# Universally measurable sets may all be $\underset{\sim}{\Delta}^1_2$

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**Abstract**

We produce a forcing extension of the constructible universe $\mathbf{L}$ in which every sufficiently regular set of any Polish space is a continuous image of a coanalytic set. In particular, we show that consistently every universally measurable set is $\underset{\sim}{\Delta}^1_2$, partially answering question CG from David Fremlin's problem list [4].

We let $\omega^\omega$ denote the Baire space, the set of functions from $\omega$ to $\omega$, and refer to its elements as *reals*. A subset of a Polish space $X$ is said to be *universally measurable* if it is measured by the completion of any $\sigma$-additive Borel measure on $X$. Equivalently, $A \subseteq X$ is universally measurable if and only if $f^{-1}[A]$ is Lebesgue measurable whenever $f\colon \omega^\omega \to X$ is a Borel function (see [6, 10], and 434D of [3], for instance). This characterization induces the corresponding notion for category : we will say that a set $A \subseteq X$ is *universally categorical* if and only if $f^{-1}[A]$ has the property of Baire whenever $f\colon \omega^\omega \to X$ is a Borel function. (The term *universally Baire* has already been established with different meaning [1], implying both universal measurability and universal categoricity.) The collections of the universally measurable subsets of $X$ and the universally categorical subsets of $X$ are both $\sigma$-algebras on $X$.

A subset $A$ of a Polish space $X$ is $\underset{\sim}{\Delta}^1_2$ if $A$ and $X \setminus A$ are continuous images of coanalytic sets. We refer the reader to [6] for background on this definition and for information on the projective sets in general. In this paper we identify a set $\mathbf{A}$ consisting of $\sigma$-algebras on $\omega^\omega$ and prove the consistency of the following statement : for every $\mathcal{A} \in \mathbf{A}$, each $A \in \mathcal{A}$ is $\underset{\sim}{\Delta}^1_2$. The set $\mathbf{A}$ contains the collection of universally measurable sets and the collection of universally categorial sets. Moreover, we will have that for each $\mathcal{A} \in \mathbf{A}$, the assertion that a given set $A \subseteq \omega^\omega$ is in $\mathcal{A}$ will be a $\underset{\sim}{\Pi}^1_2$ statement about $A$ and a $\underset{\sim}{\Sigma}^1_2$ set coding $\mathcal{A}$.

Since all uncountable Polish spaces are Borel-isomorphic (see Theorem 17.41 of [6]), the following theorem is a special case of our main theorem, where $\mathbf{V}$

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denotes the universe of all sets and $\mathbf{L}[a]$ denotes the relativization of Gödel's inner model $\mathbf{L}$ to allow a parameter for the set $a$.

**Theorem 0.1.** *If, for some $a \subseteq \omega$, $\mathbf{V}{=}\mathbf{L}[a]$, then there is a proper forcing extension in which every universally measurable subset of any Polish space is $\utilde{\Delta}^1_2$, and every universally categorical subset of any uncountable Polish space is $\utilde{\Delta}^1_2$.*

Our general theorem is Theorem 6.1 below. In the case of the Lebesgue-null ideal, Theorem 0.1 answers part of problem CG on David Fremlin's problem list [4]. Since there are only continuum many $\utilde{\Delta}^1_2$ sets, our result also strengthens (modulo the anti-large cardinal hypothesis $\mathbf{V}{=}\mathbf{L}[a]$) a previous result of the authors with Itay Neeman [8], which showed the consistency of the statement that the set of universally measurable sets has the same cardinality as $\omega^\omega$. We note that (unlike the results in [8]) some anti-large cardinal hypothesis is needed for Theorem 6.1, since the existence of infinitely many Woodin cardinals for instance implies that every projective set of reals is universally measurable [15], and there are (assuming $\mathsf{ZF}$) projective sets which are not $\utilde{\Delta}^1_2$ (see Theorem 37.7 of [6]).

# 1 Outline of the proof

The proof of Theorem 6.1 is an application of forcing machinery developed by the second author and his collaborators (especially [12, 13], but we also make use of results from [7]). It proceeds by forcing over a model of the form $\mathbf{L}[a]$ (for any $a \subseteq \omega$) with a countable support iteration of proper forcings, and makes use of the following theorem, which is Theorem III.4.1 in [14].

**Theorem 1.1.** *Suppose that $\kappa$ is a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and that $\bar{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is a countable support iteration such that each $\mathbb{P}_\alpha$ forces the corresponding $\dot{Q}_\alpha$ to be a proper forcing of cardinality less than $\kappa$. Then $\mathbb{P}_\kappa$ is $\kappa$-c.c., and for each $\alpha < \kappa$, $\mathbb{P}_\alpha$ has a dense subset of cardinality less than $\kappa$. Furthermore, for all $\alpha < \kappa$, $\mathbb{P}_\alpha$ forces that $2^{\aleph_0} < \kappa$.*

We will apply this theorem with ground models satisfying the Generalized Continuum Hypothesis, and we can let $\kappa$ be any regular cardinal with $\kappa^{<\kappa} = \kappa$ ($\omega_2$ is the most natural choice). Each step of our iterations will be an $(\omega, \infty)$-distributive partial order of cardinality continuum (see Definition 2.2), and will force the Continuum Hypothesis (CH) to hold. To see that Theorem 1.1 applies, we need to know that each $\mathbb{P}_\alpha$ preserves the statement that $2^{\aleph_0} < \kappa$. This is not hard to show directly for the partial orders we consider, but it also follows by applying Theorem 1.1 to the modified iteration where each $\dot{Q}_\alpha$ is the $\mathbb{P}_\alpha$-name for either our original $\dot{Q}_\alpha$ if CH holds, and the trivial forcing if it fails (the theorem then implies that the second case never holds). We then have from Theorem 1.1 that each $\mathbb{P}_\alpha$ (in our original, intended iteration) will have a dense subset of cardinality less than $\kappa$, and will therefore preserve the statement $2^{\aleph_1} \leq \kappa$.

The facts that (1) each $\mathbb{P}_\alpha$ preserves the statement $2^{\aleph_0} < \kappa$, (2) $\mathbb{P}_\kappa$ preserves the regularity of $\kappa$, and (3) for each $\mathcal{A} \in \mathbf{A}$ the assertion $A \in \mathcal{A}$ is $\undertilde{\Pi}^1_2$ in $A$ and a $\undertilde{\Sigma}^1_2$ set coding $\mathcal{A}$, together imply that, in the $\mathbb{P}_\kappa$-extension $\mathbf{V}[G]$, if $A \in \mathcal{A} \in \mathbf{A}$, then for club many $\alpha < \kappa$ (more importantly, at least one), $A \cap \mathbf{V}[G_\alpha]$ is in the set $\mathcal{A}$ as defined in $\mathbf{V}[G_\alpha]$ (where $G_\alpha$ denotes the restriction of $G$ to $\mathbb{P}_\alpha$).

The forcing construction in the proof of Theorem 6.1 produces a model in which every set of reals of cardinality $\aleph_1$ is $\undertilde{\Sigma}^1_2$ (the fact that each $\mathbb{P}_\alpha$ preserves the inequality $2^{\aleph_1} \leq \kappa$ makes this possible with an iteration of length $\kappa$). Given this, any $A \subseteq \omega^\omega$ with the property that $A$ and $\omega^\omega \setminus A$ are both unions of $\aleph_1$-many Borel sets is $\undertilde{\Delta}^1_2$. Given an inner model $M$ and a set $A \subseteq \omega^\omega$, say that $A$ is *$M$-Borel* if $A$ and $\omega^\omega \setminus A$ are both unions of Borel sets coded in $M$. The main theorem in this paper is established by proving that whenever $A \in \mathcal{A} \in \mathbf{A}$, $A$ is $V[G_\alpha]$-Borel for some $\alpha < \kappa$.

The paper [8] introduced the following notation : given a ground model set $A \subseteq \omega^\omega$, the *Borel reinterpretation* of $A$ in a forcing extension is the union of all the ground model Borel sets contained in $A$, each reinterpreted in the extension. This offered a characterization of the universally measurable sets as the sets $A \subseteq \omega^\omega$ with the property that the Borel reinterpretations of $A$ and $\omega^\omega \setminus A$ are complements in any extension by random forcing. Let us say that a partial order $P$ is *$\mathcal{A}$-representing* (for a $\sigma$-algebra $\mathcal{A}$ on $\omega^\omega$) if for each $A \in \mathcal{A}$ (in $\mathbf{V}$), every element of $\omega^\omega$ in any forcing extension by $P$ is in a Borel set with a code in $\mathbf{V}$ which is either contained in or disjoint from $A$. The characterization of universal measurability just given shows that random forcing is $\mathcal{A}$-representing when $\mathcal{A}$ is the $\sigma$-algebra of universally measurable sets (this fact is not used in the current paper).

**1.2 Remark.** These observations reduce the proof of the main theorem to establishing the following regarding the iterations $\langle \mathbb{P}_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ considered in this paper (iterations as in Definition 3.1 and their tails):

- each $\mathbb{P}_\alpha$ forces that $\dot{Q}_\alpha$ is a proper forcing of cardinality at most $2^{\aleph_0}$ (established in Remark 2.3);
- $\mathbb{P}_\kappa$ forces that every subset of $\omega^\omega$ of cardinality $\aleph_1$ is $\undertilde{\Sigma}^1_2$ (shown in Lemma 2.6);
- $\mathbb{P}_\kappa$ is $\mathcal{A}$-representing for each $\mathcal{A} \in \mathbf{A}$ (this is shown in Lemma 6.2).

At the end of the paper we prove one additional result not directly related to the main theorem. Given a $\mathcal{A} \in \mathbf{A}$, we say that a partial order $P$ has the *$\mathcal{A}$-reinterpretation property* if it is $\mathcal{A}$-representing, and, in addition the Borel reinterpretations of members of $\mathcal{A}$ in the ground model are in $\mathcal{A}$ as defined in forcing extensions by $P$ (many forcings have this property for the universal measurable sets, including random forcing and Sacks forcing, see [9]). Theorem 6.5 shows that the iterations considered in this paper have the $\mathcal{A}$-reinterpretation property when $\mathcal{A}$ is the set of universally measurable subsets of $\omega^\omega$. This is in some sense a negative result : an iteration forcing the statement "every universally measurable set has the property of Baire" (whose consistency is still an open question) cannot have the reinterpretation property for universally measurable sets if, for instance, it is applied to a model of Martin's Axiom (more generally, to a model with a medial limit).

## 2 Coding subsets of $\omega_1$ by reals

We let $\Omega$ denote the set of countable limit ordinals. A *ladder system* on $\omega_1$ is a sequence $\langle C_\eta : \eta \in \Omega\rangle$ such that each $C_\eta$ is a cofinal subset of $\eta$ of ordertype $\omega$. For $C$ an infinite set of ordinals and $n \in \omega$, we write $C(n)$ for the unique $\alpha \in C$ such that $|C \cap \alpha| = n$.

Given two sequences $s, t$, we write $s \lhd t$ mean that $s$ is an initial segment of $t$. Given $s \in 2^{<\omega}$, we let $[s]$ denote $\{x \in {}^\omega 2 : s \lhd x\}$.

**2.1 Definition.** We define a *dense function* to be a partial function $F\colon 2^\omega \to 2$ such that for each $s \in 2^{<\omega}$, $\{F(x) : s \lhd x \in \mathrm{dom}(F)\} = 2$.

**2.2 Definition.** We define the forcing $Q_{\bar{C},F,g}$, where

- $\bar{C} = \langle C_\eta : \eta \in \Omega\rangle$ is a ladder system on $\omega_1$;
- $F\colon 2^\omega \to 2$ is a dense partial function;
- $g$ is a function from $\Omega$ to 2.

The conditions of $Q_{\bar{C},F,g}$ are the functions $p$ such that,

- the domain of $p$ is a countable ordinal $\delta_p$;
- the range of $p$ is a subset of 2;
- for all $\eta \in (\delta_p + 1) \cap \Omega$, $\langle p(C_\eta(i)) : i < \omega\rangle \in \mathrm{dom}(F)$ and
$$F(\langle p(C_\eta(i)) : i < \omega\rangle) = g(\eta).$$

The order on $Q_{\bar{C},F,g}$ is extension.

**2.3 Remark.** Each partial order of the form $Q_{\bar{C},F,g}$ has cardinality $2^{\aleph_0}$, as does its transitive closure (so $Q_{\bar{C},F,g}$ is in $H(\mathfrak{c}^+)$). Each such partial order also forces $2^{\aleph_0} = \aleph_1$, since, by Lemma 2.4 below, it adds no new elements of $\omega^\omega$. To see that forcing with $Q_{\bar{C},F,g}$ wellorders $(2^\omega)^V$ in ordertype $\omega_1$, let $\bar{D}$ be a ladder system on $\omega_1$ such that

- $D_\eta \cap C_\eta$ is finite for each $\eta \in \Omega$ of ordertype greater than $\omega$ and
- $\sup\{D_\eta(0) : \eta \in \Omega\} = \omega_1$.

Let $G\colon \omega_1 \to 2$ be a $V$-generic function for $Q_{\bar{C},F,g}$, and note that each element of $(2^\omega)^V$ is equal to $\langle G(D_\eta(i)) : i < \omega\rangle$ for some $\eta \in \Omega$ (this follows from a standard genericity argument, and we leave the details to the reader).

**Lemma 2.4.** *If $\bar{C}$ is a ladder system on $\omega_1$, $F\colon 2^\omega \to 2$ is a partial dense function and $g$ is a function from $\Omega$ to 2, then $Q_{\bar{C},F,g}$ is proper and $(\omega,\infty)$-distributive.*

*Proof.* Let $p_0$ be a condition in $Q_{\bar{C},F,g}$. Let $X$ be a countable elementary substructure of $H(\beth_3^+)$ with $p_0, \bar{C}, F$ and $g$ in $X$. Let $\gamma = X \cap \omega_1$ and let $y$ be an element of $2^\omega$ extending

$$\langle p_0(C_\gamma(i)) : i < \omega,\, C_\gamma(i) < \delta_p \rangle$$

with $F(y) = g(\gamma)$. Let $R$ be the set of $q \le p_0$ such that

$$\langle q(C_\gamma(i)) : i < \omega,\, C_\gamma(i) < \delta_q \rangle$$

is an initial segment of $y$. For each dense subset $D$ of $Q_{\bar{C},F,g}$ in $X$ and each $p \in R \cap X$, there is a $q \le p$ in $R \cap D \cap X$. To see this, fix $D$ and $p$ and let $Y \in X$ be a countable elementary substructure of $H((\beth_2)^+)$ such that $D, p \in Y$. Extend $p$ to a condition $p' \in R \cap Y$ with $C_\gamma \cap Y \subseteq \delta_{p'}$. Then let $q \le p'$ be an element of $Y \cap D$. It follows that there exists a condition below $p_0$ which is in each dense open subset of $Q_{\bar{C},F,g}$ in $X$. $\square$

**2.5 Remark.** In the forcing extension we produce, each member of each set $\mathcal{A}_\mathcal{I}$ (where $\mathcal{I}$ is an ideal system with the absolute Fubini property as in Section 5) will be a union of $\aleph_1$ many Borel sets. Lemma 2.6 below will be used to show that in addition, in this extension, every subset of $\mathcal{P}(\omega)$ of cardinality $\aleph_1$ is $\utilde{\Sigma}^1_2$. Analytic subsets of $\omega^\omega$ are naturally coded by elements of $\mathcal{P}(\omega)$ (coding trees; see for instance Section 27A of [6]) in such a way that the set of pairs $(x,y)$ such that $x \subseteq \omega$, $y \in \omega^\omega$ and $x$ is in the set coded by $y$ is analytic. From this one gets that, in our extension, all members of $\mathcal{A}_\mathcal{I}$ are $\utilde{\Sigma}^1_2$. Since each set $\mathcal{A}_\mathcal{I}$ will be closed under complements, it will follow that, in this extension, every member of $\mathcal{A}_\mathcal{I}$ is $\utilde{\Delta}^1_2$.

Our coding of elements of $[\mathcal{P}(\omega)]^{\aleph_1}$ uses certain iterations of length $\omega$ of partial orders of the form $Q_{\bar{C},F,g}$, which we now define. Let $\pi\colon \omega \times \omega \to \omega$ be a fixed recursive bijection, and let $\pi_0$ and $\pi_1$ be functions from $\omega$ to $\omega$ such that $\pi(i) = (\pi_0(i), \pi_1(i))$ for all $i < \omega$. We choose $\pi$ so that $\pi_0(i) < i$ for all $i > 0$. We define, for each triple $(\bar{C}, F, g)$ such that

- $\bar{C}$ is a ladder system on $\omega_1$,
- $F$ is a dense function from $2^\omega$ to 2 and
- $g$ is a function from $\Omega$ to 2

the following objects recursively on $i < \omega$ (and suppress discussion of the meaning of "canonical name", trusting the reader to supply her or his preferred definition). Let

- $P_0$ be the trivial partial order;

- $\dot{g}_0$ be the canonical $P_0$-name for $g$;
- $\dot{Q}_0$ be the canonical $P_0$-name for $Q_{\bar{C},F,g}$;
- for all $i < \omega$,
  - $P_{i+1} = P_i * \dot{Q}_i$;
  - $\dot{h}_i$ be the canonical $P_{i+1}$-name for the $\dot{Q}_i$-generic function from $\omega_1$ to 2;
- for all positive $i < \omega$,
  - $\dot{g}_i$ be the canonical $P_i$-name for the set of pairs $(\alpha, k)$ such that $\alpha \in \Omega$ and $k = \dot{h}_{\pi_0(i),G_i}(\alpha + \pi_1(i))$, where $G_i$ denotes the generic filter for $P_i$;
  - $\dot{Q}_i$ be the canonical $P_i$-name for $Q_{\bar{C},F,\dot{g}_i}$.

We then let $Q^*_{\bar{C},F,g}$ denote the full (i.e., countable) support limit of the forcing iteration $\langle P_i, \dot{Q}_i : i < \omega\rangle$. The purpose of this definition is given in Lemma 2.6 below.

Given $a \subseteq \omega$, we define the *canonical ladder system relative to $a$* to be the set $\bar{C}_a = \{C^a_\alpha : \alpha < \Omega \cap \omega_1^{\mathbf{L}[a]}\}$, where each $C^a_\alpha$ is the constructibly least (in $\mathbf{L}[a]$, relative to $a$) cofinal subset of the corresponding $\alpha$ of ordertype $\omega$. This defines a ladder system in $\mathbf{L}[a]$ which is a ladder system in $\mathbf{V}$ if and only if $\omega_1^{\mathbf{L}[a]} = \omega_1$. Let $\Omega'$ be the set of countable limits of limit ordinals.

**Lemma 2.6.** *Let $a$ be a subset of $\omega$ such that $\omega_1^{\mathbf{L}[a]} = \omega_1$, and let $F\colon 2^\omega \to 2$ be a dense function whose graph is $\utilde{\Sigma}^1_2$ in $\mathbf{L}[a]$. Let $A$ be a subset of $\mathcal{P}(\omega)$ of cardinality at most $\aleph_1$. There exists a function $g\colon \Omega \to 2$ such that, if $G$ is a $V$-generic filter for $Q^*_{\bar{C}_a,F,g}$ and $M$ is an outer model of $V[G]$, then $A$ is $\utilde{\Sigma}^1_2$ in $M$.*

*Proof.* It suffices to consider the case where $A$ is nonempty. Let $g\colon \Omega \to 2$ be such that
$$A = \{\{k \in \omega : g(\beta + \omega\cdot k) = 0\} : \beta \in \Omega'\},$$
let $G$ be $V$-generic for $Q^*_{\bar{C}_a,F,g}$ and let $M$ be an outer model of $V[G]$. For each $i < \omega$, let $h_i$ be $\dot{h}_{i,G\restriction P_{i+1}}$, where $\dot{h}_i$ is as in the definition of $Q^*_{\bar{C},F,g}$. We show that, in $M$, $A$ is $\Sigma^1_2$ in $a$ and $\langle h_i\restriction\omega : i < \omega\rangle$. In particular, $A$ is the set of $x \subseteq \omega$ such that there exist

- an element $y$ of $\omega^\omega$ coding a model of the form $\mathbf{L}_\alpha[a]$, for some countable ordinal $\alpha$ (the wellfoundedness of this model being a $\Pi^1_1$ condition on $y$), such that $\omega_1^{\mathbf{L}_\alpha[a]}$ exists (i.e., some element of $\alpha$ is uncountable in $\mathbf{L}_\alpha[a]$),
- $\beta \in \Omega' \cap \omega_1^{\mathbf{L}_\alpha[a]}$,
- functions $h^*_i\colon \beta + \omega\cdot\omega \to 2$ ($i < \omega$) such that, for each $i < \omega$, $h^*_i\restriction\omega = h_i\restriction\omega$ and

- functions $g^*_i \colon \Omega \cap (\beta + \omega \cdot \omega) \to 2$ $(i < \omega)$

such that, letting $F^*$ be the function computed in $\mathbf{L}_\alpha[a]$ using a (fixed) $\underset{\sim}{\Sigma}^1_2$ definition for $F$ (which is contained in $F$ by the absoluteness of $\Pi^1_1$ relations)

1. for each $i < \omega$ and each $\gamma \in \Omega \cap (\beta + \omega \cdot \omega)$,
   - $\langle h^*_i(C^a_\gamma(j)) : j < \omega \rangle$ is in the domain of $F^*$;
   - $g^*_i(\gamma) = F^*(\langle h^*_i(C^a_\gamma(j)) : j < \omega \rangle)$;
   - if $i > 0$ then $g^*_i(\gamma) = h^*_{\pi_0(i)}(\gamma + \pi_1(i))$;
2. $x = \{k \in \omega : g^*_0(\beta + \omega \cdot k) = 0\}$.

That there exist such objects for each element of $A$ follows from the choice of $g$ and the definition of $Q^*_{\bar{C}_a,F,g}$. Given such objects, the choice of the names $\dot{g}_i$ (and item (1) above) then implies (via an inductive proof on $\gamma \in \Omega \cap (\beta + \omega \cdot \omega)$) that for each $i < \omega$, $g^*_i = \dot{g}_{i,G \restriction P_i} \restriction (\beta + \omega \cdot \omega)$ and $h^*_i = h_i \restriction (\beta + \omega \cdot \omega)$, which, again by the choice of $g$, implies that the corresponding set $x$ is in $A$. $\square$

# 3 Sequences and trees

Remarks 1.2 and 2.3 and Lemma 2.6 reduce the proof of Theorem 6.1 to showing that the forcing iterations we consider are $\mathcal{A}_\mathcal{I}$-representing, for the ideal systems $\mathcal{I}$ introduced in Section 5. These iterations will be countable support iterations where each successor step is a partial order of the form $Q_{\bar{C},F,\dot{g}_\beta}$, for a fixed ladder system $\bar{C}$ on $\omega_1$ and a fixed dense function $F$. An additional requirement on $F$ ($I$-pathology) will be introduced in Section 4. We fix the following notation.

**3.1 Definition.** Let $\bar{C}$ be a ladder system on $\omega_1$ and let $F \colon 2^\omega \to 2$ be a dense partial function. Let $\mathbf{Q}_{\bar{C},F}$ be the class of sequences of the form

$$\langle \mathbb{P}_\alpha, \dot{Q}_\beta, \dot{g}_\beta, \dot{h}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle,$$

where

- $\gamma$ is an ordinal,
- $\langle \mathbb{P}_\alpha, \dot{Q}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ is a countable support iteration,
- each $\dot{g}_\beta$ is a $P_\beta$-name for a function from $\Omega$ to 2,
- each $\dot{Q}_\beta$ is a $P_\beta$-name for the partial order $Q_{\bar{C},F,\dot{g}_\beta}$,
- each $\dot{h}_\beta$ is a $P_{\beta+1}$-name for the $\dot{Q}_\beta$-generic function.

In Section 6 we will be building a suitable $X$-generic condition, where $X$ is a countable elementary submodel of a large enough initial segment of the universe containing our iteration. The first definition below lists useful data that comes with such a situation. The second definition presents a tree of conditions which will guide us to finding our desired condition.

**3.2 Definition.** A *suitable data sequence* is sequence

$$\langle \bar{C}, F, \mathbf{q}, p, X, \bar{Y}, \bar{\imath}, \bar{D} \rangle$$

such that

- $\bar{C} = \langle C_\alpha : \alpha \in \Omega \rangle$ is a ladder system on $\omega_1$;
- $F\colon 2^\omega \to 2$ is a dense function;
- $\mathbf{q} = \langle \mathbb{P}_\alpha, Q_\beta, \dot{g}_\beta, \dot{h}_\beta : \alpha \leq \alpha_*, \beta < \alpha_* \rangle$ is in $\mathbf{Q}_{\bar{C},F}$;
- $p$ is in $\mathbb{P}_{\alpha_*}$;
- $X \prec (H((2^{2^{|\mathbf{q}|}})^+), \in)$ is countable, with $\{\bar{C}, F, \mathbf{q}, p\} \in X$;
- $\bar{Y} = \langle Y_k : k \in \omega \rangle$ is an $\in$-chain of countable elementary substructures of $H((2^{|\mathbf{q}|})^+)$, such that
  - $\{\bar{C}, F, p\} \in Y_0$,
  - $X \cap H((2^{|\mathbf{q}|})^+) = \bigcup_{k \in \omega} Y_k$,
  - for all $k \in \omega$, $C_{X \cap \omega_1} \cap (Y_{k+1} \setminus Y_k) \neq \emptyset$;
- $\bar{\imath} = \langle i_k : k \in \omega \rangle$ lists o.t.$(X \cap \alpha_*)$ without repetition in such a way that each $i_k$ is in the corresponding $Y_k$;
- $\bar{D} = \langle D_k : k \in \omega \rangle$ lists the dense open subsets of $\mathbb{P}_{\alpha_*}$ in $X$ in such a way that each $D_k$ is in the corresponding $Y_k$.

**3.3 Definition.** Suppose that

$$S = \langle \bar{C}, F, \langle \mathbb{P}_\alpha, Q_\beta, \dot{g}_\beta, \dot{h}_\beta : \alpha \leq \alpha_*, \beta < \alpha_* \rangle, p, X, \langle Y_k : k \in \omega \rangle, \langle i_k : k \in \omega \rangle, \langle D_k : k \in \omega \rangle \rangle$$

is a suitable data sequence, and let

- $\gamma$ be $X \cap \omega_1$,
- $\bar{\alpha} = \langle \alpha_i : i < i_* \rangle$ list $X \cap \alpha_*$ in increasing order,
- $\bar{C}$ be $\langle C_\alpha : \alpha \in \Omega \rangle$ and,
- for each $k \in \omega$, $m_k$ be $|C_\gamma \cap Y_k|$ and $u_k$ be $\{i_n : n < k, \alpha_{i_n} \in Y_k\}$.

A *condition tree* for $S$ is a pair $(T, \langle p_\rho : \rho \in T \rangle)$ such that

1. $T$ is a finitely branching tree of finite sequences, with a unique node of length 1;
2. each $\rho \in T$ is sequence of the form $\langle s^\rho_\ell : \ell < |\rho| \rangle$ such that
   - each $s^\rho_\ell$ is a function from $u_\ell$ to $2^{m_\ell}$;
   - whenever $\ell + 1 < |\rho|$ and $i \in u_\ell$, $s^\rho_\ell(i) \triangleleft s^\rho_{\ell+1}(i)$.

3. whenever $\rho \in T$ has length $k+1$ and $s\colon u_k \to 2^{m_{k+1}}$ is such that, for all $i \in u_k$, $s^{\rho}_k(i) \lhd s(i)$, there is exactly one $\rho' \in T$ such that:
   - $\rho \lhd \rho'$,
   - $|\rho'| = k+1$,
   - for all $i \in u_k$ $s^{\rho'}_{k+1}(i) = s(i)$;
4. for all $\rho \in T$, $p_\rho \in D_{|\rho|-1} \cap Y_{|\rho|-1}$;
5. $p_{\langle\rangle} = p$;
6. if $\rho_1 \lhd \rho_2 \in T$ then $p_{\rho_1} \geq_{\mathbb{P}_{\alpha_*}} p_{\rho_2}$;
7. if $\rho \in T$ and $|\rho| = k+1$ then for all $i \in u_k$,
$$(p_\rho \restriction \alpha_i + 1) \Vdash_{\mathbb{P}_{\alpha_i+1}} \langle \dot{h}_{\alpha_i}(C_\gamma(n)) : n < m_\ell \rangle = \check{s}^{\rho}_k(i);$$
8. if $j < i_*$ and $\rho_1, \rho_2 \in T$ have the same length and are such that $s^{\rho_1}_\ell(i) = s^{\rho_2}_\ell(i)$ whenever $\ell < |\rho_1|$ and $i \in u_\ell$, then $p_{\rho_1} \restriction \alpha_j = p_{\rho_2} \restriction \alpha_j$.

The following lemma is an adaptation of the proof of Lemma 1.8 of [12].

**Lemma 3.4.** *If $S$ is a suitable data sequence, then there exists a condition tree relative to $S$.*

*Proof.* Let $S$ be
$$\langle \bar{C}, F, \mathbf{q}, p, X, \langle Y_k : k \in \omega \rangle, \langle i_n : n < \omega \rangle, \langle D_k : k \in \omega \rangle \rangle,$$
and let

- $\gamma$ be $X \cap \omega_1$,
- $\mathbf{q} = \langle \mathbb{P}_\alpha, Q_\beta, \dot{g}_\beta, \dot{h}_\beta : \alpha \leq \alpha_*, \beta < \alpha_* \rangle$,
- $\bar{\alpha} = \langle \alpha_i : i < i_* \rangle$ list $X \cap \alpha_*$ in increasing order,
- $\bar{C}$ be $\langle C_\alpha : \alpha \in \Omega \rangle$ and
- for each $k \in \omega$, $m_k$ be $|C_\gamma \cap Y_k|$ and $u_k$ be $\{i_n : n < k, \alpha_{i_n} \in Y_k\}$.

We build $T$ and $\langle p_\rho : \rho \in T \rangle$ by recursion on the length of $\rho \in T$. For each $k \in \omega$, let $T(k)$ denote the set of sequences in $T$ of length $k$. For $k = 0$ we let $p_{\langle\rangle} = p$.

For each $k \in \omega$, let $B_k$ be the (nonempty) set of conditions $q \leq p$ in $D_k \cap Y_k$ such that for each $i \in u_k$,

- $q \restriction \alpha_i$ forces $\delta_{q(\alpha_i)}$ to be in $Y_k \cap \omega_1$;
- $q$ decides $\langle \dot{h}_{\alpha_i}(C_\gamma(n)) : n < m_k \rangle$.

Given a condition $q \in \mathbb{P}_{\alpha_*}$, $k \in \omega$ and $i \in u_k$ such that $q$ decides the value of $\langle \dot{h}_{\alpha_i}(C_\gamma(n)) : n < m_k \rangle$, we let $s^q_{k,i}$ denote the corresponding decided value. For each $\rho \in T$ of length greater than zero will be determined by a $p_\rho \in B_{|\rho|-1}$; each value $s^\rho_\ell(i)$ will be the corresponding value $s^{p_\rho}_{\ell,i}$.

To start, let $q_1$ be any member of $B_0$, and for each $i \in u_0$, let $\rho_1$ be the sequence of length 1 whose only member is $\langle s^{q_1}_{0,i} : i \in u_0 \rangle$. Let $\rho_1$ be the unique member of $T$ of length 1 and let $p_{\rho_1}$ be $q_1$.

Suppose that $k \in \omega \setminus \{0\}$ and that the members of $T(k)$ and the conditions $p_\rho \in B_k$ ($\rho \in T(k)$) have been chosen so as to satisfy Definition 3.3. Let $S$ be the set of pairs $(\rho, s)$, where $\rho \in T(k)$ and $s \colon u_k \to 2^{m_k}$ are as in item (3) of Definition 3.3. Working in $Y_k$ we pick for each pair $(\rho, s) \in S$ a condition $p_{\rho,s} \leq p_\rho$ in $B_k$ meeting conditions (7) and (8) of Definition 3.3. We do this by first choosing conditions of the form $p^0_{\rho,s} \restriction \alpha_i$ ($(\rho, s) \in S, i \in u_k$), recursively in $\{\alpha_i : i \in u_k\}$, in such a way that

- for all $(\rho, s) \in S$,
  - for each $j \in i_* \setminus u_k$, $p^0_{\rho,s}(\alpha_j) = p_\rho(\alpha_j)$;
  - for each $i \in u_{k-1}$, $p^0_{\rho,s} \restriction (\alpha_i + 1) \Vdash_{\mathbb{P}_{\alpha_i+1}} \langle \dot{h}_{\alpha_i}(C_\gamma(n)) : n < m_k \rangle = s(i)$;
  - for each $i \in u_k$, $p^0_{\rho,s} \restriction (\alpha_i + 1)$ decides the values of
  $$\langle \dot{h}_{\alpha_i}(C_\gamma(n)) : n < m_k \rangle$$
  and $\delta_{p^0_{\rho,s}}(\alpha_i)$.
- for all $(\rho_1, s_1), (\rho_2, s_2) \in S$, and all $i \in u_k$, if
  $$s_1 \restriction \{j \in u_k : \alpha_j \leq \alpha_i\} = s_2 \restriction \{j \in u_k : \alpha_j \leq \alpha_i\}$$
  then $p^0_{\rho_1,s_1} \restriction (\alpha_i + 1) = p^0_{\rho_2,s_2} \restriction (\alpha_i + 1)$.

That there exist such conditions follows from the fact that each $p_\rho$ is in $Y_{k-1}$, and, for each $i \in u_{k-1}$, $p_\rho$ forces that $\delta_{p_\rho(\alpha_i)} < Y_{k-1} \cap \omega_1$, which is below the least element of $C_\gamma$ for which the value of $\dot{h}_{\alpha_{i_n}}$ is not decided by $p_\rho$.

Finally, we choose conditions $p^\ell_{\rho,s} \leq p_{\rho,s}$ ($(\rho, s) \in S, \ell \leq |S|$, the conditions for $\ell = 0$ having been chosen) such that:

1. for all $\ell \leq |S|$, all $(\rho_1, s_1), (\rho_2, s_2) \in S$, and all $i \in u_k$, if
   $$s_1 \restriction \{j \in u_k : \alpha_j \leq \alpha_i\} = s_2 \restriction \{j \in u_k : \alpha_j \leq \alpha_i\}$$
   then $p^\ell_{\rho_1,s_1} \restriction (\alpha_i + 1) = p^\ell_{\rho_2,s_2} \restriction (\alpha_i + 1)$.
2. for all $\ell \leq |S|$, $|\{p^\ell_{\rho,s} : (\rho, s) \in S\} \cap D_k| \geq \ell$.

To do this, assuming that the conditions for some $\ell < |S|$ have been chosen, and that they are not all in $D_k$, strengthen one such condition $p^\ell_{\rho,s}$ to a condition $p^{\ell+1}_{\rho,s}$ in $D_k$. The other conditions from level $\ell$ can then all be strengthened if necessary to satisfy condition (1) by making their appropriate initial segments agree with $p^{\ell+1}_{\rho,s}$. If the conditions at level $\ell$ are all in $D_k$, then the conditions at all subsequent levels can remain the same.

The choice of these conditions $p_{\rho,s} = p^{|S|}_{\rho,s}$ then induces the corresponding members of $T(k+1)$. □

We will need to condition restrictions of our condition trees to initial segments of our iterations.

**3.5 Definition.** Suppose that $(T, \langle p_\rho : \rho \in T \rangle)$ is a condition tree relative to some suitable data sequence

$$S = \langle \bar{C}, F, \mathbf{q}, p, X, \langle Y_k : k < \omega \rangle, \langle i_n : n < \omega \rangle, \bar{D} \rangle,$$

$\gamma = X \cap \omega_1$, $\alpha_*$ is the length of $\mathbf{q}$, and that $\langle \alpha_i : i < i_* \rangle$ enumerates $X \cap \alpha_*$ in increasing order. For each $\ell \in \omega$, let $u_\ell = \{i_n : n < \ell, \alpha_{i_n} \in Y_\ell\}$. Let $\alpha_{i_*}$ denote $\alpha$.

- For each $j \leq i_*$, let
  - for each $\ell \in \omega$, $u_\ell(j)$ be $\{i \in u_\ell : \alpha_i < \alpha_j\}$;
  - for each $\rho \in T$, $\rho^{[j]}$ be
  $$\langle \rho(\ell) \restriction u_\ell(j) : \ell < |\rho| \rangle$$
  and $p^{[j]}_\rho$ be $p_\rho \restriction \alpha_j$;
  - $T^{[j]}$ be the tree of sequences $\{\rho^{[j]} : \rho \in T\}$;
  - $\bar{P}^{[j]}$ denote $\{p^j_\rho : \rho \in T\}$.
- For all $i \leq j < i_*$ and each $\rho \in T^{[j]}$, let $\rho^{[i]}$ be $\langle \rho(\ell) \restriction u_\ell(i) : \ell < |\rho| \rangle$.
- For all $i \leq j \leq i_*$, let
  - $\mathrm{proj}_{j,i} : T^{[j]} \to T^{[i]}$ be the function defined by setting
  $$\mathrm{proj}_{j,i}(\rho) = \rho^{[i]};$$
  - for each $x \in [T^{[i]}]$, $T^{[j]}(x)$ be $\{\rho \in T^{[j]} : \rho^{[i]} \triangleleft x\}$.

**3.6 Remark.** We record some observations on these definitions.

- For all $\rho \in T$ and $\ell \in \omega$, $\rho^{[i_*]} = \rho$ and $u_\ell(i_*) = u_\ell$.
- If $\rho \in T^{[j]}$ has length $k+1$, then $\rho$ has exactly $2^{|u_k(j)|(m_{k+1}-m_k)}$ many immediate successors in $T^{[j]}$.
- For all $i \leq j \leq i_*$ and $x \in [T^{[i]}]$, if $\rho \in T^{[j]}(x)$ has length $k+1$, then $\rho$ has exactly $2^{|u_k(j) \setminus u_k(i)|(m_{k+1}-m_k)}$ many successors in $T^{[j]}(x)$.

- For all $i \leq j \leq i_*$ and $k \in \omega$, the set $\mathrm{proj}_{j,i}^{-1}[\{\rho\}]$ has the same size for all $\rho \in T^{[i]}$ of length $k$ : 1, if $k \in \{0,1\}$, and $2^{|u_{k-1}(j)-u_{k-1}(i)|(m_k-m_{k-1})}$ otherwise.
- For all $i \leq p \leq j \leq i_*$ and $k \in \omega$, the set $\mathrm{proj}_{j,p}^{-1}[\{\rho\}]$ has the same size for all $\rho \in T^{[p]}(x)$ of length $k$ : 1, if $k \in \{0,1\}$, and $2^{|u_{k-1}(j)-u_{k-1}(p)|(m_k-m_{k-1})}$ otherwise.

Our reason for assuming that the ideals in our ideal systems $\mathcal{I}$ are regular (as we do in Section 5) is given in the following observation.

**3.7 Remark.** Suppose that $(T, \langle p_\rho : \rho \in T\rangle)$ is a condition tree for a suitable data sequence $S = \langle \bar{C}, F, \mathbf{q}, p, X, \bar{Y}, \bar{\imath}, \bar{D}\rangle$ is a suitable data sequence, and that $i_*$ is the ordertype of $X \cap \alpha_*$, where $\alpha_*$ is the ordertype of the iteration $\mathbf{q}$. Let $K$ be a closed subset of $[T]$. Then for all $x \in [T]$, $x \in K$ if and only if, for cofinally many $j < i_*$, $\mathrm{proj}_{i_*,j}(x) \in \mathrm{proj}_{i_*,j}[K]$.

# 4 Preserving pathology

In this section we develop suitable conditions for our dense partial functions $F$, and introduce one condition on the ideals which will appear in the ideal systems introduced in Section 5. We say that an ideal $I$ on a Polish space is *Borel* if it is generated by Borel sets (i.e., every member of the ideal is contained in a Borel member of the ideal). Given an ideal $I$ on a set $X$, we let $I^+$ denote $\mathcal{P}(X) \setminus I$, and we say that a subset of $X$ is $I$-Borel-large if intersects every Borel set in $I^+$.

**4.1 Definition.** Given a topological space $X$, an ideal $I$ on $X$, and a set $S$, we say that a partial function $F \colon X \to S$ is *$I$-pathological* if for every $s \in S$, $F^{-1}[\{s\}]$ is $I$-Borel-large.

When $I$ is the ideal of countable sets, we say that $F$ is *totally pathological.* We say that $F$ is *Lebesgue-pathological* when $I$ is the ideal of Lebesgue null sets, and *category-pathological* when $I$ is the ideal of meager sets. Note that total pathology implies both Lebesgue and category pathology.

**4.2 Remark.** A totally pathological function is dense in the sense of Definition 2.1, and the existence of a totally pathological function implies the existence of sets of reals without the perfect set property.

A standard construction, using a wellordering of the continuum and the fact that uncountable Borel sets have cardinality continuum, shows that ZFC implies the existence of (total) totally pathological functions on $2^\omega$. Moreover, if $a$ is a subset of $\omega$, then the same construction, using a $\Sigma^1_2(a)$ wellordering of $(2^\omega)^{L[a]}$, shows that there is in $\mathbf{L}[a]$ a total totally pathological function $F \colon 2^\omega \to 2$ which is $\Sigma^1_2$ in $a$.

In our proof in Section 6 we iterate forcings of the form $Q_{\bar{C},F,g}$ using a fixed function $F$ which is totally pathological and $\Sigma^1_2(a)$ in the ground model

$L[a]$. Since our iterations will add reals, the function $F$ will not remain totally pathological in the corresponding forcing extensions. For the proof of our main theorem, we need to know that $F$ remains $I$-pathological throughout the iteration. This induces a requirement on the ideals we consider.

**4.3 Definition.** A Borel ideal $I$ has the *preservation property* if every $I$-Borel-large set remains so after any countable support iteration of partial orders of the form $Q_{\bar{C},F,g}$.

**4.4 Remark.** The ideals that we consider in this paper are ideals on the set of infinite branches through some finitely branching tree of height $\omega$. There are natural versions of the meager ideal and the Lebesgue null ideal on sets of this form. Theorem 5.2 of [7] implies that these version of meager ideal have the preservation property (for a much wider class of partial orders than the ones considered here). Theorem 6.3 and Lemma 6.4 of [7] show the same thing for the ideals of Lebesgue null sets.

# 5 Ideal systems

Generalizing the class of trees introduced in Section 3, we let $\mathcal{T}$ denote the class of finitely branching trees $T$ of height $\omega$ with the property that, for each $k \in \omega$, every node on level $k$ of $T$ has the same nonzero number of successors. Given a tree $T$ in $\mathcal{T}$, we let $[T]$ denote the set of infinite branches through $T$.

We say that an ideal $I$ is *regular* if every $I$-positive Borel set contains a closed $I$-positive set. Let us say that an *ideal system* is a (formally, class-sized) function $\mathcal{I}$ which associates to every $T$ in $\mathcal{T}$ a regular Borel $\sigma$-ideal $I_T$ on $[T]$ in such a way that,

- $I_{2^{<\omega}}$ has the preservation property as defined in Definition 4.3;
- the set of pairs $(c, T) \in \omega^\omega \times (\mathcal{T} \cap \mathcal{P}(\omega^{<\omega}))$ for which $c$ is a Borel code for a member of $I_T$ is $\underset{\sim}{\Sigma}^1_2$ (see page 504 of [5], for instance, for a discussion of Borel codes, and page 490 of [5] for a discussion of the absoluteness properties of $\underset{\sim}{\Sigma}^1_2$ sets);
- for any pair $T, T'$ in $\mathcal{T}$, if $\pi\colon T \to T'$ is an isomorphism, then $I_{T'} = \pi_*[I_T]$;
- if $T \in \mathcal{T}$, $\rho \in T$ and $k \in \omega$ are such that $\rho$ is the only node on level $k$ of $T$, $T_\rho = \{\sigma : \rho^\frown\sigma \in T\}$ and $\pi\colon T_\rho \to T$ is the map sending $\sigma$ to $\rho^\frown\sigma$, then, for all $I_T = \{\pi[A] : A \in I_{T_\rho}\}$.

Given trees $T_1$ and $T_2$ in $\mathcal{T}$, we say that a map $\pi\colon T_1 \to T_2$ is a *projection map* if $\pi$ preserves length and order, and if, for each $k \in \omega$, the $\pi$-preimage of each point on level $k$ of $T_2$ has the same size. Equivalently, a projection maps is the composition of an isomorphism from $T_1$ to a product of the form $T \times T_2$ (for some $T \in \mathcal{T}$) and projection into the second coordinate. When $\pi\colon T_1 \to T_2$ is a projection map and $x$ is in $[T_2]$, we write $T^\pi_1(x)$ for the set of $\rho \in T_1$ for which $\pi(\rho) \triangleright x$. Then $T^\pi_1(x) \in \mathcal{T}$. We say that an ideal assignment $\mathcal{I}$ has the

*Fubini property* if whenever $\pi\colon T_1 \to T_2$ is a projection map, and Borel $E \subseteq [T_1]$ is $I_{T_1}$-large, the set of $x \in [T_2]$ for which $E \cap T_1^\pi(x)$ is $I_{T_1^\pi(x)}$-large is $I_{T_2}$-large. Moreover, we say that $\mathcal{I}$ has the *absolute Fubini property* if it retains the Fubini property (for all ground model trees and projection maps, but allowing new sets $E$) in any forcing extension by a countable support iteration of partial orders of the form $Q_{\bar{C},F,g}$. Even though we have restricted to ideal systems for which the set of pairs $(c,T)$ such that $c$ is a Borel code for an element of $I_T$ is $\utilde{\Sigma}^1_2$, the Fubini property, while projective, appears to be too complicated for Shoenfield absoluteness to guarantee its preservation in all outer models.

Given an ideal system $\mathcal{I}$, we let $\mathcal{A}_\mathcal{I}$ be the set of $A \subseteq \omega^\omega$ such that, for each $T \in \mathcal{T}$ and each continuous function $f\colon [T] \to \omega^\omega$, there exist Borel $B, N \subseteq [T]$ such that $N \in I_T$ and $f^{-1}[A] \triangle B \subseteq N$. It follows that $\mathcal{A}_\mathcal{I}$ is closed under complements. We let $\mathbf{A}$ be collection of sets of the form $\mathcal{A}_I$, for $I$ an ideal system with the absolute Fubini property.

Letting each ideal $I_T$ be, respectively, the ideal of Lebesgue null sets or the ideal of meager sets, we have that the collections of universally measurable subsets of $\omega^\omega$ and the universally categorical subsets of $\omega^\omega$ are in $\mathbf{A}$. This is essentially Fubini's Theorem in the first case, and the Kuratowski-Ulam theorem in the second (see Theorem 252B of [2], sections 14 and 15 of [11] or pages 104 and 53 of [6]).

# 6 The main theorem

Having defined the preservation and disintegration properties for Borel ideals, we can state our main theorem.

**Theorem 6.1.** *If, for some $a \subseteq \omega$, $\mathbf{V} = \mathbf{L}[a]$, then there is a proper forcing extension in which, for every ground model ideal system $\mathcal{I}$ satisfying the absolute Fubini property, every member of $\mathcal{A}_\mathcal{I}$ is $\utilde{\Delta}^1_2$.*

The proper forcing in our proof of Theorem 6.1 is an iteration of partial orders of the form $Q_{\bar{C},F,\dot{g}}$, for a fixed pair $(\bar{C}, F)$ such that $\bar{C}$ is a ladder system on $\omega_1$ and $F$ is a totally pathological dense function. By Remark 1.2, it suffices to prove that the tail of each such iteration is $\mathcal{A}_I$-representing, for each ground model ideal system $\mathcal{I}$ satisfying the Fubini property. Each ideal from the each such system $\mathcal{I}$ is assumed to have the preservation property, so in each intermediate model of a forcing iteration in $\mathbf{Q}_{\bar{C},F}$, $F$ is $I$-pathological (but no longer totally pathological once the iteration has added new subsets of $\omega$) if $F$ is totally pathological in the ground model. So it remains to prove the following.

**Lemma 6.2.** *Suppose that*

- *$\mathcal{I}$ is an ideal system with the Fubini property,*
- *$\bar{C}$ is a ladder function on $\omega_1$,*
- *$F\colon 2^\omega \to 2$ is a dense partial $I_{2^{<\omega}}$-pathological function,*

- $\mathbf{q} = \langle \mathbb{P}_\alpha, \dot{Q}_\beta, \dot{g}_\beta, \dot{h}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ *is a forcing iteration in* $Q_{\bar{C},F}$.

*Then* $\mathbb{P}_\gamma$ *is* $\mathcal{A}_I$*-representing.*

Fix such an $\mathcal{I}$, $\bar{C}$, $F$ and $\mathbf{q}$ from here through the proof of Lemma 6.4. Unpacking the definitions, we have to show that if

- $p$ is a condition in $\mathbb{P}_\gamma$,
- $A \subseteq \omega^\omega$ is in $\mathcal{A}_I$ and
- $\tau$ is a $\mathbb{P}_\gamma$-name for an element of $\omega^\omega$

then there exist a condition $p' \leq p$ and a Borel set $B \subseteq \omega^\omega$ such that

- $B$ is either contained in or disjoint from $A$ and
- $p' \Vdash \tau \in \check{B}$.

In fact the set we find will be a continuous image of a Borel set; as every analytic set is a union of $\aleph_1$ many Borel sets (in an absolute way, see page 201 of [6]), this suffices. To do this, fix such $p$, $A$ and $\tau$, and fix in addition $X$, $\bar{Y}$, $\bar{\imath}$ and $\bar{D}$ (with $\tau \in X$) such that $S = \langle \bar{C}, F, \mathbf{q}, p, X, \bar{Y}, \bar{\imath}, \bar{D} \rangle$ is a suitable data sequence. By Lemma 3.4, we may fix in addition a condition tree $(T, \langle p_\rho : \rho \in T \rangle)$ relative to $S$. There is then a continuous function $f \colon [T] \to \omega^\omega$ such that each value $f(x)$ is the realization of $\tau$ by $\{p_{x \restriction n} : n \in \omega\}$. Since $A$ is in $\mathcal{A}_I$, there exist then Borel sets $B$ and $N$ contained in $[T]$ such that $N \in I_T$ and $f^{-1}[A] \triangle B \subseteq N$. It suffices then to find a condition $p' \leq p$ forcing the existence (in the forcing extension) of an $x \in [T] \setminus N$ such that $\{p_{x \restriction n} : n \in \omega\}$ is a subset of the generic filter. An instance of such a condition would be a $(i_*, I_T, K, N)$-solution relative to $S$ and $(T, \langle p_\rho : \rho \in T \rangle)$, as in the Definition 6.3 below (the identity of $K$ is immaterial for our final goal, but in the inductive argument in the proof of Lemma 6.4 $K$ will be any closed $I_{T^{[j]}(\dot{\sigma}_j)}$-positive set disjoint from $\mathrm{proj}_{j,i}[N]$).

Given $j \leq i_*$, we let $\dot{\sigma}_j$ be a $\mathbb{P}_j$-name for the set of $\rho \in T^{[j]}$ for which $p_\rho \in G_j$, where $G_j$ denotes the restriction of the generic filter $G$ to $\mathbb{P}_{\alpha_j}$.

**6.3 Definition.** Suppose that $(T, \langle p_\rho : \rho \in T \rangle)$ is a condition tree relative to some suitable data sequence

$$S = \langle \bar{C}, F, \mathbf{q}, p, X, \langle Y_k : k \in \omega \rangle, \langle i_n : n < \omega \rangle, \bar{D} \rangle.$$

Let

- $\mathbf{q}$ be $\langle \mathbb{P}_\alpha, Q_\beta, \dot{g}_\beta, \dot{h}_\beta : \alpha \leq \alpha_*, \beta < \alpha_* \rangle$;
- $\langle \alpha_i : i < i_* \rangle$ enumerate $X \cap \alpha_*$ in increasing order;
- $\alpha_{i_*}$ denote $\alpha_*$;
- $I$ be an ideal on $[T]$;

- $N$ be a subset of $[T]$;
- $j$ be an element of $i_* + 1$;
- $K$ be a subset of $[T^{[j]}]$.

If $j$ is positive, we say that $q$ is a $(j, I, K, N)$-solution if

- $q \in \mathbb{P}_{\alpha_j}$ is $(X, \mathbb{P}_{\alpha_j})$-generic and $q$ forces that
  - $\dot{\sigma}_j \in K$;
  - if $j < i_*$ then $[T(\dot{\sigma}_j)] \setminus \{x \in N : \mathrm{proj}_{i_*,j}(x) = \dot{\sigma}_j\}$ is $I_{T(\dot{\sigma}_j)}$-large;
  - if $j = i_*$ then $\dot{\sigma}_j \notin N$.

If $E \subseteq [T]$ is $I$-large, then the empty condition in $\mathbb{P}_0$ is a $(0, I, K, E)$-solution.

The statement of the following lemma uses the objects introduced in this section. The case where $i = 0$ and $j = i_*$ proves Lemma 6.2 and thereby completes the proof of Theorem 6.1.

**Lemma 6.4.** *Suppose that $i \leq j \leq i_*$, $K$ is a closed $I_{T^{[j]}}$-positive subset of $[T^{[j]}]$, $N$ is an element of $I_T$, and $q$ is an $(i, I_T, K, N)$-solution forcing (in the case $j > 0$) that*

$$\{x \in K : \mathrm{proj}_{j,i}(x) = \dot{\sigma}_j\}$$

*is $I_{T^{[j]}(\dot{\sigma}_j)}$-positive. Then there is a $(j, I_T, K, N)$-solution $q' \in \mathbb{P}_{\alpha_j}$ such that $q' \restriction \alpha_i = q$.*

*Proof.* We prove this by induction primarily on $j$ and secondarily on $i$. Such a proof follows easily from the two following cases.

First, the case $j = i+1$. For notational convenience we give the proof in the subcase where $i > 0$. The case where $i = 0$ is simpler. Suppose that $G_i \subseteq \mathbb{P}_{\alpha_i}$ is a generic filter with $q \in G_i$. Then $G_i \cap X$ is $X$-generic. Let $a$ be $\dot{g}_{\alpha_i, G_i}(X \cap \omega_1)$ and let $y = \dot{\sigma}_{j,G_i}$. Then, in $\mathbf{V}[G_i]$, the set

$$\{x \in K : \mathrm{proj}_{j,i}(x) = y\} \notin I_{T^{[j]}(y)}.$$

Let $Z$ be the set of $z$ in

$$\{x \in K : \mathrm{proj}_{j,i}(x) = y\}$$

for which $\{w \in [T] \setminus N : \mathrm{proj}_{i_*,j}(w) = z\} \notin I_{T(z)}$. Since $\mathcal{I}$ has the Fubini property, $Z$ is $I_{T^{[j]}(y)}$-positive. For each $\ell \in \omega$ and each $e \in 2^{m_\ell}$ there is a unique $\sigma \in T^{[j]}(y)$ with $s^\sigma_\ell(i) = e$. Letting $k \in \omega$ be minimal with $i \in u_k$, the first $k$ levels of $T^{[j]}(y)$ have a single node. There are $2^{m_0}$ many nodes of $T^{[j]}(y)$ on level $k$, and for each $\ell \geq k$, each node on level $\ell$ has $2^{m_{\ell+1} - m_\ell}$ many successors. There exists a ground model tree $T_*$ contained in $\omega^\omega$ which is isomorphic to $T^{[j]}(y)$ starting at its shortest node with more than one immediate successor such that induced map $\pi_*$ on $[T_*]$ sends each $u \in [T_*]$ to a $z \in [T^{[j]}(y)]$ such that

$$u = \bigcup_{n \in (\ell,\omega)} s_\ell^{z\restriction n}(i).$$

Since $I_{T_*}$ has the preservation property, the restriction of $F$ to $T_*$ is $I_{T_*}$-pathological in $\mathbf{V}[G_i]$. Since $I_{T^{[j]}(y)}$ is the $\pi_*$-image of $I_{T_*}$, there is a $z \in Z$ such that

$$F(\bigcup_{n \in (\ell,\omega)} s_\ell^{z\restriction n}(i)) = a.$$

We can let $q'$ be $(q, \dot{r})$, where $\dot{r}$ is a $\mathbb{P}_{\alpha_i}$ name for $\bigcup\{s_\ell^{z\restriction n}(i) : n \in (\ell,\omega)\}$, which is an $X[G_i \cap X]$-generic condition in $\dot{Q}_{\alpha_i,G_i}$.

Fixing $i < j$, the case where $j$ is a limit ordinal follows from repeated application of the assumption that the lemma holds for all $j'$ in the interval $(i,j)$. Let $\langle j_n : n \in \omega\rangle$ be an increasing cofinal sequence in the interval $(i,j)$, with $j_0 = i$. To go from $j_n$ to $j_{n+1}$, apply the lemma with an $I_{T^{[j_{n+1}]}}$-positive closed set $K'$ such that, for each $z \in K'$ the set $\{x \in K : \mathrm{proj}_{j,j_{n+1}}(x) = z\}$ is $I_{T^{[j]}(z)}$-positive (which exists by the Fubini property). The case $(i,j)$ then follows from Remark 3.7, and the assumption that each ideal $I_{T^{[j]}(y)}$ is a $\sigma$-ideal. In the case where $j = i_*$, we can first shrink $K$ to make it disjoint from $N$. $\square$

Finally, we show that the Borel reinterpretations of universally measurable sets under the forcing extensions considered here are again universally measurable.

**Theorem 6.5.** *Let $\bar{C}$ be a ladder system on $\omega_1$ and let $F\colon 2^\omega \to 2$ be a null-pathological dense partial function. Let*

$$\langle \mathbb{P}_\alpha, \dot{Q}_\beta, \dot{g}_\beta, \dot{h}_\beta : \alpha \le \gamma, \beta < \gamma\rangle,$$

*be an element of $\mathbf{Q}_{\bar{C},F}$. If $A \subseteq \omega^\omega$ is universally measurable, then the Borel reinterpretation of $A$ is universally measurable in any forcing extension by $\mathbb{P}_\gamma$.*

*Proof.* It follows from Lemma 6.2 that the Borel reinterpretations of $A$ and $\omega^\omega \setminus A$ will be complements in any forcing extension by $\mathbb{P}_\gamma$. We have to show that if

- $p$ is a condition in $\mathbb{P}_\gamma$,
- $A \subseteq \omega^\omega$ is a universally $\mathcal{A}_I$-set and
- $\tau$ is a $\mathbb{P}_\gamma$-name for a Borel measure on $\omega^\omega$

then there exist a condition $p' \le p$ and a $\mathbb{P}_\gamma$-name $\dot{B}$ for a Borel subset $\omega^\omega$ such that $p'$ forces the symmetric difference of $\dot{B}$ and the Borel reinterpretation of $A$ to be Lebesgue-null. To do this, fix such $p$, $A$ and $\tau$, and fix in addition $X$, $\bar{Y}$, $\bar{\imath}$ and $\bar{D}$ (with $\tau \in X$) such that $S = \langle \bar{C}, F, \mathbf{q}, p, X, \bar{Y}, \bar{\imath}, \bar{D}\rangle$ is a suitable data sequence. By Lemma 3.4, we may fix in addition a condition tree $(T, \langle p_\rho :$

$\rho \in T\rangle$) relative to $S$. There is then a continuous function $f\colon [T] \to \omega^\omega$ such that each value $f(x)$ is the realization of $\tau$ by $\{p_{x \restriction n} : n \in \omega\}$. Then each $f(x)$ is a Borel measure on $\omega^\omega$. Let $\lambda$ be Lebesgue measure for $[T]$ and let $\nu$ be the measure on $\omega^\omega$ defined by setting $\nu(B)$ to be $\int f(x)(B)\, d\lambda$. Since $A$ is universally measurable, there exist Borel sets $B$ and $N$ contained in $[T]$ such that $N$ is $\nu$-null and $A \triangle B \subseteq N$. We can then apply Lemma 6.4 in the case where $i = 0$, $j = i_*$ and, for each $x \in E$, $N$ is $f(x)$-null. The resulting condition $q'$ then forces that (the reinterpretations of) $B$ and $N$ will witness that $A$ is measurable relative to the realization of $\tau$. $\square$

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