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Infinitary Logics and A.E.C.

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Abstract

We prove here that every a.e.c. with LST number $\leqslant \kappa$ and vocabulary τ of cardinality $\leqslant \kappa$ can be defined in the logic $\mathbb{L}_{\beth_2(\kappa)^+,\kappa^+}(\tau)$. In this logic an a.e.c. is therefore an EC class unlike in the Presentation Theorem. Furthermore, we study a connection between the sentences defining an a.e.c. and the infinitary logic L^1_κ.

Introduction

The Presentation Theorem [5] is central to the development of stability for abstract elementary classes: notably, it enables Ehrenfeucht-Mostowski techniques for classes that have large enough models. This has as almost immediate consequence the link between categoricity and stability, and constitutes the beginning of relatively advanced classification/stability theory in that wider setting.

The Presentation Theorem provides a way to capture an a.e.c. as a PC-class: by expanding its vocabulary of the AEC with infinitely many function symbols, an a.e.c. may be axiomatized by an infinitary formula. Although for the stability-theoretical applications mentioned this expansion is harmless, the question as to whether it is possible to axiomatize an a.e.c. with a (necessarily infinitary) sentence *in the same vocabulary* of the a.e.c. has been asked in various ways in the past. Here we provide a positive solution: given an a.e.c $\mathcal{K}$ we provide an infinitary sentence *in the same original vocabulary* $\varphi_{\lambda+2,0}$ whose models are exactly those in $\mathcal{K}$. Therefore, unlike the situation

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in the Presentation Theorem, here the class turns out to be an EC Class, not a PC class.

The main idea is that a "canonical tree of models", each of size the LST-number of the class, the tree of height ω ends up providing enough tools; the sentence essentially describes all possible maps from elements of this tree into arbitrary potential models in the class. A combinatorial device (a partition theorem theorem on well-founded trees due to Komjath and the first author [3]) is necessary for our proof.

We prove the following two theorems:

Theorem (Theorem 2.1)**.** *Let* $\lambda = \beth_2(|\tau| + \kappa)^+$, *where* $\kappa = \mathrm{LST}(\mathcal{K})$. *Then there is a sentence* $\psi_{\mathcal{K}}$ *in the logic* $\mathbb{L}_{\lambda^+,\kappa^+}(\tau)$ *such that* $\mathcal{K} = \mathrm{Mod}(\psi_{\mathcal{K}})$.

and

Theorem (Here, a reduced version of Theorem 3.1)**.** *If* $M_1 \subseteq M_2$ *are* $\tau = \tau_{\mathcal{K}}$*-structures, then TFAE:*

- $M_1 \prec_{\mathcal{K}} M_2$
- *if* $\bar{a} \in {}^{\kappa\geqslant}(M_1)$ *then there are* $\bar{b}$, N *and* f *such that*
  1. $\bar{b} \in {}^{\kappa\geqslant}(M_1)$ *and* $N \in \mathcal{M}_1$
  2. $\mathrm{Rang}(\bar{a}) \subseteq \mathrm{Rang}(\bar{b})$
  3. f *is an isomorphism from* N *onto* $M_1 \restriction \mathrm{Rang}(\bar{b})$
  4. $M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa\rangle]$.

1 Canonical trees for a.e.c.'s

Fix $\mathcal{K}$ for the remainder of this paper an a.e.c. with vocabulary $\tau = \tau(\mathcal{K})$ and $\mathrm{LST}(\mathcal{K}) = \kappa$. Let λ be the cardinal $\beth_2(\kappa + |\tau|)^+$.

Without loss of generality we may assume that all models in $\mathcal{K}$ are of cardinality $\geqslant \kappa$. Furthermore, we will use an "empty model" called M_{empt} with the property that $M_{\mathrm{empt}} \prec_{\mathcal{K}} M$ for all $M \in \mathcal{K}$.

Notation 1.1. *We fix the following notation, models and elements in the rest of this paper.*

- *We first fix a sequence of (different) elements* $(a^*_\alpha \mid \alpha < \kappa \times \omega)$ *in some model in* $\mathcal{K}$.
- *Given a model* M, *we denote by* $\bar{\alpha}_M$ *a sequence of ordinals* $\langle \alpha_k[M] \mid k < n\rangle$ *for some* $n < \omega$, *where for each* $k < n$, $\alpha_k[M] < \kappa$.

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- $S_{\bar{\alpha}[M]} := \bigcup_{k<n}[\kappa \times k, \kappa \times k + \alpha_k[M])$.

We now define the **canonical tree** of $\mathcal{K}$:

- $\mathcal{M}_n := \{M \in \mathcal{K} \mid$ for some $\bar{\alpha} = \bar{\alpha}_M$ of length n, M has universe $\{a^*_\alpha \mid \alpha \in S_{\bar{\alpha}[M]}\}$ and $m < n \Rightarrow M \restriction S_{\bar{\alpha}\restriction m[M]} \prec_{\mathcal{K}} M\}$ (and $\mathcal{M}_0 = \{M_{\mathrm{empt}}\}$),
- $\mathcal{M} = \mathcal{M}_{\mathcal{K}} := \bigcup_n \mathcal{M}_n$; this is a tree with ω levels under $\prec_{\mathcal{K}}$ (equivalenty under $\subseteq$).

And some further notation for the rest of the proof:

Notation 1.2. *1.* $\bar{x}_n := \langle x_\alpha \mid \alpha < \kappa \times n\rangle$,

2. $\bar{x}_{=n} := \langle x_\alpha \mid \alpha \in [\kappa \times n, \kappa \times (n+1))\rangle$.

We now define by induction on $\gamma < \lambda^+$ formulas

$$\varphi_{M,\gamma,n}(\bar{x}_n),$$

for every n and $M \in \mathcal{M}_n$ (when $n = 0$ we may omit M).

Case 1 : $\gamma = 0$

If $n = 0$ then the formula $\varphi_{0,0}$ is $\top$ (the sentence denoting "truth"). Assume $n > 0$. Then

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

where $\mathrm{Diag}^n_\kappa(M)$ is the set $\{\varphi(x_{\alpha_0}, \dots, x_{\alpha_{n-1}}) \mid \alpha_0, \dots, \alpha_{n-1} \in S_{\bar{\alpha}[M]}$, $\varphi(y_0, \dots, y_{n-1})$ is an atomic or a negation of an atomic formula and $M \models \varphi(a^*_{\alpha_0}, \dots, a^*_{\alpha_{n-1}})\}$.

Case 2 : γ a limit ordinal

Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

Case 3 : $\gamma = \beta + 1$

Let $\varphi_{M,\gamma,n}(\bar{x}_n)$ be the formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{M}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta\right]$$

Note: all the formulas constructed belong to $\mathbb{L}_{\lambda^+,\kappa^+}(\tau)$. When $n = 0$ our formulas are really *sentences* $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$. These sentences may be understood as "external approximations" to the a.e.c. $\mathcal{K}$. Our first aim is to prove how these approximations end up characterizing the a.e.c. $\mathcal{K}$.

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2 Characterizing $\mathcal{K}$ by its canonical sentence

In this section we prove the first main theorem:

Theorem 2.1. *There is a sentence $\psi_{\mathcal{K}}$ in the logic $\mathbb{L}_{\lambda^+,\kappa^+}(\tau)$ such that $\mathcal{K} = \mathrm{Mod}(\psi_{\mathcal{K}})$.*

Our first aim in this section is to prove that every model $M \in \mathcal{K}$ satisfies $\varphi_{\gamma,0}$, for all $\gamma < \lambda^+$.

In order to achieve this, we prove the following (more elaborate) statement, by induction on γ.

Claim 2.2. *If $M \in \mathcal{K}$, $n < \omega$, $N \in \mathcal{M}_n$, $f : N \to M$ is a $\prec_{\mathcal{K}}$-embedding (if $n = 0$, f is empty) <u>then</u> $M \models \varphi_{N,\gamma,n}[\langle f(a^*_\alpha \mid \alpha < \kappa \times n)\rangle]$.*

Proof Let first $\gamma = 0$. Then we have either $n = 0$ in which case trivially $M \models \varphi_{0,0}(= \top)$ or $n > 0$. In the latter case $\varphi_{N,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(N)$; if $f : N \to M$ is a $\prec_{\mathcal{K}}$-embedding, M satisfies this sentence as it satisfies each of the formulas $\varphi(y_0, \dots y_{n-1})$ satisfied in N by the images of the $\prec_{\mathcal{K}}$-map f.

The case γ limit ordinal is an immediate consequence of the induction hypothesis.

Let now $\gamma = \beta + 1$ and assume that for every $M \in \mathcal{K}$, $n < \omega$, $N \in \mathcal{M}_n$, if $f : N \to M$ is a $\prec_{\mathcal{K}}$-embedding then $M \models \varphi_{N,\beta,n}[\langle f(a^*_\alpha \mid \alpha < \kappa \times n)\rangle]$. Now, fix $M \in \mathcal{K}$, $n < \omega$, $N \in \mathcal{M}_n$ and $f : N \to M$ a $\mathcal{K}$-embedding. We want to check that $M \models \varphi_{N,\gamma,n}[\langle f(a^*_\alpha) \mid \alpha < \kappa \times n\rangle]$, i.e. we need to verify that

$$M \models \forall \bar z_{[\kappa]} \bigvee_{\substack{N' \succ_{\mathcal{K}} N \\ N' \in \mathcal{M}_{n+1}}} \exists \bar x_{=n} \Big[\varphi_{N',\beta,n+1}(\bar x_n {}^\frown \bar x_{=n}) \wedge \bigwedge_{\alpha < \alpha_{n+1}[N']} \bigvee_{\delta \in S[N']} z_\alpha = x_\delta\Big]$$

when $\bar x_n$ is replaced in M by $\langle f(a^*_\alpha) \mid \alpha < \kappa \times n\rangle$.

So let $\bar c_{[\kappa]} \in M$. By the LST axiom, there is some $M' \prec_{\mathcal{K}} M$ containing both $\bar c_{[\kappa]}$ and $\langle f(a^*_\alpha) \mid \alpha < \kappa \times n\rangle$, with $|M'| = \kappa$. By the isomorphism axioms there is $N' \succ_{\mathcal{K}} N$, $N' \in \mathcal{M}_{n+1}$, isomorphic to M' through an isomorphism f' extending f. We may now apply the induction hypothesis to N', f': since $f' : N' \to M$ is a $\prec_{\mathcal{K}}$-embedding, we have that $M \models \varphi_{N',\beta,n+1}[\langle a^*_\alpha \mid \alpha < \kappa \times (n+1)\rangle]$. But this enables us to conclude: N' is a witness in the disjunction on models $\prec_{\mathcal{K}}$-extending N, and the existential $\exists \bar x_{=n}$ is witnessed by $\langle a^*_\alpha \mid \alpha \in [\kappa \times n, \kappa \times (n+1))\rangle$. As the original M' had been chosen to include the sequence $\bar c_{[\kappa]}$, the last part of the formula holds. $\square_{\text{Claim 2.2}}$

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Now we come to the main point:

Claim 2.3. *If* M *is a* τ*-model and* $M \models \varphi_{\lambda+2,0}$ *then* $M \in \mathcal{K}$.

PROOF Let $\mathcal{N} := \{N_* \subseteq M \mid N_*$ has cardinality κ and for some $N \in \mathcal{M}_1$ there is a bijective $f : \kappa \to N_*$ such that $M \models \varphi_{N,\lambda,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa\rangle]\}$. In particular, the previous f is an isomorphism from N to N_*.

We prove first

$$N^*_1 \subseteq N^*_2 \quad (N^*_\ell \in \mathcal{N}) \quad \text{ then } N^*_1 \prec_{\mathcal{K}} N^*_2. \tag{1}$$

To see this, choose $(N^\ell_\eta, f^\ell_\eta)$ for $\ell = 1, 2$ and $\eta \in \mathrm{ds}(\lambda) := \{\nu \mid \nu$ a decreasing sequence of ordinals $< \lambda\}$ by induction on $\ell g(\eta)$ such that

1. $N^\ell_\eta \in \mathcal{M}_{\ell g(\eta)+1}$
2. f^ℓ_η embeds N^ℓ_η into M: $f^\ell_\eta(N^\ell_\eta) \subseteq M$
3. $M \models \varphi_{N^\ell_\eta,\mathrm{last}(\eta),\ell g(\eta)+1}[\langle f^\ell_\eta(a^*_\alpha \mid \alpha < \kappa \times (\ell g(\eta) + 1))\rangle]$ where $\mathrm{last}(\langle\rangle) = \lambda$, $\mathrm{last}(\nu^\frown\langle\alpha\rangle) = \alpha$
4. if $\nu \triangleleft \eta$ then $N^\ell_\nu \prec_{\mathcal{K}} N^\ell_\eta$ and $f^\ell_\nu \subseteq f^\ell_\eta$
5. if $\ell g(\eta) = 0$ then $f^\ell_\eta(N^\ell_\eta) = N^*_\ell$
6. $N^1_\eta \subseteq N^2_\eta$ and $\nu \triangleleft \eta \Rightarrow N^2_\nu \subseteq N^1_\eta$.

Carrying the induction is standard: if $\ell g(\eta) = 0$ let f^ℓ_η be a one-to-one function from $a^*_\alpha \mid \alpha < \kappa$ onto N^*_ℓ; as $\|N^*_\ell\| = \kappa$ there is a model N^ℓ_η with universe $a^*_\alpha \mid \alpha < \kappa$ such that f^ℓ_η is an isomorphism from N^ℓ_η onto $N^*\ell$.
If $\ell g(\eta) = n = m + 1$ we first choose (f^1_η, N^1_η). We use

for $\ell = 1$ and from the inductive definition of $\varphi_{N^1_{\eta\restriction m,\mathrm{last}(\eta\restriction m),m}}$ with $\bar{z}_{[\kappa]} \mapsto \langle f^2_{\eta\restriction m}(a^*_\alpha) \mid \alpha < \kappa \times m\rangle$ (using renaming), the $\bar{x}_{=n}$ gives us the map f^1_η, with domain N^1_η

Now to choose (f^2_η, N^2_η) we use the inductive definition of $\varphi_{N^2_{\eta\restriction m,\mathrm{last}(\eta\restriction m),m}}$ with $\bar{z}_{[\kappa]} \mapsto \langle f^1_{\eta\restriction n}(a^*_\alpha) \mid \alpha < \kappa \times n\rangle$; as before, the sequence $\bar{x}_{=n}$ gives us the map f^2_η, with domain N^2_η.

Let us now check why having carried the induction suffices.

We apply the partition theorem on well founded trees due to Komjath and the first author [3]. In [2], Gruenhut and the first author provide the following useful form.

Theorem 2.4 (Komjath-Shelah, [3])**.** *Let* α *be an ordinal and* μ *a cardinal. Set* $\lambda = \left(|\alpha|^{\mu^{\aleph_0}}\right)^+$ *and let* $F(\mathrm{ds}(\lambda^+)) \to \mu$ *be a colouring of the tree of finite*

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descending sequences of ordinals $< \lambda$. *Then there are an embedding* $\varphi : \mathrm{ds}(\alpha) \to \mathrm{ds}(\lambda)$ *and a function* $c : \omega \to \mu$ *such that for every* $\eta \in \mathrm{ds}(\alpha)$ *of length* $n+1$

$$F(\varphi(\eta)) = c(n).$$

We apply it with number of colours μ equal to $\kappa^{|\tau|+\kappa} = 2^\kappa$; therefore $(2^\kappa)^{\aleph_0} = 2^\kappa$. We thus obtain a sequence $(\eta_n)_{n<\omega}$, $\eta_n \in \mathrm{ds}(\lambda)$ such that:

$$k \leqslant m \leqslant n, \ell \in \{1,2\} \Rightarrow N^\ell_{\eta_m \restriction k} = N^\ell_{\eta_n \restriction k}.$$

We therefore obtain $(N^\ell_k, g^\ell_{k,n})_{k \leqslant n}$ such that

- $N^1_k \subseteq N^2_k \subseteq N^1_{k+1}$
- $g^\ell_{k,n}$ is an isomorphism from N^ℓ_k onto $N^\ell_{\eta_n \restriction k}$
- $g^1_{k,n} \subseteq g^2_{k,n} \subseteq g^1_{k+1,n}$.

Hence $N^\ell_n \prec_{\mathcal{K}} N^\ell_{n+1}$ and so $\langle N^\ell_n \mid n < \omega \rangle$ is $\prec_{\mathcal{K}}$-increasing. Let $N_\ell := \bigcup_n N^\ell_n$. Then clearly $N_1 = N_2$; call this model N. Since we then have $N^1_n \prec_{\mathcal{K}} N$, $N^2_n \prec_{\mathcal{K}} N$ and $N^1_n \subseteq N^2_n$ by the coherence axiom for A.E.C.s we have that $N^1_n \prec_{\mathcal{K}} N^2_n$. In particular, when $n = 0$ we get that $N^*_1 \prec_{\mathcal{K}} N^*_2$.

Finally, we also have that

$$\mathcal{N} \text{ is cofinal in } [M]^{\leqslant \kappa}. \tag{2}$$

This is true, since $M \models \varphi_{\lambda+2,0}$

Finally, putting together (1) and (2), we conclude that every τ-model M such that $M \models \varphi_{\lambda+2,0}$ must be in the class (use the union axiom for a.e.c.'s). $\square_{\text{Lemma 2.3}}$

Lastly, we complete the proof of Theorem 2.1: Claims 2.2 and 2.3 provide the definability in the class, as clearly $\varphi_{\gamma,0} \in \mathbb{L}_{\lambda^+,\kappa^+}(\tau_{\mathcal{K}})$.

$\square_{\text{Theorem 2.1}}$

3 Strong embeddings and definability

We now focus on the relation $\prec_{\mathcal{K}}$ of our a.e.c. $\mathcal{K}$: we characterize it in $\mathbb{L}_{\lambda^+,\kappa^+}$. We prove an analog of a "Tarski-Vaught" criterion for a.e.c.'s.

Theorem 3.1. *Let* $\mathcal{K}$ *be an a.e.c.,* $\tau = \tau(\mathcal{K})$, $\kappa = \mathrm{LST}(\mathcal{K})$, $\lambda = \beth_2(\kappa + |\tau|)$. *Then, given* τ*-models* $M_1 \subseteq M_2$, *the following are equivalent:*

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(A) $M_1 \prec_{\mathcal{K}} M_2$

(B) if $\bar{a}_\ell \in {}^{\kappa \geqslant}(M_\ell)$ *for* $\ell = 1, 2$ *and* $\gamma < \lambda$ *then there are* $\bar{b}_\ell$, N_ℓ *and* f_ℓ *for* $\ell = 1, 2$ *such that:*
for $\ell = 1, 2$,

- *(a)* $\bar{b}_\ell \in {}^{\kappa \geqslant}(M_\ell)$ *and* $N_\ell \in \mathcal{M}_\ell$
- *(b)* $\mathrm{Rang}(\bar{a}_\ell) \subseteq \mathrm{Rang}(\bar{b}_\ell)$
- *(c)* f_ℓ *is an isomorphism from* N_ℓ *onto* $M \restriction \mathrm{Rang}(\bar{b}_\ell)$
- *(d)* $\mathrm{Rang}(\bar{b}_1) \subseteq \mathrm{Rang}(\bar{b}_2)$
- *(e)* $N_1 \subseteq N_2$
- *(f)* $M_\ell \models \varphi_{N\ell,\gamma,\ell}[\langle f_\ell(a^*_\alpha) \mid \alpha < \kappa\ell \rangle]$.

(C) if $\bar{a} \in {}^{\kappa \geqslant}(M_1)$ *then there are* $\bar{b}$, N *and* f *such that*

- *(a)* $\bar{b} \in {}^{\kappa \geqslant}(M_1)$ *and* $N \in \mathcal{M}_1$
- *(b)* $\mathrm{Rang}(\bar{a}) \subseteq \mathrm{Rang}(\bar{b})$
- *(c)* f *is an isomorphism from* N *onto* $M_1 \restriction \mathrm{Rang}(\bar{b})$
- *(d)* $M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa \rangle \rangle]$.

Proof <u>*(A)⇒ (B)*</u>: Let $\bar{a}_\ell \in {}^{\kappa \geqslant}(M_\ell)$ for $\ell = 1, 2$ and let $\gamma < \lambda$. Choose first $N^*_1 \prec_{\mathcal{K}} M_1$ of cardinality $\leqslant \kappa$ including $\mathrm{Rang}(\bar{a}_1)$ and next, choose $N^*_2 \prec_{\mathcal{K}} M_2$ including $N_1 \cup \bar{a}_2$. Let $\bar{b}_\ell$ enumerate N^*_ℓ and let (N_1, f_1, N_2, f_2) be such that

1. $N_1 \in \mathcal{M}_1$, $N_2 \in \mathcal{M}_2$, $N_1 \subseteq N_2$ and
2. f_ℓ is an isomorphism from N_ℓ onto N^*_ℓ for $\ell = 1, 2$.

This is possible: since $M_1 \prec_{\mathcal{K}} M_2$ and $N^*_\ell \prec_{\mathcal{K}} M_\ell$ for $\ell = 1, 2$, we also have that $N^*_1 \prec_{\mathcal{K}} N^*_2$. Therefore there are corresponding models $N_1 \subseteq N_2$ in the canonical tree, at levels 1 and 2 (as these must satisfy $N_1 \prec_{\mathcal{K}} N_2$).

We then have that $f_\ell : N_\ell \to M_\ell$ is a $\mathcal{K}$-embedding from elements N_1 and N_2 in the canonical tree $\mathcal{M}$. By Claim 2.2, we may conclude that

$$M_1 \models \varphi_{N_1,\gamma,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa \rangle]$$

and

$$M_2 \models \varphi_{N_2,\gamma,2}[\langle f(a^*_\alpha) \mid \alpha < \kappa \times 2 \rangle],$$

for each $\gamma < \kappa$.

<u>*(B)⇒ (C)*</u>: let $\bar{a} \in {}^{\kappa \geqslant}(M_1)$. We need $\bar{b}$, $N \in \mathcal{M}_1$ and $f : N \to M_1 \restriction \mathrm{Rang}(\bar{b})$ such that

$$M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a^*_\alpha \mid \alpha < \kappa) \rangle]. \tag{3}$$

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(B) provides a model $N = N_1 \in \mathcal{M}_1$ and elements $\bar{b} = \bar{b}_1$, as well as an isomorphism $f : N \to \mathrm{Rang}(\bar{b})$. We now check that (B) also implies 3.

Recall the definition of $\varphi_{N,\lambda+1,1}$ (as applied to $[\langle f(a^*_\alpha \mid \alpha < \kappa)\rangle]$). This formula holds in M_2 if for every $\bar{c}_{[\kappa]}$ (of size κ) in M_2, **for some** $\prec_{\mathcal{K}}$-extension N' of N in $\mathcal{M}_2$ we have that

$$M_2 \models \exists \bar{x}_{=2} \varphi_{N',\lambda,2}[\langle f(a^*_\alpha \mid \alpha < \kappa)\rangle ^\frown \bar{x}_{=2}] \tag{4}$$

and the elements $\bar{c}_{[\kappa]}$ are "covered" by the list of elements (of length $\kappa \times 2$) $\langle f(a^*_\alpha \mid \alpha < \kappa)\rangle ^\frown \bar{x}_{=2}$. But the remaining part of clause (B) provides just this: there is *some* $N' = N_2 \in \mathcal{M}_2$, extending $N = N_1$ such that for each $\gamma < \lambda$, and an isomorphism f' from N' into some $\prec_{\mathcal{K}}$-submodel N^* of M_2 containing $\mathrm{Rang}(\bar{c}_{[\kappa]})$ such that $M_2 \models \varphi_{N',\gamma,2}[\langle f'(a^*_\alpha \mid \alpha < \kappa \times 2)\rangle]$. The submodel N' witnesses the disjunction on models and $\langle f'(a^*_\alpha) \mid \alpha \in [\kappa, \kappa \times 2)\rangle$ witnesses the existential $\bar{x}_{=2}$.

(C)⇒ (A): assuming (C) means that for every κ-tuple $\bar{a}$ from M_1 there are a model $N \in \mathcal{M}_1$, a κ-tuple $\bar{b}$ from M_1 containing $\bar{a}$ and an isomorphism from N onto $M_1 \restriction \mathrm{Rang}(\bar{b})$ such that

$$M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa\rangle].$$

This means that for each $\bar{c}$ included in M_2 (of length κ) there are some extension N' of N with $N' \in \mathcal{M}_2$ and some $\bar{d}$ included in M_2, of length κ, such that

$$M_2 \models \varphi_{N',\lambda,2}[\langle f(a^*_\alpha) \mid \alpha < \kappa\rangle ^\frown \bar{d}]$$

and such that $\mathrm{Rang}(\bar{c}) \subseteq \mathrm{Rang}([\langle f(a^*_\alpha)\rangle] ^\frown \bar{d}])$.

So let, for each $\bar{a} \in {}^{\kappa \geqslant}(M_1)$,

- $N^*_{\bar{a}} \prec_{\mathcal{K}} M_1$ be such that $|N_{\bar{a}}| = \kappa$,
- $\mathrm{Rang}(\bar{a}) \subseteq N^*_{\bar{a}}$.

Furthermore let for each such $\bar{a}$,

$$\mathcal{N}_{\bar{a}} := \{N^* \subseteq M_2 \mid N_{\bar{a}} \prec_{\mathcal{K}} N^* \text{ and } \|N^*\| = \kappa\}.$$

Then for each $\bar{a}$, $\mathcal{N}_{\bar{a}}$ is cofinal among subsets of M_2 of cardinality κ. By the union axiom, $N_{\bar{a}} \prec_{\mathcal{K}} \bigcup \mathcal{N}_a = M_2$.

On the other hand, allowing $\bar{a}$ to range over $[M_1]^{\leqslant \kappa}$, we have a system $\mathcal{N}^0$ of $\subseteq$-submodels of M_1 whose union is M_1 (again by (C)). Using a well-founded tree argument as in the proof of 1, this is really a cofinal $\prec_{\mathcal{K}}$ system, and therefore for each $\bar{a}$, $N_{\bar{a}} \prec_{\mathcal{K}} \bigcup \mathcal{N}^0 = M_1$ and we may conclude that $M_1 \prec_{\mathcal{K}} M_2$.

$\square_{\text{Theorem 3.1}}$

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4 Around the logic of an a.e.c.

The logic L^1_κ from Shelah's paper [6] satisfies Interpolation and a weak form of compactness: strong undefinability of well-order. Furthermore, it satisfies a Lindström-like maximality theorem for these properties (as well as union of ω-chains of models). The logic L^1_κ, however, lacks a well-defined syntax. Väänänen and Villaveces [7] have produced a logic with a clearly defined (and relatively symple) syntax, whose Δ-closure (a notion appearing first in [4]) is L^1_κ, and which satisfies several of the good properties of that logic (of course, strong undefinability of well-order but also closure under unions of chains). Also, Dzamonja and Väänänen have linked chain logic [1] to L^1_κ.

All of these logics are close to our constructions in this paper: the sentence $\varphi_{\lambda+2,0}$ belongs to $\mathbb{L}_{\lambda^+,\kappa^+}$ and L^1_μ lies in between two logics of the form $L_{\mu,\aleph_0}$ and $L_{\mu,\mu}$. Our sentence $\varphi_{\lambda+2,0}$ belongs to L^1_μ. However, it is not clear if this is the minimal logic for which this is the case.

The question of which is the internal logic of an a.e.c. remains still partially open.

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