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# ON CARDINAL INVARIANTS OF THE CONTINUUM

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## 0. Introduction.

For a survey on this area, see van-Douwen [D] and Balcar and Simon [BS].

Nyikos has asked us whether there may be (in our terms) an undominated family $\subseteq {}^{\omega}\omega$ of power $\aleph_1$, while there is no splitting family $\subseteq [\omega]^{\omega}$ of power $\aleph_1$. He observed that it seems necessary to prove, assuming CH, the existence of a P-point without a Ramsey ultrafilter below it (in the Rudin-Keisler order). We give here a positive answer, using a countable support iteration of length $\aleph_2$ of a special forcing notion whose definition takes some space. This forcing notion makes the "old" $[\omega]^{\omega}$ an unsplitting family. The proof of this is quite easy, but we have more trouble proving that the "old" ${}^{\omega}\omega$ is not dominated, and then we have to prove that this is preserved by the iteration. We prove a more general preservation lemma. From the forcing notion (and, in fact, using a simpler version), we can construct a P-point as above.

Then E. Miller told us he is more interested in having in this model "no MAD has power $\leq \aleph_1$ (MAD stands for "a maximal almost disjoint family of infinite subsets of $\omega$"). A variant of our forcing can "kill" a MAD and the forcing has the desired properties if we first add $\aleph_1$ Cohen reals.

In the first section we prove a preservation lemma for countable support iterations whose main instance is that no new $f \in {}^{\omega}\omega$ dominates all old

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ones, and prove the consistency of ZFC + $2^{\aleph_0} = \aleph_2 + \mathfrak{d} = \mathfrak{s} > \mathfrak{b}$ where $\mathfrak{d}$ is the minimal power of a dominating subfamily of ${}^\omega\omega$ (see 1.1), $\mathfrak{s}$ is the minimal power of a splitting subfamily of $[\omega]^\omega$ (see 1.3), and $\mathfrak{b}$ is the minimal power of an undominated subfamily of ${}^\omega\omega$.

However, a main point was left out in Section 1: the definition of the forcing we iterate, and the proof of its relevant properties: that it adds a subset $\underline{r}$ of $\omega$ such that $\{A \in V: A \subseteq \omega, \underline{r} \subseteq^* A\}$ is an ultrafilter in the Boolean algebra $\mathcal{P}(\omega)^V$; but in a strong sense it does not add a function $\underset{\sim}{f} \in {}^\omega\omega$ dominating all old members of ${}^\omega\omega$. Note that Mathias forcing adds a subset $\underline{r}$ of $\omega$ as required above, but also adds an undesirable $\underset{\sim}{f}$.

In those sections we also prove the consistency of ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ + $\aleph_2 = \mathfrak{s} = \mathfrak{a} > \mathfrak{b} = \aleph_1$, where $\mathfrak{a} = \min\{|A| : A$ a maximal family of almost disjoint subsets of $\omega\}$. In the third section we show that in the model we have constructed, there is a MAD (maximal family of pairwise almost disjoint infinite subsets of $\omega$) of power $\aleph_1$ (hence $\mathfrak{a} = \aleph_1$). This answers a question of Balcar and Simon: they defined

$\mathfrak{a}_s = \min\{|A| :$ A is a maximal family of almost disjoint subsets of $\omega\times\omega$, which are graphs of partial function from $\omega$ to $\omega\}$.

They have proved $\mathfrak{s} \leqslant \mathfrak{a}_s$ and $\mathfrak{a} \leqslant \mathfrak{a}_s \leqslant 2^{\aleph_0}$, so our result implies that $\mathfrak{a} < \mathfrak{a}_s$ is consistent.

In the fourth section we present a proof[1] of the consistency of $\aleph_1 = \mathfrak{s} < \mathfrak{b} = \aleph_2$ by finite support iteration of Hechler forcing.

In the fifth section we prove the consistency (with ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$) of $\aleph_1 = \mathfrak{h} < \mathfrak{s} = \mathfrak{b} = \aleph_2$ (where $\mathfrak{h}$ is the minimal cardinal $\kappa$ for which $\mathcal{P}(\omega)$/finite is a $(\kappa, 2^{\aleph_0})$-distributive Boolean algebra).

So the order relationships between the cardinals mentioned above are

[1]This was proved several years ago by Balcar and Simon (this result is mentioned in Remark 4.7 in p.18 [BPS]). However, as we have already written up the proof and as they used a different model (add $\aleph_1$ random reals to a model satisfying MA), we retain this section.

$$\begin{array}{ccccccccc}
 & & \mathfrak{s} & \longrightarrow & \mathfrak{d} & \longrightarrow & & & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
\aleph_1 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{a}_s
\end{array}$$

(where arrow means "$\leqslant$ is provable is ZFC") (see [D] for results not mentioned above, and on two other cardinal invariants).

## 1. The Iteration.

In this section we define some properties, prove a preservation lemma and then prove our theorem except for one crucial point -- the existence of specific forcings which are the individual steps in our iteration.

1.1. Notation: a) ${}^{\omega}\omega$ is the set of functions from $\omega$ to $\omega$.

b) $<^*$ is the partial order defined on ${}^{\omega}\omega$ as: $f <^* g$ iff for all but finitely many $n < \omega$, $f(n) < g(n)$. In this case we say that $g$ dominates $f$. We say that $g$ dominates a family $F \subseteq {}^{\omega}\omega$ if $g$ dominates every $f \in F$.

c) $[\omega]^{\omega}$ is the family of infinite subsets of $\omega$. We say $A \subseteq^* B$ if $A - B$ is finite.

1.2. Definition:

1) A family $F \subseteq {}^{\omega}\omega$ is dominating if every $g \in {}^{\omega}\omega$ is dominated by some $f \in F$.

2) A family $F \subseteq {}^{\omega}\omega$ is unbounded (or undominated) if no $g \in {}^{\omega}\omega$ dominates it.

1.3. Definition:

1) A family $\mathcal{P} \subseteq [\omega]^{\omega}$ is a splitting family if for every $A \in [\omega]^{\omega}$ for some $X \in \mathcal{P}$ $A \cap X$ and $A - X$ are infinite.

2) We call $\mathcal{P}$ MAD if it is a subfamily of $[\omega]^{\omega}$, its members are pairwise almost disjoint (= has finite intersections) and is maximal with respect to those two properties.

1.4. Definition:

1) A forcing notion $P$ is almost ${}^{\omega}\omega$-bounding if for every $P$-name of a function from $\omega$ to $\omega$ and $p \in P$ <u>for some</u> $g: \omega \to \omega$ (from V!) <u>for every</u> infinite $A \subseteq \omega$ (again $A$ from $V$) there is $p'$, $p \leq p' \in P$ such that

$$p' \Vdash_P \text{"for infinitely many } n \in A, \ \underset{\sim}{f}(n) < g(n)\text{"}$$

2) A forcing notion $P$ is weakly bounding (or $F$-weakly bounding, where $F \subseteq ({}^{\omega}\omega)^V$) if $({}^{\omega}\omega)^V$ (or $F$) is an unbounded family in $V^P$.

<u>1.5. Claim</u>:

1) If a forcing notion $P$ is weakly bounding, and $\underset{\sim}{Q}$ ($\in V^P$) is almost ${}^{\omega}\omega$-bounding <u>then</u> their composition $P*\underset{\sim}{Q}$ is weakly bounding.

2) If $Q$ is almost ${}^{\omega}\omega$-bounding, $F \subseteq {}^{\omega}\omega$ an unbounded family (from V) <u>then</u> $F$ is still an unbounded family in $V^Q$.

We shall want to prove that e.g. the limit of a countable support iteration of almost (${}^{\omega}\omega$)-bounding forcing notions is weakly bounding. This will show us in the proof of the main theorem that the family of "old" functions in ${}^{\omega}\omega$ is unbounded. To this end we prove a more general preservation theorem closely connected to [Sh1, VI] and [Sh2, 1.3].

1.6. <u>Definition</u>:

1) We say $W$ is absolute if it is a <u>definition</u> (possibly with parameters) of a set so that if $V^1 \subseteq V^2$ are extensions of $V$ (but still models of ZFC) and $x \in V^1$ then $V^1 \vDash$ "$x \in W$" iff $V^2 \vDash$ "$x \in W$". Note that a relation is a particular case of a set. It is well known that $\underset{=}{\Pi}^1_2$ relations on reals and generally $\kappa$-Souslin relations are absolute.

2) We say that a player absolutely wins a game if the definition of legal move, the outcomes and the strategy (which need not be a function with a unique outcome) are absolute and its being a winning strategy is preserved by extensions of V.

3) We can relativize absoluteness to a family of extensions.

Remark: E.g. if $\bar{R}$ is $\underset{=}{\Sigma}^1_2$, the strategy is $\underset{=}{\Sigma}^1_1$ and the outcome of a play is $\underset{=}{\Pi}^1_2$.

1.7. Notation: $R$ will usually denote an absolute two-place relation on ${}^{\omega}\omega$ (so when we extend the universe, we reinterpret $R$, but we know that the interpretations are compatible). Sometimes $R$ is an absolute three-place relation on ${}^{\omega}\omega$ and then we write $xR^zy$ instead of $R(x,y,z)$.

Let $\bar{R}$ denote $\langle R_n: n < \omega\rangle$ (each $R_n$ as above) so $\bar{R}^m = \langle R^m_n: n < \omega\rangle$. We identify $\langle R: n < \omega\rangle$ with $R$.

Let $\eta < \nu$ mean $\eta$ is an initial segment of $\nu$; $P_1 < P_2$ means $P_1$ is a submodel of $P_2$ (as partial orders) and every maximal antichain of $P_1$ is a maximal antichain of $P_2$.

Let $\mathcal{S}_{<\kappa}(A) = \{B \subseteq A: |B| < \kappa\}$ and if $\kappa$ is regular uncountable $\mathcal{D}_{<\kappa}(A)$ is the filter on $\mathcal{S}_{<\kappa}(A)$ generated by the sets $G(M) = \{|N|: N \prec M, \|N\| < \kappa\}$ for $M$ a model with universe $A$ and $< \kappa$ relations.

1.8. Definition:

1) For $F \subseteq {}^{\omega}\omega$ and $R$ (two place), we say that $F$ is $R$-bounding if $(\forall f \in {}^{\omega}\omega)(\exists g \in F)[f\ R\ g]$.

2) For $F \subseteq {}^{\omega}\omega$, $\bar{R}$ (each $R_n$ two place) and $S \subseteq \mathcal{S}_{<\aleph_1}(F)$ the pair $(F,\bar{R})$ is $S$-nice if

α) $F$ is $\bar{R}$-bounding which means it is $R_n$-bounding for each $n$.

β) For any $N \in S$, for some $g \in F$, for every $n_0, m_0$ player II has a winning strategy for the following game which lasts $\omega$ moves and which is absolute for extensions preserving (α). On the $k$th move: player I chooses $f_k \in {}^{\omega}\omega$, $g_k \in F \cap N$, such that $f_k \restriction m_{\ell+1} = f_\ell \restriction m_{\ell+1}$ for $0 < \ell < k$ and $f_k\ R_{n_k}\ g_k$ then player II chooses $m_{k+1} > m_k$ and $n_{k+1} > n_k$. In the end player II wins if $\bigcup_k f_k \restriction m_k\ R_{n_0}\ g$.

3) We say $(F,\bar{R})$ is $S/\mathcal{D}_{<\aleph_0}(F)$-nice if the set of $N$ for which (β) holds or $N \notin S$ belongs to $\mathcal{D}_{<\aleph_0}(F)$.

4) We omit $S$ when this holds for some $S \in \mathcal{D}_{<\aleph_0}(F)$.

5) We say "almost S-nice" if in 2) ($\beta$) we just demand that player I has no winning strategy in any extension of V.

Remark: We can use ${}^{\omega}\lambda$ instead ${}^{\omega}\omega$.

Sometimes we need a more general framework (but the reader may skip it, later replacing $H_z$, $R^z_n$ by $F$, $R_n$).

1.9. Notation. If $H$ is a set of pairs, let Rang $H = \{y: (\exists x)\langle x,y\rangle \in H\}$ Dom $H = \{x: (\exists y)\ \langle x,y\rangle \in H\}$, $H_x = \{y: \langle x,y\rangle \in H\}$.

We shall treat a set $F$ as $\{\langle x,x\rangle: x \in F\}$.

1.10. Definition.

1) For a set $H \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$, and $\bar{R}$ and $S \subseteq \mathcal{S}_{<\aleph_1}(F)$ we say that $(H,\bar{R})$ is S-nice if

$\alpha$) For every $z \in \mathrm{Dom}H$, $H_z$ is $\bar{R}^z$-bounding, i.e. $(\forall n)(\forall f \in {}^{\omega}\omega)(\exists g \in H_z)[f\ R^z_n\ g]$ letting $\bar{R}^z = \langle R^z_n: n < \omega\rangle$.

$\beta$) For any $N \in S$ for some $g \in$ Rang $H$ for every $z_0 \in \mathrm{Rang}(H\cap N)$ and for every $n_0,m_0$ player II absolutely modulo $\alpha$) wins the following game which lasts $\omega$ moves. In the kth move: player I chooses $f_k \in {}^{\omega}\omega$, $g_k \in \mathrm{Rang}(H\cap N)$ such that $f_k\restriction m_{\ell+1} = f_\ell\restriction m_{\ell+1}$ for $0 < \ell < k$ and $f_k\ R^{z_k}_{n_k}\ g_k$ then player II chooses $m_{k+1} > m_k$, $n_{k+1} > n_k$ and $z_{k+1} \in \mathrm{Dom}(H\cap N)$. At the end of play, player II wins iff $(\bigcup_k f_k\restriction m_{k+1})\ R^{z_0}_{n_0}\ g$.

2) We write "almost S-nice" if in ($\beta$) player I has no winning strategies and this is absolute. Let us give few examples.

1.11. Claim: Let $F \subseteq {}^{\omega}\omega$ be an unbounded set, such that $(\forall f_0,\dots,f_n,\dots \in F)(\exists g \in F)[\bigwedge_{n<\omega} f_n <^* g]$ and $f\ R\ g$ iff $g \not<^* f$.

Then $(F,R)$ is nice.

Proof: We have to describe $g$ and an absolute winning strategy for $N$. Choose $g\in F$, $(\forall f\in N)\ f <^* g$. As for the strategy, $n_\ell$ is irrelevant, we just choose $m_{k+1} = \min\{m$: there are at least $k$ numbers $i < n$ such that $g(i) > f_k(i)\}$.

1.12. Claim: Suppose $P \subseteq [\omega]^\omega$ is a P-filter (i.e. it is a filter and for any $A_n \in P$ $(n < \omega)$ for some $A^* \in P$, $(\forall n)[A^* \subseteq^* A_n]$) with no intersection (i.e. there is no $X \in [\omega]^\omega$, $X \subseteq^* A$ for every $A \in P$).

Let $R$ be: $xRy$ iff $x \notin [\omega]^\omega$ or $y \notin [\omega]^\omega$ or $y \not\subseteq^* x$. (We identify $x \subseteq \omega$ with its characteristic function).

Then $(P,R)$ is nice.

Proof: Now $(\alpha)$ is obvious. In $(\beta)$ choose $g = A^* \in F$ such that $(\forall A \in N)\ A^* \subseteq^* A$.

Again the only non-obvious point is the winning strategy; again $n_k$ is irrelevant and player II chooses $m_k = \min\{m: f_k \cap m \cap g$ has power $> k\}$.

1.13. Lemma:

1) Suppose $\langle P_j, \underset{\sim}{Q}_i : i<\delta,\ j\leq\delta\rangle$ is a countable support iteration of proper forcing.

Suppose further that $S \subseteq \mathcal{S}_{<\aleph_1}(H)$ is stationary (i.e. $\neq \emptyset \bmod \mathcal{D}_{<\aleph_1}(H)$), in $V$, $(H,\bar{R})$ is $S/\mathcal{D}_{<\aleph_1}(H)$-nice and for every $i < \delta$, in $V^{P_i}$ $H$ is $\bar{R}$-bounding.

Then in $V^{P_\delta}$, $H$ is $\bar{R}$-bounding.

2) We can replace $S/\mathcal{D}_{<\aleph_1}(H)$-nice by almost $S/\mathcal{D}_{<\aleph_1}(H)$-nice.

Remark:

1) For the case which we really need in 1.15, you can read the proof with $n_0 = 0$, $F$ instead $H$, $R$ instead $R^n_{z_n}$.

2) The proof gives somewhat more than the lemma, i.e. it applies to more cases. "$H$ is $\bar{R}$-bounding" means that $(\alpha)$ of 1.10 holds.

Proof: 1) If $cf\delta > \aleph_0$, then any real in $V^{P_\delta}$ belongs to $V^{P_j}$ for some $j<\delta$ (see [Sh1, III, 4.4]); hence there is nothing to prove, so we shall assume $cf\delta = \omega$. By [Sh1, III, 3.3], w.l.o.g. $\delta = \omega$.

Suppose $p \in P_\omega$, $z_0 \in \mathrm{Dom}\, H$, $n_0 < \omega$ and $\Vdash_{P_\omega}$ "$\underset{\sim}{f} \in {}^\omega\omega$" ; we shall find $r$, $p \leqslant r \in P_\omega$ and $g \in H_{z_0}$ such that $r \Vdash_{P_\omega}$ "$f\ R^{z_0}_{n_0}\ g$". Let $N$ be a countable elementary submodel of $(H(\lambda),\in)$ ($\lambda$ regular large enough) to which $\langle P_j, Q_i : i<\omega, j\leqslant\omega\rangle$, $p$, $\underset{\sim}{f}$, $z_0$, $S$, $H$ belong as well as the parameters involving the definitions of the $R_n$'s. The set of such $N$ belongs to $\mathcal{D}_{<\aleph_1}(H(\lambda))$, hence for some such $N$, $N \cap H \in S$.

As in [Sh1, III 3.2], w.l.o.g. $\underset{\sim}{f}(n)$ is a $P_n$-name; and we let $p = \langle p^0_n : n<\omega\rangle \Vdash_{P_n}$ "$p^0_n \in \underset{\sim}{Q}_n$". Let $g \in H_{z_0}$ be as in Def. 1.8 (for $N \cap H$).

We shall now define by induction on $k<\omega$ $q_k$, $p_k$, $\underset{\sim}{p}_k$, $g_k$, $z_k$, $m_k$, $n_k$ such that

1) $q_k \in P_k$ is $(N,P_k)$-generic

2) $q_k \restriction n = q_n$ for $n < k$

3) $p_k \in P_\omega$

4) $q_k \geqslant p_k \restriction k$

5) $p_{k+1} \restriction k = p_k \restriction k$, $p_{n+1} \geqslant p_n$

6) $q_k \Vdash_{P_k}$ "$p_k \in N$"

7) $z_k \in \mathrm{Dom}(H \cap N)$ is a $P_k$-name

8) $m_k < m_{k+1}$ are $P_k$-names of natural numbers

Note that 1) implies that $N \cap H$ belongs to the club of $\mathcal{S}_{<\aleph_1}(H)$ involving "$(H,\bar{R})$ is $S/\mathcal{D}_{<\aleph_1}(H)$-nice".

For $k = 0$, $q_0 = \emptyset$, $p_0 = p$.

For $k+1$, we work in $V[\underset{\sim}{G}_k]$, $\underset{\sim}{G}_k$ a generic subset of $P_k$, $q_k \in G_k$. So $p_k \in N[G_k]$ $p_k \restriction k \in G_k$. In $N[G_k]$ we can find an increasing sequence of conditions $p_{k,i} \in P_\omega/P_n$ for $i < \omega$, such that $p_{k,i} \in N[G_k]$, $p_{k,i}$ forces values for $\underset{\sim}{f}(j)$, $j \leqslant i$. So for some function $f_k \in N[G_k]$, $p_{k,i} \Vdash_{P_\omega/P_k}$ "$\underset{\sim}{f} \restriction i = f_k \restriction i$". As $N[\underset{\sim}{G}_k] \prec (H(\lambda)[G_k],\in)$ (see [Sh1 III 2.11, p. 89]) for some

$g_k \in N \cap H_{z_k}$, $N[G_k] \models$ "$f_k \, R^{z_k}_{n_k} \, g_k$". Now we use the absolute strategy (from Def 1, for $N \cap H$) to choose $z_{k+1}$, $n_{k+1}$, $m_{k+1}$ (the strategy's parameters may not be in $N$, but the result is) and we want to have $p_{k+1} = p_{k,m_{k+1}}$. However all this was done in $V[\underset{\sim}{G}_k]$, so we have only a suitable $P_k$-name. In the end, let $r \in P_\omega$ be defined by $r \restriction k = q_k \restriction k$ for each $k$; by requirement (2) this suffices. Suppose $r \in G_\omega \subseteq P_\omega$, $G_\omega$ generic. Then in $V[G_\omega]$ we have made a play of the game from Def. 1.10, player II using his winning strategy so $(\bigcup_k f_k \restriction k)[G_\omega] R^{z_0}_{n_0} \, g$ holds in $V[G_\omega]$, but clearly $p_{k,n_k} \leqslant p_{k+1} \leqslant r$ hence $p_{k,n_k} \in G_\omega$ hence $(\underset{\sim}{f}_k \restriction k)[G_\omega] = (\underset{\sim}{f} \restriction k)[G_\omega]$, so $\underset{\sim}{f}[G_\omega] = \bigcup_k (f_k \restriction k)[G_\omega]$. So $\underset{\sim}{f}[G_\omega] \, R^{z_0}_{n_0} \, g$ holds in $V[G_\omega]$. So $r$ forces the required information.

We shall prove later (in 2.13)

1.14 <u>Main Lemma</u>. There is a forcing notion $Q$ such that

(a) $Q$ is proper

(b) $Q$ is almost ${}^\omega\omega$-bounding

(c) $|Q| = 2^{\aleph_0}$

(d) In $V^Q$ there is an infinite set $A^* \subseteq \omega$ such that for every infinite $B \subseteq \omega$ from $V$ $A^* \cap B$ or $A^* - B$ is finite.

1.14A <u>Remark</u>. For 1.15 it is enough to prove 1.14 assuming CH.

1.15 <u>Main Theorem</u>. Assume $V \models CH$.

1) Then for some forcing notion $P^*$ ($P^*$ is proper, satisfies the $\aleph_2$-c.c., is weakly bounding and)

(*) In $V^{P^*}$, $2^{\aleph_0} = \aleph_2$, there is an unbounded family of power $\aleph_1$, but no splitting family of power $\aleph_1$.

2) We can also demand that in $V^{P^*}$ there is no MAD of power $\aleph_1$ (see Def. 1.3(2)).

<u>Proof</u>.

1) We define a countable support iteration of length $\aleph_2$: $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_2 \rangle$

with (direct) limit $P^* = P_{\omega_2}$. Now each $\underset{\sim}{Q}_\alpha$ is the $Q$ from 1.14 for $V^{P_\alpha}$, so $V^{P_\alpha} \models$ "$|\underset{\sim}{Q}_\alpha| = 2^{\aleph_0}$". As $V \models CH$ we can prove by induction on $\alpha$ that $\Vdash_{P_\alpha}$ "CH" (see [Sh1, Th. 4.1, p. 96]). We also know that $P^*$ satisfies the $\aleph_2$-c.c. (see [Sh1, Th. 4.1, p. 96]). If $P$ is a family of subsets of $\omega$ of power $\leqslant \aleph_1$ in $V^{P^*}$ then for some $\alpha$, $P \in V^{P_\alpha}$, and forcing by $Q_\alpha$ gives a set $A^*_\alpha$ exemplifying $P$ is not a splitting family. So from all the conclusions of 1.15 only the existence of an undominated family of power $\aleph_1$ remains. Now we shall prove that $F = ({}^\omega\omega)^V$ is as required. It has power $\aleph_1$ as $V \models CH$. We prove that it is an undominated family in $V^{P_\alpha}$ by induction on $\alpha \leqslant \omega_2$. For $\alpha = 0$ this is trivial; $\alpha = \beta+1$: as $\underset{\sim}{Q}_\beta$ is almost ${}^\omega\omega$-bounding (see 1.14) and by Fact 1.5(1); if $cf\ \alpha \geqslant \aleph_0$ by Lemma 1.13.

2) Similar. We use a countable support iteration $\langle P_j, \underset{\sim}{Q}_i : i < \omega_2, j \leqslant \omega_2 \rangle$ such that:

(a) for every $i < \omega_2$, and MAD $\langle A_\alpha : \alpha < \omega_1 \rangle \in V^{P_i}$, for some $j > i$, <u>either</u> $\underset{\sim}{Q}_{2j}$ = adding $\aleph_1$-Cohen reals, and $\underset{\sim}{Q}_{2j+1} = \{p \in \underset{\sim}{Q}^{V^{P_{2j+1}}} : p \geqslant p_{2j+1}\}$ where in $V^{P_{2j+1}}$, $p_{2j+1} \Vdash_{\underset{\sim}{Q}}$ "$\langle A_\alpha : \alpha < \omega_1 \rangle$ is not a MAD" <u>or</u> $\underset{\sim}{Q}_{2j}$ = adding $\aleph_1$-Cohen reals, $\underset{\sim}{Q}_{2j+1} = Q[I_{2j+1}]^{V^{P_{2j+1}}}$ where $I_{2j+1}$ is the ideal which $\langle A_\alpha : \alpha < \omega_1 \rangle$ and the cofinite sets generate

(b) For $j$ even $\underset{\sim}{Q}_j$ is adding $\aleph_1$ Cohen reals

(c) For $j$ odd, $\underset{\sim}{Q}_j$ is $\underset{\sim}{Q}$ or $Q[I]$, or $\{p \in \underset{\sim}{Q} : p \geqslant p_j\}$, but always it is ${}^\omega\omega$-bounding.

Use 2.16, 2.17.

<u>Remark</u>. Really the conclusion of 1.5 is satisfied by each $Q_\alpha$ and is preserved by countable support iteration of proper forcing.

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## 2. The Forcing.

**2.1 Definition.** 1) Let $K_n$ be the family of pairs $(s,h)$, $s$ a finite set, $h$ a partial function from $\mathcal{P}(s)$ (the family of subsets of $s$) to $n+1$ such that

(a) $h(s) = n$

(b) if $h(t) = \ell+1$ $(t \subseteq s)$, $t = t_1 \cup t_2$ then $h(t_1) \geqslant \ell$ or $h(t_2) \geqslant \ell$.

2) $K_{\geqslant n}$, $K_{\leqslant n}$, $K_{(n,m)}$ are defined similarly, and $K = \cup K_n$.

We call $s$ the domain of $(s,h)$ and write $a \in (s,h)$ instead of $a \in s$. We call $(s,h)$ standard if $s$ is a finite subset of the family of hereditarily finite sets. We use the letter $d$ to denote such pairs. We call $(s,h)$ simple if $h(t) = [\log_2(t)]$ for $t \subseteq s$.

**2.2 Definition.**

1) Suppose $(s_\ell,h_\ell) \in K_{s(\ell)}$ for $\ell = 0,1$. We say $(s_0,h_0) \leqslant^d (s_1,h_1)$ (or $(s_1,h_1)$ refines $(s_0,h_0)$) if:

$$s_0 = s_1 \text{ and for } t_1 \subseteq t_2 \subseteq s_0,\ [h_1(t_1) < h_1(t_2) \Rightarrow h_0(t_1) < h_0(t_2)]$$

(so $n(0) \leqslant n(1)$) and $\mathrm{Dom}(h_1) \subseteq \mathrm{Dom}(h_0)$.

2) We say $(s_0,h_0) \leqslant^e (s_1,h_1)$ if for some $s_0' \in \mathrm{Dom}\, h_0$, $(s_0', h_0 \restriction \mathcal{P}(s_0')) = (s_1,h_1)$.

3) We say $(s_0,h_0) \leqslant (s_1,h_1)$ if for some $(s',h')$, $(s_0,h_0) \leqslant^e (s',h') \leqslant^d (s_1,h_1)$.

**2.3 Fact:** The relations $\leqslant^d$, $\leqslant^e$, $\leqslant$ are partial orders of $K$.

**2.4 Definition.**

1) Let $L_n$ be the family of pairs $(S,H)$ such that:

a) $S$ is a finite tree with a root.

b) $H$ is a function whose domain is $\mathrm{in}(S)$ = the set of non-maximal points of $S$ and value $H_x$ for $x \in \mathrm{in}(S)$.

c) For $x \in in(S)$, $(Suc_S(x), H_x) \in K_{\geq n}$ where $Suc_S(x)$ is the set of immediate successors of $x$ in $S$ with $H_x(Suc_S(x)) \geq n$.

2) We say $(S^0,H^0) \leq (S^1,H^1)$ if $S^0 \supseteq S^1$, they have the same root, $in(S^1) = S^1 \cap in(S^0)$ and for every $x \in in(S^1)$, $(Suc_{S^0}(x),H^0_x) \leq (Suc_{S^1}(x),H^1_x)$.

3) Let $int(S) = S - in(S)$, $lev(S,H) = \max\{n: (S,H) \in L_n\}$. $x \in (S,H)$ means $x \in S$. A member of $L_n$ is standard if $int(S) \subseteq \omega$ and $in(S)$ consists of hereditarily finite sets not in $\omega$. Let for $x \in S$, $(S,H)^{[x]} = (S^{[x]}, H \restriction S^{[x]})$ where $S^{[x]}$ is $S \restriction \{y \in S: S \models x \leq y\}$.

4) If $\underset{=}{t} \in L_n$, $\underset{=}{t} = (S^{\underset{=}{t}}, H^{\underset{=}{t}})$.

2.5 <u>Fact</u>. The relation $\leq$ is a partial order of $L = \bigcup_n L_n$.

2.6 <u>Fact</u>. If $(S,H) \in L_n$ then $(S',H') = half(S,H)$ belongs to $L_{[(n+1)/2]}$ where $S' = S$, $H'_s(A) = [H_s(A) - lev(S,H)/2]$ and $Dom(H'_s) = \{A: H_s(A) \geq lev(S,H)/2\}$.

2.7 <u>Fact</u>. If $(S,H) \in L_{n+1}$, $int(S) = A_0 \cup A_1$ <u>then</u> there is $(S^1,H^1) \geq (S,H)$, $(S^1,H^1) \in L_n$ and $[int(S^1) \subseteq A_0$ or $int(S^1) \subseteq A_1]$.

<u>Proof</u>. Easy by induction on the height of the tree.

2.8 <u>Definition</u>. We define the forcing-notion $Q$:

1) $p \in Q$ <u>if</u> $p = (W,T)$ where $W$ is a finite subset of $\omega$, $T$ is a countable (infinite) set of pairwise disjoint standard members of $L$ and $T - L_n$ is finite for each $n$; let $cnt(T) = \bigcup_{(H,S) \in T} int(S,H) = cnt(p)$.

2) Given $t_1 = (S_1,H_1),\ldots,t_k = (S_k,H_k)$ all from $L$ such that $S_i \cap S_j = \emptyset$ $(i \neq j)$, and given $t = (S,H)$ from $L$, $t$ is <u>built</u> from $t_1,\ldots,t_k$ if: There are incomparable nodes $a_1,\ldots,a_k$ of $S$ such that every node of $S$ is comparable with some $a_i$, and such that, letting $S(a_i) = \{b \in S: b \geq_S a_i\}$, $(S_i,H_i) = (S(a_i), H \restriction S(a_i))$.

3) $(W^0,T^0) \leq (W^1,T^1)$ iff: $W^0 \subseteq W^1 \subseteq W^0 \cup cnt(T^0)$, and: letting $T^0 = \{\underset{=}{t}^0_0, \underset{=}{t}^0_1,\ldots\}$, $T^1 = \{\underset{=}{t}^1_0, \underset{=}{t}^1_2,\ldots\}$, there are finite, non-empty, pairwise disjoint subsets of $\omega$, $B_0,B_1,\ldots$, and there are $\hat{\underset{=}{t}}_i \geq \underset{=}{t}^0_i$ for all

$i \in \cup_j B_j$, such that for each $n$ only finitely many of the $\hat{\underset{=}{t}}_i$ are inside $L_n$, and such that for each $j$, letting $B_j = \{i_1,\dots,i_k\}$, $\underset{=}{t}^1_j$ is built from $\hat{\underset{=}{t}}_{i_1},\dots,\hat{\underset{=}{t}}_{i_k}$.

4) We call $(W,T)$ standard if $T = \{\underset{=}{t}_n : n<\omega\}$, $\max(W) < \min[\mathrm{int}(\underset{=}{t}_n)]$, $\max[\mathrm{int}(\underset{=}{t}_n)] < \min[\mathrm{int}(\underset{=}{t}_{n+1})]$ and $\mathrm{lev}(\underset{=}{t}_n)$ is strictly increasing.

2.9 <u>Definition</u>: For $p = (W,T)$ we write $W = W^p$, $T = T^p$. We say $q$ is a pure extension of $p$ ($\leq$ pure) if $q \geq p$, $W^q = W^p$. We say $p$ is pure if $W^p = \emptyset$, and $p \leq^* q$ if omitting finitely many members of $T^q$ makes $q \geq p$.

2.10 <u>Definition</u>: For an ideal $I$ of $\mathcal{P}(\omega)$ (which includes all finite sets) let $Q[I]$ be the set of $p \in Q$ such that for every $A \in I$, for infinitely many $t \in T^p$, $\mathrm{int}(\underset{=}{t}) \cap A = \emptyset$.

2.11 <u>Fact</u>: 1) If $p \in Q$, $\tau_n$ $(n<\omega)$ are Q-names of ordinals, then there is a pure standard extension $q$ of $p$ such that: letting $T^q = \{\underset{=}{t}_n : n < \omega\}$ for every $n < \omega$, $W \subseteq \max[\mathrm{int}(\underset{=}{t}_n)] + 1$, let $q^n_W = (W, \{\underset{=}{t}_\ell : \ell > n\})$. Then for $k \leq n$: $q^n_W$ forces a value on $\tau_k$ iff some pure extension of $q^n_W$ forces a value on $\tau_k$.

2) $Q$ is proper (in fact $\alpha$-proper for every $\alpha < \omega_1$).

3) $\Vdash_Q$ "$\{n: (\exists p \in \underset{\sim}{G}_Q)[n \in W^p]\}$ is an infinite subset of $\omega$ which $\mathcal{P}(\omega)^V$ does not split."

<u>Proof</u>: Easy (for 3) use 2.7).

2.12 <u>Lemma</u>: Let $q$, $\tau_n$ be as in 2.11. Then for some pure standard extension $r$ of $q$, letting $T^r = \{\underset{=}{t}'_n : n<\omega\}$, ($\mathrm{lev}(\underset{=}{t}'_n)$ strictly increasing, of course) the following holds.

(*) <u>For</u> <u>every</u> $n < \omega$, $W \subseteq [\max(\mathrm{int}(\underset{=}{t}'_n)) + 1]$, and $\underset{=}{t}''_{n+1} \geq t'_{n+1}$ (so we ask only $\mathrm{lev}(\underset{=}{t}''_{n+1}) \geq 0$) <u>there</u> <u>is</u> $W' \subseteq \mathrm{int}(\underset{=}{t}''_{n+1})$, s.t. $(W \cup W', \{\underset{=}{t}_\ell : \ell > n + 1\})$ forces a value on $\tau_m$ $(m \leq n)$ (we can allow $n = -1$ letting $\max \mathrm{int}(\underset{=}{t}'_{-1}) + 1$ be $\max\{W^q \cup \{-1\}\}$).

This lemma follows easily from claim 2.14 (see below) (choose by it the $\underset{=}{t}'_n$ by induction on $n$) and is enough for proving Lemma 1.14.

2.13 Proof of Lemma 1.14: By 2.11, (a) and (d) (of 1.14) holds, and (c) is trivial. For proving (b) (i.e., $Q$ is almost ${}^{\omega}\omega$-bounding) let $\underset{\sim}{f} \in {}^{\omega}\omega$, $p \in Q$ be given. Let $\tau_n = \underset{\sim}{f}(n)$ and apply 2.11(1), 2.12 getting $r \geq p$. We now have to define $g \in {}^{\omega}\omega$ (as required in Def 1.1). $g(n) = \max\{k$: for some $W \subseteq [(\max(\underset{=}{t}'_{n+1}) + 1]$, $(W, \{\underset{=}{t}'_{\ell}: \ell > n+1\}) \Vdash$ "$\underset{\sim}{f}(n) = k$"$\}$. Let $A \subseteq \omega$ be infinite, and we define $p' = (W^p, \{\underset{=}{t}'_{n+1}: n \in A\})$, so $p' \geq r \geq p$. Now check.

2.14 Claim: Let $(\emptyset, T)$ be a pure condition, and let $W$ be a family of finite subsets of $\mathrm{cnt}(T)$ so that

(*) for every $(\emptyset, T') \geq (\emptyset, T)$, there is a $w \subseteq \mathrm{cnt}(T')$, $w \in W$.

Let $k < \omega$. Then there is $\underset{=}{t} \in L_k$ appearing in some $(\emptyset, T') \geq (\emptyset, T)$ such that: $\underset{=}{t}' \geq \underset{=}{t} \Rightarrow (\exists w \in W)[w \subseteq \mathrm{int}(\underset{=}{t}')]$.

Proof: Let $T = \{\underset{=}{t}_n: n < \omega\}$. For notational simplicity, w.l.o.g. let $W$ be closed upward.

Stage A: There is $n$ such that for every $\underset{=}{t}'_{\ell} \geq \mathrm{half}(\underset{=}{t}_{\ell})$ $(\ell < n)$, $\bigcup_{\ell<n} \mathrm{int}(\underset{=}{t}'_{\ell}) \in W$. This is because the family of $\langle \underset{=}{t}'_{\ell}: \ell<n \rangle$, $n < \omega$, $\underset{=}{t}'_n \geq \mathrm{half}(\underset{=}{t}_{\ell})$ form an $\omega$-tree with finite branching and for every infinite branch $\langle \underset{=}{t}'_{\ell}: \ell < \omega \rangle$, by (*) there is a member $\langle \underset{=}{t}'_{\ell}: \ell<n \rangle$ with $\bigcup_{\ell<n} \mathrm{int}(\underset{=}{t}'_{\ell}) \in W$. [Why? Define $(S^{\ell}, H^{\ell}) \in L$ such that $\underset{=}{S}^{\ell} = S^{\underset{=}{t}'_{\ell}}$ and $H^{\ell}_x(A) = H^{\underset{=}{t}_{\ell}}_x(A)$ when $x \in \mathrm{in}(S^{\ell})$, $A \subseteq \mathrm{Suc}_{(S^{\ell})}(x)$, so $\langle (S^{\ell}, H^{\ell}): \ell < \omega \rangle \in Q$, $(\emptyset, T') \leq (\emptyset, \{(S^{\ell}, H^{\ell}): \ell < \omega\})$. Now apply (*).] By Konig's lemma we finish.

Stage B: There are $n(0) < n(1) < n(2) < \cdots$ such that for every $m$ and $\underset{=}{t}'_{\ell} \geq \mathrm{half}(\underset{=}{t}_{\ell})$ for $n(m) \leq \ell < n(m+1)$, the set $\bigcup\{\mathrm{int}(\underset{=}{t}'_{\ell}): n(m) \leq \ell < n(m+1)\} \in W$. The proof is by repeating stage A.

Stage C: There are $m(0) < m(1) < \cdots$ such that: if $i < \omega$, for a function with domain $[m(i), m(i+1))$, $h(j) \in [n(j), n(j+1))$, $\underset{=}{t}'_{\ell} \geq \mathrm{half}(\underset{=}{t}_{\ell})$ for all relevant $\ell$ then $\bigcup\{\underset{=}{t}'_{h(j)}: j \in [m(i), m(i+1))\}$ belongs to $W$.

The proof is parallel to that of A.

Stage D: We define a partial function $H$ from finite subsets of $\omega$ to $\omega$: $H(u) \geq 0$ if for every $\underset{=}{t}'_\ell \geq \mathrm{half}(\underset{=}{t}_\ell)$ $(\ell \in u)$, $(\bigcup_{\ell \in u} \mathrm{int}(\underset{=}{t}'_\ell)) \in W$.

$H(u) \geq m + 1$ if $[u = u_1 \cup u_2 \rightarrow H(u_1) \geq m \vee H(u_2) \geq m]$.

Now we have shown that $H([n(i), n(i+1))) \geq 0$, and $H([n(m(i)), n(m(i+1)))) \geq 1$.

It clearly suffices to find $u$, $H(u) \geq k$. [We then define $\underset{=}{t} = (S, H)$ as follows: $S = \bigcup_{\ell \in u} S^{\underset{=}{t}_\ell} \cup \{u\}$, $u$ is the root and the order restricted to $S^{\underset{=}{t}_\ell}$ is as in $\underset{=}{t}_\ell$; for $x \in S^{\underset{=}{t}_\ell}$, $H_x = H_x^{\underset{=}{t}_\ell}$ and $H_u(A) = H(A)$.] We prove the existence of such $u$ by induction on $k$, (e.g., simultaneously for all $T'$, $(\emptyset, T') \geq (\emptyset, T)$.

The rest of this section deals with $Q[I]$.

2.15 Notation: Let $Q^0$ be the forcing of adding $\aleph_1$ Cohen reals $\langle r_i : i < \omega_1 \rangle$, $r_i \in {}^\omega\omega$. Let $I \in V$ be an ideal of $\mathcal{P}(\omega)$, including all finite subsets of $\omega$ but $\omega \notin I$ and generated by a MAD $\langle A_i : i < \omega_1 \rangle$ (the $\omega_1$ is not necessary - just what we use).

2.16 Claim: In $V^{Q^0}$: 1) If $p \in Q[I]$ and $\tau_n (n<\omega)$ are $Q[I]$-names of ordinals then there is a pure standard extension $q$ of $p$ such that: $q \in Q[I]$, and letting $T^q = \{\underset{=}{t}_n : n < \omega\}$, for every $n < \omega$ and $W \subseteq [\max \mathrm{int}(\underset{=}{t}_n) + 1]$ let $q^n_W = (W, \{t_\ell : n < \ell < \omega\})$, then ($q^n_W \in Q[I]$, of course, and) for every $k \leq n$ $q^n_W$ forces a value on $\tau_k$ iff some pure extension of $q^n_W$ in $Q[I]$ forces a value on $\tau_k$.

2) $Q[I]$ is proper, moreover $\alpha$-proper for every $\alpha < \omega_1$.

3) $\Vdash_{Q[I]}$ "$\{n : (\exists p \in G_{Q[I]})\, n \in W^p\}$ is an infinite subset of $\omega$ which is almost disjoint from every $A \in I$."

4) $Q[I]$ is almost ${}^\omega\omega$-bounding or in $V^{Q^0}$ for some $p \in Q[I]$, $p \Vdash$ "$\langle A_i : i < \omega_1 \rangle$ is not a MAD."

Proof: 1) Let $\lambda$ be regular large enough, $N$ a countable elementary submodel of $(H(\lambda), \in, V \cap H(\lambda))$ to which $I$, $\langle r_i : i < \omega_1 \rangle$, $Q[I]$, $p$, and $\langle \tau_n : n < \omega \rangle$ belong. Let $\delta = N \cap \omega_1$ (so $\delta \notin N$).

We define by induction on $n < \omega$, $q^n \in Q[I] \cap N$, $\underset{\sim}{t}_n$ and $k_n < \omega$ such that:

a) each $q^n$ is a pure extension of $p$.

b) $q^n \geq q^\ell$ for $\ell < n$ and if $w \subseteq k_n$, $m < n + 1$ and some pure extension of $(w, T^{q^n})$ forces a value on $\tau(m)$, then $(w, T^{q^n})$ does it.

c) $k_n > k_\ell$ and $k_n > \max \operatorname{int} \underset{\sim}{t}_\ell$ for $\ell < n$.

d) every $\ell \in \operatorname{cnt}(q^n)$ is $> k_n$.

e) $\underset{\sim}{t}_n \in T^{q^n}$ and $\operatorname{lev}(\underset{\sim}{t}_n) > n$ and $\min \operatorname{int}(\underset{\sim}{t}_n)$ is $> k_n$.

There is no problem in doing this: we first choose $k_n$, then $q^n$ and at last $\underset{\sim}{t}_n$. We want in the end to let $T^q = \{\underset{\sim}{t}_n : n < \omega\}$. One point is missing. Why does $q = (w^p, T^q)$ belong to $Q[I]$ (not just to $Q$)? But we can use some function in $V[\langle r_i : i < \delta \rangle]$ to choose $k_n$, $q^n$, and then let $\underset{\sim}{t}_n$ be the $r_\delta(n)$-th member of $T^{q^n}$ which satisfies the requirement (in some fixed well ordering from $V$ of the hereditarily finite sets). As $I \in V$ and $r_\delta \in {}^\omega\omega$ is Cohen generic over $V[\langle r_i : i < \delta \rangle]$, this should be clear.

2), 3) easy.

4) Assume that in $V^{Q^0}$, $\Vdash_Q$ "$\langle A_i : i < \omega_1 \rangle$ is a MAD". Like in 2.13 it suffices to prove the parallel of 2.12, 2.14.

As for the proof of 2.14 for $Q[I]$ for stage A note that if $\underset{\sim}{t}'_n \geq \operatorname{half}(\underset{\sim}{t}_n)$ for $n < \omega$, then $(\emptyset, \{(S^\ell, H^\ell) : \ell < \omega\}) \in Q[I]$ (check Definition 2.10). Stage B is similar. For stage C we have to use the specific character of $I$ - generated by a MAD. By 2.16A without loss of generality there are distinct $i_n < \omega_1$ such that $B_n = \{\ell < \omega : \operatorname{int}(\underset{\sim}{t}_\ell) \subseteq A_{i_n}\}$ is infinite for each $n$, and without loss of generality $[m(\ell), m(\ell+1)) \cap B_k \neq 0$ for $k < \ell$. Now we restrict ourselves to functions $h$ such that $h(j) \in B_{j-[\sqrt{j}]}$.

As for the proof of 2.12 from 2.14 (for $Q[I]$) we again have to choose the sequence $\langle \underset{\sim}{t}'_n : n < \omega \rangle$ using some Cohen generic $r_\delta$.

2.16A <u>Fact</u>: Suppose (in $V_1$) $\langle A_i : i < \omega_1 \rangle \in V_1$ is a MAD, $\Vdash_Q$ "$\langle A_i : i < \omega_1 \rangle$ is a MAD". Let $I$ be the ideal generated by $\{A_i : i < \omega\}$ and the finite subsets of $\omega$. Then $(W,\{\underset{\sim}{t}_n : n<\omega\})$ is a standard condition in $Q[I]$ iff it is a standard condition in $Q$ and there are finite pairwise disjoint $u_\ell \subseteq \omega_1 (\ell<\omega)$ such that for each $\ell$, for infinitely many $n<\omega$, $\mathrm{int}(\underset{\sim}{t}_n) \subseteq \bigcup_{i\in u_\ell} A_i$ iff there are singletons $u_\ell$ as above.

Proof. The third condition implies trivially the second. We shall prove [second $\Rightarrow$ first] and then [first $\Rightarrow$ third]. Suppose there are $u_\ell$ $(\ell<\omega)$ as above. Then every $B\in I$ is included in $\bigcup_{i\in u} A_i \cup \{0,\cdots,n\}$ for some finite $u\subseteq\omega_1$ and $n<\omega$. But for some $\ell$, $u_\ell$ is disjoint from $u$, hence $B\cap(\bigcup_{i\in u_\ell} A_i)$ is finite. We know for infinitely many $n<\omega$, $\mathrm{int}(\underset{\sim}{t}_n) \subseteq \bigcup_{i\in u_\ell} A_i$, and the $\mathrm{int}(\underset{\sim}{t}_n)$ $(n<\omega)$ are pairwise disjoint, hence for infinitely many $n<\omega$, $\mathrm{int}(\underset{\sim}{t}_n)\cap B=\emptyset$, as required.

For the other direction suppose $p=(W,\{\underset{\sim}{t}_n : n<\omega\})\in Q[I]$. We define by induction on $m$ a finite $u_m\subseteq\omega_1$, disjoint from $\bigcup_{\ell<m} u_\ell$, such that $I_m = \{n<\omega: \mathrm{int}(\underset{\sim}{t}_n)\subseteq \bigcup_{i\in u_m} A_i\}$ are infinite. For $m=0$, we know $p\in Q$, $\langle A_i : i<\omega_1\rangle$ is a MAD even after forcing by $Q$, so by 2.11(3) there are $p'=(W',\{\underset{\sim}{t}'_n : n<\omega\})\in Q$, $p\leqslant p'$ and $i_0<\omega_1$ such that

$$p' \Vdash \text{“}\{n: (\exists q\in \underset{\sim}{G}_Q)\,[n\in W^q]\}\cap A_{i_0} \text{ is infinite”}.$$

By 2.7, w.l.o.g. $\bigcup_{n<\omega} \mathrm{cnt}(\underset{\sim}{t}'_n)\subseteq A_{i_0}$. Let $u_0=\{i_0\}$. For $m>0$ start with $(W,\{\underset{\sim}{t}_n : \mathrm{cnt}(\underset{\sim}{t}_n)\cap(\bigcup_{\ell\leqslant m}\ \bigcup_{i\in u_\ell} A_i)=\emptyset\})$.

A trivial remark is

2.17 Fact: Cohen forcing and even the forcing for adding $\lambda$ Cohen reals (by finite information) is almost $({}^\omega\omega)$-bounding.

## 3. On $\mathfrak{s}>\mathfrak{b}=\mathfrak{a}$.

3.1 Theorem: Assume $V\models CH$. Then for some forcing notion $P^*$ ($P$ is proper, satisfies the $\aleph_2$-c.c., is weakly bounding and):

(*) In $V^{P^*}$ $2^{\aleph_0}=\aleph_2$, there is an unbounded family of power $\aleph_1$ and also a MAD of power $\aleph_1$, but there is no splitting family of power $\aleph_1$.

Proof: The forcing $\langle P_\alpha, \underset{\sim}{Q}_\alpha : a < \omega_2\rangle$, $P^*$ are as in the proof of 1.15(1). So the only new point is the construction of a MAD of power $\aleph_1$. This will be done in $V$; unfortunately the proof of its being MAD in $V^{P^*}$ does not seem to follow from 1.13 (though the proof is similar).

Let $\{\langle B^i_n : n < \omega\rangle : i < \aleph_1\}$ enumerate (in $V$) all sequences $\langle B_n : n < \omega\rangle$ of finite pairwise disjoint nonempty subsets of $\omega$ (remember CH holds in $V$). Next choose a MAD $\langle A_\alpha : \alpha < \aleph_1\rangle$ such that

(*) if $\delta$ is a limit ordinal, $i < \delta$, and for every $k < \omega$, $\alpha_1, \cdots, \alpha_k < \delta$ for infinitely many $n < \omega$, $B^i_n \cap (A_{\alpha_1} \cup \cdots \cup A_{\alpha_k}) = \emptyset$ then for infinitely many $n < \omega$, $B^i_n \subseteq A_\delta$.

Let $\lambda$ be regular large enough. For a generic $G_\alpha \subseteq P_\alpha$ $(\alpha \leqslant \omega_2)$, $N \prec (H(\lambda)[G_\alpha], \in)$ is called good if it is countable, $G_\alpha$, $\langle P_j, \underset{\sim}{Q}_i : i < \alpha, j \leqslant \alpha\rangle$, $\langle A_i : i < \omega_1\rangle$, $\langle\langle B^i_n : n < \omega\rangle : i < \omega_1\rangle \in N$ and for every sequence $\langle B_n : n < \omega\rangle \in N$ of finite non-empty pairwise disjoint subsets of $\omega$, letting $\delta = N \cap \omega_1$, if $(\forall k<\omega)(\forall \alpha_1 \cdots \alpha_k < \delta)(\exists^\infty n<\omega)[B_n \cap (A_{\alpha_1} \cup \cdots \cup A_{\alpha_1}) = \emptyset]$ then $(\exists^\infty n)[B_n \subseteq A_\delta]$.

We shall prove by induction on $\alpha \leqslant \omega_2$,

$(\mathrm{st})_\alpha$ for every $\beta < \alpha$, $N \prec (H(\lambda), \in)$ to which $\langle P_j, \underset{\sim}{Q}_i : i<\alpha, j\leqslant\alpha\rangle$, and $\alpha, \beta$ belongs and generic $G_\beta \subseteq P_\beta$ if $N[G_\beta] \cap \omega_1 = N \cap \omega_1$, $N[G_\beta]$ is good, and $p \in N[G_\beta] \cap P_\alpha/G_\beta$ then there is $q \in P_\alpha/G_\beta$, $q \geqslant p$, $q$ $(N[G_\beta], P_\alpha/G_\beta)$-generic and whenever $G_\alpha \subseteq P_\alpha$ is generic, $G_\beta \subseteq G_\alpha$, $q \in G_\alpha$, $N[G_\alpha]$ is good.

This is proved by induction. The case $\alpha = \omega_2$, $\beta = 0$ gives the desired conclusion (as we find a good $N \prec (H(\lambda), \in)$ to which a $P_{\omega_2}$-name of an infinite subset of $\omega$ disjoint to every $A_i$ belongs). The case $\alpha = 0$ is trivial (saying nothing) and the case $\alpha$ limit is similar to the proof of 1.13 (and, say, 1.11). In the case $\alpha$ successor, by using the induction hypothesis we can assume $\alpha = \beta + 1$.

By renaming $V[G_\beta]$, $N[G_\beta]$ as $V$, $N$, we see that it is enough to prove for any good $N$ and $p \in Q \cap N$ (remember $Q_\beta = Q^{V[G_\beta]}$) there is $q \geqslant p$ which is $(N,Q)$-generic and $q \Vdash_Q$ "$N[G]$ is good".

Let $\delta = N \cap \omega_1$, and let $\delta = \{\tau(\ell): \ell<\omega\}$. Let $\{\tau_\ell: \ell<\omega\}$ be a list of all Q-names of ordinals which belong to $N$, and $\{\langle \underset{\sim}{B}^\ell_n: n<\omega\rangle: \ell<\omega\}$ be a list of all Q-names of $\omega$-sequences of pairwise disjoint non-empty finite subsets of $\omega$ which belong to $N$. For notational simplicity only, assume $p$ is pure.

We shall define by induction on $\ell < \omega$ pure $p_\ell = (\emptyset, \{\underline{\underline{t}}^\ell_n: n<\omega\})$ and $k_\ell < \omega$ such that:

a) $p_\ell \in N$, $p_\ell$ standard (so max int $\underline{\underline{t}}^\ell_n <$ min int $\underline{\underline{t}}^\ell_{n+1}$)

b) $p_0 = p$, $p_{\ell+1} \geq p_\ell$, $k_{\ell+1} > k_\ell$

c) $\underline{\underline{t}}^\ell_n = \underline{\underline{t}}^{\ell+1}_n$ for $n \leq \ell$

d) $p_{\ell+1} \Vdash_Q$ "$\tau_\ell \in C$" for some countable set of ordinals which belongs to $N$.

e) for every $w_0 \subseteq (\max[\text{int } \underline{\underline{t}}^\ell_\ell] + 1)$, $m < \ell$, and $\underline{\underline{t}} \geq \underline{\underline{t}}^{\ell+1}_{\ell+1}$ there is $w_1 \subseteq \text{int}(\underline{\underline{t}})$ such that $(w_0 \cup w_1, \{\underline{\underline{t}}^{\ell+1}_i: \ell + 1 < i < \omega\}) \Vdash_Q$ "$(\exists j<\omega)[\underset{\sim}{B}^m_j \subseteq [k_\ell, k_{\ell+1})$, $\underset{\sim}{B}^m_j$ is disjoint from $A_{\tau(0)} \cup \cdots \cup A_{\tau(\ell)}$ and $\underset{\sim}{B}^m_j \subseteq A_\delta$".

Let $p^m_\ell = (\emptyset, \{\underline{\underline{t}}^{\ell,m}_n: n<\omega\})$.

Suppose $p_\ell$ is defined. By 2.12 there is a pure $p^0_\ell \geq p_\ell$ in $N$ such that $\underline{\underline{t}}^{\ell,0}_i = \underline{\underline{t}}^\ell_i$ for $i \leq \ell$, $p^0_\ell \Vdash$ "$\tau_\ell \in C$" for some countable set of ordinals from $N$.

Next by 2.12 we can find a pure $p^1_\ell \geq p^0_\ell$, $\underline{\underline{t}}^{\ell,1}_i = \underline{\underline{t}}^\ell_i$ for $i \leq \ell$ and $k_{\ell,i} (i<\omega)$ such that:

(i) $k_{\ell,0} = k_\ell$, $k_{\ell,i+1} > k_{\ell,i}$

(ii) for every $m < i$ and $W_0 \subseteq (\max[\text{int } \underline{\underline{t}}^{\ell,i}_{\ell+i}]+1)$ and $\underline{\underline{t}} \geq \underline{\underline{t}}^{\ell,0}_{\ell+i+1}$ for some $W_1 \subseteq \text{cnt}(\underline{\underline{t}})$, $(W_0 \cup W_1, \{\underline{\underline{t}}^{\ell,1}_n: \ell+i+1 < n < \omega\}) \Vdash_Q$ "$(\exists j<\omega)[\underset{\sim}{B}^m_j \subseteq [k_{\ell,i}, k_{\ell,i+1})$, $\underset{\sim}{B}^m_j$ is disjoint from $A_{\tau(0)} \cup \cdots \cup A_{\tau(\ell+i)}]$".

Now apply the goodness of $N$ to the sequence $\langle [k_{\ell,i}, k_{\ell,i+1}) - A_{\tau(0)} \cup \cdots \cup A_{\tau(\ell)}: i<\omega\rangle$, so for some $i$, $[k_{\ell,i}, k_{\ell,i+1}) - A_{\tau(0)} \cup \cdots \cup A_{\tau(\ell)} \subseteq A_\delta$. Let $\underline{\underline{t}}^{\ell+1}_n = \underline{\underline{t}}^\ell_n$ for $n \leq \ell$, $\underline{\underline{t}}^{\ell+1}_n = \underline{\underline{t}}^{\ell,1}_{n+i}$ for $n > \ell$.

So we have defined $p_{\ell+1}$ satisfying (a) - (e). So we can define $p_\ell$ for $\ell < \omega$ and now $q = (0, \{\underline{\underline{t}}^n_n: n < \omega\})$ is as required.

## 4. Splitting number smaller than unbounding number is consistent.

4.1 Definition: $Q^d$ will be the following (well known as Hechler's forcing) forcing notion: the conditions are the pairs $p = (f,g)$, $f$ a finite function from some $n$ to $\omega$, $g \in {}^\omega\omega$, and $(f^0,g^0) \leqslant (f^1,g^1)$ iff $f^0 \subseteq f^1$ and $[m \in \text{Dom } f^1 - \text{Dom } f^0 \Rightarrow f^1(m) \leqslant g^0(m)]$ and $(\forall m)(g^0(m) \leqslant g^1(m))$.

Let $f = f^p$, $g = g^p$.

Let $\underset{\sim}{r}$ be the function $\underset{\sim}{r}(n) = m$ iff $(\exists p \in G_Q)\ f^p(n) = m$.

4.2 Lemma: Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \delta\rangle$ be a finite support iteration, each $Q_i$ being $Q^d$ in $V^{P_i}$, and $P = \lim \bar{Q}$, $\text{cf}\delta > \aleph_0$ and

(*) there are, in $V$, no projective sets $D_m \subseteq [\omega]^\omega$, each is a filter and $(\forall A \subseteq \omega)\ (\exists n)\ [A \in D_n \vee \omega - A \in D_n]$.

Then

(1) $P$ satisfies the countable chain condition, $(2^{\aleph_\alpha})^{V^P}$ is the minimal cardinal in $V \geqslant 2^{\aleph_0} + |\delta|$) and of cofinality $> \aleph_\alpha$.

(2) $\Vdash_P$"$\mathfrak{b} = \mathfrak{d} = \text{cf}\delta$", in fact the generic $r_i \in {}^\omega\omega$ of $Q_i$ dominates $({}^\omega\omega)^{V^{P_i}}$.

(3) $\Vdash_{P_\delta}$"$\mathfrak{s} = (2^{\aleph_0})^V$", in fact $\mathcal{P}(\omega)^V$ is a splitting family in $V^P$.

Proof: We leave (1), (2) to the reader, and concentrate on (3). Suppose $p \in P$, $\underset{\sim}{A}$ a P-name, and $p \Vdash_P$"$\underset{\sim}{A}$ is an infinite subset of $\omega$ not split by $\mathcal{P}(\omega)^V$ ".

We can define by induction on $n < \omega$ a countable family $R_n$ of conditions from $P$ s.t.

(1) $p \in R_0$

(2) For each $m < \omega$, for some maximal antichain $I_m$ of $P$, $(\forall q \in I_m)\ (q \Vdash_P$"$m \in \underset{\sim}{A}$" or $q \Vdash_P$"$m \notin \underset{\sim}{A}$") and $I_m \subseteq R_0$.

(3) For each $n < \omega$, $q \in R_n$, $m < \omega$ and $\alpha \in \text{Dom } q$, for some maximal antichain $I_{q,\alpha} \subseteq R_{n+1}$ of $P_\alpha$, for every $r \in I_{q,\alpha}$, for some $f \in V$ and $k$, $r \Vdash_{P_\alpha}$"$f^{q(\alpha)} = f$ and $g^{q(\alpha)}(m) = k$".

We call $R \subseteq P$ closed if for every $q \in R$, $m < \omega$ and $\alpha \in \mathrm{Dom}\, q$ there is $I_{q,\alpha} \subseteq R$ as in (3). So clearly $\bigcup_{n<\omega} R_n$ is closed.

The countability of the $I$'s follows from the c.c.c. and we can carry this proof as each $q \in P$ has a finite domain $\subseteq \delta$, $q(\alpha)$ a $P_\alpha$- name of a member of $Q^d$.

Now let $W = \bigcup\{\mathrm{Dom}\, q: q \in R_n, n < \omega\}$, and let $P^* = \langle r \in P$: $r$ belongs to some closed $R_r \subseteq P$ s.t. $\bigcup_{q \in R_r} \mathrm{Dom}\, q \subseteq W\}$. By [Sh3, 6.5], $P^* < P$; hence $V^P = (V^{P^*})^{P/P^*}$, so let $G \subseteq P$ be generic, $p \in G$; then $G \cap P^*$ is a generic subset of $P^*$ and $\underset{\sim}{A}[G] \in V^{P^*}$. By a trivial absoluteness argument in $V^{P^*}$, $\underset{\sim}{A}[G]$ is not split by $\mathcal{P}(\omega)^V$. Observe also that $P^*$ is isomorphic to $P_\alpha$ where $\alpha$ is the order type of $W$. As $W$ is countable, $\alpha$ is countable. So we can find directed subsets $\Gamma_n$ of $P^*$ such that $\bigcup \Gamma_n$ is a dense subset of $P^*$ [ $\bigcup_{n<\omega} \Gamma_n$ is the set of $q \in P^*$ such that each $f^{q(\alpha)}$ is an actual function and put $q_1$, $q_2$ in the same $\Gamma_n$ iff $\mathrm{Dom}\, q_1 = \mathrm{Dom}\, q_2$ and $f^{q_1(\alpha)} = f^{q_2(\alpha)}$ for every $\alpha$ in their domain].

Define $D_n = \{B \in \mathcal{P}(\omega)$: for some $q \in \Gamma_n$, $q \geqslant p$, $q \Vdash_{P^*} "\underset{\sim}{A} \subseteq^* B"\}$. As $\Gamma_n$ is directed, $D_n$ is a filter, and by the choice of $p$ and $A$ each member of $D_n$ is infinite. Also for every infinite $B \subseteq \omega$ $(B \in V)$, $p \Vdash_{P^*}$ "$\underset{\sim}{A} \subseteq^* B$ or $\underset{\sim}{A} \cap B$ is finite"; hence there is $q \geqslant p$ s.t. $q \Vdash_{P^*}$" $\underset{\sim}{A} - B$ is finite" or $q \Vdash_{P^*}$"$\underset{\sim}{A} \cap B$ is finite" without loss of generality, for some $n$, $q \in \Gamma_n$. Hence $B \in D_n$ or $\omega - B \in D_n$. As easily each $D_n$ is projective we get a contradiction to (*).

4.3 <u>Claim</u>: If $\langle r_i: i < \omega_1\rangle$ is a sequence of $\aleph_1$ Cohen reals (i.e., this is a generic set for the appropriate forcing $P^0$) then $V[r_i: i < \omega_1]$ satisfies (*).

<u>Proof</u>: Let $D_n$ form a counterexample, $G$ in $V[G]$, $G \subseteq P^0$ generic. Clearly for some $i$, the parameters appearing in the definition of the $D_n$ belong to $V[r_j: j < i]$. So w.l.o.g. $i = 0$, and we can consider $r_i$ as a function from $\omega$ to $\{0,1\}$. So for some $\ell \in \{0,1\}$ and $n < \omega$,

$\{m: r_0(m) = \ell\} \in D_n$ (in $V[r_i : i < \omega_1]$), hence this is forced by some $p \in P^0$. Choose $n(*)$ large enough so that $p$ gives no information on $r_0(m)$ for $m \geq n(*)$. Define $r_i'$: $r_i'(n) = r_i(n)$ except when $i = 0 \wedge n \geq n(*)$ in which case $r_i'(n) = 1 - r_i(n)$. It is easy to check that also $\langle r_i' : i < \omega_1\rangle$ comes from some generic $G' \subseteq P^0$, and $p \in G'$. Clearly $V[G] = V[G'] = V[r_i : i < \omega_1]$. As $p \Vdash_{P^0}$ "$\{m: r_i(m) = \ell\} \in D_n$" also (looking at $V[G']$), $\{m: r_i'(m) = \ell\} \in D_n$. But $\{m: r_i(m)=\ell\} \cap \{m: r_i'(m)=\ell\} \subseteq \{0,\cdots,n(*)-1\}$, hence is finite, contradicting "$D_n \subseteq [\omega]^\omega$ is a filter".

4.4 <u>Conclusion</u>: It is consistent with ZFC that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \mathfrak{h} = \mathfrak{b} > \mathfrak{s}$ if ZFC is consistent.

<u>Remarks</u>: 1) We can get other values for $\mathfrak{h} > \mathfrak{s}$.

2) I think we can prove the case of (*) we need without having to force it.

<u>Proof</u>: Start with $V = L$, add $\aleph_1$ Cohen reals [so by 4.3, (*) of 4.2 holds] and then force by $P$ from 4.2 for $\delta = \omega_2$. By 4.2 we get a model as required.

5. <u>On</u> $\mathfrak{h} < \mathfrak{s} = \mathfrak{b}$.

5.1 <u>Definition</u>: Let $\mathfrak{h}$ be the minimal cardinal $\lambda$ such that there is a tree $T$ with $\lambda$ levels and $A_t \in [\omega]^\omega$ for $t \in T$, $[t < s \Rightarrow A_s \subseteq^* A_t]$ and $(\forall B \in [\omega]^\omega)(\exists t \in T)[A_t \subseteq^* B]$.

See [BPS] on it (and why it exists).

5.2 <u>Theorem</u>: Assume $V \models CH$.

For some proper forcing $P$ of power $\aleph_2$ satisfying the $\aleph_2$-c.c., in $V^P$ $\mathfrak{h} = \aleph_1$, $\mathfrak{b} = \mathfrak{s} = \aleph_2$ (and $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$).

<u>Proof</u>: We shall use the direct limit $P$ of the iteration $\langle P_i, \underset{\sim}{Q}_i : i < \omega_2\rangle$ where:

1) letting $i = (\omega_1)^2 + j$, $j < (\omega_1)^2$, if $j \neq 0, \omega_1, \omega_1 + 1$ then $\underset{\sim}{Q}_i$ is Cohen forcing; if $j = \omega_1$ then $\underset{\sim}{Q}_i$ is $Q$ from Def. 2.8 (in $V^{P_i}$), and if $j = \omega_1 + 1$ then $\underset{\sim}{Q}_i$ is $Q^d$ (see Def. 4.1). For $j = 0$ see the end of the proof.

2) We use the variant of countable support iteration defined in [Sh1, III p. 96,7], i.e., using only hereditarily countable names (we could have used Mathias forcing instead of the $Q$ from 2.8). Clearly $|P| = \aleph_2$, $P$ satisfies the $\aleph_2$-c.c. and is proper (see [Sh1, III p. 96,7]), hence forcing by $P$ preserves cardinals. Clearly in $V^P$ $\mathfrak{s}$, $\mathfrak{b} \geq \aleph_2$, and $2^{\aleph_0} = \aleph_2$; hence in $V^P$, $\mathfrak{s} = \mathfrak{b} = \aleph_2$, and always $\mathfrak{h} \geq \aleph_1$. So the only point left is $V^P \models$ "$\mathfrak{h} \leq \aleph_1$".

We define by induction on $i < \omega_2$, a $P_{\alpha(i)}$-name $\underset{\sim}{\eta}_i$, $\underset{\sim}{A}_i$, $\nu_i$ such that

(a) $\alpha(i) = (\omega_1)^3(i+1)$

(b) $\underset{\sim}{\eta}_i \in \bigcup_{\beta<\omega_1} {}^{\beta}(\omega_2)$ and for every successor $\beta < \ell(\underset{\sim}{\eta}_i)$ $[\underset{\sim}{\eta}_i \restriction \beta \in \{\underset{\sim}{\eta}_j : j<i\}]$ (i.e., those things are forced).

(c) $\underset{\sim}{\eta}_j < \underset{\sim}{\eta}_i \Rightarrow \underset{\sim}{A}_i \subseteq^* \underset{\sim}{A}_j$ $(j<i)$ and $\underset{\sim}{A}_i$ is an infinite subset of $\omega$.

(d) if $\underset{\sim}{A} \subseteq \omega$ is infinite and $A \in V^{P_j}$ then for some $i < j + \omega_1$, $\underset{\sim}{A} \subseteq \underset{\sim}{A}_j$

(e) $\underset{\sim}{A}_i$ includes no infinite set from $V^{P_{\alpha(j)}}$ when $j < i$, and is a subset of the generic real of $Q_{\omega_1^3 i+3}$

There is no problem to do this if you know the well known way to build trees exemplifying the definition of $\mathfrak{h}$ (see Balcar <u>et al</u>. [BPS]), provided that no $\omega_1$-branch has an intersection. I.e., for no $\eta \in {}^{\omega_1}(\omega_2)$ and $B \in [\omega]^{\omega}$ (in $V^{P_{\omega_2}}$) $B \subseteq^* A_{i_\alpha}$ where $\eta \restriction (\alpha+1) = \eta_{i_\alpha}$ for $\alpha < \omega_1$. Let $i(*) = \bigcup_{\tau<\omega_1} \alpha(i_\tau)$, in $V^{P_{i(*)}}$ there is no intersection by (e) (though maybe $\eta \notin V^{P_{i(*)}}$). So it is enough to prove this for a fixed $i(*)$.

We can look at the iteration $\langle P'_\beta, \underset{\sim}{Q}_\tau : i(*) < \tau < \omega_2, i(*) \leq \beta \leq \omega_1 \rangle$, $P'_\beta = P_\beta / P_{i(*)}$. Let $G_1 \subseteq P_{i(*)}$ be generic, $V_1 = V[G]$. Note that every element of $P'_{\omega_2}$ can be represented by a countable function from ordinals ($< \omega_2$) to hereditarily countable sets. The set of elements of $P'_{\omega_2}$ as well as its partial order are definable from ordinal parameters only (all this in $V[G]$). Suppose $p \in P'_{\omega_2}$ forces $\underset{\sim}{B}$ (a $P'_{\omega_2}$-name of a subset of $\omega$) and $\underset{\sim}{i}_\tau$ ($\tau < \omega_1$) to be as above. So for some $j(*) < i(*)$ $p \in V[G \cap P_{j(*)}]$.

There is $p_1$, $p \leqslant p_1 \in P'_{\omega_2}$, $p_1 \Vdash$ "$\underset{\sim}{i}_\tau = i$" for some $\tau, i$, $j(*) < \omega_1^2 i < i(*)$ so $p_1 \Vdash$ "$\underset{\sim}{B} \subseteq r_i$" where $r_i$ is the generic real the set $G \cap Q_{\omega_1^2 i+3}$ gives. Now using automorphisms of the forcing $P_{i(*)}/P_{j(*)}$ we see that there is $p_2$, $p \leqslant p_2 \in P'_{\omega_2}$ such that $p_2 \Vdash$ "$\underset{\sim}{B}$ is almost disjoint from $r_i$". From this we can conclude that $p \Vdash$ "$\bigcup_{\tau<\omega_1} \underset{\sim}{\eta}_{i_\tau} \notin V[G]$" (otherwise some $p_0 \geqslant p$ forces a particular value and repeat the argument above for $p_0$).

Looking at $Q_{i(*)}$ (see below) we see that it does not add any $\omega_1$-branch to $T = \{\underset{\sim}{\eta}_i : \alpha(i) < i(*)\}$. Let $G_2 \subseteq P_{i(*)+1}$ be generic and we shall work in $V_2 = V[G_2]$, and assume $p \in P_{\omega_2}/P_{i(*)+1}$ (i.e., $P_{\omega_2}/G_2$) force $\underset{\sim}{B}$, $\underset{\sim}{i}_\tau$ ($\tau < \omega_1$) to be as above. Let $N$ be a countable elementary submodel of $H((2^{\aleph_0})^+)^{V_2}$ to which $p$, $P_{\omega_2}/P_{i(*)+1}$, $\underset{\sim}{B}$, and $\langle \underset{\sim}{i}_\tau : \tau < \omega_1 \rangle$ belong. Now each $Q_i$ is strongly proper and so is $P_{\omega_2}/P_{i(*)+1}$ (see [Sh1]). It is enough to find $q \geqslant p$ (in $P_{\omega_2}/P_{i(*)+1}$) which forces that for every $\eta \in T$, $\ell(\eta) = \delta \overset{\text{def}}{=} N \cap \omega_1$,

$$q \Vdash \text{"for some } \tau < \delta,\ \underset{\sim}{\eta}_{i_\tau} \not< \eta\text{"}$$

By the definition of strongly proper and of $Q_{i(*)}$ this is possible.

How is $Q_{i(*)}$ defined? Let it be $\{(\langle I_\ell : \ell < n \rangle, w) : n < \omega,\ I_\ell$ a finite antichain in ${}^{\omega>}\omega$, $w$ a finite subset of ${}^{\omega}\omega\}$. The order is $(\langle I^0_\ell : \ell < n^0 \rangle, w^0) \leqslant (I^1_\ell : \ell < n^1 \rangle, w^1 \rangle)$ iff $n^0 \leqslant n^1$, $I^0_\ell \subseteq I^1_\ell$ for $\ell < n^0$, $w^0 \subseteq w^1$ <u>and</u> for every $\eta \in w^1 - w^0$, $n^0 \leqslant \ell < n^1$, no member of $I^1_\ell$ is an initial segment of $\eta$.

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