the left hand side. In particular, using a construction of [KST19], we get another ordering of the ten entries in Cichoń's diagram.
- S. 5 We list some open questions regarding alternative orders of Cichoń's diagram with ten values.

1. The LCU and COB properties and σ-closed elementary submodels

Let R be a binary relation on some basic set Y. The cardinal $\mathfrak{b}_R$, the *bounding number* of R, is the minimal size of an unbounded family. I.e.,

$$\mathfrak{b}_R := \min\big\{|B| : \ B \subseteq Y, \ (\forall g \in Y)\,(\exists f \in B)\,\neg(fRg)\big\},$$

Dually, $\mathfrak{d}_R$, the *dominating number* of R, is the minimal size of a dominating family. I.e.,

$$\mathfrak{d}_R := \min\big\{|D| : \ D \subseteq Y, \ (\forall f \in Y)\,(\exists g \in D)\, fRg\big\}.$$

We will use these notions in two situations:

On the one hand, R may be a directed partial order (or a linear order) without largest element, such as $([X]^{<\kappa}, \subseteq)$ or $(\kappa, \in)$. Then we will call $\mathfrak{b}_R$ the *completeness of* R and denote it by $\mathrm{comp}(R)$; and we call $\mathfrak{d}_R$ the *cofinality of* R and denote it by $\mathrm{cof}(R)$. Note that R is $<\lambda$-directed iff $\lambda \leq \mathrm{comp}(R)$ (as we assume that R is directed). If in addition R is linear without a maximal element, then $\mathrm{cof}(R) = \mathrm{comp}(R)$ is an infinite regular cardinal.

On the other hand, R may be a (possibly non-transitive) Borel relation on the reals (more generally: a sufficiently absolute definition of a binary relation on the reals), and we get the cardinal characteristics of the continuum $\mathfrak{b}_R$ and $\mathfrak{d}_R$. Note that $(\mathfrak{b}_R, \mathfrak{d}_R) = (\mathfrak{d}_{R^\perp}, \mathfrak{b}_{R^\perp})$, where we define the dual relation $R^\perp$ by $xR^\perp y$ iff $\neg(yRx)$. All entries of Cichoń's diagram are of this form, for quite natural relations R. (For more details, see the references after Theorem 2.4.)

In the following we give definitions of LCU and COB which are notational variants[2] of the definitions given in [GKS19, Def. 1.8 & 1.15].

We investigate relations on the reals, and fix ω^ω as representation of the reals. (This choice is irrelevant, and we could use any of the other usual representations as well. We just pick one so that we can later refer to the reals as a well defined object, and so that we can e.g. use $(\forall x \in \omega^\omega)$ in formulas.)

Definition 1.1. Assume R is a binary relation on ω^ω which is Borel, or just sufficiently absolutely defined.[3]

- For a directed partial order $(S, \leq_S)$ without maximal elements, the "cone of bounds" property $\mathsf{COB}(P, S)$ says: There is a sequence[4] $(g_s)_{s\in S}$ of P-names of reals such that for any P-name f of a real there is an $s \in S$ such that
$$P \Vdash (\forall t \geq_S s)\, f R g_t.$$
- For a linear order L without largest element, the "linear cofinal unbounded" property $\mathsf{LCU}_R(P, L)$ is defined as:
  There is a sequence $(c_\alpha)_{\alpha\in L}$ of P-names of reals such that for each P-name g of a real there is an $\alpha_0 \in L$ such that
$$P \Vdash (\forall \alpha \geq_L \alpha_0)\, \neg(c_\alpha R g).$$

(When writing $P \Vdash fRg$, we of course mean that we evaluate the definition of R in the extension.)

Actually, LCU is a special case of COB:

$$\mathsf{LCU}_R(P, \kappa) \text{ is equivalent to } \mathsf{COB}_{R^\perp}(P, \kappa) \tag{1.2}$$

(again, $R^\perp$ denotes the dual of R). However, LCU and COB will play different roles in our arguments, so we prefer to have different notations for these two concepts.

The following is basically the same as [GKS19, Lem. 1.9 & 1.16] (see also [GKMS19, Fact 2.14]):

[2]There are other variants of these definitions that do not mention forcings ([GKMS19, Def. 2.11]) but are applied to the extension $V[G]$. These variants are basically equivalent.

[3]The discussion after (1.4) shows which amount of absoluteness is sufficient for us. We will need non-Borel relations only in Subsection 4.1.

[4]If S is a (partially) ordered set, we sometimes use "a sequence indexed by S" as synonym for "a function with domain S".

Lemma 1.3. *(1) Let S be a $<\lambda$-directed partial order without a largest element, and let $A \subseteq S$ be cofinal. Then $\mathsf{COB}_R(P,S)$ is equivalent to $\mathsf{COB}_R(P,A)$, and implies*

$$P \Vdash \big(\mathfrak{b}_R \geq \lambda \ \&\ \mathfrak{d}_R \leq |A|\big).$$

(2) Let L be linear without a largest element and set $\lambda := \mathrm{cof}(L)$. (So λ is an infinite regular cardinal.) Then $\mathsf{LCU}_R(P,L)$ is equivalent to $\mathsf{LCU}_R(P,\lambda)$, and implies[5]

$$P \Vdash \big(\mathfrak{b}_R \leq |\lambda| \ \&\ \mathfrak{d}_R \geq \lambda\big).$$

Proof. Regarding the equivalence: Let $(g_s)_{s\in S}$ witness $\mathsf{COB}_R(P,S)$. Then $(g_s)_{s\in A}$ witnesses $\mathsf{COB}_R(P,A)$. On the other hand, if $(g'_s)_{s\in A}$ witnesses $\mathsf{COB}_R(P,A)$, then we assign to every $s \in S$ some $a(s) \in A$ above s, and set $g''_s := g'_{a(s)}$. Then $(g''_s)_{s\in S}$ witnesses $\mathsf{COB}_R(P,S)$.

From now on assume that $(g_s)_{s\in A}$ witnesses $\mathsf{COB}_R(P,A)$. Regarding $\mathfrak{d}_R$, note that $\{g_s : s \in A\}$ is forced to be dominating.

Regarding $\mathfrak{b}_R$, assume that p_0 forces that $X \subseteq \omega^\omega$ is of size less than (the ordinal) λ. Fix $p_1 \leq p_0$, $\kappa < \lambda$ and P-names $(f_\alpha)_{\alpha\in\kappa}$ of reals such that $p_1 \Vdash X = \{f_\alpha : \alpha \in \kappa\}$. For each α let s_α be an element of S satisfying the COB requirement for f_α. As S is $<\lambda$-directed, there is some $t \in S$ above all s_α, i.e., $P \Vdash f_\alpha R g_t$ for all $\alpha \in \kappa$. Accordingly, p_0 cannot force X to be unbounded.

The claims on LCU follow from the ones on COB by (1.2) (together with the fact that for linear orders L, $\mathrm{comp}(L) = \mathrm{cof}(L)$ and that $(\mathfrak{b}_R, \mathfrak{d}_R) = (\mathfrak{d}_{R^\perp}, \mathfrak{b}_{R^\perp})$). □

In the following results we show that when we restrict a poset P to a σ-closed elementary submodel N of some $H(\chi)$, then the LCU and COB properties still hold (when we intersect the parameter with N as well). These are simple technical tools we will use to prove the main results.

Assume that κ is regular, P κ-cc, $N \preccurlyeq H(\chi)$ is $<\kappa$-closed and $P \in N$. Then $P \cap N$ is again κ-cc and thus a complete subforcing of P. So given a $P \cap N$-generic G over V, there is a P-generic G^+ over V extending G. Note that G^+ is P-generic over N as well, and that $N[G^+] \preccurlyeq H^{V[G^+]}(\chi)$.

There is a correspondence of $P \cap N$-names σ for reals and P-names $\tau \in N$ for reals, such that $\sigma[G] = \tau[G^+]$ and for all $p \in P \cap N$ and sufficiently absolute φ,

$$p \Vdash_P \varphi(\tau) \text{ iff } p \Vdash_{P\cap N} \varphi(\sigma). \tag{1.4}$$

In a bit more detail: A "nice Q-name for a ζ-subset" (for an ordinal ζ) is a sequence $\bar h := \big((h_n, A_n)\big)_{n<\zeta}$ such that A_n is a maximal antichain in Q and $h_n : A_n \to 2$ (evaluated in the generic extension as $\{n \in \zeta : (\exists a \in G_Q \cap A_n)\, h_n(a) = 1\}$). As $P \cap N \lessdot P$, every nice $P \cap N$-name $\bar h$ for a ζ-subset is also a nice P-name, and furthermore $\bar h \in N$ whenever $\zeta < \kappa$ (as N is $<\kappa$-closed). On the other hand, if $\zeta < \kappa$ then every nice P-name $\bar h$ for a ζ-subset which is in N is actually a nice $P\cap N$-name. Note that if φ is Borel, then we are done with showing (1.4). For a more general formula φ, note that we have just shown that $N[G^+] \cap 2^{<\kappa} = V[G] \cap 2^{<\kappa}$, and using an absolute bijection between $2^{<\kappa}$ and $H(\kappa)$, we get that $N[G^+] \cap H(\kappa) = V[G] \cap H(\kappa)$. So (1.4) holds whenever φ is, e.g., (provably) absolute between the universe and $H(\chi)$ (for $\chi = \kappa$ as well as for χ sufficiently large), where φ may use elements of $H(\kappa)$ (or names for such elements) as parameters.

[5] We actually do mean $\mathfrak{d}_R \geq \lambda$ and not just $\mathfrak{d}_R \geq |\lambda|$, i.e., if λ is not a cardinal in the extension anymore, then we have $\mathfrak{d}_R \geq |\lambda|^+$. But this is irrelevant in our application, as P will preserve λ.

Lemma 1.5. *Assume P is κ-cc for some uncountable regular κ and $N \preccurlyeq H(\chi)$ is $<\kappa$-closed. Then $P \cap N$ is a κ-cc complete subforcing of P. Assume in the following that P, S, L, κ, R are in N.*

(1) $\mathsf{COB}_R(P, S)$ *implies* $\mathsf{COB}_R(P \cap N, S \cap N)$.
   So if we set $\lambda_1 := \mathrm{comp}(S \cap N)$ *and* $\lambda_2 := \mathrm{cof}(S \cap N)$, *then* $\mathsf{COB}_R(P, S)$ *implies* $P \cap N \Vdash \mathfrak{b}_R \geq \lambda_1$ & $\mathfrak{d}_R \leq |\lambda_2|$.
(2) $\mathsf{LCU}_R(P, L)$ *implies* $\mathsf{LCU}_R(P \cap N, L \cap N)$.
   So if we set $\lambda := \mathrm{cof}(L \cap N)$, *then* $\mathsf{LCU}_R(P, L)$ *implies* $P \cap N \Vdash \mathfrak{b}_R \leq |\lambda|$ & $\mathfrak{d}_R \geq \lambda$.

Proof. Let $(f_s)_{s\in S}$ witness $\mathsf{COB}_R(P, S)$ in N. Then $(f_s)_{s\in S\cap N}$ witnesses $\mathsf{COB}_R(P\cap N, S\cap N)$: Assume $g \in V$ is a $P\cap N$-name for a real. As above we interpret it as a P-name in N. So N thinks there is some $s \in S$ such that for all $t \geq_S s$, $P \Vdash gRf_t$. So by absoluteness (1.4), for every $t \geq_S s$ in N we get $P \cap N \Vdash gRf_t$.

Again, (2) is a special case of (1). □

Lemma 1.6. *Let $\kappa \leq \lambda \leq \theta$ be cardinals with κ and λ uncountable regular, S a directed set without maximal elements, ζ a regular cardinal, and let P be a κ-cc poset. Assume $(N_i)_{i<\lambda}$ is an increasing sequence of $<\kappa$-closed elementary submodels of $H(\chi)$, where χ is a fixed, sufficiently large*[6] *regular cardinal. Assume that $|N_i| = \theta$, that $\theta \cup \{\theta, P, R, S, \zeta\} \subseteq N_i$, and that $N_i \in N_{i+1}$ for any $i < \lambda$. Set $N := \bigcup_{i<\lambda} N_i$ (which is also a $<\kappa$-closed elementary submodel).*

(1) $\mathrm{cof}(\zeta \cap N) = \zeta'$, *where* $\zeta' := \begin{cases} \zeta & \text{if } \zeta \leq \theta, \\ \lambda & \text{otherwise.} \end{cases}$
   In particular $\mathsf{LCU}_R(P, \zeta)$ *implies* $\mathsf{LCU}_R(P \cap N, \zeta')$.
(2) $\mathrm{comp}(S \cap N) \geq \min(\kappa, \mathrm{comp}(S))$.
(3) *If* $\mathrm{cof}(S) \leq \theta$, *then* $S \cap N$ *is cofinal in* S, *and in particular* $S \cap N$ *has the same cofinality and completeness as* S.
(4) *If* $\mathrm{comp}(S) > \theta$, *then* $\mathrm{cof}(S \cap N) = \lambda$.
   In particular $\mathsf{COB}_R(P, S)$ *implies* $\mathsf{COB}_R(P \cap N, \lambda)$.

Proof. For (2), the assumptions of Lemma 1.5 are sufficient: Assume that $A \subseteq S\cap N$ has size less than $\min(\kappa, \mathrm{comp}(S))$. As N is $<\kappa$-closed, $A \in N$. By absoluteness, N knows that the set A (which is smaller than $\mathrm{comp}(S)$ after all) has an upper bound, so there is an upper bound of A in $S \cap N$.

(3) only requires that $\theta \cup \{\theta\} \subseteq N$ and $|N| = \theta$: In N, let $A \subseteq S$ be a cofinal subset of size $\mathrm{cof}(S)$. Since $\mathrm{cof}(S) \leq \theta \subseteq N$, we have $A \subseteq N$, so $A \subseteq S \cap N$ is cofinal in S. And it is clear that any cofinal subset of a partial order has the same completeness and cofinality as the order itself.

For (4), fix $i < \lambda$. Since $|N_i| \leq \theta < \mathrm{comp}(S)$, there is some $\alpha_i \in S$ bounding $N_i \cap S$. In fact, we can find such α_i in $S \cap N_{i+1}$ because $N_i \in N_{i+1}$. Hence, $(\alpha_i)_{i<\lambda}$ is a cofinal increasing sequence of $S \cap N$, so $\mathrm{cof}(S \cap N) = \lambda$. The claim on COB follows from Lemmas 1.5(1) and 1.3(1).

For (1), if $\zeta > \theta$ then, by (4) applied to $S = \zeta$, $\mathrm{cof}(\zeta \cap N) = \lambda$; if $\zeta \leq \theta$ then $\zeta \cap N = \zeta$, so $\zeta' = \zeta$. The claim on LCU follows from Lemmas 1.5(2) and 1.3(2). □

[6] It is enough to assume θ, ζ, S and 2^P are in $H(\chi)$.

2. The forcing for the left hand side

We set $(\mathfrak{b}_i, \mathfrak{d}_i)$ to be the following pairs of dual characteristics in Cichoń's diagram:

$$(\mathfrak{b}_i, \mathfrak{d}_i) = \begin{cases} (\,\mathrm{add}(\mathcal{N})\,,\ \mathrm{cof}(\mathcal{N})\,) & \text{for } i = 1, \\ (\,\mathrm{cov}(\mathcal{N})\,, \mathrm{non}(\mathcal{N})\,) & \text{for } i = 2, \\ (\quad \mathfrak{b} \quad,\quad \mathfrak{d} \quad) & \text{for } i = 3, \\ (\mathrm{non}(\mathcal{M}),\ \mathrm{cov}(\mathcal{M})) & \text{for } i = 4. \end{cases} \tag{2.1}$$

We will use for each i two Borel relations[7] on ω^ω, R_i^{LCU} and R_i^{COB}, in such a way that ZFC proves

$$\mathfrak{b}_{R_i^{\mathsf{COB}}} \leq \mathfrak{b}_i \leq \mathfrak{b}_{R_i^{\mathsf{LCU}}} \text{ and } \mathfrak{d}_{R_i^{\mathsf{COB}}} \geq \mathfrak{d}_i \geq \mathfrak{d}_{R_i^{\mathsf{LCU}}}. \tag{2.2}$$

We write LCU_i instead of $\mathsf{LCU}_{R_i^{\mathsf{LCU}}}$ and COB_i instead of $\mathsf{COB}_{R_i^{\mathsf{COB}}}$.

It is useful to have relations satisfying (2.2), because in this way we get:

Corollary 2.3. *$\mathsf{LCU}_i(P, \kappa)$ for κ regular implies $P \Vdash \mathfrak{b}_i \leq |\kappa|\ \&\ \mathfrak{d}_i \geq \kappa$.*

$\mathsf{COB}_i(P, S)$ for $\mathrm{comp}(S) = \kappa_1$ and $\mathrm{cof}(S) = \kappa_2$ implies $P \Vdash \mathfrak{b}_i \geq \kappa_1\ \&\ \mathfrak{d}_i \leq |\kappa_2|$.

Theorem 2.4. *Assume GCH and fix regular cardinals $\aleph_1 < \mu_1 < \mu_2 < \mu_3 < \mu_4 < \mu_\infty$ such that each μ_n is the successor of a regular cardinal.*

We can choose $R_i^{\mathsf{LCU}}, R_i^{\mathsf{COB}}$ satisfying (2.2) and construct a ccc poset P such that the following holds for $i = 1, 2, 3, 4$:

(a) *If $i < 4$ then, for all regular κ such that $\mu_i \leq \kappa \leq \mu_\infty$, $\mathsf{LCU}_i(P, \kappa)$ holds. In the case $i = 4$, $\mathsf{LCU}_i(P, \mu_4)$ and $\mathsf{LCU}_i(P, \mu_\infty)$ hold.*
(b) *There is a directed order S_i with $\mathrm{comp}(S_i) = \mu_i$ and $\mathrm{cof}(S_i) = \mu_\infty$ such that $\mathsf{COB}_i(P, S_i)$ holds.*

Accordingly, P forces

$$\mathrm{add}(\mathcal{N}) = \mu_1 < \mathrm{cov}(\mathcal{N}) = \mu_2 < \mathfrak{b} = \mu_3 < \mathrm{non}(\mathcal{M}) = \mu_4 < \mathrm{cov}(\mathcal{M}) = \mu_\infty = \mathfrak{c}.$$

This theorem is proved in [GKS19]; we will not repeat the proof here but instead point out where to find the definitions and proofs in the cited papers (the *italic* labels in the following paragraph refer to the cited paper):

Def. 1.2 defines relations called R_i for $i = 1, \dots, 4$. These R_i are, apart from $i = 2$, the "canonical" relations for $\mathfrak{b}_i, \mathfrak{d}_i$. They play the role of R_i^{LCU} and, apart from $i = 2$, also of R_i^{COB}. R_2^{COB} is implicitly defined in *Def. 1.17* as the canonical relation: $x R_2^{\mathsf{COB}} y$ iff y is not in the Borel null set coded by x. *Lem. 1.3* corresponds to (2.2) in this work, and *Thm. 1.35* is our Theorem 2.4.

Remark 2.5. In [BCM18, Thm. 5.3] a different construction is presented, which gives a stronger conclusion and requires the weaker assumption that $\aleph_1 \leq \mu_1 < \mu_2 < \mu_3 < \mu_4 < \mu_\infty = \mu_\infty^{<\mu_3}$ are just regular cardinals. If we use this paper, then $R_i^{\mathsf{COB}} = R_i^{\mathsf{LCU}} = R_i$ for all i, see [BCM18, Exm. 2.16] (where R_i corresponds to item $(5 - i)$).

[7] Actually, in most cases we will use the same R_i^{LCU} and R_i^{COB}, which is moreover the "canonical" choice for $(\mathfrak{b}_i, \mathfrak{d}_i)$. See the explanation that follows Theorem 2.4.

3. Cichoń's maximum without large cardinals

Theorem 3.1. *Assume GCH and $(\mu_n)_{1\le n\le 9}$ is a weakly increasing sequence of cardinals with μ_n regular for $n\le 8$ and $\mu_9^{\aleph_0}=\mu_9$. Then there is a ccc poset P^{fin} forcing that*

$$\aleph_1 \le \mathrm{add}(\mathcal{N}) = \mu_1 \le \mathrm{cov}(\mathcal{N}) = \mu_2 \le \mathfrak{b} = \mu_3 \le \mathrm{non}(\mathcal{M}) = \mu_4 \le \\ \le \mathrm{cov}(\mathcal{M}) = \mu_5 \le \mathfrak{d} = \mu_6 \le \mathrm{non}(\mathcal{N}) = \mu_7 \le \mathrm{cof}(\mathcal{N}) = \mu_8 \le \mathfrak{c} = \mu_9.$$

Full GCH is not actually required, see Remark 3.5.

Note that the μ_n are required to be only weakly increasing, i.e., we can replace each $\le$ in the inequality of characteristics by either $<$ or $=$ at will. So we get the consistency of 2^9 many different "sub-constellations" in Cichoń's diagram. Of course several of these have been known to be consistent before (even without large cardinals). E.g., the sub-constellation where we always choose "=" is just CH.

Proof. We fix an increasing sequence of cardinals (see Figure 2)

$$\aleph_1 \le \lambda_7 \le \lambda_5 \le \lambda_3 \le \lambda_1 \le \lambda_0 \le \lambda_2 \le \lambda_4 \le \lambda_6 \le \lambda_\infty < \\ < \theta_7 < \theta_6 < \theta_5 < \theta_4 < \theta_3 < \theta_2 < \theta_1 < \theta_0 < \theta_\infty, \tag{3.2}$$

such that the following holds:

(1) All cardinals are regular, with the possible exception of λ_∞,
(2) $\lambda_\infty = \lambda_\infty^{\aleph_0}$.
(3) GCH, plus θ_n is the successor of a regular cardinal for $n = 6, 4, 2, 0, \infty$.
I.e., the assumptions for Theorem 2.4 are satisfied if we set

$$\mu_i := \theta_{8-2i} \text{ for } i = 1, 2, 3, 4, \text{ and } \mu_\infty := \theta_\infty. \tag{3.3}$$

So we can apply Theorem 2.4, resulting in the forcing P^{pre}. (Thus P^{pre} forces the situation shown in the upper Cichoń diagram of Figure 2.)

We will now construct a forcing $P^{\mathrm{fin}} = P^{\mathrm{pre}} \cap N^*$ (a complete subforcing of P^{pre}) which forces $(\mathfrak{b}_i, \mathfrak{d}_i) = (\lambda_{8-2i+1}, \lambda_{8-2i})$ for all $i = 1, \dots, 4$, and $\mathfrak{c} = \lambda_\infty$ (i.e., the situation shown in the lower Cichoń diagram of Figure 2).

We fix $N_{n,\alpha}$ for $0 \le n \le 7$, $\alpha \in \lambda_n$, as well as N_8, satisfying the following for any $n \le 7$:

- Each $N_{n,\alpha}$ as well as N_8 is an elementary submodel of $H(\chi)$ and contains (as elements) the sequences of θ's and λ's, as well as P^{pre} and S_i (the directed orders provided by Theorem 2.4) for $i = 1, 2, 3, 4$.
- $N_{n,\alpha}$ contains $(N_{m,\beta})_{m<n,\,\beta\in\lambda_m}$ as well as $(N_{n,\beta})_{\beta<\alpha}$.
  N_8 contains $(N_{m,\beta})_{m\le 7,\,\beta\in\lambda_m}$.
- $|N_{n,\alpha}| = \theta_n$, and $N_{n,\alpha}$ is $<\theta_n$-closed (thus $\theta_n \subseteq N_{n,\alpha}$).[8]
- We set $N_n := \bigcup_{\alpha\in\lambda_n} N_{n,\alpha}$. Note that N_n is $<\lambda_n$-closed and has size θ_n.
- N_8 is $<\aleph_1$-closed and has size λ_∞.
- We set $N^* := N_0 \cap \dots \cap N_7 \cap N_8$.
- For $0 \le m \le 8$, we set $P_m := P^{\mathrm{pre}} \cap N_0 \cap \dots \cap N_m$ and $P^{\mathrm{fin}} := P_8 = P^{\mathrm{pre}} \cap N^*$.
  Note that $N_0 \cap \dots \cap N_m$ is again an elementary submodel of $H(\chi)$,[9] and accordingly each P_m is a complete subforcing of P^{pre}.

[8] For $n \le 6$, $<\theta_{n+1}^+$-closed is enough; for $n = 7$, $<\lambda_7$-closed is sufficient.
[9] If $M, N \preccurlyeq H(\chi)$ and $M \in N$ then $M \cap N \preccurlyeq M$ and $M \cap N \preccurlyeq N$.

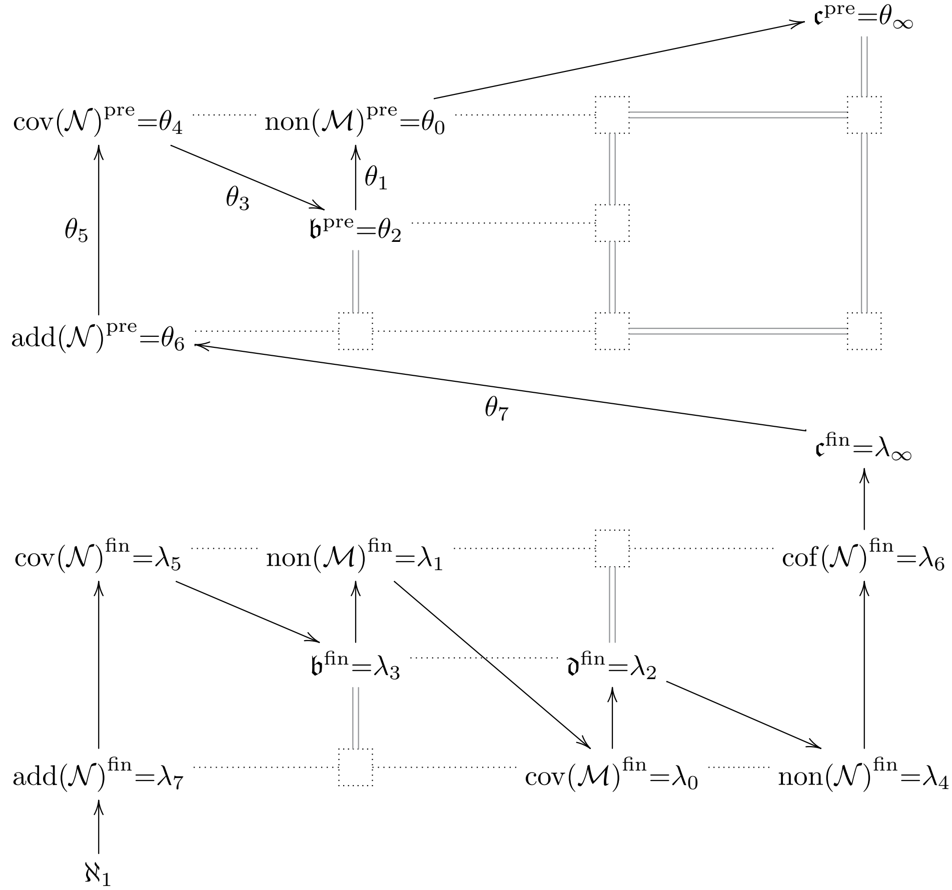

FIGURE 2. Our setup. The cardinals λ_n and θ_n are increasing along the arrows (strictly increasing above λ_∞). The preparatory forcing $P^{\rm pre}$ forces $\mathfrak{x} = \mathfrak{x}^{\rm pre}$ for each left hand side characteristic $\mathfrak{x}$ (and forces the whole right side to be θ_∞); while the final forcing $P^{\rm fin}$ forces $\mathfrak{x} = \mathfrak{x}^{\rm fin}$ for every characteristic $\mathfrak{x}$ (on either side). I.e., the upper Cichoń's diagram shows the situation forced by $P^{\rm pre}$, and the lower diagram shows the one forced by $P^{\rm fin}$.

Regarding LCU: We fix $i \in \{1,2,3,4\}$ (the case $i = 3$, as an example, is described more explicitly below). Let us call the set of regular cardinals κ satisfying $\mathsf{LCU}_i(P,\kappa)$ the "LCU_i-spectrum of P", and let $X_i^{\rm pre}$ be the LCU_i-spectrum of $P^{\rm pre}$. So

$$\{\theta_\infty, \theta_0, \dots, \theta_{8-2i}\} \subseteq X_i^{\rm pre}$$

- In the first step $n = 0$, let us consider the LCU_i-spectrum X_i^0 of P_0: As $\theta_\infty \in X_i^{\rm pre}$, we get $\lambda_0 \in X_i^0$, and as $\theta_0, \dots, \theta_{8-2i}$ are in $X_i^{\rm pre}$, they are in X_i^0 as well (both according to Lemma 1.6(1), using $\kappa = \lambda_i$).
- For the next step $n = 1$, we similarly get that the LCU_i-spectrum X_i^1 of P_1 contains λ_0, λ_1, and, if $i \neq 4$, also $\theta_1, \dots, \theta_{8-2i}$.

- In this way we get that the final LCU_i-spectrum X_i^{fin} of P^{fin} contains $\lambda_0, \dots, \lambda_{8-2i+1}$.
- This implies (by Corollary 2.3) that P^{fin} forces

$$\mathfrak{b}_i \leq \min(\lambda_0, \dots, \lambda_{8-2i+1}) = \lambda_{8-2i+1} \text{ and } \mathfrak{d}_i \geq \max(\lambda_0, \dots, \lambda_{8-2i+1}) = \lambda_{8-2i}.$$

So we get half of the desired inequalities.

This may be more transparent if we consider an explicit example, say $i = 3$. In each line of the following table, each cardinal in the right column is guaranteed to be an element of the LCU_3 spectrum of the forcing notion in the left column:

P^{pre}			θ_∞,	θ_0,	θ_1,	θ_2
P_0	$=$	$P^{\mathrm{pre}} \cap N_0$	λ_0,	θ_0,	θ_1,	θ_2
P_1	$=$	$P^{\mathrm{pre}} \cap N_0 \cap N_1$	λ_0,	λ_1,	θ_1,	θ_2
P_2	$=$	$P^{\mathrm{pre}} \cap N_0 \cap N_1 \cap N_2$	λ_0,	λ_1,	λ_2,	θ_2
P_3	$=$	$P^{\mathrm{pre}} \cap N_0 \cap N_1 \cap N_2$	λ_0,	λ_1,	λ_2,	λ_3
$\vdots$				$\vdots$		
P^{fin}			λ_0,	λ_1,	λ_2,	λ_3

Since λ_3 is the smallest of these 4 cardinals, and λ_2 the largest, we get that P^{fin} forces $\mathfrak{b}_i \leq \lambda_3$ and $\mathfrak{d}_i \geq \lambda_2$.

Regarding COB: Again we fix $i \in \{1, 2, 3, 4\}$. Let $m := 8 - 2i$. In particular, $0 \leq m \leq 6$, m is even, so according to (3.2) we have $\lambda_{m+1} \leq \lambda_m$ and

$$\lambda_m = \max_{0 \leq n \leq m} (\lambda_n) = \max_{0 \leq n \leq m+1} (\lambda_n). \tag{3.4}$$

Recall that $\mathsf{COB}_i(P^{\mathrm{pre}}, S_i)$ holds where $\mathrm{comp}(S_i) = \theta_m$ and $\mathrm{cof}(S_i) = \theta_\infty$ (cf. Theorem 2.4 and (3.3)).

We claim that

$$T := S_i \cap N_0 \cap \dots \cap N_{m+1}$$

satisfies

$$\mathrm{comp}(T) \geq \min_{0 \leq n \leq m+1} (\lambda_n) = \lambda_{m+1} \quad \text{and} \quad \mathrm{cof}(T) \leq \max_{0 \leq n \leq m+1} (\lambda_n) = \lambda_m.$$

Completeness is clear by applying Lemma 1.6(2) iteratively: $\mathrm{comp}(S_i) > \lambda_0$, so $\mathrm{comp}(S_i \cap N_0) \geq \lambda_0$. Then $\mathrm{comp}(S_i \cap N_0 \cap N_1) \geq \min\{\lambda_0, \lambda_1\}$, and so on.

Regarding the cofinality:

- Let Λ be the product $\prod_{n=0}^{m} \lambda_n$. So $|\Lambda| = \lambda_m$ by (3.4).
  For $\eta \in \Lambda$, set $N^\eta := \bigcap_{n=0}^{m} N_{n,\eta(n)}$. Note that $N_0 \cap \dots \cap N_m = \bigcup_{\eta \in \Lambda} N^\eta$, and that Λ is an element, and thus a subset, of each elementary submodel.[10]
- For $\eta \in \Lambda$, set $T_\eta := S_i \cap N^\eta$. Since N_η is $<\theta_m$-closed and $\mathrm{comp}(S_i) \geq \theta_m$, we get $\mathrm{comp}(T_\eta) \geq \theta_m > \theta_{m+1}$. Hence, by Lemma 1.6(4) applied to $N = N_{m+1}$, $S = T_\eta$, $\kappa = \lambda = \lambda_{m+1}$ and $\theta = \theta_{m+1}$, we conclude $\mathrm{cof}(T_\eta \cap N_{m+1}) = \lambda_{m+1}$.
  Choose $C_\eta \subseteq T_\eta$ cofinal in $T_\eta \cap N_{m+1}$ of size λ_{m+1}. Hence, $C := \bigcup_{\eta \in \Lambda} C_\eta$ is cofinal in T because $T = \bigcup_{\eta \in \Lambda} T_\eta \cap N_{m+1}$, so $\mathrm{cof}(T) \leq |C| \leq |\Lambda| \cdot \lambda_{m+1} = \lambda_m \cdot \lambda_{m+1} = \lambda_m$ by (3.4).

[10]Element is clear, as all N's contain the sequence of λ's. Subset follows from the fact that each N contains λ_∞ and thus λ_m as a subset, and that $|\Lambda| = \lambda_m$.

Now we show, by induction on $n \geq m+1$, that $S_i \cap N_0 \cap \cdots \cap N_n$ has completeness $\geq \lambda_{m+1}$ and cofinality $\leq \lambda_m$. The step $n = m+1$ was done above; for the steps $n > m+1$, by induction we know that $S' := S_i \cap N_0 \cap \cdots \cap N_{n-1}$ has cofinality at most λ_m and completeness at least λ_{m+1}. So by Lemma 1.6(3), the same holds for $S' \cap N_n$.

To summarize: For any $i = 1, \ldots, 4$, the cofinality of $S_i \cap N^*$ is at most λ_{8-2i}, and the completeness at least λ_{8-2i+1}. By Lemmas 1.5(2) and 2.3(2) we get

$$P^{\text{fin}} \Vdash \mathfrak{b}_i \geq \lambda_{8-2i+1} \ \& \ \mathfrak{d}_i \leq \lambda_{8-2i}.$$

So we get the remaining inequalities we need.

Regarding the continuum: There is a sequence $(x_\xi)_{\xi<\theta_\infty}$ of (nice) P^{pre}-names of reals that are forced to be pairwise different due to absoluteness (1.4). Note that this sequence belongs to N^*, so $(x_\xi)_{\xi\in\theta_\infty\cap N^*}$ is a sequence of P^{fin}-names of reals that are forced (by P^{fin}) to be pairwise different. Hence, P^{fin} forces $\mathfrak{c} \geq |\theta_\infty \cap N^*| = \lambda_\infty$.[11] The converse inequality also holds because $|P^{\text{fin}}| = \lambda_\infty = \lambda_\infty^{\aleph_0}$. □

Remark 3.5. If we base the left-hand forcing P^{pre} on [BCM18] (see Remark 2.5), then our proof (when we change item (3) on p. 8 to the assumptions listed in Remark 2.5) shows that GCH can be weakened to the following: There are at least 9 cardinals $\theta > \mu_9$ satisfying $\theta^{<\theta} = \theta$. Or, to be even be more pedantic: There are regular cardinals $\theta_7 < \ldots < \theta_0 < \theta_\infty$ larger than μ_9 such that $\theta_7^{<\mu_1} = \theta_7$, $\theta_\infty^{<\theta_2} = \theta_\infty$ and $\theta_i^{\theta_{i+1}} = \theta_i$ for $i \neq 7, \infty$.

4. Variants

4.1. **Another order.** The paper [KST19] constructs (assuming GCH) a ccc forcing notion P which forces another ordering of the left hand side. More concretely, P is ccc and it has LCU and COB witnesses for the following:[12]

$$\text{add}(\mathcal{N}) < \mathfrak{b} < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \mathfrak{c}.$$

If we use this forcing P instead of P^{pre}, then the same argument shows that we can find a complete subforcing P^{fin} that extends the order to the right hand side:

Theorem 4.1. *Assume GCH and let $(\mu_n)_{1\leq n\leq 9}$ be a weakly increasing sequence of cardinals with μ_n regular for $n \leq 8$ and $\mu_9^{\aleph_0} = \mu_9$. Then there is a ccc poset P^{fin} forcing that*

$$\aleph_1 \leq \text{add}(\mathcal{N}) = \mu_1 \leq \mathfrak{b} = \mu_2 \leq \text{cov}(\mathcal{N}) = \mu_3 \leq \text{non}(\mathcal{M}) = \mu_4 \leq$$
$$\leq \text{cov}(\mathcal{M}) = \mu_5 \leq \text{non}(\mathcal{N}) = \mu_6 \leq \mathfrak{d} = \mu_7 \leq \text{cof}(\mathcal{N}) = \mu_8 \leq \mathfrak{c} = \mu_9.$$

[11] This argument can be written in terms of the LCU property for the identity relation on ω^ω: As $\mathsf{LCU}_{\text{Id}}(P^{\text{pre}}, \kappa)$ holds for all regular $\kappa \leq \lambda_\infty$ (even up to θ_∞), we get $\mathsf{LCU}_{\text{Id}}(P^{\text{fin}}, \kappa)$ for all these cardinals, which implies $\lambda_\infty \leq \mathfrak{c}$.

[12] In (2.1), the order/numbering of $(\mathfrak{b}, \mathfrak{d})$ and $(\text{cov}(\mathcal{N}), \text{non}(\mathcal{N}))$ is swapped; for this new ordering we again get Theorem 2.4. We use the same R^{LCU}- and R^{COB}-relations as in [GKS19], except for the R^{LCU}-relation for the pair $(\text{cov}(\mathcal{N}), \text{non}(\mathcal{N}))$: Now we have to use a relation which is an ω_1-union of Borel relations (which was originally defined in [KO14] and fit into a formal preservation framework in [CM19]; see details in [KST19, Def. 2.3]). This is the only place in this paper where we have to use a non-Borel relation R; but this is no problem as R is sufficiently absolute in the sense described after (1.4).

Remark 4.2. As in Remark 3.5, full GCH is not needed, but it is enough that there are 9 regular cardinals larger than μ_9 satisfying some arithmetical properties. However, it is not enough that $\theta^{<\theta} = \theta$ for these 9 cardinals, but it is required in addition that one of them is $\aleph_1$-inaccessible.[13] For details, refer to [Mej19b, GKMS19].

4.2. **A weaker notion than COB sufficient for the proof.** Several papers about constellations of Cichoń's diagram preceding [GKS19, BCM18], such as [Bre91, Mej13, GMS16], have considered similar, but simpler, forcing constructions. While LCU witnesses are added in the same way, these do not provide for COB. Instead, a weaker property, which we call DOM below, is implicit in these constructions. We now show that this notion is sufficient to carry out the proof of the main result.

Definition 4.3. Let R be a relation on ω^ω and let κ be a cardinal.

1. A set $A \subseteq \omega^\omega$ is *κ-R-dominating* if, whenever $F \subseteq \omega^\omega$ has size $<\kappa$, there is some real $a \in A$ dominating over F, that is, $(\forall x \in F) x R a$. Dually, we say that A is *κ-R-unbounded* if it is κ-$R^\perp$-dominating.
2. Assume that R is sufficiently absolutely defined and let P be a forcing notion. We define $\mathsf{DOM}_R(P, \kappa, S)$ to mean the following: There is a sequence $(f_\alpha)_{\alpha\in S}$ of P-names of reals such that, whenever $\gamma < \kappa$ and $(x_\xi)_{\xi<\gamma}$ is a sequence of P-names of reals, there is some $\alpha \in S$ such that $P \Vdash (\forall \xi < \gamma) x_\xi R f_\alpha$.

(Note that $\mathsf{DOM}_R(P, \kappa, S)$ is stronger than just saying "P adds a κ-R-dominating family".)

The following is straightforward:

- $\mathsf{COB}_R(P, S)$ implies $\mathsf{DOM}_R(P, \mathrm{comp}(S), \mathrm{cof}(S))$.
- If κ is regular then $\mathsf{LCU}_R(P, \kappa)$ implies $\mathsf{DOM}_{R^\perp}(P, \kappa, \kappa)$.
- $\mathsf{DOM}_R(P, \kappa, S)$ implies $P \Vdash \big(\kappa \le \mathfrak{b}_R \;\&\; \mathfrak{d}_R \le |S|\big)$.
  (This generalizes Lemma 1.3.)

For this weaker notion we have the following result similar to Lemma 1.6.

Lemma 4.4. *With the same hypothesis as in Lemma 1.6, assuming also $\nu \in N$:*

(1) $\mathsf{DOM}_R(P, \nu, S)$ *implies* $\mathsf{DOM}_R(P \cap N, \min\{\kappa, \nu\}, S \cap N)$.
(2) If $|S| \le \theta$ *then* $\mathsf{DOM}_R(P, \nu, S)$ *implies* $\mathsf{DOM}_R(P \cap N, \nu, S \cap N)$.
(3) If $\nu > \theta$ *then* $\mathsf{DOM}_R(P, \nu, S)$ *implies* $\mathsf{COB}_R(P \cap N, \lambda)$.
In particular, if S is directed and $\mathrm{comp}(S) > \theta$ *then* $\mathsf{COB}_R(P, S)$ *implies* $\mathsf{COB}_R(P \cap N, \lambda)$.

Proof. In the following, assume that $(f_\alpha)_{\alpha\in S}$ witnesses $\mathsf{DOM}_R(P, \nu, S)$.

(1) If $(x_\xi)_{\xi<\gamma}$ is a sequence of P-names of reals and $\gamma < \min\{\kappa, \nu\}$ then the sequence is in N, so there is some $\alpha \in S \cap N$ such that $P \Vdash x_\xi R f_\alpha$ for all $\xi < \gamma$. By absoluteness, $P \cap N$ forces the same.

(2) is clear because $S \subseteq N$ (as $|S| \le \theta$ and $S \in N$).

(3) Fix $i < \lambda$. Since $|N_i| \le \theta < \nu$, there is some $\alpha_i \in S$ such that $P \Vdash x R f_{\alpha_i}$ for all $x \in N_i$ that are P-names for reals. In fact, we can find such α_i in $S \cap N_{i+1}$. Hence, $(f_{\alpha_i})_{i<\lambda}$ witnesses $\mathsf{COB}_R(P \cap N, \lambda)$. □

As in [Mej19b], a simpler version of P^{pre} can be constructed in such a way that

[13] Recall that a cardinal θ is *κ-inaccessible* if $\mu^\nu < \theta$ for every $\mu < \theta$ and $\nu < \kappa$.

(a) of Theorem 2.4 holds, and
(b') For $i = 1, 2, 3, 4$ there is some set S_i of size μ_∞ such that $\mathsf{DOM}_{R_i^{\mathsf{COB}}}(P, \mu_i, S_i)$ holds.

Thanks to Lemma 4.4 (in particular item (3)), the same proof of Theorem 3.1 can be carried out in this simpler context.

5. Open questions

[GKS19, Sect. 3] asks the following questions: Can you show the consistency of Cichoń's maximum ...

✓(a) ... without using large cardinals?
✓(b) ... for specific (regular) values, such as $\mu_i = \aleph_{i+1}$?
∼(c) ... for other orderings of the ten entries?
∼(d) ... together with further distinct values of additional ("classical") cardinal characteristics?

This work, more concretely Theorem 3.1, solves questions (a) and (b).
A first result for (d), namely adding $\aleph_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} < \mathrm{add}(\mathcal{N})$, is done in [GKMS19] (which also gives a more complicated construction to achieve (b)).

Of course, it would be interesting to add more characteristics. For example, we can ask:

Question 1. *Can we add the splitting number $\mathfrak{s}$ and the reaping number $\mathfrak{r}$?*

The pair $(\mathfrak{s}, \mathfrak{r})$ might be most promising among the classical characteristics, as it is of the form $(\mathfrak{b}_R, \mathfrak{d}_R)$ for a Borel relation R which is well understood.

Question (c) remains largely open. There are four possible configurations where $\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$, and at the moment only 2 are known to be consistent (see Theorems 3.1 and 4.1).

Question 2. *Are the following two constellations consistent?*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < \mathfrak{c},$$

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}.$$

It is not clear whether our method in Section 3 can be applied to solve this question (the same applies to Boolean ultrapowers), since we start with a poset forcing an order of the left side of Cichoń's diagram and our method only manages to *dualize* this order to the right side (e.g., if on the left we force $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$, then on the right we can only expect to force the dual inequality $\mathfrak{d} < \mathrm{non}(\mathcal{N})$).
The case when $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M})$ seems to be more complex.[14] We do not even know how to force the consistency of $\aleph_1 < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M})$. J. Brendle however does, see [Bre19] for slides of a presentation of his method of "shattered iterations". Brute force counting shows that there are 57 configurations of ten different values in Cichoń's diagram (satisfying the obvious inequalities) where $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M})$, but none of them have been proved to be consistent so far.

Question 3. *Is any constellation of Cichoń's maximum consistent where* $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M})$*?*

[14] Recall that finite support iterations add Cohen reals at limit steps, so they force $\mathrm{non}(\mathcal{M}) \le \mathrm{cov}(\mathcal{M})$ (when the length has uncountable cofinality).

References

[Bar84] Tomek Bartoszyński, *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. **281** (1984), no. 1, 209–213.

[BCM18] Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía, *Filter-linkedness and its effect on preservation of cardinal characteristics*, preprint, arXiv:1809.05004, 2018.

[BJ95] Tomek Bartoszyński and Haim Judah, *Set theory*, A K Peters, Ltd., Wellesley, MA, 1995, On the structure of the real line.

[BJS93] Tomek Bartoszyński, Haim Judah, and Saharon Shelah, *The Cichoń diagram*, J. Symbolic Logic **58** (1993), no. 2, 401–423.

[Bre91] Jörg Brendle, *Larger cardinals in Cichoń's diagram*, J. Symbolic Logic **56** (1991), no. 3, 795–810.

[Bre19] Brendle, Jörg, *Forcing and cardinal invariants, parts 1 and 2*, tutorial at Advanced Class - Young Set Theory Workshop 2019, July 28th and 29th 2019, Slides: https://sites.google.com/view/estc2019/advanced-class-yst/program.

[CKP85] J. Cichoń, A. Kamburelis, and J. Pawlikowski, *On dense subsets of the measure algebra*, Proc. Amer. Math. Soc. **94** (1985), no. 1, 142–146.

[CM19] Miguel A. Cardona and Diego A. Mejía, *On cardinal characteristics of Yorioka ideals*, Mathematical Logic Quarterly **65** (2019), no. 2, 170–199.

[Coh63] Paul Cohen, *The independence of the continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148.

[FFMM18] Vera Fischer, Sy D. Friedman, Diego A. Mejía, and Diana C. Montoya, *Coherent systems of finite support iterations*, J. Symb. Log. **83** (2018), no. 1, 208–236.

[FGKS17] Arthur Fischer, Martin Goldstern, Jakob Kellner, and Saharon Shelah, *Creature forcing and five cardinal characteristics in Cichoń's diagram*, Arch. Math. Logic **56** (2017), no. 7-8, 1045–1103.

[Git19] Moti Gitik, *A remark on large cardinals in "Cichon Maximum"*, unpublished note, April 18th 2019, http://www.math.tau.ac.il/~gitik/sstr.pdf.

[GKMS19] Martin Goldstern, Jakob Kellner, Diego Mejía, and Saharon Shelah, *Controlling cardinal characteristics without adding reals*, preprint, arXiv:1904.02617v1, 2019.

[GKS19] Martin Goldstern, Jakob Kellner, and Saharon Shelah, *Cichoń's maximum*, Ann. of Math. **190** (2019), no. 1, 113–143, arXiv:1708.03691.

[GMS16] Martin Goldstern, Diego Alejandro Mejía, and Saharon Shelah, *The left side of Cichoń's diagram*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 4025–4042.

[Göd40] Kurt Gödel, *The Consistency of the Continuum Hypothesis*, Annals of Mathematics Studies, no. 3, Princeton University Press, Princeton, N. J., 1940.

[JS90] Haim Judah and Saharon Shelah, *The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing)*, J. Symbolic Logic **55** (1990), no. 3, 909–927.

[Kam89] Anastasis Kamburelis, *Iterations of Boolean algebras with measure*, Arch. Math. Logic **29** (1989), no. 1, 21–28.

[KO14] Shizuo Kamo and Noboru Osuga, *Many different covering numbers of Yorioka's ideals*, Arch. Math. Logic **53** (2014), no. 1-2, 43–56.

[KST19] Jakob Kellner, Saharon Shelah, and Anda Tănasie, *Another ordering of the ten cardinal characteristics in cichoń's diagram*, Comment. Math. Univ. Carolin. **60** (2019), no. 1, 61–95.

[KTT18] Jakob Kellner, Anda Ramona Tănasie, and Fabio Elio Tonti, *Compact cardinals and eight values in Cichoń's diagram*, J. Symb. Log. **83** (2018), no. 2, 790–803.

[Kun11] Kenneth Kunen, *Set theory*, Studies in Logic (London), vol. 34, College Publications, London, 2011.

[Mej13] Diego Alejandro Mejía, *Matrix iterations and Cichon's diagram*, Arch. Math. Logic **52** (2013), no. 3-4, 261–278.

[Mej19a] Diego A. Mejía, *Matrix iterations with vertical support restrictions*, Proceedings of the 14th and 15th Asian Logic Conferences (Byunghan Kim, Jörg Brendle, Gyesik Lee, Fenrong Liu, R Ramanujam, Shashi M Srivastava, Akito Tsuboi, and Liang Yu, eds.), World Sci. Publ., 2019, pp. 213–248.

[Mej19b] Diego A. Mejía, *A note on "Another ordering of the ten cardinal characteristics in Cichoń's Diagram" and further remarks*, Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku **2141** (2019), 1–15, arXiv:1904.00165.

[Mil81] Arnold W. Miller, *Some properties of measure and category*, Trans. Amer. Math. Soc. **266** (1981), no. 1, 93–114.

[Mil84] ———, *Additivity of measure implies dominating reals*, Proc. Amer. Math. Soc. **91** (1984), no. 1, 111–117.

[RS83] Jean Raisonnier and Jacques Stern, *Mesurabilité et propriété de Baire*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 7, 323–326.

[RS85] ———, *The strength of measurability hypotheses*, Israel J. Math. **50** (1985), no. 4, 337–349.

Institut für Diskrete Mathematik und Geometrie, TU Wien, 1040 Vienna, Austria.
Email address: martin.goldstern@tuwien.ac.at
URL: http://www.tuwien.ac.at/goldstern/

Institut für Diskrete Mathematik und Geometrie, TU Wien, 1040 Vienna, Austria.
Email address: kellner@fsmat.at
URL: http://dmg.tuwien.ac.at/kellner/

Creative Science Course (Mathematics), Faculty of Science, Shizuoka University, Ohya 836, Suruga-ku, Shizuoka-shi, Japan 422-8529.
Email address: diego.mejia@shizuoka.ac.jp
URL: http://www.researchgate.com/profile/Diego_Mejia2

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA.
Email address: shlhetal@math.huji.ac.il
URL: http://shelah.logic.at