

ON κ -HOMOGENEOUS, BUT NOT κ -TRANSITIVE PERMUTATION GROUPS

SAHARON SHELAH AND LAJOS SOUKUP

Abstract. A permutation group G on a set A is κ -homogeneous iff for all $X, Y \in [A]^\kappa$ with $|A \setminus X| = |A \setminus Y| = |A|$ there is a $g \in G$ with $g[X] = Y$. G is κ -transitive iff for any injective function f with $\text{dom}(f) \cup \text{ran}(f) \in [A]^{\leq \kappa}$ and $|A \setminus \text{dom}(f)| = |A \setminus \text{ran}(f)| = |A|$ there is a $g \in G$ with $f \subset g$.

Giving a partial answer to a question of P. M. Neumann [?] we show that there is an ω -homogeneous but not ω -transitive permutation group on a cardinal λ provided

- (i) $\lambda < \omega_\omega$, or
- (ii) $2^\omega < \lambda$, and $\mu^\omega = \mu^+$ and \square_μ hold for each $\mu \leq \lambda$ with $\omega = \text{cf}(\mu) < \mu$, or
- (iii) our model was obtained by adding ω_1 many Cohen generic reals to some ground model.

For $\kappa > \omega$ we give a method to construct large κ -homogeneous, but not κ -transitive permutation groups. Using this method we show that there exists κ^+ -homogeneous, but not κ^+ -transitive permutation groups on κ^{+n} for each infinite cardinal κ and natural number $n \geq 1$ provided $V = L$.

§1. Introduction. Denote by $S(A)$ the group of all permutations of the set A . The subgroups of $S(A)$ are called *permutation groups on A* .

We say that a permutation group G on A is κ -homogeneous iff for all $X, Y \in [A]^\kappa$ with $|A \setminus X| = |A \setminus Y| = |A|$ there is a $g \in G$ with $g[X] = Y$.

We say that a permutation group G on A is κ -transitive iff for any injective function f with $\text{dom}(f) \cup \text{ran}(f) \in [A]^{\leq \kappa}$ and $|A \setminus \text{dom}(f)| = |A \setminus \text{ran}(f)|$ there is a $g \in G$ with $f \subset g$.

In this paper we give a partial answer to the following question which was raised by P.N. Neumann in [?, Question 3]:

Suppose that $\kappa \leq \lambda$ are infinite cardinals. Does there exist a permutation group on λ that are κ -homogeneous, but not κ -transitive?

In section 2 we show that there exist ω -homogeneous, but not ω -transitive permutation groups on $\lambda < \omega_\omega$ in ZFC, and on any infinite λ if $V = L$ (see Theorem 2.2.5).

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In section 3 we develop a general method to obtain large κ -homogeneous, but not κ -transitive permutation groups for arbitrary $\kappa \geq \omega$ (see Theorem 3.3.4). Applying our method we show that if $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and \square_ν holds for each $\kappa \leq \nu < \lambda$, then there is a κ -homogeneous, but not κ -transitive permutation group on λ (Corollary 3.3.12).

Finally in section 4, using some lemmas from section 3, we prove that after adding ω_1 Cohen reals in the generic extension for each infinite λ there exist ω -homogeneous, but not ω -transitive permutation groups on λ (Theorem 4.4.1).

Our notation is standard.

DEFINITION 1.1.1. If λ is fixed and $f \in S(A)$ for some $A \subset \lambda$, we take

$$f^+ = f \cup (\text{id} \upharpoonright (\lambda \setminus A)) \in S(\lambda).$$

Given a family of functions, \mathcal{G} , we say that a function y is \mathcal{G} -large iff

$$|y \setminus \bigcup \mathcal{H}| = |y|$$

for each finite $\mathcal{H} \subset \mathcal{G}$.

We say that a permutation group on A is κ -intransitive iff there is a \mathcal{G} -large injective function y with $\text{dom}(y) \cup \text{ran}(y) \in [A]^\kappa$ and $|A \setminus \text{dom}(y)| = |A \setminus \text{ran}(y)| = |A|$.

A κ -intransitive group is clearly not κ -transitive.

§2. ω -homogeneous but not ω -transitive.

DEFINITION 2.2.1. Given a set A we say that a family $\mathcal{A} \subset [A]^\omega$ is nice on A iff \mathcal{A} has an enumeration $\{A_\alpha : \alpha < \mu\}$ such that

- (N1) \mathcal{A} is cofinal in $\langle [A]^\omega, \subset \rangle$,
- (N2) for each $\beta < \mu$ there is a countable set $I_\beta \in [\beta]^\omega$ such that for all $\alpha < \beta$ there is a finite set $J_{\alpha,\beta} \in [I_\beta]^{<\omega}$ such that

$$A_\alpha \cap A_\beta \subset \bigcup_{\zeta \in J_{\alpha,\beta}} A_\zeta.$$

THEOREM 2.2.2. Assume that λ is an infinite cardinal, and $\mathcal{A} \subset [\lambda]^\omega$ is a nice family on λ . Then for each $A \in \mathcal{A}$ there is an ordering \leq_A on A such that

- (1) $\text{tp}(A, \leq_A) = \omega$ for each $A \in \mathcal{A}$,
- (2) if $A, B \in \mathcal{A}$, then there is a partition $\{C_i : i < n\}$ of $A \cap B$ into finitely many subsets such that $\leq_A \upharpoonright C_i = \leq_B \upharpoonright C_i$ for all $i < n$.

PROOF. Fix an enumeration $\{A_\beta : \beta < \mu\}$ of \mathcal{A} witnessing that \mathcal{A} is nice.

We will define \leq_{A_β} by induction on $\beta < \mu$.

Assume that \leq_{A_α} is defined for $\alpha < \beta$.

By (N2) we can fix a countable set $I_\beta = \{\beta_i : i < \omega\} \in [\beta]^\omega$ such that for all $\alpha < \beta$ there is $n_\alpha < \omega$ such that

$$A_\alpha \cap A_\beta \subset \bigcup_{i < n_\alpha} A_{\beta_i}.$$

Choose an order \leq_{A_β} on A_β such that

(i) for each $i < \omega$ writing $D_i = A_{\beta_i} \setminus \bigcup_{j < i} A_{\beta_j}$ we have

$$\leq_{A_\beta} \upharpoonright (A_\beta \cap D_i) = \leq_{A_{\beta_i}} \upharpoonright (A_\beta \cap D_i);$$

(ii) $tp(A_\beta, \leq_{A_\beta}) = \omega$.

By induction on β we show that (2) holds for β .

Assume that (2) holds for $\beta' < \beta$.

To check (2) for β fix $\alpha < \beta$.

To define \leq_β we considered a set $I_\beta = \{\beta_i : i < \omega\} \in [\beta]^\omega$ such that we had $n_\alpha < \omega$ with

$$A_\alpha \cap A_\beta \subset \bigcup_{i < n_\alpha} A_{\beta_i}.$$

For $i < n_\alpha$ let $C'_i = A_\alpha \cap A_\beta \cap D_i$, where $D_i = A_{\beta_i} \setminus \bigcup_{j < i} A_{\beta_j}$. Then $\{C'_i : i < n_\alpha\}$ is a partition of $A_\alpha \cap A_\beta$ and

$$\leq_{A_\beta} \upharpoonright C'_i = \leq_{A_{\beta_i}} \upharpoonright C'_i$$

by (i). By the inductive hypothesis, $A_{\beta_i} \cap A_\alpha$ has a partition into finitely many pieces $\{C_{i,j} : j < k_i\}$ such that $\leq_{A_\alpha} \upharpoonright C_{i,j} = \leq_{A_{\beta_i}} \upharpoonright C_{i,j}$. Then the partition

$$\{C'_i \cap C_{i,j} : i < n, j < k_i\}$$

of $A_\alpha \cap A_\beta$ works for α and β . Indeed,

$$\leq_{A_\alpha} \upharpoonright C'_i \cap C_{i,j} = \leq_{A_{\beta_i}} \upharpoonright C'_i \cap C_{i,j} = \leq_{A_\beta} \upharpoonright C'_i \cap C_{i,j}.$$

⊔

THEOREM 2.2.3. *Assume that λ is an infinite cardinal, $\mathcal{A} \subset [\lambda]^\omega$ is a cofinal family and for each $A \in \mathcal{A}$ we have an ordering \leq_A on A such that*

- (1) $tp(A, \leq_A) = \omega$ for each $A \in \mathcal{A}$,
- (2) if $A, B \in \mathcal{A}$, then there is a partition $\{C_i : i < n\}$ of $A \cap B$ into finitely many subsets such that $\leq_A \upharpoonright C_i = \leq_B \upharpoonright C_i$ for all $i < n$.

Then there is a permutation group on λ that is ω -homogeneous and ω -intransitive.

PROOF. For $A \in \mathcal{A}$ let

$$\mathcal{G}_A = \{f^+ \in S(\lambda) : f \in S(A) \wedge \text{there is a finite partition } \{C_i : i < n\} \text{ of } A \text{ such that } f \upharpoonright C_i \text{ is } \leq_A\text{-order preserving}\}.$$

Let G be the permutation group on λ generated by

$$\bigcup \{\mathcal{G}_A : A \in \mathcal{A}\}.$$

CLAIM 2.2.3.1. G is ω -homogeneous.

Indeed, let $X, Y \in [\lambda]^\omega$ with $|\lambda \setminus X| = |\lambda \setminus Y| = \lambda$. Pick $A \in \mathcal{A}$ such that $X \cup Y \subset A$ and $|A \setminus X| = |A \setminus Y| = \omega$.

Let c be the unique \leq_A -monotone bijection between X and Y and d be the unique \leq_A -monotone bijection between $A \setminus X$ and $A \setminus Y$. Then taking $g = c \cup d$ we have $g^+ \in \mathcal{G}_A \subset G$ and $g^+[X] = Y$.

CLAIM 2.2.3.2. G is ω -intransitive.

Pick $A \in \mathcal{A}$ and choose $B \in [A]^\omega$ such that $|A \setminus B| = \omega$.

Let b_0, b_1, \dots be the \leq_A -increasing enumeration of B . Define a bijection $y : B \rightarrow \omega$ as follows: for $i < \omega$ and $j < 2^i$ let

$$y(b_{2^i+j}) = b_{2^{i+1}-j}.$$

Observe that if c is \leq_A -monotone then

$$|\{i < \omega : |\{j < 2^i : c(b_{2^i+j}) = r(b_{2^i+j})\}| \geq 2\}| \leq 1.$$

Indeed, if $|\{j < 2^i : c(b_{2^i+j}) = y(b_{2^i+j})\}| \geq 2$, then c should be \leq_A -decreasing, and if $|\{i : |\{j < 2^i : c(b_{2^i+j}) = y(b_{2^i+j})\}| \geq 2\}| \geq 2$, then y should be \leq_A -increasing.

So y can not be covered by finitely many \leq_A -monotone functions. But for any $h \in G$, $h \cap (A \times A)$ can be covered by finitely many \leq_A -monotone functions by (2) and by the construction of G .

Thus y is G -large. ⊥

To obtain nice families we recall some topological results. We say that a topological space X is *splendid* (see [?]) iff it is countably compact, locally compact, locally countable such that $|\overline{A}| = \omega$ for each $A \in [X]^\omega$.

We need the following theorem:

THEOREM (Juhász, Nagy, Weiss, [?]). *If*

- (i) $\kappa < \omega_\omega$, or
- (ii) $2^\omega < \kappa$, $\text{cf}(\kappa) > \omega$ and $\mu^\omega = \mu^+$ and \square_μ hold for each $\mu < \kappa$ with $\omega = \text{cf}(\mu) < \mu$,

then there is a splendid space X of size κ .

Remark. In [?, Theorem 11] the authors formulated a bit weaker result: *if $V = L$ and $\text{cf}(\kappa) > \omega$ then there is a splendid space X of size κ .* However, to obtain that results they combined ‘‘Lemmas 7, 9 and 16 with the remark after Theorem 8’’ and their arguments used only the assumptions of the theorem above.

LEMMA 2.2.4. *If X is a splendid space, \mathcal{U} is the family of compact open subsets of X , and $Y \subset X$, then $\mathcal{U}[Y = \{U \cap Y : U \in \mathcal{U}\}]$ is nice on Y .*

PROOF. Let $A \in [Y]^\omega$. Then \overline{A} is countable, so it is compact. Since a splendid space is zero-dimensional, A can be covered by

finitely many compact open set, and so A can be covered by an element of \mathcal{U} . Thus $\mathcal{U}[Y$ is cofinal in $\langle [Y]^\omega, \subset \rangle$.

To check (N2) observe that every $U \in \mathcal{U}$ is a countable compact space, so it is homeomorphic to a countable successor ordinal. Thus U has only countably many compact open subsets. Hence $\mathcal{U} \upharpoonright U$ is countable which implies (N2) in the following stronger form:

(N2⁺) for each $\beta < \mu$ there is a set $I_\beta \in [\beta]^\omega$ such that for all $\alpha < \beta$ there is $\zeta_\alpha \in I_\beta$ such that

$$A_\alpha \cap A_\beta = A_{\zeta_\alpha} \cap A_\beta.$$

⊢

Remark. By [?, Corollary 2.2], if $(\omega_{\omega+1}, \omega_\omega) \rightarrow (\omega_1, \omega)$ holds, then the cardinality of a splendid space is less than ω_ω . So we need some new ideas if we want to construct arbitrarily large nice families in ZFC.

THEOREM 2.2.5. *If λ is an infinite cardinal, and*

- (i) $\lambda < \omega_\omega$, or
- (ii) $2^\omega < \lambda$, and $\mu^\omega = \mu^+$ and \square_μ hold for each $\mu \leq \lambda$ with $\omega = \text{cf}(\mu) < \mu$.

then there is an ω -homogeneous and ω -intransitive permutation group on λ .

PROOF. Applying the Juhasz-Nagy-Weiss theorem for $\kappa = \lambda$ if $\text{cf}(\lambda) > \omega$, and for $\kappa = \lambda^+$ if $\lambda > \text{cf}(\lambda) = \omega$, we obtain a splendid space on $\kappa \geq \lambda$. So, by Lemma 2.2.4, we obtain a nice family on \mathcal{A} on λ .

Thus, putting together Theorems 2.2.2 and 2.2.3 we obtained the desired permutation group on λ . ⊢

§3. κ -homogeneous but not κ -transitive for $\kappa > \omega$. Write $\mathcal{A}[X = \{A \cap X : A \in \mathcal{A}\}$ and $\mathcal{A}[*X = \{\bigcap \mathcal{A}' \cap X : \mathcal{A}' \in [\mathcal{A}]^{<\omega}\}$.

DEFINITION 3.3.1. Let $\kappa < \lambda$ be cardinals. We say that a cofinal family $\mathcal{A} \subset [\lambda]^\kappa$ is *locally small* iff $|\mathcal{A}[A| \leq \kappa$ for all $A \in \mathcal{A}$.

DEFINITION 3.3.2. If X, Y are subsets of ordinals with the same order types, then let $\rho_{X,Y}$ be the unique order preserving bijection between X and Y .

DEFINITION 3.3.3. If \mathcal{F} is a set of functions, an $\mathcal{F} \cup \{x\}$ -term t is a sequence $\langle h_0, \dots, h_{n-1} \rangle$, where $h_i = x$ or $h_i = x^{-1}$ or $h_i = f_i$ or $h_i = f_i^{-1}$ for some $f_i \in \mathcal{F}$. If g is function we use $t[g]$ to denote the function $h'_0 \circ h'_1 \circ \dots \circ h'_{n-1}$, where

$$h'_i = \begin{cases} f_i & \text{if } h_i = f_i, \\ f_i^{-1} & \text{if } h_i = f_i^{-1}, \\ g & \text{if } h_i = x, \\ g^{-1} & \text{if } h_i = x^{-1}. \end{cases}$$

If \mathcal{H} is a set of $\mathcal{F} \cup \{x\}$ -terms, then write

$$\mathcal{H}[g] = \{t[g] : t \in \mathcal{H}\}.$$

We say that an $\mathcal{F} \cup \{x\}$ -term t is an \mathcal{F} -term iff neither x nor x^{-1} are in the t . If t is a \mathcal{F} -term, then the function $t[g]$ does not depend on g , so we will write $t[]$ instead of $t[g]$ in that situation.

We say that a term t' is a *subterm* of a term $t = \langle h_0, \dots, h_{n-1} \rangle$ iff $t' = \langle h_{i_0}, h_{i_1}, \dots, h_{i_k} \rangle$, where $i_0 < i_1 < \dots < i_k < n$.

The set of all $\mathcal{F} \cup \{x\}$ -terms is denoted by $TERM(\mathcal{F} \cup \{x\})$.

The set of all \mathcal{F} -terms is denoted by $TERM(\mathcal{F})$.

THEOREM 3.3.4. *Assume that $2^\kappa = \kappa^+$ and there is a cofinal, locally small family $\mathcal{A} \subset [\lambda]^\kappa$. Then there is a permutation group G on λ which is κ -homogeneous, but not κ -transitive.*

Before proving this theorem we need some preparation.

LEMMA 3.3.5. *Assume that*

- (1) λ is a cardinal, \mathcal{H} is a finite set of $S(\lambda) \cup \{x\}$ -terms, and \mathcal{H} is closed for subterms,
- (2) g is an injective function, $\text{dom}(g) \cup \text{ran}(g) \subset \lambda$,
- (3) $\alpha, \alpha^* \in \lambda$ such that

$$\langle \alpha, \alpha^* \rangle \notin \bigcup \mathcal{H}[g],$$

- (4) $\zeta_0 \in \lambda \setminus \text{dom}(g)$ and $\zeta_1 \in \lambda \setminus \text{ran}(g)$,
- (5) $\eta_0 \in \lambda \setminus \text{ran}(g)$ and $\eta_1 \in \lambda \setminus \text{dom}(g)$ such that

$$\eta_0, \eta_1 \notin \{t[g](\alpha), t[g]^{-1}(\alpha^*) : t \in \mathcal{H}\}.$$

Let $g_0 = g \cup \{\langle \zeta_0, \eta_0 \rangle\}$ and $g_1 = g \cup \{\langle \eta_1, \zeta_1 \rangle\}$. Then

$$\langle \alpha, \alpha^* \rangle \notin \mathcal{H}[g_0] \cup \mathcal{H}[g_1].$$

PROOF. We prove only $\langle \alpha, \alpha^* \rangle \notin \mathcal{H}[g_0]$. The proof of the other statement is similar.

Assume on the contrary that $\langle \alpha, \alpha^* \rangle \in \mathcal{H}[g_0]$.

Pick the shortest term $t = \langle f_0, \dots, f_n \rangle$ from \mathcal{H} such that $t[g_0](\alpha) = \alpha^*$.

Write $\alpha_{n+1} = \alpha$ and $\alpha_i = \langle f_i, \dots, f_n \rangle [g_0](\alpha)$ for $0 \leq i \leq n$. Hence $\alpha_0 = \alpha^*$.

Let i maximal such that α_i is ζ_0 or η_0 . Since $t[g](\alpha)$ can not be α^* by (3), i is defined.

Since $\alpha_i = \langle f_i, \dots, f_n \rangle [g](\alpha)$, it follows that $\alpha_i \neq \eta_0$ by (5). So $\alpha_i = \zeta_0$.

Let j minimal such that α_j is ζ_0 or η_0 . Since $\alpha_j = (\langle f_0, \dots, f_{j-1} \rangle [g])^{-1}(\alpha^*)$, it follows that $\alpha_j \neq \eta_0$ by (5). So $\alpha_j = \zeta_0$ by (5). Thus $\alpha_i = \alpha_j = \zeta_0$, and so

$$\alpha^* = \langle f_0, \dots, f_{j-1}, f_i, \dots, f_n \rangle [g_0](\alpha).$$

Since $j < i$, the term $t' = \langle f_0, \dots, f_{j-1}, f_i, \dots, f_n \rangle$ is shorter than t and still $\alpha^* = t'[g_0](\alpha)$. So the length of t was not minimal. Contradiction. \dashv

LEMMA 3.3.6. *Assume that*

- (1) $y \in S(\kappa)$,
- (2) $A \in [\lambda]^\kappa$, and $B, C \in [A]^\kappa$ such that $|A \setminus B| = |A \setminus C| = \kappa$,
- (3) $\mathcal{F} \in [S(\lambda)]^\kappa$ such that

$$|y \setminus \bigcup \mathcal{H}[]| = \kappa$$

whenever \mathcal{H} is a finite set of \mathcal{F} -terms.

Then there is $g \in S(A)$ such that

- (i) $g[B] = C$,
- (ii)

$$|y \setminus \mathcal{H}[g^+]| = \kappa$$

whenever \mathcal{H} is a finite set of $\mathcal{F} \cup \{x\}$ -terms.

PROOF OF LEMMA 3.3.6. Write

$$\text{TASK}_0 = A \times \{\text{dom}, \text{ran}\} \text{ and } \text{TASK}_1 = [TERM(\mathcal{F} \cup \{x\})]^{<\omega} \times \kappa.$$

Let $\{I_0, I_1\} \in [[\kappa]^\kappa]^2$ be a partition of κ , and fix enumerations $\{T_i : i \in I_0\}$ of TASK_0 , and $\{T_i : i \in I_1\}$ of TASK_1 .

By transfinite induction, for $i < \kappa$ we will construct a function g_i and if $i = j + 1$ for some $j \in K_1$ then we also pick an ordinal $\alpha_{j+1} \in \kappa$ for such that

- (a) g_i is an injective function, $\text{dom}(g_i) \cup \text{ran}(g_i) \subset A$,
- (b) $g_i[B] \subset C$ and $g_i[A \setminus B] \subset A \setminus C$;
- (c) $|g_i| \leq i$;
- (d) if $i = j + 1$, $j \in I_0$ and $T_j = \langle \zeta, \text{dom} \rangle$, then $\zeta \in \text{dom}(g_i)$;
- (e) if $i = j + 1$, $j \in I_0$ and $T_j = \langle \zeta, \text{ran} \rangle$, then $\zeta \in \text{ran}(g_i)$;
- (f) if $i = j + 1$, $j \in I_1$ and $T_j = \langle \mathcal{H}_j, \chi_j \rangle$, then
 - (i) $\alpha_{j+1} \in \kappa \setminus \{\alpha_{j'+1} : j' \in I_1 \cap j\}$, and
 - (ii) $t[g_i \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined and $t[g_i \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$ for each $t \in \mathcal{H}_j$.

Let $g_0 = \emptyset$.

If i is limit, then let $g_i = \bigcup_{j < i} g_j$.

Assume that $i = j + 1$.

CLAIM 3.3.6.1.

$$|y \setminus \bigcup \mathcal{H}[g_j \cup \text{id}_{\lambda \setminus A}]| = \kappa. \quad (\dagger)$$

for each finite set \mathcal{H} of $\mathcal{F} \cup \{x\}$ -terms.

PROOF OF THE CLAIM. Fix \mathcal{H} . We can assume that \mathcal{H} is closed for subterms. By (3) we have $|y \setminus \bigcup \mathcal{H}[]| = \kappa$, and

$$y \cap \bigcup \mathcal{H}[] = y \cap \bigcup \mathcal{H}[\text{id}_{\lambda \setminus A}] \quad (\circ)$$

because \mathcal{H} is closed for subterms. Since $|g_j| < \kappa$, we have

$$|t[g_j \cup \text{id}_{\lambda \setminus A}] \setminus t[\text{id}_{\lambda \setminus A}]| < \kappa. \quad (\bullet)$$

for each $t \in \mathcal{H}$. Putting together $|y \setminus \bigcup \mathcal{H}[]| = \kappa$, (o) and (•) we obtain (†). ⊣

Case 1. $j \in I_0$ and so $T_j = \langle \zeta_j, x_j \rangle \in A \times \{\text{dom}, \text{ran}\}$.

Assume first that $x_j = \text{dom}$. If $\zeta_j \in \text{dom}(g_j)$, let $g_i = g_j$. If $\zeta_j \notin \text{dom}(g_j)$, then pick $\eta \in C$ if $\zeta_i \in B$, and pick $\eta \in A \setminus C$ if $\zeta_i \in A \setminus B$ such that and $\eta \notin \text{ran}(g_j)$.

Let $g_i = g_j \cup \langle \zeta_i, \eta \rangle$. Then g_i satisfies (a)–(f).

The case $x_j = \text{ran}$ is similar.

Case 2. $j \in I_1$ and so $T_j = \langle \mathcal{H}_j, \chi_j \rangle \in [TERM(\mathcal{F} \cup \{x\})]^{<\omega} \times \kappa$.

We can assume that \mathcal{H}_j is closed for subterms.

By Claim 3.3.6.1, we have

$$|y \setminus \bigcup \mathcal{H}_j[g_j \cup \text{id}_{\lambda \setminus A}]| = \kappa.$$

So we can pick $\alpha_{j+1} \in \kappa \setminus \{\alpha_{j'+1} : j' \in I_1 \cap j\}$ such that

(*) for each $t \in \mathcal{H}_j$ either $t[g_j \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1})$ is undefined or $t[g_j \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$.

Now in finitely many steps, using Lemma 3.3.5, we can extend the function g_j to a function g_i such that

(*) $t[g_i \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined and $t[g_i \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$ for each $t \in \mathcal{H}_j$.

Indeed, if $t[g' \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1})$ is not defined, where $t = \langle t_0, \dots, t_n \rangle$ then there is $i < n$ such that either

$$\zeta_i = \langle t_{i+1}, \dots, t_n \rangle [g' \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1}) \text{ is defined, } t_i = x \text{ and } \zeta_i \in A \setminus \text{dom}(g')$$

or

$$\zeta_i = \langle t_{i+1}, \dots, t_n \rangle [g' \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1}) \text{ is defined, } t_i = x^{-1} \text{ and } \zeta_i \in A \setminus \text{ran}(g').$$

In both cases, using Lemma 3.3.5, we can extend g' to g'' such that $\langle t_i, \dots, t_n \rangle [g'' \cup \text{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined and $\langle \alpha_{j+1}, y(\alpha_{j+1}) \rangle \notin \bigcup \mathcal{H}_j[g'' \cup \text{id}_{\lambda \setminus A}]$.

After the inductive construction, the function $g = \bigcup_{i < \kappa} g_i$ meets the requirements. ⊣

LEMMA 3.3.7. *Assume that $2^\kappa = \kappa^+$ and there is a cofinal, locally small subfamily $\mathcal{C} \subset [\lambda]^\kappa$. Then there is a family $\mathcal{D} \subset [\lambda]^\kappa \times [\lambda]^\kappa$ such that*

(1) *if $\langle A, B \rangle \in \mathcal{D}$, then $B \cup \kappa \subset A$ and $|A \setminus B| = \kappa$.*

Moreover, writing $\mathcal{A} = \{A : \langle A, B \rangle \in \mathcal{D}\}$ and $\mathcal{B} = \{B : \langle A, B \rangle \in \mathcal{D}\}$

(2) *\mathcal{A} is a cofinal, locally small subfamily of $[\lambda]^\kappa$,*

(3) *\mathcal{B} is cofinal in $\langle [\lambda]^\kappa, \subset \rangle$,*

(4) *$\{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\} \subset \mathcal{B}$.*

PROOF OF LEMMA 3.3.7. Fix a locally small, cofinal subfamily $\mathcal{C} \subset [\lambda]^\kappa$. We can assume that $|\{C \in \mathcal{C} : D \subset C\}| = |\mathcal{C}|$ for all $D \in [\lambda]^\kappa$.

Write $\mu = |\mathcal{C}|$. Then $2^\kappa = \kappa^+ \leq \mu$. So we can construct \mathcal{D} by induction such that $\mathcal{A} \subset \mathcal{C}$, $\kappa \subset \bigcap \mathcal{A}$ and $\mathcal{B} = \mathcal{C} \cup \{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\}$. \dashv

After that preparation we prove the main theorem of this section.

PROOF OF THEOREM 3.3.4. Fix \mathcal{D} , \mathcal{A} and \mathcal{B} as in Lemma 3.3.7.

For $\langle A, B \rangle \in \mathcal{D}$ consider the structure $\mathcal{M}_{\langle A, B \rangle} = \langle A, <, B, \{A \cap X : A \in \mathcal{A}\} \rangle$.

Fix $\mathcal{D}' \in [\mathcal{D}]^{\kappa^+}$ such that writing $\mathcal{A}' = \{A' : \langle A', B' \rangle \in \mathcal{D}'\}$ and $\mathcal{B}' = \{B' : \langle A', B' \rangle \in \mathcal{D}'\}$ we have

- (a) $\forall \langle A, B \rangle \in \mathcal{D} \exists \langle A', B' \rangle \in \mathcal{D}'$ such that $\rho_{A, A'}$ is an isomorphism between $\mathcal{M}_{\langle A, B \rangle}$ and $\mathcal{M}_{\langle A', B' \rangle}$.
- (b) $\{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\} \subset \mathcal{B}'$.

Pick $K \in [\kappa]^\kappa$ with $|\kappa \setminus K| = \kappa$. Choose $y \in S(\kappa)$ such that $y(\alpha) \neq \alpha$ for each $\alpha \in \kappa$.

LEMMA 3.3.8 (Key lemma). *There are functions $\mathcal{F} = \{f_{\langle A, B \rangle} : \langle A, B \rangle \in \mathcal{D}'\}$ such that*

- (a) $f_{\langle A, B \rangle} \in S(A)$,
- (b) $f_{\langle A, B \rangle}[B] = K$,

moreover, taking

$$\mathcal{S} = \{\rho_{C_0, C_1} : \langle A_0, B_0 \rangle, \langle A_1, B_1 \rangle \in \mathcal{D}', C_0 \in \mathcal{A}[*A_0], C_1 \in \mathcal{A}[*A_1], \rho_{C_0, C_1}[\mathcal{A}[C_0]] = \mathcal{A}[C_1]\},$$

if \mathcal{H} is a finite collection of $\mathcal{F} \cup \mathcal{S}$ -terms, then

$$|y \setminus \bigcup \mathcal{H}[y]| = \kappa.$$

Before proving the Key lemma, we show how the Key Lemma completes the proof of Theorem 3.3.4.

So assume that the Key lemma holds.

For each $\langle A, B \rangle \in \mathcal{D}$ pick $\langle A', B' \rangle \in \mathcal{D}'$ such that $\rho_{A, A'}$ is an isomorphism between $\mathcal{M}_{\langle A, B \rangle}$ and $\mathcal{M}_{\langle A', B' \rangle}$. We assume that $\langle A', B' \rangle = \langle A, B \rangle$ for $\langle A, B \rangle \in \mathcal{D}'$.

Let

$$g_{\langle A, B \rangle} = \rho_{A', A} \circ f_{\langle A', B' \rangle} \circ \rho_{A, A'} \in S(A).$$

Let G be the permutation group on λ generated by

$$\mathcal{G} = \{g_{\langle A, B \rangle}^+ : \langle A, B \rangle \in \mathcal{D}\}.$$

LEMMA 3.3.9. *G is κ -homogeneous.*

PROOF OF LEMMA 3.3.9. It is enough to show that for each $X \in [\lambda]^\kappa$ there is $g \in G$ with $g[X] = K$.

So fix $X \in [\lambda]^\kappa$. Pick $\langle A, B \rangle \in \mathcal{D}$ such that $X \subset B$.

Then

$$\begin{aligned} Z = g_{\langle A, B \rangle}[X] &\subset g_{\langle A, B \rangle}[B] = (\rho_{A', A} \circ f_{\langle A', B' \rangle} \circ \rho_{A, A'})[B] \\ &= (\rho_{A', A} \circ f_{\langle A', B' \rangle})[B'] = \rho_{A', A}[K] = K. \end{aligned}$$

Since $|Z| = |\kappa \setminus Z| = \kappa$, there is C such that $\langle C, Z \rangle \in \mathcal{D}'$. Then $f_{\langle C, Z \rangle}[Z] = K$. Thus $g_{\langle C, Z \rangle}^+[Z] = K$ because $\langle C', Z' \rangle = \langle C, Z \rangle$ and so $f_{\langle C, Z \rangle} = g_{\langle C, Z \rangle}$.

Thus $K = (g_{\langle C, Z \rangle}^+ \circ g_{\langle A, B \rangle}^+)[X]$. ⊣

LEMMA 3.3.10. G is not κ -transitive.

PROOF OF LEMMA 3.3.10. We prove that $y \notin h$ for any $h \in G$. Assume that

$$h = (g_0^+)^{\ell_0} \circ (g_1^+)^{\ell_1} \circ \dots \circ (g_{n-1}^+)^{\ell_{n-1}},$$

where $g_i = g_{\langle A_i, B_i \rangle} = \rho_{A'_i, A_i} \circ f_{A'_i, B'_i} \circ \rho_{A_i, A'_i}$ and $\ell_i \in \{-1, 1\}$ for $i < n$.

Since $g_i^+ \setminus g_i$ is the identity function on $\lambda \setminus A_i$, we have

$$h \subset \bigcup \{ (g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \dots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}} : \\ k < n, i_0 < i_1 < \dots < i_{k-1} < n \}.$$

Fix $k \leq n$ and $i_0 < i_1 < \dots < i_{k-1} < n$.

Observe that if $\ell_i = -1$ then

$$(g_i)^{\ell_i} = (\rho_{A'_i, A_i} \circ f_{A'_i, B'_i} \circ \rho_{A_i, A'_i})^{-1} = \rho_{A'_i, A_i} \circ (f_{A'_i, B'_i})^{-1} \circ \rho_{A_i, A'_i}.$$

So

$$(g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \dots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}} = \\ \rho_{A'_{i_0}, A_{i_0}} \circ (f_{A'_{i_0}, B'_{i_0}})^{\ell_{i_0}} \circ \rho_{A_{i_0}, A'_{i_0}} \circ \rho_{A'_{i_1}, A_{i_1}} \circ (f_{A'_{i_1}, B'_{i_1}})^{\ell_{i_1}} \circ \rho_{A_{i_1}, A'_{i_1}} \circ$$

For $j < k$ let

$$\rho_j^* = \rho_{A_{i_j}, A'_{i_j}} \circ \rho_{A'_{i_{j+1}}, A_{i_{j+1}}}.$$

Observe that

$$\rho_j^* = \rho_{\rho_{A_{i_{j+1}}, A'_{i_{j+1}}}[A_{i_j} \cap A_{i_{j+1}}], \rho_{A_{i_j}, A'_{i_j}}[A_{i_j} \cap A_{i_{j+1}}]} \in \mathcal{S}.$$

(See Figure 1.)

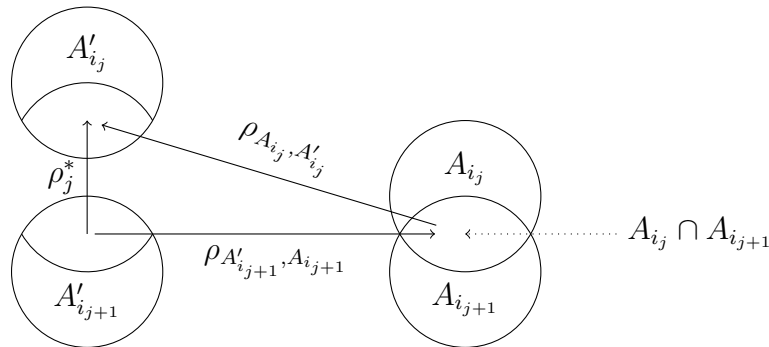


FIGURE 1. The function ρ_j^*

Thus

$$\begin{aligned} (g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \dots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}} = \\ \rho_{A_{i_0}, A'_{i_0}} \circ (f_{A'_{i_0}, B'_{i_0}})^{\ell_0} \circ \rho_0^* \circ (f_{A'_{i_1}, B'_{i_1}})^{\ell_1} \circ \rho_1^* \circ \dots \\ \circ (f_{A'_{i_{k-1}}, B'_{i_{k-1}}})^{\ell_{i_{k-1}}} \circ \rho_{A'_{i_{k-1}}, A_{i_{k-1}}}. \end{aligned}$$

Since $\rho_{A_\ell, A'_\ell} \upharpoonright \kappa = \text{id} \upharpoonright \kappa$, we have

$$\begin{aligned} ((g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \dots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}}) \cap \kappa \times \kappa \subset \\ (f_{A'_{i_0}, B'_{i_0}})^{\ell_0} \circ \rho_0^* \circ (f_{A'_{i_1}, B'_{i_1}})^{\ell_1} \circ \rho_1^* \circ \dots \\ \circ (f_{A'_{i_{k-1}}, B'_{i_{k-1}}})^{\ell_{i_{k-1}}} \end{aligned}$$

But $(f_{A'_{i_0}, B'_{i_0}})^{\ell_0} \circ \rho_0^* \circ (f_{A'_{i_1}, B'_{i_1}})^{\ell_1} \circ \rho_1^* \circ \dots \circ (f_{A'_{i_{k-1}}, B'_{i_{k-1}}})^{\ell_{i_{k-1}}} = t[]$ for the $\mathcal{F} \cup \mathcal{S}$ -term $t = \langle (f_{A'_{i_0}, B'_{i_0}})^{\ell_0}, \rho_0^*, (f_{A'_{i_1}, B'_{i_1}})^{\ell_1}, \rho_1^*, \dots, (f_{A'_{i_{k-1}}, B'_{i_{k-1}}})^{\ell_{i_{k-1}}} \rangle$.

Since there are only finitely many sequences $i_0 < \dots < i_{k-1} < n$, we obtain that $h \cap \kappa \times \kappa$ is covered by the union of finitely many $\mathcal{F} \cup \mathcal{S}$ -terms.

But y is not covered by the union of finitely many $\mathcal{F} \cup \mathcal{S}$ -terms. So y witnesses that G is not κ -transitive. \dashv

PROOF OF THE KEY LEMMA 3.3.8. Write $\mathcal{D}' = \{\langle A_\alpha, B_\alpha \rangle : \alpha < \kappa^+\}$.

By transfinite induction, we define functions $\{f_\alpha : \alpha < \kappa^+\}$ such that taking

$$\mathcal{F}_{<\beta} = \{f_\gamma : \gamma < \beta\}$$

and

$$\mathcal{S}_{<\beta} = \{\rho_{C_0, C_1} : \delta, \gamma < \beta, C_0 \in \mathcal{A}[\delta], C_1 \in \mathcal{A}[\gamma], \rho_{C_0, C_1}[\mathcal{A} \upharpoonright C_0] = \mathcal{A} \upharpoonright C_1\},$$

we have

- (i) $f_\alpha \in \mathcal{S}(A_\alpha)$,
- (ii) $f_\alpha[B_\alpha] = K$,
- (iii) if \mathcal{H} is a finite collection of $\mathcal{F}_{<\alpha+1} \cup \mathcal{S}_{<\alpha+1}$ -terms, then

$$|y \setminus \mathcal{H}[]| = \kappa.$$

Assume that we have constructed f_β for $\beta < \alpha$. Then we have:

if \mathcal{H} is a finite collection of $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha}$ -terms, then $|y \setminus \mathcal{H}[]| = \kappa$. (*)

To continue the construction we need a bit more.

CLAIM 3.3.10.1. *If \mathcal{H} is a finite collection of $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha+1}$ -terms, then*

$$|y \setminus \mathcal{H}[]| = \kappa.$$

PROOF. First observe that if $\rho_i = \rho_{A_i, A_i^*}$ for $i < 2$, then

$$\rho_1 \circ \rho_0 = \rho_{\rho_0^{-1}[A_0^* \cap A_1], \rho_1[A_0^* \cap A_1]}. \quad (\ddagger)$$

Let

$$t = \langle t_0, t_1, \dots, t_n \rangle$$

be an element of \mathcal{H} . Since $\rho_{C_0, C_1} \upharpoonright \kappa = \text{id} \upharpoonright \kappa$, $t \upharpoonright \kappa \cap \kappa \times \kappa = \langle t_1, \dots, t_n \rangle \upharpoonright \kappa \cap \kappa \times \kappa$ if $t_0 \in \mathcal{S}_{<\alpha+1}$. So we can assume that $t_0 \in \mathcal{F}_{<\alpha}$. Similar argument give that we can assume that $t_n \in \mathcal{F}_{<\alpha}$.

Now assume that

$$\langle t_i, \dots, t_j \rangle = \langle f_{\alpha_i}, \rho_{C_{i+1}, D_{i+1}}, \rho_{C_{i+2}, D_{i+2}}, \dots, \rho_{C_{j-1}, D_{j-1}}, f_{\alpha_j} \rangle$$

Then, by (\ddagger)

$$\rho_{C_{i+1}, D_{i+1}} \circ \rho_{C_{i+2}, D_{i+2}} \circ \dots \circ \rho_{C_{j-1}, D_{j-1}} = \rho_{E_i, E_j}.$$

for some $E_i \in \mathcal{A}[C_{i+1}]$ and $E_j \in \mathcal{A}[D_{j-1}]$.

Thus we can assume that $j = i + 2$ and

$$\langle t_i, t_{i+1}, t_{i+2} \rangle = \langle f_{\alpha_0}, \rho_{E_0, E_1}, f_{\alpha_1} \rangle.$$

Now

$$f_{\alpha_0} \circ \rho_{E_0, E_1} \circ f_{\alpha_1} = f_{\alpha_0} \circ \rho_{A_{\alpha_0} \cap E_0, A_{\alpha_1} \cap E_1} \circ f_{\alpha_1}$$

and $\rho_{A_{\alpha_0} \cap E_0, A_{\alpha_1} \cap E_1} \in \mathcal{S}_{<\alpha}$.

Thus there is a $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha}$ -terms s_t such that

$$t \upharpoonright \kappa \cap (\kappa \times \kappa) = s_t \upharpoonright \kappa \cap (\kappa \times \kappa).$$

Since $|y \setminus \bigcup \{s_t \upharpoonright \kappa : t \in \mathcal{H}\}| = \kappa$ by $(*)$, the Claim holds. \dashv

Since the claim holds, we can apply Lemma 3.3.6 for the family $\mathcal{F} = \mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha+1}$ to obtain f_α as g .

So we proved the Key Lemma 3.3.8. \dashv

So we proved theorem 3.3.4 \dashv

The following theorem is hidden in [?]:

THEOREM 3.3.11. *If $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and \square_ν holds for each $\kappa \leq \nu < \lambda$, then there is a cofinal, locally small family in $[\lambda]^\kappa$.*

Indeed, in subsection 2.4 of [?] the author defines the *weakly rounded* subsets of $\lambda = \kappa^{+n}$, in Lemma 2.4.1 he shows that the family of weakly rounded sets is cofinal, finally on page 52 he proves a Claim which clearly implies that the family of weakly rounded sets is locally small.

Putting together Theorems 3.3.4 and 3.3.11 we obtain the following corollary.

COROLLARY 3.3.12. *If $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and \square_ν holds for each $\kappa \leq \nu < \lambda$, then there is a κ -homogeneous, but not κ -transitive permutation group on λ .*

§4. ω -homogeneous but not ω -transitive permutation groups in the Cohen model. For $f \in S(\kappa)$ let $\text{supp}(f) = \{\alpha : f(\alpha) \neq \alpha\}$. Write

$$S_\omega(\lambda) = \{f \in S(\lambda) : |\text{supp}(f)| \leq \omega\}.$$

THEOREM 4.4.1. *If $P = \text{Fin}(2^\omega, 2)$ then*

$V^P \models$ “for each $\lambda \geq \omega_1$ there is an ω -homogeneous and ω -intransitive permutation group on λ .”

The proof of this theorem is based on the following Lemma. Let us recall that if $g \in S(\omega_1)$ then $g^+ = g \cup (\text{id} \upharpoonright (\lambda \setminus \omega_1))$.

LEMMA 4.4.2. *Assume that $V_0 \subset V_1$ are ZFC models and $\lambda \geq \omega_2$ is a cardinal in V_1 . If*

1. $\forall X \in ([\lambda]^\omega)^{V_1} \exists Y \in ([\lambda]^\omega)^{V_0} X \subset Y$,
2. $V_1 \models G$ is an ω -homogeneous permutation group on ω_1 ,
 $G \supset S_\omega(\omega_1)^{V_0}$, and $r \in S(\omega)$ is G -large,

then in V_1 the permutation group G^* on λ generated by

$$\{g^+ : g \in G\} \cup S_\omega(\lambda)^{V_0}$$

is ω -homogeneous, and r is G^* -large.

PROOF. We will work in V_1 .

First we show that G^* is ω -homogeneous.

If $X, Y \in [\lambda]^\omega$ first pick $X_0, Y_0 \in [\lambda]^\omega \cap V_0$ with $X \subset X_0$ and $Y \subset Y_0$ such that $|X_0 \setminus X| = |Y_0 \setminus Y| = \omega$. Fix $f, h \in S_\omega(\lambda)^{V_0}$ with $f[X_0] = \omega$ and $h[Y_0] = \omega$. Since G is ω -homogeneous, there is $g \in G$ with $g[f[X]] = h[Y]$. Then $(h^{-1} \circ g^+ \circ f)[X] = Y$ and $h^{-1} \circ g^+ \circ f \in G^*$.

Before proving that r is G^* -large we need some preparation. Write

$$G^+ = \{g^+ : g \in G\}.$$

CLAIM 4.4.2.1. *If $h_0, \dots, h_k \in S_\omega(\lambda)^{V_0}$ and $A \in [\omega_1]^\omega$ then there is $h \in S_\omega(\omega_1)^{V_0}$ such that*

$$(h_0 \circ \dots \circ h_k) \cap (A \times A) \subset h.$$

PROOF OF THE CLAIM 4.4.2.1. By (1) we can assume that $A \in V_0$, and so $h' = (h_0 \circ \dots \circ h_k) \cap (A \times A) \in V_0$. Since h' is a countable injective function with $\text{dom}(h') \cup \text{ran}(h') \subset \omega_1$ it can be extended to a permutation $h \in S_\omega(\omega_1)^{V_0}$. \dashv

If \mathcal{F} is a set of functions, let

$$\langle \mathcal{F} \rangle_{\text{gen}} = \{f_0 \circ \dots \circ f_{n-1} : n \in \omega, f_i \in \mathcal{F} \text{ for } i < n\}.$$

CLAIM 4.4.2.2. *For each $t \in \langle G^+ \cup S_\omega(\lambda)^{V_0} \rangle_{\text{gen}}$ there is a finite set $\mathcal{H} \subset \langle G \cup S_\omega(\lambda)^{V_0} \rangle_{\text{gen}}$ such that*

$$t \subset \bigcup \mathcal{H}.$$

PROOF OF THE CLAIM 4.4.2.2. If $t = f_0 \circ \dots \circ f_{n-1}$, let

$$\mathcal{H} = \{\text{id}_\lambda\} \cup \{f'_{i_0} \circ \dots \circ f'_{i_j} \circ \dots \circ f'_{i_k} : k \leq n, i_0 < \dots < i_j < \dots < i_k < n\},$$

where $f'_i = f_i$ if $f_i \in S_\omega(\lambda)^{V_0}$, and $f'_i = g$ if $f_i = g^+$ for some $g \in G$, and id_λ denotes the identity function on λ .

Pick $\alpha \in \lambda$ such that $t(\alpha) \neq \alpha$.

Write $\alpha_n = \alpha$ and $\alpha_i = f_i(\alpha_{i+1})$ for $i = n-1, \dots, 0$. Let $0 \leq i_0 < i_1 < \dots < i_\ell < n$ be the increasing enumeration of the set $\{i < n : \alpha_i \neq \alpha_{i+1}\}$. Let $s = f'_{i_0} \circ \dots \circ f'_{i_\ell}$. Then $s \in \mathcal{H}$ and $s(\alpha) = t(\alpha)$. \dashv

CLAIM 4.4.2.3. For each $s \in \langle G \cup S_\omega(\lambda)^{V_0} \rangle_{\text{gen}}$ and countable set $A \in [\omega_1]^\omega$ there is $u \in \langle G \cup S_\omega(\omega_1)^{V_0} \rangle_{\text{gen}}$ such that

$$s \cap (A \times A) \subset u.$$

PROOF OF THE CLAIM 4.4.2.3. Since both G and $S_\omega(\lambda)^{V_0}$ are groups we can assume that

$$s = g_0 \circ h_0 \circ \dots \circ g_n \circ h_n,$$

where $g_i \in G$ and $h_i \in S_\omega(\lambda)^{V_0}$.

Write $A_n = A$, and let $B_i = h_i[A_{i+1}] \cap \omega_1$ and $A_i = g_i[B_i]$ for $i = n-1, \dots, 0$.

By Claim 4.4.2.1 for each i there is $h'_i \in S_\omega(\omega_1)^{V_0}$ such that $h_i \cap (A_{i+1} \times B_i) \subset h'_i$.

Let $u = g_0 \circ h'_0 \circ \dots \circ g_n \circ h'_n$.

We show that $s \cap (A \times A) \subset u$.

Fix $\alpha \in A$. Let $\alpha_n = \alpha$ and for $i = n-1, \dots, 0$ let $\beta_i = h_i(\alpha_{i+1})$ and $\alpha_i = g_i(\beta_i)$. If $s(\alpha)$ is defined and $s(\alpha) \in A$, then for each $i < n$ we have $\beta_i \in B_i$ and $\alpha_i \in A_i$, and so $u(\alpha)$ is also defined and $u(\alpha) = s(\alpha)$. \dashv

Putting together Claims 4.4.2.2 and 4.4.2.3 we obtain that

CLAIM 4.4.2.4. For each $g \in G^*$ there is a finite subset H_g of G such that

$$g \cap (\omega \times \omega) \subset \bigcup \{h \upharpoonright \omega : h \in H_g\}.$$

Claim 4.4.2.4 yields that r is G^* -large.

So we proved the G^* is ω -intransitive which completes the proof of the lemma. \dashv

By Lemma 4.4.2 the following theorem yields theorem 4.4.1.

THEOREM 4.4.3. If $P = \text{Fin}(2^\omega, 2)$ then $V^P \models$ "there is an ω -homogeneous and ω -intransitive permutation group G on ω_1 with $G \supset S_\omega(\omega_1)^V$ ".

PROOF. Given sets X and Y let us denote by $\text{Bij}_p(X, Y)$ the set of all finite bijections between subsets of X and Y .

We will define an iterated forcing system with finite support

$$\langle P_\nu : 0 \leq \nu \leq 2^\omega, Q_\nu : -1 \leq \nu < 2^\omega \rangle$$

and an increasing sequence of permutation groups $\langle G_\nu : \nu < 2^\omega \rangle$, $G_\nu \triangleleft S(\omega)^{V^{P_\nu}}$, simultaneously.

Take $G_0 = S_\omega(\omega_1)^V$ and $P_0 = \mathcal{Q}_{-1} = \text{Bij}_p(\omega, \omega)$. Denote by r the generic permutation of ω given by the V -generic filter over P_0 . By standard density arguments it is easy to see that r is G_0 -large. Now we carry out the inductive construction as follows:

- for each $\nu < 2^\omega$ we pick $X_\nu, Y_\nu, Z_\nu \in ([\omega_1]^\omega)^{V^{P_\nu}}$ with $X_\nu \cup Y_\nu \subset Z_\nu$ and $|Z_\nu \setminus X_\nu| = |Z_\nu \setminus Y_\nu| = \omega$,
- put

$$Q_\nu = \{p_0 \cup p_1 : p_0 \in \text{Bij}_p(X_\nu, Y_\nu), p_1 \in \text{Bij}_p(Z_\nu \setminus X_\nu, Z_\nu \setminus Y_\nu)\},$$

$$\mathcal{Q}_\nu = \langle Q_\nu, \supset \rangle \text{ and } g_\nu = \bigcup \mathcal{G}_\nu, \text{ where } \mathcal{G}_\nu \text{ is the } \mathcal{Q}_\nu\text{-generic filter over } V^{P_\nu},$$
- take $G_{\nu+1}$ as the subgroup of $S(\omega_1)^{V^{P_{\nu+1}}}$ generated by $G_\nu \cup \{g_\nu^+\}$.
- for limit ν let $G_\nu = \bigcup_{\zeta < \nu} G_\zeta$.

We use a bookkeeping function to ensure that every pair $X, Y \in ([\omega]^\omega)^{V^{P_{2^\omega}}}$ will be chosen as X_ν, Y_ν in some step. Then $G = \bigcup_{\nu < 2^\omega} G_\nu$ will be ω -homogeneous.

So the question is whether we guarantee that r is G_ν -large during the induction.

If ν is a limit ordinal, then $G_\nu = \bigcup_{\zeta < \nu} G_\zeta$, so if r is G_ζ -large for $\zeta < \nu$, then r is G_ν -large as well.

Assume now that r is G_ν -large and prove that r is $G_{\nu+1}$ -large as well.

The following lemma clearly implies this statement. In this lemma we use some notations introduced in Definition 3.3.3 in the previous section.

LEMMA 4.4.4. *If \mathcal{H} is a finite set of $G_\nu \cup \{x\}$ -terms, $p \in Q_\nu$, M is a natural number, then there is a condition $q \leq p$ in Q_ν and there is $\alpha \in \omega \setminus M$ such that $t[q](\alpha)$ is defined for each $t \in \mathcal{H}$ and $t[q](\alpha) \neq r(\alpha)$.*

PROOF OF THE LEMMA. We can assume that \mathcal{H} is closed for subterms.

We know that $|r \setminus \bigcup \mathcal{H}[]| = \omega$ because r is G_ν -large.

Since \mathcal{H} is closed for subterms,

$$y \cap \bigcup \mathcal{H}[] = y \cap \bigcup \mathcal{H}[\text{id}_{\omega_1 \setminus Z_\nu}].$$

Since $|p| < \omega$, we have

$$|y \setminus \bigcup \mathcal{H}[p \cup \text{id}_{(\lambda \setminus Z_\nu)}]| = \omega.$$

So we can pick $\alpha \in \omega \setminus M$ such that

- (*) for each $t \in \mathcal{H}$ either $t[p \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha)$ is undefined or $t[p \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha) \neq r(\alpha)$.

Now in finitely many steps, using Lemma 3.3.5, we can extend the function $p \in Q_\nu$ to a function $q \in Q_\nu$ such that

(*) $t[q \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha)$ is defined and $t[q \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha) \neq r(\alpha)$ for each $t \in \mathcal{H}$.

Indeed, if $t[q' \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha)$ is not defined, where $t = \langle t_0, \dots, t_n \rangle$ then there is $i < n$ such that either

$\zeta_i = \langle t_{i+1}, \dots, t_n \rangle [q' \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha)$ is defined, $t_i = x$ and $\zeta_i \notin \text{dom}(q')$

or

$\zeta = \langle t_{i+1}, \dots, t_n \rangle [q' \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha)$ is defined, $t_i = x^{-1}$ and $\zeta_i \notin \text{ran}(q')$.

In both cases, using Lemma 3.3.5, we can extend q' to q'' such that $\langle t_i, \dots, t_n \rangle [q'' \cup \text{id}_{\lambda \setminus Z_\nu}](\alpha)$ is defined and $\langle \alpha, r(\alpha) \rangle \notin \mathcal{H}[q'' \cup \text{id}_{\lambda \setminus Z_\nu}]$. So we proved Lemma 4.4.4. \dashv

So r is $G_{\nu+1}$ -large.

Thus, by transfinite induction, we proved that r is G -large which completes the proof of the theorem. \dashv

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INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, BUDAPEST, HUNGARY

E-mail: soukup@renyi.hu

URL: <http://www.renyi.hu/~soukup>