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Note

Note on a Min-Max Problem of Leo Moser

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ABSTRACT

Moser asks how a pair of $(n + 1)$-sided dice should be loaded (identically) so that on throwing the dice the frequency of the most frequently occurring sum is as small as possible. G. F. Clements finds a relative minimum, conjecturing that it is always the solution. This conjecture is disproved for $n = 3$.

For a fixed integer $n \geqslant 1$ let $A = (a_0, ..., a_n)$ denote a point in Euclidean $(n + 1)$-space. Let

$$p(A, x) = \sum_{j=0}^{n} a_j x^j,$$

$$q(A, x) = p^2(A, x) = \sum_{j=0}^{2n} c_j(A)\, x^j,$$

and let

$$M(A) = \max_{0 \leqslant j \leqslant 2n} c_j(A).$$

Moser in [1] asks for the minimum of M, subject to the conditions

$$a_j \geqslant 0, \qquad j = 0, 1, ..., n, \qquad \sum_{j=0}^{n} a_j = 1.$$

Clements, in [2], conjectures that $\bar{A} = (\bar{a}_0, ..., \bar{a}_n)$ where

$$\bar{a}_i = K(n) \binom{-1/2}{j} (-1)^j, \qquad j = 0, ..., n,$$

and

$$K(n) = \left(\sum_{j=0}^{n} \binom{-1/2}{j} (-1)^j\right)^{-1}$$

is the solution, as $c_0(\bar{A}) = c_1(\bar{A}) = \cdots = c_n(\bar{A})$. He proves it for $n = 1, 2$ and shows that $\bar{A}$ is a relative minimum.

Now if A is a solution, then some of the $c_j(A)$ are equal to $M(A)$. If we assume that $c_0(A),\dots, c_n(A)$ are equal to $M(A)$ we obtain Clements' solution. In my opinion it is more natural to look for a solution A such that $c_j(A) = M(A)$ for some other set of indices j, say for $[n/2] \leqslant j \leqslant [n/2] + n$, but I did not succeed in finding such an A.

Let us examine in that way the most simple case not solved by Clements, that is, $n = 3$. We try to find an A such that

$$\begin{aligned} c_1(A) = c_2(A) &= c_3(A) = c_4(A) = M(A);\\ 2a_0a_1 = c_1(A) &= 2a_0a_2 + a_1^2 = c_2(A)\\ &= 2a_0a_3 + 2a_1a_2 = c_3(A)\\ &= 2a_1a_3 + a_2^2 = c_4(A). \end{aligned}$$

Let $x = a_1/a_0$. Then

$$a_2 = [c_2(A) - a_1^2]/2a_0 = [2a_0a_1 - a_1^2]/2a_0 = a_0(2x - x^2)/2$$

and

$$\begin{aligned} a_3 &= [c_3(A) - 2a_1a_2]/2a_0\\ &= [2a_0a_1 - 2a_1a_2]/2a_0\\ &= a_0[x - x(2x - x^2)/2]\\ &= a_0(x^3 - 2x^2 + 2x)/2; \end{aligned}$$

$$\begin{aligned} 2a_0^2x &= 2a_0a_1 = c_1(A) = c_4(A) = 2a_1a_3 + a_2^2\\ &= a_0^2[2x \cdot (x^3 - 2x^2 + 2x)/2 + (2x - x^2)^2/4]. \end{aligned}$$

Therefore

$$5x^3 - 12x^2 + 12x - 8 = 0.$$

Here $x = 3/2$ is a good approximation. Remembering that

$$\sum_{i=0}^{3} a_i = 1$$

we have $A = (16/61, 24/61, 6/61, 15/61)$ and $M(A) = 3.16^2/61^2$, which is less than the result of Clements' conjecture, $16^2/35^2$.

REMARK. Mr. Pinhas Shapira and I disproved the conjecture also for $n = 4, 5$ by numerical computations.

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References

1. L. Moser, *Report of the Institute in the Theory of Numbers*, University of Colorado, June 21–July 17, 1959.
2. G. F. Clements, On a Min-Max Problem of Leo Moser, *J. Combinatorial Theory* **4** (1968), 36–39.