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REMARKS ON $\aleph_1$-CWH NOT CWH FIRST COUNTABLE SPACES

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Abstract. CWH, CWN stand for collectionwise Hausdorff and collectionwise normal respectively. We analyze the statement "there is a $\lambda - CWH$ not CWH first countable (Hausdorff topological) space". We prove the existence of such a space under various conditions, show its equivalence to: there is a λ-CWN not CWN first countable space and give an equivalent set theoretic statement; the nicest version we can obtain is in 4.8.

§0 INTRODUCTION

This paper is concerned with several "almost, but not quite"-type questions for first countable Hausdorff spaces, e.g.:

(1) Is there an $\aleph_1$-metrizable but not metrizable such space?
(2) Is there such space which is $\aleph_1$-CWH but not CWH?
(3) Similarly for CWN.

We feel that these questions are of considerable interest, especially the first one which is somewhat of a classical problem in set-theoretic topology.

Generalizations to $\lambda > \aleph_1$ are considered, and the analogy with CWN is explored. Here, CWH stands for collectionwise Hausdorff, and CWN for collectionwise normal. For the purposes of this paper, a space is always a first countable Hausdorff space.

Note that λ-metrizable $\Rightarrow$ λ-CWH.

Our motivation is to prove that, to a certain degree, the above three questions are equivalent. Note that in the class of Moore spaces, the equivalence of metrizability and CWN is well known (see [F184]).

We give below a summary of some of the results.

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My thanks also go to Franklin Tall for showing me the problem in 1992, to the participants of the seminar in Jerusalem in July '92 and Fall '93 for their remarks, and to Mirna Džamonja for many corrections and writing up the first part of the introduction.

In the first section it is shown that instances of non trivial cardinal arithmetic (for example $\neg SCH$) imply examples for (1). For this we note that for singular λ of countable cofinality, a λ-metrizable not metrizable space can be constructed from the assumption that there are λ^+ functions $\eta_\alpha \in {}^\omega\lambda$ such that for $\beta < \lambda^+$ there are pairwise disjoint end segments for $\langle \eta_\alpha : \alpha < \beta \rangle$.

The strength of this assumption is discussed, for example if it were to fail for all relevant λ, then for every singular μ we would have $\mathrm{pp}(\mu) = \mu^+$. This in turn has many implications, like that $\diamondsuit^*(\{\delta < \mu^+ : \mathrm{cf}(\delta) \neq \mathrm{cf}(\mu)\})$ holds for μ singular strong limit.

Also in the first section is a construction the method of which is going to play a major role in the rest of the paper. We wish to construct an $\aleph_1$-CWH not CWH space from the existence of a strong limit λ of uncountable cofinality with $2^\lambda = \lambda^+$. We succeed to obtain "not CWH" at this stage, but we need something more to prove the "$\aleph_1$-CWH".

Spaces which are $\aleph_1$-CWH but not CWH are further discussed in §2. It is proved that MA + $\neg$CH implies the existence of an $\aleph_1$-metrizable but not metrizable space. Consideration is given to a combinatorial property $INCWH(\lambda)$, defined in 2.4. If one can show that there is an $\lambda > \aleph_1$ such that $INCWH(\lambda)$ holds, then the original problem (1) is solved. It is further shown that $INCWH(\lambda)$ implies the existence of an $(<\lambda)$-CWH but not λ-CWH space, and more. The proof uses the construction mentioned in the discussion of the first section. Further variants of $INCWH(\lambda)$ are introduced.

Discussion of the variants of freeness continues in the third section. It is shown that the version introduced earlier, $INCWH(\lambda)$ can be further weakened, to a property called $INCWH^4(\lambda)$, to obtain a principle equivalent to the existence of a space X with λ points which is $(<\lambda)$-CWH, but not λ-CWH.

Having thus hopefully convinced the reader that freeness has a lot to do with the original problem, we give a detailed discussion of the general concept of freeness and its connection to topological properties we discussed earlier. This is the subject of the fourth section. The method of constructing topological spaces introduced in the first section is further explored now. Some equivalences are given and in particular, the CWN-spaces enter the arena. Freely stated, Theorem 4.8 shows that basically, the existence of a space which is (λ)-CWH, ${}^*CWN$ respectively, but not λ-CWH, ${}^*CWN$ respectively are equivalent, and in addition equivalent to the existence of a $(<\lambda)$-free not free family of functions with domains countable sets of ordinals and range $\subseteq \omega$. (The theorem gives in fact more but does not distinguish λ_1, λ_2 if $\lambda_1^{\aleph_0} = \lambda_2^{\aleph_0}$.)

In the fifth section of the paper, we continue the investigation of variants of freeness.

The author had a flawed proof of the existence of spaces as above in ZFC, for some $\lambda > \aleph_1$, in June of 1992; still we decided that there is some interest in the correct part and some additions.

Further work on the variants of freeness, as well as their connection with metrizability, is in preparation.

* * *

We shall deal mainly with first countable topological spaces.
All spaces will be Hausdorff.

0.1 Definition. 1) A space X is metrizable if the topology on X is induced by a metric.
2) A space X is $(<\lambda)$-metrizable if for each $Y \subseteq X$ such that $|Y| < \lambda$, the induced topology on Y is metrizable. Let μ-metrizable mean $(<\mu^+)$-metrizable.
3) A space X is CWH (collectionwise Hausdorff) if for every subspace Y on which the induced topology is discrete (i.e. every subset is open) there is a sequence $\langle u_y : y \in Y\rangle$ of pairwise disjoint open subsets of X, such that for every $y \in Y$ we have $y \in u_y$.
4) A space X is $(<\lambda)$-CWH if for every $Y \subseteq X$ of cardinality $<\lambda$, Y (with the induced topology) is CWH.
$\mu - CWH$ means $(<\mu^+) - CWH$.
5) A space X is CWN (collectionwise normal) if: whenever $\langle Y_i : i < \alpha\rangle$ is a sequence of pairwise disjoint subsets of X, and each Y_i is clopen in $X \restriction (\bigcup_{j<\alpha} Y_j)$, then we can find pairwise disjoint open $\langle \mathcal{U}_i : i < \alpha\rangle$ in X such that $Y_i \subseteq \mathcal{U}_i$.
6) A space is $(<\lambda)-^* \ CWN$ if every subspace with $<\lambda$ points is CWN (we use the $*$ because another notion is: there is a bound $\alpha < \lambda$ such that all relevant subspaces are of size $<\alpha$).
$\mu -^* CWN$ means $(<\mu^+) -^* CWN$.

0.2 Question. (ZFC) 1) Are there $\aleph_1$-metrizable not metrizable (first countable Hausdorff topological) spaces?
2) Are there $\aleph_1 - CWH$ not CWH first countable spaces?
We shall also consider analogous questions with $\aleph_1$ replaced by any $\lambda > \aleph_0$.
Note: λ-metrizable $\Rightarrow \lambda - CWH$. Also, metrizable $\Rightarrow CWN \Rightarrow CWH$.

0.3 Observation. Assume X is a space with character $\chi \leq \lambda$ (i.e. every point has a neighborhood basis of cardinality $\leq \chi$).
Then:

(a) X is $\lambda - CWH$ iff for every subset Y of cardinality $\leq \lambda$ on which the induced topology is discrete there is a sequence $\langle u_y : y \in Y\rangle$ of pairwise disjoint open subsets of X such that $y \in u_y$.
(b) In (a), for any fixed $\chi \leq \mu \leq \lambda$, we can restrict ourselves (on both sides) to discrete subsets of cardinality μ.

Proof. (a) The implication $\Leftarrow$ is immediate. For the implication $\Rightarrow$ assume that $Y \subseteq X, |Y| \leq \lambda$ and $X \restriction Y$ is the discrete topology. Let $\langle \mathcal{U}^y_i : i < i^y \leq \chi\rangle$ be a neighborhood basis in X for $y \in Y$; choose for $y^1, y^2 \in Y$ and $i_1 < i^{y^1}, i_2 < i^{y^2}$ a point $z[y^1, y^2, i_1, i_2]$ which is in $\mathcal{U}^{y^1}_{i_1} \cap \mathcal{U}^{y^2}_{i_2}$, if this intersection is non-empty. By the assumption $X \restriction Y_1$ is CWH, where
$Y_1 = Y \cup \{z[y^1, y^2, i_1, i_2]\colon y^1 \in Y, y^2 \in Y, i_1 < i^{y^1}, i_1 < i^{y^2}\}$.
(b) Follows from the proof of (a). $\square_{0.3}$

§1 Analysis of "$\aleph_1 - CHW$ but not CHW"

1.1 Lemma. *1) Assume*

$(*)_\lambda$ *$cf(\lambda) = \aleph_0 < \lambda$, $\eta_\alpha \in {}^\omega\lambda$ for $\alpha < \lambda^+$, and for each $\beta < \lambda^+$, we can find pairwise disjoint end segments for $\langle \eta_\alpha : \alpha < \beta\rangle$ (e.g. $\exists h_\beta : \beta \to \omega$ such that*

$$\alpha_1 < \alpha_2 < \beta \wedge k > h_\beta(\alpha_1) \wedge k > h_\beta(\alpha_2) \Rightarrow \eta_{\alpha_1} \restriction k \neq \eta_{\alpha_2} \restriction k).$$

Then 1) the space ${}^{\omega>}\lambda \cup \{\eta_\alpha : \alpha < \lambda^+\}$ with the topology given below is

(α) *first countable and Hausdorff*
(β) *$\lambda - CWH$, even λ-metrizable*
(γ) *not $\lambda^+ - CHW$.*

The topology is the obvious one: each $\eta \in {}^{\omega>}\lambda$ is isolated, and for each $\alpha < \lambda^+$, the neighborhood basis of η_α is $\{\{\eta_\alpha \restriction \ell : k < \ell \leq \omega\} : k < \omega\}$.
2) Hence, the space is not metrizable but is λ-metrizable.

Proof. Straightforward. $\square_{1.1}$

1.2 Conclusion. 1) If the answer to 0.2(1) or 0.2(2) is "no", then $(*)_\lambda$ of 1.1 is not true for any λ.
2) If $(*)_\lambda$ of 1.1 fails for all λ, then

$(*)$ $cf(\lambda) = \aleph_0 < \lambda \Rightarrow pp(\lambda) = \lambda^+$

(by [Sh:355],1.5A).
3) If 2)'s conclusion holds, then for every λ singular we have $pp(\lambda) = \lambda^+$. (By [Sh:371],1.10 or [Sh:371],1.10A(6) or [Sh:355],2.4(1)), hence for every pair $\theta < \mu$ we have $cf([\mu]^{\leq\theta}, \subseteq) = \text{cov}(\mu, \theta^+, \theta^+, 2) \leq \mu^+$ (by [Sh:400],1.8), in fact

$$cf([\mu]^{\leq\theta}, \subseteq) = \begin{cases} \mu & \text{if} \quad cf(\mu) > \theta \\ \mu^+ & \text{if} \quad cf(\mu) \leq \theta. \end{cases}$$

4) If 3)'s conclusion holds then:
$(*)$ if λ is singular strong limit then

(a) $2^\lambda = \lambda^+$
hence
(b) $\diamondsuit^*_{S_\lambda}$ where $S_\lambda = \{\delta < \lambda^+ : cf(\delta) \neq cf(\lambda)\}$, and $\diamondsuit^*_S$ means that there is a $\langle \mathcal{P}_\delta : \delta \in S\rangle$ satisfying $\mathcal{P}_\alpha \subseteq [\alpha]^\lambda$, $|\mathcal{P}_\alpha| = \lambda$ such that

$$(\forall X \subseteq \lambda^+)(\exists \text{ club } C)\Big[\bigwedge_{\delta \in S \cap C} (X \cap \delta) \in \mathcal{P}_\delta\Big]$$

((b) holds by [Sh:108]; see there on earlier work of Gregory).
Note that clearly $\diamondsuit^*_S$ & $S_1 \subseteq S \Rightarrow \diamondsuit^*_{S_1}$.
(5) Not only $pp(\lambda) > \lambda^+$ and $\lambda > \aleph_0 = cf(\lambda)$ implies $(*)_\lambda$ (from 1.11); but assume

we have $\langle \lambda_n : n < cf(\lambda) = \aleph_0 \rangle$, $\sum\limits_n \lambda_n = \lambda$, $\lambda_n = cf(\lambda_n)$, $\mathrm{tcf}(\Pi\lambda_n/J^{bd}_\omega) = \lambda^+$, as exemplified by $\bar{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ such that

> $\oplus$ if $\aleph_0 < cf(\delta) \le \kappa < \lambda$, then there is a closed unbounded $A \subseteq \delta$ and $n_\alpha < cf(\lambda)$ for $\alpha \in A$ such that $n_\alpha, n_\beta < n < cf(\lambda) \Rightarrow f_\alpha(n) < f_\beta(n)$.

Then using $\oplus$ we get $(*)_\lambda$ of 1.1, so we get a $\kappa-CWH$, κ-metrizable first countable space which is not metrizable nor CWH (so not λ^+-metrizable and not λ^+-CWH). In fact $\lambda > cf(\lambda) = \aleph_0$, $cf([\lambda]^{\aleph_0}, \subseteq) > \lambda^+$ is sufficient too (see [Sh:355],§6).

1.3 Construction. Assume $\lambda = \beth_{\omega_1}$ (or just λ is a strong limit, $cf(\lambda) \neq \aleph_0$), $2^\lambda = \lambda^+$ and S is a stationary subset of λ^+ such that

$$S \subseteq \{\delta < \lambda^+ : cf(\delta) = \aleph_0 \text{ and } \lambda^2 \text{ divides } \delta\}.$$

We shall build a space with the set of points $\{x_\alpha, y_\alpha : \alpha < \lambda^+\}$. Each x_α will be isolated in X and each y_α will have a countable neighborhood basis in X. We shall have $\{u_{\alpha,n} : n < \omega\}$ as a neighborhood base of y_α with $u_{\alpha,n}$ decreasing in n and $u_{\alpha,n} = \{y_\alpha\} \cup \{x_\beta : f_\alpha(\beta) > n\}$ where f_α is a function from λ^+ to ω which we shall define below.
Note that each y_α is isolated in the space restricted to $\{y_\alpha : \alpha < \lambda^+\}$.
The only thing left is to define f_α for $\alpha < \lambda^+$.
We set $f_\alpha(\beta) = 0$ except in some specified cases. For the space to be Hausdorff it is enough to have:
for $\alpha < \beta$ there is an $m = m(\alpha, \beta) < \omega$ such that
$\neg(\exists\gamma)[f_\alpha(\gamma) \ge m \ \& \ f_\beta(\gamma) \ge m]$. We shall make a stronger condition:

$(*)_0$ $\alpha < \beta \Rightarrow (\exists^{\le 1}\gamma)[f_\alpha(\gamma) \ge 1 \ \& \ f_\beta(\gamma) \ge 1]$.

Remember that, as remarked in 1.2(4), it is reasonable to assume

> $(*)_1$ $\diamondsuit_S$ holds since $2^\lambda = \lambda^+$ and $cf(\lambda) > \aleph_0$ (or by the proof of 1.2 for arbitrarily large $\mu < \lambda$, $(*)_\mu$ of 1.1 holds). So there is a $\langle g_\alpha : \alpha \in S \rangle$ with $g_\alpha : \alpha \to \omega$, such that
>
> $$(\forall g \in {}^{\lambda^+}\omega)(\exists^{\text{stat}}\alpha \in S)(g_\alpha = g \restriction \alpha).$$

Now, if the space is CWH then there is a $g : \lambda^+ \to \omega$ such that $\langle u_{\alpha,g(\alpha)} : \alpha < \lambda^+ \rangle$ are pairwise disjoint.

We define by induction on α a limit $< \lambda^+$ the value of $f_i(j)$ for $i, j < \alpha$. Denote $f^\alpha_i = f_i \restriction \alpha$ and denote the sequence $\langle f^\alpha_i : i < \alpha \rangle$ by $\bar{f}^\alpha$, so if $i < \alpha$, then f^α_i is a sequence in ${}^\alpha\omega$ and f^α_i is an initial segment of f^β_i when $\alpha < \beta$. Usually we just give value zero to $f_i(j)$.
If $\alpha = \omega$, or α is limit there are no problems.
If $cf(\alpha) > \aleph_0$, and $\bar{f}^\alpha$ is defined, we define $\bar{f}^{\alpha+\omega}$ by letting all the new values be equal to zero. If $\alpha \in S$, and g_α looks as a candidate for g, i.e. $\langle u^\alpha_{i,g_\alpha(i)} : i < \alpha \rangle$ are pairwise disjoint, where $u^\alpha_{i,k} =: \{x_\beta : \beta < \alpha \ \& \ f_i(\beta) > k\}$, and if for some m

it happens that $\mathrm{otp}(\{\beta < \alpha : g_\alpha(\beta) = m\}) = \alpha$, then choose for the minimal such $m = m_\alpha$

(a) $\beta_n^\alpha < \beta_{n+1}^\alpha \cdots < \alpha = \bigcup_n \beta_n^\alpha$

(b) $g_\alpha(\beta_n^\alpha) = m$

and define $\bar{f}^{\alpha+\omega}$ (extending $\bar{f}^\alpha$) by

$$f_\alpha^{\alpha+\omega}(\alpha + n) = n$$

and

$$f_{\beta_n^\alpha}^{\alpha+\omega}(\alpha + n) = m + 1$$

(other values of $\bar{f}^{\alpha+\omega}$ are zero). If g_α fails the conditions above or if $\alpha \notin S$ but $cf(\alpha) = \aleph_0$, choose $m_\alpha = -1$, β_n^α satisfying conditions (a) above and extend $\bar{f}^\alpha$ as just described.

So, if g_α satisfies all the conditions above, we cannot extend g_α to a g which is as required (for CWH) and defined on λ^+ : if $g(\alpha) = k$ we get

$$x_{\alpha+k+18} \in \left(u_{\alpha,k+1} \cap u_{\beta_{k+18}^\alpha, g_\alpha(\beta_{k+18}^\alpha)}\right).$$

So the space is not CWH (hence not metrizable). For simplicity, we can request that $\beta_n^\alpha \notin \bigcup_{\gamma \in S} [\gamma, \gamma + \omega)$. Hence, $(*)_0$ holds, so the space is Hausdorff.

Suppose the space is not $\aleph_1 - CWH$. So for some $\mathcal{U} \in [\lambda^+]^{\aleph_1}$

$$X \restriction \{x_\alpha, y_\alpha : \alpha \in \mathcal{U}\}$$

is not CWH.

So without loss of generality if

$$\alpha \in S \cap \mathcal{U}$$

then

$$\alpha + n \in \mathcal{U} \text{ and}$$
$$\beta_n^\alpha \in \mathcal{U}.$$

So

$\otimes$ for every $g : \mathcal{U} \to \omega$ (candidate to give the separation), <u>we get</u>: for some $\alpha \in S \cap \mathcal{U}$, $(\exists^\infty n)\, g(\beta_n^\alpha) \le m_\alpha$.

$\square_{1.3}$

1.4 Comments. (0) Unfortunately, we have not proved "X is $\aleph_1 - CWH$".
(1) The space constructed in 1.3 has neighborhood bases consisting of countable sets, like the ones considered in the earlier consistency results from [JShS:320]. However, above a superrcompact pp is trivial, no such phenomenon arrises here.
(2) But $\Vdash_{\mathrm{Levy}(\aleph_1,\lambda^+)}$ "X is not $\aleph_1 - CWH$" may fail unless we put more restrictions on the β_n^α. See (3).
(3) If we build X as above, let $\mathbb{P} = \mathrm{Levy}(\aleph_1, \lambda^+)$ and there is a $\mathbb{P}$-name $\underset{\sim}{g}$ such that

$$\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{g} : \lambda^+ \to \omega \text{ witnesses that } X \text{ is } CWH\text{”},$$

then X is $\aleph_1 - CWH$.
[Why? given a $Y \in [\lambda^+]^{\aleph_1}$, we can find $\langle p_i : i < \omega_1\rangle$ increasing in $\mathbb{P}$ such that $\bigwedge_{\alpha\in Y} \bigvee_i p_i \Vdash$ "$\underset{\sim}{g}(\alpha) =$ something"].
(4) It is well known that if $\lambda = cf(\lambda) > \aleph_0$ and $S \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}$ is stationary not reflecting, then there is a $(< \lambda) - CWH$ not CWH space. A space like this can be constructed in a fashion similar to that of 1.3. Namely, we can choose for each $\delta \in S$ an increasing sequence $\langle \alpha_n^\delta : n < \omega\rangle$ which converges to δ. We require $\alpha_n^\delta \notin S$, say α_n^δ a successor ordinal. We define

$$f(\beta,\delta) = \begin{cases} n & \text{if} \quad \delta \in S \text{ and } \beta = \alpha_n^\delta \\ 0 & \text{otherwise.} \end{cases}$$

We set $X = \{y_\delta : \delta \in S\} \cup \{x_\beta : \beta \in \lambda\}$ set x_β isolated and let $u_{\delta,n} = \{y_\delta\} \cup \{x_\beta : \beta < \delta \ \&\ f(\beta,\delta) \geq n\}$ be a neighborhood base for $y_\delta (n < \omega)$.

To see that X is not CWH, do as above, and to see it is $(< \lambda) - CWH$, use induction on $\alpha < \lambda$. since S is not reflecting, at $\alpha \in S$ we can choose a cofinal sequence which avoids S, and apply the induction hypothesis.

1.5 Definition. We say that the space X is $\lambda - WCWH$ if for any discrete set of λ points, some subset of cardinality λ can be separated by disjoint open sets. X is $WCWH$ if X is $|X|$-$WCWH$.

1.5A Remark. By a theorem of Foreman and Laver for first countable spaces we have the consistency of: $\aleph_1 - WCWH \Rightarrow \aleph_2 - WCWH$.

Namely in [FoLa88], starting with a huge embedding $j : V \to M$ with critical point κ and $j(\kappa) = \lambda$, the following is obtained:

There is a forcing notion $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ such that $\mathbb{P}$ is κ-c.c.,$|\mathbb{P}| = \kappa$, $V[G_{\mathbb{P}}] \models$ "$\kappa = \omega_1$", $\mathbb{R} \in V[G_{\mathbb{P}}]$ is λ-c.c., of cardinality λ and $(< \kappa)$-closed and $V[G_{\mathbb{P}*\underset{\sim}{\mathbb{R}}}] \models$ "$\lambda = \omega_2$". In addition, there is a regular embedding $h : (\mathbb{P} * \underset{\sim}{\mathbb{R}}) \to j\mathbb{P}$ with $h(p) = p$ for all $p \in \mathbb{P}$ and the master condition property holds for h, $j\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{R}}$. Finally, if G is $(\mathbb{P} * \underset{\sim}{\mathbb{R}})$-generic, then in $V[G], j\mathbb{P}/h''(G)$ is κ-centered.

The consistency of $\aleph_1 - WCWH \Rightarrow \aleph_2 - WCWH$ for first countable spaces clearly follows from the above result of [FoLa88]. For the convenience of the reader we include the following easy Claim 1.5B which shows this implication. In fact, M. Foreman informs us that from other results in [FoLa88], the implication is even easier.

1.5B Claim. *Suppose X is a first countable topological space and $|X| = \kappa^+$, while $Y_0 \subseteq X$ is a discrete subspace of X, with $|Y_0| = \kappa^+$. If $\mathbb{P}$ is a κ-centered forcing notion such that*

$$\Vdash_{\mathbb{P}} \text{“There is a } Y \subseteq Y_0 \text{ with } |Y| = |X| \text{ and } Y \text{ is separated in } X\text{”},$$

then
in V, there is a $Y \subseteq Y_0, |Y| = |Y_0|$ and Y is separated in X.

Proof. Without loss of generality, the set of points of Y_0 is κ^+, and we denote $\lambda = \kappa^+$. We may fix a set $\{\underset{\sim}{x}_\gamma : \gamma < \lambda\}$ of $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \text{“}\{\underset{\sim}{x}_\gamma : \gamma < \lambda\} \text{ is separated } \subseteq \lambda \text{ and has cardinality } \lambda\text{”}.$$

We can also assume that there are no repetitions among the $\underset{\sim}{x}_\gamma$, and that $\underset{\sim}{x}_\gamma \geq \gamma$. Suppose that in V, the neighborhood bases for points in Y_0 are given by

$$\langle \langle u^n_y : n < \omega \rangle : y \in Y_0 \rangle .$$

So, without loss of generality $\{u^{\underset{\sim}{n}(\gamma)}_{\underset{\sim}{x}_\gamma} : \gamma < \lambda\}$ are pairwise disjoint, in $V^{\mathbb{P}}$.

Now, let $\mathbb{P} = \bigcup\limits_{i<\kappa} \mathbb{P}_i$ where each $\mathbb{P}_i$ is directed.

For each $\alpha < \lambda$, there is a condition p_α forcing a value to $\underset{\sim}{x}_\alpha, \underset{\sim}{n}(\alpha)$ say $\beta_\alpha, m(\alpha)$. So, there is an $i(*) < \kappa$ such that $A = \{\alpha : p_\alpha \in \mathbb{P}_{i(*)}\}$ is unbounded in λ. Therefore, $\{\beta_\alpha : \alpha \in A\}$ is separated by

$$\{u^{m(\alpha)}_{\beta_\alpha} : \alpha \in A\}.$$

(So, having that any two members of $\mathbb{P}_i$ are compatible, or that out of any λ elements of $\mathbb{P}$ there are λ pairwise compatible, i.e. $\mathbb{P}$ is λ-Knaster, suffices). $\square_{1.5.B}$

On the other hand, e.g.

1.6 Claim. *There is a first countable Hausdorff space X which is $(2^{\aleph_0})^+ - WCWH$ but is not $WCWH$.*

Proof. Let $\lambda = \sum\limits_{n<\omega} \lambda_n$, $\lambda_n^{\aleph_0} < \lambda_{n+1}$. Let $\langle \eta_\alpha : \alpha < \lambda^+ \rangle$, $\eta_\alpha \in {}^\omega\lambda$, $\alpha < \beta$ and $\eta_\alpha <_{J^{bd}_\omega} \eta_\beta$.

We define the topological space X on ${}^{\omega>}\lambda \cup \{\eta_\alpha : \alpha < \lambda^+\}$ as in 1.1.

Proof that X is not $\lambda^+ - WCWH$: if $\mathcal{U} \in [\lambda^+]^{\lambda^+}$, $\langle \eta_\alpha : \alpha \in \mathcal{U} \rangle$ cannot be separated as $|\{\eta_\alpha \restriction \ell : \ell < \omega, \alpha \in \mathcal{U}\}| \leq \lambda$.

If $\mathcal{U} \in [\lambda^+]^{(2^{\aleph_0})^+}$, without loss of generality $\mathrm{otp}(\mathcal{U}) = (2^{\aleph_0})^+$; set $\mathcal{U} = \{\alpha_\zeta : \zeta < (2^{\aleph_0})^+\}$. Now for some $Y \in [(2^{\aleph_0})^+]^{(2^{\aleph_0})^+}$ and n, $\langle \eta_{\alpha_\zeta} \restriction [n,\omega) : \zeta \in Y\rangle$ is strictly increasing (not just modulo J^{bd}_ω but in every coordinate (see [Sh:111] or [Sh:355], §1, also [Sh:355],§6; see also [Sh:400],§5, [Sh:430], §6)). $\square_{1.6}$

1.7 Remark. We can prove other Claims similar to 1.6 (see the references above).

§2 On not CWH, $\aleph_1 - CWH$ Spaces

2.1 Definition. For an ordinal γ let us define

$(*)^1_\gamma$ there is an $S \subseteq \{\delta < \gamma : cf(\delta) = \aleph_0\}$ and, for $\delta \in S$, a sequence $\langle \beta^\delta_n : n < \omega\rangle$ strictly increasing with limit δ, and a $m_\delta < \omega$, such that $(\forall g \in {}^\gamma\omega)(\exists \delta \in S)(\exists^\infty n)[g(\beta^\delta_n) \leq m_\delta]$.

2.2 Claim. *(1) If the answer to 0.2 is no (or much less), then for some $\gamma < \omega_2$, $(*)^1_\gamma$ holds.*
(2) If $MA + \neg CH$, then $\gamma < 2^{\aleph_0} \Rightarrow \neg(*)^1_\gamma$.
(3) Without loss of generality, in $(*)^1_\gamma$, each β^δ_n is a successor ordinal.

Proof. 1) By the proof of 1.3 and 1.2. take $\gamma = otp(u)$, where u is like at the end of 1.3.
(2) Check. Use the natural forcing $\{p : p$ is a finite function from γ to $\omega\}$ with $p \leq q$ iff $p \subseteq q$ & $(\forall\delta)(\delta \in S \cap \mathrm{Dom}(p) \rightarrow (\forall n)[\beta^\delta_n \in \mathrm{Dom}(q)\setminus\mathrm{Dom}(p) \rightarrow q(\beta^\delta_n) > n])$.
(3) Check. $\square_{2.2}$

2.2A Conclusion. If $MA + \neg CH$ <u>then</u> the answer to 0.2 is yes. In fact, there is an $\aleph_1$-metrizable (hence $\aleph_1$-CWH) not CWH (hence not metrizable) first countable space.

Proof. By 2.2(1) and 2.2(2).

2.3 Claim. *If $(*)^1_\gamma$ for some $\gamma < \omega_2$, then $(*)^1_{\omega_1}$.*

Proof. Choose $\gamma^* < \omega_2$ minimal such that $(*)^1_{\gamma^*}$. Clearly $\gamma^* \geq \omega_1$.

If $\gamma^* = \omega_1$ we are done. So assume $\gamma^* > \omega_1$, and we shall get a contradiction. We fix an $S \subseteq \gamma^*$ and m_δ, $\langle \beta^\delta_n : n < \omega\rangle$ for $\delta \in S$, which exemplify $(*)^1_{\gamma^*}$. Note that for every $\gamma < \gamma^*$ there is a $g_\gamma \in {}^\gamma\omega$ such that:

$\otimes$ if $\delta \in S \cap \gamma$ <u>then</u> $\{n : g_\gamma(\beta^\delta_n) \leq m_\delta\}$ is finite.

Case 1. $\gamma^* = \gamma + 1$ and $\gamma \notin S$.
Extend g_γ by $\{\langle\gamma, 0\rangle\}$.

Case 2. $\gamma^* = \gamma + 1$ and $\gamma \in S$.
Define $g \in {}^{\gamma^*}\omega$:

$$\begin{array}{ll} \underline{\text{if}}\ \beta \in \gamma, \beta \notin \{\beta_n^\gamma : n < \omega\} & \underline{\text{then}}\ g(\beta) = g_\gamma(\beta) \\ \underline{\text{if}}\ \beta = \gamma & \underline{\text{then}}\ g(\beta) = 0 \\ \underline{\text{if}}\ \beta = \beta_n^\gamma & \underline{\text{then}}\ g(\beta) = \text{ Max}\{g_\gamma(\beta), n+8, m_\gamma + 8\}. \end{array}$$

So g gives a contradiction.

Case 3. $cf(\gamma^*) = \aleph_0$.

Let $\gamma^* = \bigcup\limits_{n<\omega} \gamma_n$, $\gamma_0 = 0$, $\gamma_n < \gamma_{n+1}$, and each γ_{n+1} is a successor of a successor ordinal.

Let $g = \cup\{g_{\gamma_{n+1}} \restriction [\gamma_n, \gamma_{n+1}) : n < \omega\}$ - it gives a contradiction.

Case 4. $cf(\gamma^*) = \omega_1$.

Let $\langle \gamma_i : i < \omega_1 \rangle$ be increasing continuous with limit γ^*, $\gamma_0 = 0$, γ_{i+1} a successor of a successor ordinal.
Let $S^* =: \{\gamma_i : \gamma_i \in S$ (so i is a limit ordinal)$\}$.

Subcase A. γ^*, $\langle < \beta_n^\gamma : n < \omega >: \gamma \in S^* \rangle$, $\langle m_\gamma : \gamma \in S^* \rangle$ do not exemplify $(*)^1_{\gamma^*}$.
So some $g^* \in {}^{\gamma^*}\omega$ shows this. Define g by:

$$\underline{\text{if}}\ \beta \in [\gamma_i, \gamma_{i+1})\ \underline{\text{then}}\ g(\beta) = \text{ Max}\{g_{\gamma_{i+1}}(\beta), g^*(\beta)\}$$

So g gives a contradiction.

Subcase B. $\langle < \beta_n^\gamma : n < \omega >: \gamma \in S^* \rangle$, $\langle m_\gamma : \gamma \in S^* \rangle$ exemplify $(*)^1_{\gamma^*}$.
If S^* is not stationary, then we get a contradiction as in case 3, noting that in this case we can without loss of generality assume $\bigwedge\limits_i \gamma_i \notin S^*$. Therefore we may note that S^* is stationary, even though this will not be used in the rest of the proof. Let $\gamma^* = \bigcup\limits_{i<\omega_1} a_i$ with a_i countable increasing continuous, <u>such that</u> $a_0 = \emptyset$, $a_i \cap \{\gamma_j : j < \omega_1\} = \{\gamma_j : j < i\}$, $a_i \subseteq \gamma_i$ and $\gamma_j \in a_i \wedge j \in S^* \Rightarrow \bigwedge\limits_n \beta_n^{\gamma_j} \in a_i$.
For $i \in S^*$ let $u_i =: \{n < \omega : \beta_n^{\gamma_i} \in a_i\}$.
Note

> $\otimes$ if $i \in S^*$ and $j < i$, then $\{n \in u_i : \beta_n^{\gamma_i} \in a_j\}$ is finite, as it is included in $\{n < \omega : \beta_n^{\gamma_i} < \gamma_j\}$. [Why? Remember $a_j \subseteq \gamma_j$].

Let $S^{**} = \{i \in S^* : u_i$ is infinite and i is a limit ordinal$\}$. So we already know

> $\oplus$ for every $g \in {}^{\gamma^*}\omega$, for some $i \in S^*$, for infinitely many $n < \omega$ we have $g(\beta_n^{\gamma_i}) \leq m_{\gamma_i}$.

We claim

$\oplus^+$ for every $g \in {}^{\gamma^*}\omega$ for some $i \in S^{**}$, for infinitely many $n \in u_i$ we have $g(\beta_n^{\gamma_i}) \leq m_{\gamma_i}$.

Otherwise, for some $g^* \in {}^{\gamma^*}\omega$ this fails and we define g:

<u>let</u> $\beta \in a_{i+1} \backslash a_i$ (there is one and only one such i),
<u>then</u> $g(\beta) = \operatorname{Max}\{g^*(\beta), m_{\gamma_i} + 8, m_{\gamma_i+1} + 8\}$

As g gives a contradiction to $\oplus$, clearly $\oplus^+$ holds.

Now let h be a one to one function from ω_1 onto γ^* such that for i limit, h maps $\{j : j < i\}$ onto a_i.
Let for $i \in S^{**}$, $\{j_n^i : n < \omega\}$ enumerate $\{j < i : h(j) \in \{\beta_n^{\gamma_i} : n \in u_i\}\}$, and $m_i^* = m_{\gamma_i}$ for $i \in S^{**}$.
Now $\langle < j_n^i : n < \omega >: i \in S^{**}\rangle$, $\langle m_i^* : i \in S^{**}\rangle$ exemplify that γ^* could have been chosen to be $= \omega_1$, as required. $\square_{2.3}$

We define the combinatorial property we actually use

2.4 Definition. 1) $INCWH(\lambda) = INCWH^1(\lambda)$ means:

λ is regular $> \aleph_0$ and for some stationary $S \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}$ we have $\langle m_\delta, < \beta_n^\delta : n < \omega >: \delta \in S\rangle$ such that:
$m_\delta < \omega$, $\beta_n^\delta < \beta_{n+1}^\delta < \delta = \bigcup\limits_{n<\omega} \beta_n^\delta$, each β_n^δ is a successor <u>and</u>:

(a) for every $g \in {}^\lambda\omega$, for some $\delta \in S$, for infinitely many n we have $g(\beta_n^\delta) \leq m_\delta$.

$(b)_\lambda$ for every $\mathcal{U} \subseteq \lambda$, $|\mathcal{U}| < \lambda$, for some $g \in {}^{\mathcal{U}}\omega$, for every $\delta \in S \cap \mathcal{U}$, for every $n < \omega$ large enough, $g(\beta_n^\delta) > m_\delta$.

2) We can replace m_δ by $\langle m_n^\delta : n < \omega\rangle$, requesting $g(\beta_n^\delta) \leq m_\delta^n$ in (a) and $g(\beta_n^\delta) > m_\delta^n$ in $(b)_\lambda$. In this way we obtain an apparently weaker property, which we call $INCWH^2(\lambda)$.
For other versions of the principle, as well as the connections between the various versions, see §3.

2.4A Discussion. 1) If $INCWH(\lambda)$, then there is a space (as in 1.3) which is Hausdorff first countable with λ points, not metrizable, not even CWH, but every subspace of smaller cardinality is metrizable. So, the notation, "$INCWH$" is derived from "incompactness for CWH", where incompactness is understood in the model-theorethic sense.
2) So if we prove $(\exists\lambda > \aleph_1)INCWH(\lambda)$ we have solved the original problem 0.2.
3) $(b)_\kappa$ means that we require $|\mathcal{U}| < \kappa$. Note that $(b)_{\aleph_1}$ holds trivially and that $\mu \leq \kappa$ & $(b)_\kappa \Rightarrow (b)_\mu$.

More formally

2.5 Claim. *If $INCWH(\lambda)$ then $SINCWH(\lambda)$ (even exemplified by a $(<\lambda)$-metrizable space), where:*

2.6 Definition. $SINCWH(\lambda)$ means that there is a first countable T_2-space X with λ points which is $(<\lambda)-CWH$ (i.e. for every discrete subset of cardinality $<\lambda$ we can choose pairwise disjoint open neighborhoods separating the subspace) but not $\lambda - CWH$.

Proof of 2.5. Assuming $INCWH(\lambda)$ we build a space X witnessing $SINCWH(\lambda)$. The points of X are y_α $(\alpha<\lambda)$ and $x_{\alpha,\beta}$ $(\beta<\alpha<\lambda)$ with $x_{\alpha,\beta}$ isolated and y_α which have neighborhood bases $\langle u_{\alpha,n} : n<\omega\rangle$:

$$\underline{\text{if}} \quad \alpha\in S \qquad u_{\alpha,n}=\{y_\alpha\}\cup\{x_{\alpha,\beta} : \text{for some } k>n \text{ we have } \beta=\beta^\alpha_k\}$$

$$\underline{\text{if}}\ \alpha\notin S \qquad u_{\alpha,n}=\{y_\alpha\}\cup\{x_{\delta,\alpha} : \alpha<\delta\in S, \text{ for some } k \text{ we have } \alpha=\beta^\delta_k, \text{ and } k\le m_\delta\}.$$

Here, S is a fixed stationary $\subseteq\{\delta<\lambda : cf(\delta)=\aleph_0\}$ which exemplifies $INCWH(\lambda)$, together with $\langle m_\delta, \langle\beta^\delta_n : n<\omega\rangle : \delta\in S\rangle$.

<u>Checking of "X not CWH"</u>

Let $Y=\{y_\alpha : \alpha<\lambda\}$.

Note that $X\restriction Y$ is a discrete subspace of X. Note that $\{u_{\alpha,n} : n<\omega\}$ is the neighborhood basis of y_α. Suppose that there is $\langle u_{\alpha,g(\alpha)} : \alpha<\lambda\rangle$, a sequence of pairwise disjoint sets, for some $g\in {}^\lambda\omega$. As $u_{\alpha,g(\alpha)}\cap u_{\beta,g(\beta)}=\emptyset$ for $\alpha\ne\beta(<\lambda)$ clearly for $\alpha\in S$ and $\beta=\beta^\alpha_k$ we get $k>g(\alpha)\Rightarrow g(\beta)>m_\alpha$ (since otherwise $x_{\alpha,\beta}\in u_{\alpha,g(\alpha)}\cap u_{\beta,g(\beta)}$, why? $x_{\alpha,\beta}\in u_{\alpha,g(\alpha)}$ as $k>g(\alpha)$ and $x_{\alpha,\beta}\in u_{\beta,g(\beta)}$ as $g(\beta)\le m_\alpha$).

So g contradicts (a) of $INCWH(\lambda)$.

<u>Checking of "X is $(<\lambda)-CWH$"</u>

Let $Z\subseteq X, |Z|<\lambda$ and $X\restriction Z$ is discrete. Let $Z_0=\{x_{\alpha,\beta} : \beta<\alpha<\lambda\}\cap Z, Z_1=\{y_\alpha : \alpha\in\lambda\backslash S\}\cap Z$, $Z_2=\{y_\alpha : \alpha\in S\}\cap Z$, so $\langle Z_1,Z_2,Z_3\rangle$ is a partition of Z. Let $\mathcal{U}=\{\alpha\in S : y_\alpha\in Z_2\}$, so $|\mathcal{U}|<\lambda$ and $\mathcal{U}\subseteq\lambda$ hence by the assumption, there is a $g_0\in {}^{\mathcal{U}}\omega$ as in $(b)_\lambda$.

We define u_z, a neighborhood of z for $z\in Z$ (remembering Z is discrete):

$$\begin{aligned}
&\text{if } z=x_{\alpha,\beta}\in Z_0, u_z=\{x_{\alpha,\beta}\}\\
&\text{if } z=y_\alpha\in Z_1, u_z=u_{\alpha,n(\alpha)} \text{ where}\\
&\qquad n(\alpha)=\ \mathrm{Min}\{n : n\ge g(\alpha)+8 \text{ and } u_{\alpha,n}\cap Z_0=\emptyset\}\\
&\text{if } z=y_\delta\in Z_2, u_z=u_{\delta,n(\delta)} \text{ where}\\
&\qquad n(\delta)=\ \mathrm{Min}\{n : n\ge m_\delta+8 \text{ and } u_{\delta,n}\cap Z_0=\emptyset\}.
\end{aligned}$$

Now check that $\{u_z : z\in Z\}$ separates the points of Z. $\square_{2.5}$

Note that 2.5 also follows from 3.6 + 3.8 below.

2.7 Claim. *Assume λ and $\langle m_\delta, \langle \beta^\delta_n : n < \omega\rangle : \delta \in S\rangle$ are as in 2.4, but we require λ just to be an ordinal, and weaken $(b)_\lambda$ to*

> $(b)_\kappa$ *for every $\mathcal{U} \subseteq \lambda$, $|\mathcal{U}| < \kappa$, for some $g \in {}^{\mathcal{U}}\omega$ for every $\delta \in S \cap \mathcal{U}$, for every n large enough $g(\beta^\delta_n) > m_\delta$.*

<u>Then</u> if λ satisfies this weakened version of $INCWH$, then for some regular μ, $\kappa \le \mu \le \lambda$ we have $INCWH(\mu)$.

Proof. If we allow μ in the definition of $INCWH(\mu)$ to be an ordinal: straightforward (and suffices for our main interest). Namely, we choose a $\mathcal{U}$ such that

(α) $\mathcal{U} \subseteq \lambda$,
(β) there is no $g \in {}^\lambda\omega$ such that for every $\delta \in S \cap \mathcal{U}$ for every n large enough $g(\beta^\delta_n) > m_\delta$,
(γ) under $(\alpha) + (\beta)$ the order type of $\mathcal{U}$ is minimal.

Clearly $\text{otp}(\mathcal{U}) \le \lambda$ and $\text{otp}(\mathcal{U}) \ge \kappa$. By the same proof as 2.3, $\text{otp}(\mathcal{U})$ is a regular cardinal, we call it μ and with the a_i's as in the proof of 2.3, we get $INCWH(\mu)$. $\square_{2.7}$

2.8 Conclusion. If $\lambda = cf(\lambda) > \aleph_0$ and $\diamondsuit_{\{\delta<\lambda: cf(\delta)=\aleph_0\}}$, then for some regular uncountable $\lambda' \le \lambda$ (but not necessarily $\lambda' > \aleph_1$!), we have $INCWH(\lambda')$.

Proof. By the proof of 1.3 there is a sequence $\langle m_\delta, \langle \beta^\delta_n : n < \omega\rangle : \delta \in S\rangle$ exemplifying *(a)* of 2.4, with $S = \{\delta < \lambda : cf(\delta) = \aleph_0\}$. We know that $(b)_{\aleph_1}$ holds. Now use 2.7.

2.9 Observation. If $S_1 \subseteq S_2 \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}$, $\langle m_\delta, \langle \beta^\delta_n : n < \omega\rangle : \delta \in S_1\rangle$ witness $INCWH(\lambda)$, <u>then</u> we can find a $\langle m'_\delta, \langle \gamma^\delta_n : n < \omega\rangle : \delta \in S_2\rangle$ witnessing $INCWH(\lambda)$.

Proof. By [Sh:351], proof of 4.4(2) we can find $\langle \eta_\delta : \delta \in S_2 \backslash S_1\rangle$ such that:
η_δ is an increasing ω-sequence of successor ordinals with limit δ such that

$$\eta_{\delta_1}(n_1) = \eta_{\delta_2}(n_2) \Rightarrow n_1 = n_2 \ \& \ \eta_{\delta_1} \restriction n_1 = \eta_{\delta_2} \restriction n_2.$$

Now we define $m^2_\delta, \beta^{2,\delta}_n$ for $\delta \in S_2, n < \omega$: if $\delta \in S_1$ then $m^2_\delta = m_\delta$, $\beta^{2,\delta}_n = 2\beta^\delta_n$ and if $\delta \in S_2 \backslash S_1$ then $m_\delta = 3$, $\beta^{2,\delta}_n = 2\eta_\delta(n) + 1$.
Now check. $\square_{2.9}$

2.10 Remark. We can replace in our discussion $\aleph_0$ as the character of the spaces under consideration by θ. Towards this we define a family of spaces.

2.11 Definition. $X \in \mathcal{T}_\theta^\ell$ <u>if</u> X is a Hausdorff space with each point x having a neighborhood basis $\{u_{x,\alpha} : \alpha < \alpha^*\}$ such that:

(a) $\ell = 0$ and $\alpha^* \leq \theta$ or
(b) $\ell = 1, \alpha^* \leq \theta$ and $\langle u_{x,\alpha} : \alpha < \alpha^* \rangle$ is decreasing, or
(c) $\ell = 2, \alpha^* = \theta$, and $\langle u_{x,\alpha} : \alpha < \alpha^* \rangle$ is decreasing.

If $\ell = 2$, we may omit it.

2.12 Definition. We define also the principles
$INCWH(\lambda, \theta) = INCWH^1(\lambda, \theta)$ and $INCWH^2(\lambda, \theta)$ as in 2.4 (with ω replaced by θ).
E.g. $INCWH^2(\lambda, \theta)$ means that there are S and $\langle \varepsilon_i^\delta : i < \theta \rangle, \langle \beta_i^\delta : i < \theta \rangle$ for $\delta \in S$ such that:

$(A)(i)$ $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$
(ii) β_i^δ is a successor ordinal (this is not a serious obstracle, we in fact want $\beta_i^\delta \notin S$)
(iii) $i < j < \theta, \delta \in S \Rightarrow \beta_i^\delta < \beta_j^\delta < \delta$
(iv) $\delta = \bigcup_{i<\theta} \beta_i^\delta$
(v) $\varepsilon_i^\delta < \theta$

$(B)(i)$ for every $g \in {}^\lambda\theta$ for some $\delta \in S$ we have $\theta = \sup\{i < \theta : g(\beta_i^\delta) \leq \varepsilon_i^\delta\}$
(ii) for $X \in [\lambda]^{<\lambda}$ there is $g \in {}^\lambda\theta$ such that for every $\delta \in S \cap X$ we have $\theta > \sup\{i : g(\beta_i^\delta) \leq \varepsilon_i^\delta\}$.

2.13 Claim.

(α) *if $\lambda > cf(\lambda) = \theta$, $pp(\lambda) > \lambda^+$ (or the parallel of 1.2(5)), then*
   $\otimes$ *there is an $X \in \mathcal{T}_\theta^2$, $|X| = \lambda^+$, X is $\lambda - CWH$, X has a discrete subspace of size λ^+, but for some $X' \subseteq X$, $|X'| = \lambda$, $cl(X') = X$ (so $|cl(X')| > \lambda$) (this is a strong form of X is not $\lambda^+ - CWH$).*

(β) *if $\lambda > cf(\lambda) = \theta$, λ is a strong limit and $2^\lambda = \lambda^+$, then: $INCWH(\lambda', \theta)$ for some $\lambda' = cf(\lambda') \in [\theta^+, \lambda^+]$.*

Proof. Similar to the above, replacing $\aleph_0$ with θ. $\square_{2.13}$

§3 Variants of Freeness

3.1 Definition. 1) $INCwh(\lambda) = INCwh^1(\lambda)$ is defined as in 2.4 except that $\langle \beta_n^\delta : n < \omega \rangle$ is not required to be increasing with limit δ, just $[n \neq m \Rightarrow \beta_n^\delta \neq \beta_m^\delta]$.
2) $INCwh^2(\lambda)$ is defined as in (1) but we use $\langle m_n^\delta : n \in \omega \rangle$ rather than a single m_δ. (Compare 2.4(2).)

3.2 Claim. *0) $INCWH^\ell(\lambda) \Rightarrow INCwh^\ell(\lambda)$, $INCWH^1(\lambda) \Rightarrow INCWH^2(\lambda)$, $INCwh^1(\lambda) \Rightarrow INCwh^2(\lambda)$.*
1) $INCwh^2(\mathfrak{b})$ (where
$\mathfrak{b} =$ Min$\{|F| : f \subseteq {}^\omega\omega$ and for no $g \in {}^\omega\omega$ do we have for every $f \in F$ that $f <^* g\}$.
As usual, $f <^* g$ means that $\{n : f(n) \geq g(n)\}$ is finite).
2) Assume $\lambda \leq 2^{\aleph_0}$ and for $\alpha < \lambda$, f_α is a partial function from ω to ω, Dom(f_α) is infinite and $U \subseteq \lambda$ & $|U| < \lambda \Rightarrow (\exists f \in {}^\omega\omega) \bigwedge_{\alpha \in U} f_\alpha \leq^* f$ but for no $f \in {}^\omega\omega$, $\bigwedge_{\alpha<\lambda} f_\alpha <^* f$, then $INCwh^2(\lambda)$.
3) It does not matter in 3.1 if we demand "β^δ_n is a successor ordinal".

Proof. 0) Check.
1) By 2).
2), 3) Check. $\square_{3.2}$

3.2A Questions.
1) Are there examples like in 2.6 for λ singular e.e., does $SINCWH(\lambda)$ hold for λ singular?
2) Suppose that in 3.2(2) we allow for each α a filter F_α on Dom(f_α) generated by $\aleph_0$ sets and we require $\bigwedge_{\alpha \in U} \{\beta \in \text{Dom}(f_\alpha) : f_\alpha(\beta) < f(\beta)\} \in F_\alpha$; is this equivalent to $INCwh^2(\lambda)$?

3.3 Claim. *Assume $INCwh^2(\kappa)$, $\lambda > \kappa$ and λ is regular, $S \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}$ is stationary and $\diamondsuit_S$ holds.*
Then (1) there is a $\langle\langle m^\delta_n, \beta^\delta_n : n < \omega\rangle : \delta \in S\rangle$ as in 2.4(2), but only (a) and (b)$_\kappa$ hold.
2) For some regular $\lambda' \in [\kappa, \lambda]$, we have $INCWH^2(\lambda')$.
3) We can replace $INCwh^2(\lambda')$, $INCWH^2(\lambda')$ by $INCwh^1(\lambda)$, $INCWH^1(\lambda')$ respectively.

Proof. Now (2) follows from (1) as in 2.7 and we leave (3) to the reader. The proof of 3.3(1) is like the construction of 1.3 with one twist. Let $h : \lambda \to \kappa$ be such that for every $\zeta < \kappa$, the set $h^{-1}(\{\zeta\})$ has cardinality λ. Let $\langle\langle m^\zeta_n, {}^*\beta^\zeta_n : n < \omega\rangle : \zeta \in S^*\rangle$ witness $INCwh^2(\kappa)$.

Let $\langle g_\delta : \delta \in S\rangle$ witness $\diamondsuit_S$ i.e. $g_\delta \in {}^\delta\omega$ and for every $g \in {}^\lambda\omega$ for stationarily many $\delta \in S$ we have $g_\delta = g \restriction \delta$.
For each $\delta \in S$ we define a function $g^*_\delta \in {}^\kappa\omega$:

$$g^*_\delta(\zeta) = \text{Min}\Big\{m : \text{for arbitrarily large } \alpha < \delta \text{ we have} : m = g_\delta(\alpha) \text{ and } h(\alpha) = \zeta\Big\}, \text{ if defined.}$$

If for some $\zeta < \kappa$, $g^*_\delta(\zeta)$ is not defined (i.e. there is no such m) - we do nothing. If $g^*_\delta \in {}^\kappa\omega$ is defined we know that for some $\zeta(\delta) \in S^*$,

$$(\exists^\infty n)(g^*_\delta({}^*\beta_n^{\zeta(\delta)}) \leq m_n^{\zeta(\delta)}).$$

(Such a $\zeta(\delta)$ exists by the choice of $\left\langle \langle m_n^\zeta, {}^*\beta_n^\zeta : n < \omega\rangle : \zeta \in S^*\right\rangle$). We fix such a δ.

Now, choose γ_n^δ such that:

(a) $\gamma_n^\delta < \delta, h(\gamma_n^\delta) = {}^*\beta_n^{\zeta(\delta)}, g_\delta(\gamma_n^\delta) = g^*_\delta({}^*\beta_n^{\zeta(\delta)})$
(b) $\delta = \bigcup\limits_{n<\omega} \gamma_n^\delta$ and $\gamma_n^\delta < \gamma_{n+1}^\delta$.

$\langle \gamma_n^\delta : n < \omega\rangle$ exist by the definition of q^*_δ.

We claim $\left\langle \langle m_n^{\zeta(\delta)}, \gamma_n^\delta : n < \omega\rangle : \delta \in S\right\rangle$ witness the conclusion. Looking at Definition 2.4, we see that the preliminary properties hold. Ordinals γ_n^δ are not necessarily successors, but this does not matter by 3.2(3).
We have to prove clause (a) of 2.4, we well as $(b)_\kappa$ of 2.7.

Proof of (a). Let $g \in {}^\lambda\omega$. For each $\zeta < \kappa$, the set $\{\alpha < \lambda : h(\alpha) = \zeta\}$ has cardinality λ, so

$$g^*(\zeta) = \mathrm{Min}\{m : (\exists^\lambda \alpha)[h(\alpha) = \zeta \wedge g(\alpha) = m]\}$$

is well defined. Let

$$A =: \{(\zeta, m) : (\exists^\lambda \alpha < \lambda)[g(\alpha) = m \wedge h(\alpha) = \zeta] \text{ and } \zeta < \kappa, m < \omega\}.$$

Then

$$E =: \Big\{\delta < \lambda : \text{for every } (\zeta, m) \in A, \text{ for unboundedly many } \alpha < \delta \\ \text{we have } g(\alpha) = m, h(\alpha) = \zeta, \text{ and for every} \\ (\zeta, m) \in (\kappa \times \omega)\backslash A, \text{ we have} \\ \delta > \sup\{\alpha < \lambda : g(\alpha) = g^*(\zeta) \wedge h(\alpha) = \zeta\}\Big\}$$

is a club of λ.
For stationarily many $\delta \in S$, $g_\delta \subseteq g$ so there is such a $\delta \in E \cap S$.
Now check: $g^*_\delta = g^*$ (g^*_δ was defined earlier). The rest is also easy to check.

Proof of $(b)_\kappa$ i.e. $(<\kappa)$-freeness. Let $U \subseteq \lambda$, $|U| < \kappa$, hence $V = \{h(\alpha) : \alpha \in U\}$ is a subset of κ of cardinality $< \kappa$, so by the choice of $\langle m_n^\delta, {}^*\beta_n^\delta : n < \omega, \delta \in S^*\rangle$ there is a $f^* : V \to \omega$ exemplifying $(b)_\kappa$ for $\langle m_n^\delta, {}^*\beta_n^\delta : n < \omega, \delta \in S^*\rangle$.
Choose $f : U \to \omega$ by $f(\alpha) = f^*(h(\alpha))$, now f exemplifies $(b)_\kappa$ for $\left\langle \langle m_n^{\zeta(\delta)}, \gamma_n^\delta : n < \omega\rangle : \delta \in S\right\rangle$. $\square_{3.3}$

3.4 Conclusion. $(\exists\lambda \geq \mu)INCWH^\ell(\lambda)$ is equivalent to $(\exists\lambda \geq \mu)INCwh^\ell(\lambda)$ (for $\ell = 1, 2$).

3.5 Definition. 1) $INCWH^3(\lambda)$ means: there are $S \subseteq \lambda$ and $f : \lambda \times \lambda \to \omega$ such that: **IF** we define the spaces as before, i.e.

- the points of $X = X_{f,S}$ are $y_\alpha, x_{\alpha,\beta}, (\alpha < \beta < \lambda)$
- each $x_{\alpha,\beta}$ is isolated
- the sets

$$u_{\alpha,n} = \{y_\alpha\} \cup \{x_{\alpha,\beta} : f(\alpha,\beta) \geq n, \alpha < \beta, \alpha \notin S, \beta \in S\}$$
$$\cup \{x_{\beta,\alpha} : f(\beta,\alpha) \geq n, \beta < \alpha, \beta \notin S, \alpha \in S\}$$

for $n < \omega$ is a neighborhood base at y_α, **THEN**:

(a) $\alpha < \beta < \lambda$, $u_{\alpha,n} \cap u_{\beta,m} \neq \emptyset \Rightarrow x_{\alpha,\beta} \in u_{\alpha,n} \cap u_{\beta,m} \Rightarrow \beta \in S \wedge \alpha \notin S$
(b) for every $\alpha < \beta < \lambda$ for some n we have: $u_{\beta,n} \cap u_{\alpha,n} = \emptyset$,

and

(c) the space X is not CWH but is $(< \lambda) - CWH$.
Note that (a) follows directly from the definition of $u_{\alpha,n}$'s.

2) $INCWH^4(\lambda)$ means: there is a symmetric two-place function f from $\lambda \times \lambda$ to $F =: \{v : v \subseteq \omega \times \omega$ is finite, and $(n, m) \in v, n' \leq n, m' \leq m \Rightarrow (n', m') \in v\}$ which is not free (i.e. for any $g : \lambda \to \omega$ for some $\alpha < \beta$ we have $(g(\alpha), g(\beta)) \in f(\alpha, \beta)$), but is λ-free (i.e. for every $A \subseteq \lambda$, $|A| < \lambda$, there is a $g : A \to \omega$ with no such $\alpha < \beta$ which are from A).

The point is that (and also see 3.7)

3.6 Claim. *1) $INCWH^1(\lambda) \Rightarrow INCWH^2(\lambda) \Rightarrow INCWH^3(\lambda) \Rightarrow INCWH^4(\lambda)$.*

Proof. 1) $INCWH^1(\lambda) \Rightarrow INCWH^2(\lambda)$ is obvious from the definition.
$INCWH^2(\lambda) \Rightarrow INCWH^3(\lambda)$ follows from the proof of 3.7 below.
Suppose that X is defined as in the definition of $INCWH^3(\lambda)$, using some $f^* : \lambda \times \lambda \to \omega$ and S which exemplify $INCWH^3(\lambda)$.
We define $f : \lambda \times \lambda \to F$ by:
if $\beta < \alpha < \lambda$, then:

$$f(\alpha,\beta) = f(\beta,\alpha), \text{ and}$$

$$f(\beta,\alpha) = \{(n,m) : u_{\alpha,n} \cap u_{\beta,m} \neq \emptyset\}.$$

To check that f is as required we simply use the fact that X is $(< \lambda) - CWH$ but not CWH. $\square_{3.6}$

3.7 Claim. $SINCWH(\lambda) \Leftrightarrow INCWH^4(\lambda)$.

Proof $\Rightarrow$. Let the space X exemplify $SINCWH(\lambda)$. Let $\{y_\alpha : \alpha < \lambda\} \subseteq X$ exemplify "X not $\lambda - CWH$" i.e. it is discrete not separated and $\alpha \neq \beta \Rightarrow y_\alpha \neq y_\beta$.

Let $u_{\alpha,n} \supseteq u_{\alpha,n+1}, \{u_{\alpha,n} : n < \omega\}$ be a neighborhood basis of y_α. Now for each α, n, β, m choose if possible $x_{\alpha,n,\beta,m} \in u_{\alpha,n} \cap u_{\beta,m}$. Let $f(\alpha,\beta) = \{(n,m) : x_{\alpha,n,\beta,m}$ is defined$\}$. This f exemplifies $INCWH^4(\lambda)$ (remember in the definition of freeness, in 3.5(2), we consider only $\alpha < \beta$).

$\Leftarrow$ We define the space X with the points $y_\alpha, x_{\alpha,\beta}$ $(\alpha < \beta < \lambda)$ in which each $x_{\alpha,\beta}$ is isolated and the neighborhood basis for y_α is given by (for $n \in \omega$)

$$\begin{aligned} u_{\alpha,n} = \{y_\alpha\} &\cup \{x_{\alpha,\beta} : \alpha < \beta \text{ and } \exists m \geq n((n,m) \in f(\alpha,\beta))\} \\ &\cup \{x_{\beta,\alpha} : \beta < \alpha \text{ and } \exists m \geq n((m,n) \in f(\alpha,\beta)\}. \end{aligned}$$

Here, f is the function which exemplifies $INCWH^4(\lambda)$. We show that X is $(< \lambda) - CWH$ and not CWH. Suppose that X is CWH and that $u_{\alpha,g(\alpha)} (\alpha \in \lambda)$ exemplify this. Let $\alpha < \beta$ be such that $(g(\alpha), g(\beta)) \in f(\alpha,\beta)$. Then $x_{\alpha,\beta} \in u_{\alpha,g(\alpha)} \cap u_{\beta,g(\beta)}$, contradiction. On the other hand, if $A \subseteq \lambda$ and $|A| < \lambda$, let $g : A \rightarrow \omega$ be such that for no $\alpha < \beta$ from A, do we have $(g(\alpha), g(\beta)) \in f(\alpha,\beta)$. Then for $\alpha < \beta \in A$, we have $u_{\alpha,g(\alpha)} \cap u_{\beta,g(\beta)} = \emptyset$. $\square_{3.7}$

We finish this section by the following

3.8 Claim. *In 2.5 we can weaken $INCWH^1(\lambda)$ to $INCWH^2(\lambda)$.*

Proof. Suppose that $\langle \langle m_n^\delta, \beta_n^\delta : n < \omega \rangle : \delta \in S \rangle$ exemplify $INCWH^2(\lambda)$ and define the space X as in 2.5, except that the neighborhood basis for y_α when $\alpha \notin S$ is given by (for $n < \omega$)

$$\begin{aligned} u_{\alpha,n} = \{y_\alpha\} \cup \{x_\delta, \alpha : &\alpha < \delta \in S \text{ and for some } k \\ &\text{we have } \alpha = \beta_k^\delta \text{ and } n \leq m_n^\delta\}. \end{aligned}$$

$\square_{3.8}$

Comment. The $INCWH^\ell(\lambda)$ are not so artificial: $SINCWH(\lambda)$ is equivalent to $INCWH^4(\lambda)$.

§4 General Set Theoretic Spectrum of Freeness

4.0 Definition. For $\lambda > cf(\lambda) = \theta$ let $(*)_\lambda$ mean: there is a $\{\eta_\alpha : \alpha < \lambda^+\} \subseteq {}^\theta\lambda$ which is λ-free in the sense of $1.1(1)(*)_\lambda$: for any α for some $g \in {}^\theta\alpha$, $\langle \{\eta_\beta(i) : i \in [g(\beta), \theta)\} : \beta < \alpha \rangle$ are pairwise disjoint and for simplicity we require $\eta_{\alpha_1}(\zeta_1) = \eta_{\alpha_2}(\zeta_2) \Rightarrow \zeta_1 = \zeta_2$

4.1 Definition. We define various versions of the spectrum of freeness.
1) For θ a regular cardinal and $\sigma \geq 1$ (if $\sigma = 1$ we omit it) let:

$SP_{\theta,\sigma} = \Big\{ \lambda$: there is a family H such that :

(a) every $h \in H$ is a partial function from ordinals to θ

(b) $h \in H \Rightarrow |\mathrm{Dom}(h)| = \theta$

(c) every $H' \subseteq H$ of cardinality $< \lambda$ is σ-free which means that it can be represented as a union $\bigcup\limits_{i<i(*)} H'_i$ where $i(*) < 1 + \sigma$, and each H'_i is free. For H'_i to be free means that there is a g, a function from ordinals to θ such that $(\forall h)(\exists \xi < \theta)[h \in H'_i \rightarrow (\forall \alpha \in \mathrm{Dom}(h)[h(\alpha) \leq g(\alpha) \vee h(\alpha) \leq \xi]$

(d) H is not σ-free, $|H| = \lambda \Big\}$

2)

$SPd_{\theta,\sigma} = \{\lambda$: there is an H satisfying (a)-(d) above and (e) each $h \in H$ is one to one$\}$.

3)

$SPw_{\theta,\sigma} = \Big\{ \lambda$: there is a family H such that :

(a) if $(h, \bar{u}) \in H$ then h is a function from ordinals to θ

(b) if $(h, \bar{u}) \in H$, then $\bar{u} = \langle u_\varepsilon : \varepsilon < \theta \rangle$ is a decreasing sequence of subsets of $\mathrm{Dom}(h)$

(c) every pair (H', Z'), with $Z' \subseteq$ ordinals, $|Z'| < \lambda$ and $H' \subseteq H$ of cardinality $< \lambda$ is σ-free, which means $H' \times Z'$ can be represented as $\bigcup\limits_{i<i(*)} H'_i \times Z'_i$ where $i(*) < 1 + \sigma$ and each (H'_i, Z'_i) is free. This means that there are functions g, f with $g : H'_i \rightarrow \theta$ and f from ordinals to θ such that for every $(h, \bar{u}) \in H'_i$, for some $\zeta < \theta$ for every $\alpha \in u_\zeta \cap Z'_i \cap \mathrm{Dom}(h)$ we have $h(\alpha) \leq \max\{f(\alpha), g(h, \bar{u})\}$.

(d) (H, λ) is not σ-free, $|H| = \lambda \Big\}$

The reader can restrict himself to the case $\sigma = 1$ (also in Definition 4.2(3)).

4.1A Observation. 0) In Definition 4.1(1), if each $h \in H$ converges to θ (i.e. $\forall \zeta < \theta |\{\alpha : h(\alpha) < \zeta\}| < \theta$), in clause (c) of 4.1(1) we can just demand $(\forall h)[h \in H' \rightarrow \theta > |\{\alpha : h(\alpha) > g(\alpha)\}|]$.
1) In Definition 4.1(1) without loss of generality $\bigcup_{h \in H} \mathrm{Dom}(h) \subseteq \lambda$ and in 4.1(3) without loss of generality $\bigcup_{(h,\bar{u}) \in H} \mathrm{Dom}(h) \subseteq \lambda$. Also, without loss of generality $\mathrm{Dom}(g) = \lambda$.
2) Note $\theta^+ \cap SP_\theta = \emptyset$
[why? if $H = \{h_\zeta : \zeta < \zeta^* \leq \theta\}$, let $\bigcup_\zeta \mathrm{Dom}(h_\zeta) = \{\alpha_i : i < \theta\}$, and let

$$g(\alpha_i) = \sup\{h_\zeta(\alpha_i) : \zeta < i,\ \alpha_i \in \mathrm{Dom}(h_\zeta)\}.$$

This g exemplifies that H is free.] This also follows from 4.1(B1) and 4.2(2).
3) $SP_\theta \cap [\theta^+, 2^\theta] \neq \emptyset$ [this follows from 4.1A(4) below (and 4.3(1))].
4) We let

$$\begin{aligned}\mathfrak{b}[\theta] = \mathrm{Min}\{|F| : F \subseteq {}^\theta\theta, \text{ and for no } g \in {}^\theta\theta \text{ do we have} \\ (\forall f \in F)(\exists \zeta < \theta)(f \restriction [\zeta, \theta) < g \restriction [\zeta, \theta))\}\end{aligned}$$

if $\sigma \leq \theta^+$ then clearly $\mathfrak{b}[\theta] \in SP_{\theta,\sigma}$.
5) In Definition 4.1(3) without loss of generality for $(h, \bar{u}) \in H$, we have $\bigcap_{\zeta < \theta} u_\zeta = \emptyset$.
Also without loss of generality, for $(h, \bar{u}) \in H$ we have
$u_\zeta = \{\alpha \in \mathrm{Dom}(h) : h(\alpha) \geq \zeta\}$ (we say: $\bar{u}$ is standard for h).
6) Suppose that H is as in 4.1(3) and $[(h, \bar{u}) \in H \Rightarrow |u_0| < \lambda = \mathrm{cf}(\lambda)]$ or $\sup\{|u_0| : (h, \bar{u}) \in H\} < \lambda$; and assume that for every $(h, \bar{u}) \in H$ we know that $\bar{u}$ is standard. Then clause (c) means:
For every $H' \subseteq H$ with $|H'| < \lambda$, there are sets H'_i for $i < i(*) < 1 + \sigma$ such that $H' = \bigcup_{i < i(*)} H'_i$ and for each $i < i(*)$, there is a function f_i from ordinals to θ with the following property.
For every $(h, \bar{u}) \in H'_i$

$$\exists \xi < \theta \exists \zeta < \theta \forall \alpha \in u_\xi [h(\alpha) \leq \max\{\zeta, g_i(\alpha)\}].$$

7) Note also that we can without loss of generality assume that
$Z' \subseteq \bigcup_{(h,\bar{u}) \in H'} \mathrm{Dom}(h)$, for 4.1(3)c).
8) We usually restrict our attention to the case $\sigma \leq \theta^+$. Actually, the main interest is in $\sigma = 1$.
On the other hand, for $\sigma \leq \theta$, every H which satisfies (a) and (b) of Definition 4.1(1) is σ-free iff it is free.
[Why? If H is free than it is definitely σ-free. If $H = \bigcup_{i < i(*)} H_i$ for some $i(*) < 1 + \sigma$, and H_i is free as exemplified by g_i, <u>then</u>

$$g \stackrel{\mathrm{def}}{=} \sup\{g_i : i < i(*)\}$$

is a function from ordinals to θ which exemplifies H is free.]

4.1B Notation. For $a \subseteq \theta \times \theta$: we say that a is pie if
$(\zeta_1, \xi_1) \neq (\zeta_2, \xi_2) \in a \Rightarrow \neg(\zeta_1, \xi_1) \leq (\zeta_2, \xi_2)$ coordinatewise.
$\mathrm{Pie}(\theta \times \theta) = \{a : a \subseteq \theta \times \theta$ and a is pie (hence finite)$\}$
$C\ell(a) = \{(\zeta, \xi) \in (\theta \times \theta) : (\exists x \in a)(x \leq (\zeta, \xi)$ coordinatewise$\}$, for $a \subseteq \theta \times \theta$.

4.2 Definition. 1) For θ a regular cardinal and $\sigma \geq 1$ (if $\sigma = 1$ we omit it) let:

$SQ_{\theta,\sigma} = \Big\{\lambda$: there is a family H such that :

(a) every $h \in H$ is a partial function defined on the ordinals

(b) $h \in H \Rightarrow |\mathrm{Dom}(h)| = \theta$, $\mathrm{Rang}(h) \subseteq \mathrm{Pie}(\theta \times \theta)$

(c) every $H' \subseteq H$ of cardinality $< \lambda$ is σ-free which means that it can be represented as a union $\bigcup\limits_{i<i(*)} H'_i$ where $i(*) < 1 + \sigma$, and each H'_i is free. For H'_i to be free means that there is a g, a function from ordinals to θ such that $(\forall h)(\exists \zeta < \theta)[h \in H'_i \rightarrow (\forall \alpha \in \mathrm{Dom}(h))[(g(\alpha), \zeta) \in C\ell(h(\alpha))]$

(d) H is not σ-free, $|H| = \lambda\Big\}$.

2)

$SQd_{\theta,\sigma} = \Big\{\lambda$: there is an H satisfying (a)-(d) above and

(e) each $h \in H$ is simple, which means: there is an enumeration $\mathrm{Dom}(h) = \{\alpha_\zeta : \zeta < \theta\}$ with no repetitions, such that for each $\zeta < \theta$ for some $\beta_\zeta, \gamma_\zeta < \theta$ we have $C\ell(h(\alpha_\zeta)) = \{(\zeta_1, \zeta_2) : (\zeta_1, \zeta_2) \not\leq (\beta_\zeta, \gamma_\zeta)\}$ and $\langle \gamma_\zeta : \zeta < \theta \rangle$ is strictly increasing and $\bigcup\limits_{\xi<\zeta} \beta_\xi < \gamma_\zeta\Big\}$.

Note that $(\zeta_1, \zeta_2) \not\leq (\beta_\zeta, \gamma_\zeta)$ means that either $\zeta_1 > \beta_\zeta$ or $\zeta_2 > \gamma_\zeta$.

3)

$$SQw_{\theta,\sigma} = \Big\{ \lambda : \text{there is a family } H \text{ such that :}$$

(a) if $(h, \bar{u}) \in H$ then h is a function from ordinals to $\mathrm{Pie}(\theta \times \theta)$

(b) if $(h, \bar{u}) \in H$, then $\bar{u} = \langle u_\varepsilon : \varepsilon < \theta \rangle$ is a decreasing sequence of subsets of $\mathrm{Dom}(h)$

(c) every pair (H', Z'), with $Z' \subseteq$ ordinals, $|Z'| < \lambda$ and $H' \subseteq H$ of cardinality $< \lambda$ is σ-free, which means $H \times Z$ can be represented as $\bigcup_{i<i(*)} H'_i \times Z'_i$ where $i(*) < 1 + \sigma$ and each (H'_i, Z'_i) is free. This means that there are functions g, f with $g : H'_i \to \theta$ and f from ordinals to θ such that for every $(h, \bar{u}) \in H'_i$, for some $\zeta < \theta$ for every $z \in u_\zeta \cap Z'_i \cap \mathrm{Dom}(h)$ we have $(g(h), f(z)) \in C\ell(h(z))$

(d) (H, λ) is not σ-free, $|H| = \lambda$

$$(e)\ (h, \bar{u}) \in H \Rightarrow \bigcap_{\varepsilon<\theta} u_\varepsilon = \emptyset \Big\}.$$

4.2(A) Remark. 1) In 4.2(3)c), we can assume that $Z' \subseteq \bigcup_{h \in H'} \mathrm{Dom}(h)$.
2) As in 4.1, we consider normally only the case $\sigma \leq \theta^+$.
3) $SPx_{\theta,\sigma}$ can be understood as a particular case of $SQx_{\theta,\sigma}$, where $\mathrm{Rang}(h)$ is restricted to $\{(\zeta, \zeta) : \zeta < \theta\}$. Here, $x \in \{w, d\}$ or x is omitted.

4.2B Fact. 1) $\lambda \in SP_{\theta,\sigma}$ implies that $\lambda \in SQ_{\theta,\sigma}$
$\lambda \in SPd_{\theta,\sigma}$ implies that $\lambda \in SQd_{\theta,\sigma}$, and .
$\lambda \in SPw_{\theta,\sigma}$ implies that $\lambda \in SQw_{\theta,\sigma}$.
2) $\lambda \in SQd_{\theta,\sigma}$ implies that $\lambda \in SP_{\theta,\sigma}$.

Proof. 1) If H exemplifies that $\lambda \in SP_{\theta,\sigma}$, let $H^\otimes = \{h^\otimes : h \in H\}$, where for $h \in H$, $h^\otimes$ is a function with domain $\mathrm{Dom}(h)$ and

$$h^\otimes(\alpha) = \{(h(\alpha), h(\alpha))\}.$$

Similarly for $SPd_{\theta,\sigma}$.
If H exemplifies that $\lambda \in SQw_{\theta,\sigma}$, let $H^\otimes = \{(h^\otimes, \bar{u}) : (h, \bar{u}) \in H\}$.
2) See §5, Remark 5.8.

4.2C Notation. For a function h from a subset of ordinals to Pie$(\theta \times \theta)$, we say that h converges to θ, if

$$(\forall \beta < \theta)(\exists \alpha)(\forall \gamma \in \mathrm{Dom}(h) \backslash \alpha)$$
$$[(\varepsilon_1, \varepsilon_2) \in h(\gamma) \Rightarrow \varepsilon_1 > \beta \text{ and } \varepsilon_2 > \beta].$$

4.2D Observation. 0) In Definition 4.2(1), if each $h \in H$ converges to θ, in clause (c) of 4.2(1) we can just demand
$(\forall h)[h \in H' \to \theta > |\{\alpha : \exists(\varepsilon_1, \varepsilon_2) \in h(\alpha)[\varepsilon_1 > g(\alpha) \vee \varepsilon_2 > g(\alpha)]\}|]$.
1) In Definition 4.2(1) without loss of generality $\bigcup_{h \in H} \mathrm{Dom}(h) \subseteq \lambda$ and in 4.2(3) without loss of generality $\bigcup_{(h,\bar{u}) \in H} \mathrm{Dom}(h) \subseteq \lambda$. Also, without loss of generality, $\mathrm{Dom}(g) = \lambda$.
2) Note $\theta^+ \cap SQ_\theta = \emptyset$
[why? if $H = \{h_\zeta : \zeta < \zeta^* \leq \theta\}$, $\bigcup_\zeta \mathrm{Dom}(h_\zeta) = \{\alpha_i : i < \theta\}$, let $g(\alpha_i) = \sup\{\max\{h_\zeta(\alpha_i)_1, h_\zeta(\alpha_i)_2 : \zeta < i,\ \alpha_i \in \mathrm{Dom}(h_\zeta) \text{ and } h_\zeta(\alpha_i) = (h_\zeta(\alpha_i)_1, h_\zeta(\alpha_i)_2)\}\}$.]
3) $SQ_\theta \cap [\theta^+, 2^\theta] \neq \emptyset$ [this follows from 4.1A(3) and 4.2B(1)]. Actually, $\mathfrak{b}[\theta] \in SQ_\theta$.
4) In 4.2(3)c), if $|u_0| < \lambda = \mathrm{cf}(\lambda)$ for $(h, \bar{u}) \in H$, we obtain the following property. For every $H' \subseteq H$ of cardinality $< \lambda$, there are sets H'_i for $i < i(*) < 1 + \sigma$, such that there are functions $\{g_i : i < i(*)\}, g_i : H'_i \to \theta$ satisfying: if $(h, \bar{u}) \in H'_i$, then $(\exists \zeta < \theta)(\exists \xi < \theta)(\forall \alpha \in u_\zeta)[(g_i(\alpha), \xi) \in C\ell(h(\alpha)]$.

4.3 Claim. *1) If there is an H as in (a), (b) of 4.1(1) which is $(< \mu) - \sigma$-free not $\lambda - \sigma$-free <u>then</u> there is a $\lambda' \in [\mu, \lambda] \cap SP_{\theta,\sigma}$. Similarly for Definitions 4.1(2), 4.1(3) (see also Claim 4.3A).*
2) If $pp_{\Gamma(\theta)}(\lambda) > \lambda^+$, $\lambda > cf(\lambda) = \theta$ (or just $(*)_\lambda$ of 4.0) and $\lambda \geq \sigma$ <u>then</u> $SP_{\theta,\sigma} \cap [\lambda^+, \lambda^\theta] \neq \emptyset$.[1]

Proof. 1) Straightforward.
2) Let $\{\eta_\alpha : \alpha < \lambda^+\} \subseteq {}^\theta\lambda$ be λ-free, without loss of generality $\langle \{\eta_\alpha(\zeta) : \alpha < \lambda^+\} : \zeta < \theta \rangle$ are pairwise disjoint and let

$$H = \Big\{ h : \text{for some } \alpha < \lambda^+ \text{ and } a \subseteq \lambda^+, \mathrm{otp}(a) = \theta, \mathrm{Dom}(h) = a,$$
$$h \text{ is strictly increasing and for } \beta \in a$$
$$h(\beta) = \sup\{\varepsilon : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon)\} \Big\}.$$

Now H is not free: if $g : \lambda^+ \to \theta$, then for some $\varepsilon^* < \theta$, $A = \{\alpha < \lambda^+ : g(\alpha) = \varepsilon^*\}$ is of cardinality λ^+. Choose by induction on $\zeta < \lambda^+$ an ordinal $\alpha^*_\zeta < \lambda^+$ increasing with ζ such that

[1] $\Gamma(\theta)$ refers to the class of θ-complete ideals on θ which $\supseteq J^{bd}_\theta$. pp_Γ is pp taken only over the ideals in Γ

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$$\bigcup\{\text{Rang}(\eta_\alpha) : \alpha \in A \cap \alpha^*_{\zeta+1} \backslash \alpha^*_\zeta\} = \bigcup\{\text{Rang}(\eta_\alpha) : \alpha \in A \backslash \alpha^*_\zeta\}.$$

Next choose $\alpha \in A\backslash\alpha^*_\theta$ and $\beta_\zeta \in A \cap [\alpha^*_\zeta, \alpha^*_{\zeta+1})$ for $\zeta < \theta$ such that $\eta_{\beta_\zeta}(\zeta) = \eta_\alpha(\zeta)$ (note that the existence of such β_ζ follows from the definition of α^*_ζ and the assumption $\zeta \neq \xi \Rightarrow \eta_\alpha(\zeta) \neq \eta_\beta(\xi)$ for ζ, $\xi < \theta$ and α, $\beta < \lambda^+$) and let $a = \{\beta_\zeta : \zeta < \theta\}$, $h \in {}^a\theta$, $h(\beta_\zeta) = \sup\{\varepsilon : \eta_\alpha(\varepsilon) = \eta_{\beta_\zeta}(\varepsilon)\} \geq \zeta$, so $h \in H$. As $\beta_\zeta \in A$, $g(\beta_\zeta) = \varepsilon^*$ =constant, so if $\xi < \theta$, $\{\beta \in \text{Dom}(h) : h(\beta) \geq g(\beta), \xi\}$ includes $\{\beta_\zeta : \xi, \varepsilon^* < \zeta < \theta\}$, which is a contradiction.

On the other hand, H is λ^+-free. For suppose $H' \subseteq H$, $|H'| \leq \lambda$. For $h \in H'$ choose α_h, a_h witnessing $h \in H$. Then $b = \bigcup\{\{\alpha_h\} \cup a_h : h \in H'\}$ is a subset of λ^+ of cardinality $\leq \lambda$, hence we can find $\langle \varepsilon_\alpha : \alpha \in b\rangle$ such that $\langle \text{Rang}(\eta_\alpha \restriction [\varepsilon_\alpha, \theta)) : \alpha \in b\rangle$ is a sequence of pairwise disjoint subsets of λ. Let us define a $g : \lambda^+ \to \theta$ such that $\alpha \in b \Rightarrow g(\alpha) = \varepsilon_\alpha$. Now if $h \in H'$, let $a_h = \{\beta_\zeta : \zeta < \theta\}$ (increasing with ζ), so

$$h(\beta_\zeta) = \sup\{\varepsilon : \eta_{\alpha_h}(\varepsilon) = \eta_{\beta_\zeta}(\varepsilon)\}$$

so $h(\beta_\zeta) \leq \max\{\varepsilon_{\alpha_h}, \varepsilon_{\beta_\zeta}\} = \max\{g(\alpha_h), g(\beta_\zeta)\}$.
So choose $\xi = g(\alpha_h)$ and we get the desired conclusion.
To finish we use part (1). $\square_{4.3}$

4.3A Claim. *1) If there is an H as in (a), (b) of 4.2(1) which is $(< \mu) - \sigma$-free not $\lambda - \sigma$-free <u>then</u> there is a $\lambda' \in [\mu, \lambda] \cap SQ_{\theta,\sigma}$. Similarly for 4.2(2), 4.2(3).*
2) If $pp_{\Gamma(\theta)}(\lambda) > \lambda^+$, $\lambda >$ cf$(\lambda) = \theta$ (or just $(*)_\lambda$ of 4.0) and $\lambda \geq \sigma$ <u>then</u> $SQ_{\theta,\sigma} \cap [\lambda^+, \lambda^\theta] \neq \emptyset$.

Proof. 1) Straightforward.
2) This follows from 4.3(2) and 4.1B(1). $\square_{4.3A}$

4.4 Claim. *1) The following implications hold for any λ:*

$$(a) \Rightarrow (b) \Leftrightarrow (b)^+ \Leftrightarrow (c) \Leftrightarrow (c)^+ \Rightarrow (d),$$

where

- (a) $\lambda \in SQ_{\aleph_0}$
- (b) *There is a $(< \lambda)$-CWH not λ-CWH first countable space.*
- $(b)^+$ *There is a space like in (b), which is in addition $(< \lambda)$-metrizable.*
- (c) *There is a $(< \lambda) -^* CWN$ not $^*CWN$ first countable space with λ points.*
- $(c)^+$ *There is a space like in (c), which is in addition $(< \lambda)$-metrizable.*
- (d) $\lambda \in SQw_{\aleph_0}$.

2) $\lambda \in SQd_{\theta,\sigma} \Rightarrow \lambda \in SQ_{\theta,\sigma} \Rightarrow \lambda \in SQw_{\theta,\sigma} \Rightarrow [\lambda, \lambda^\theta] \cap SQd_{\theta,\sigma} \neq \emptyset$ for $\sigma \leq \theta^+$.
3) $\lambda \in SP_{\theta,\sigma} \Rightarrow \lambda \in SPw_{\theta,\sigma} \Rightarrow [\lambda, \lambda^\theta] \cap SPd_{\theta,\sigma} \neq \emptyset$ for $\sigma \leq \theta^+$.
4) We can replace $\aleph_0$, "first countable Hausdorff topological spaces" <u>by</u> $\theta, \mathcal{T}_\theta$ respectively (of course, $\lambda > \theta$).

Proof. 1),4)
(a) implies (b), $(b)^+$, (c), $(c)^+$.

So assume H exemplifies that $\lambda \in SQ_\theta$; without loss of generality $\mathrm{Dom}(h) \subseteq \lambda$ for $h \in H$. We can use the space

$$X = \{y_i : i < \lambda\} \cup \{z_h : h \in H\} \cup \{x_{h,i} : h \in H \text{ and } i \in \mathrm{Dom}(h)\},$$

and for $\zeta < \theta$ let

$$u_\zeta[z_h] = \{z_h\} \cup \{x_{h,i} : i \in \mathrm{Dom}(h), (\zeta,\zeta) \notin C\ell(h(i))\},$$

$$u_\zeta[y_i] = \{y_i\} \cup \{x_{h,i} : h \in H, i \in \mathrm{Dom}(h), (\zeta,\zeta) \notin C\ell(h(i))\}$$

and $x_{h,i}$ is isolated.

Suppose $H' \subseteq H$, $|H'| < \lambda$ and let
$X[H'] = \{y_i : i < \lambda\} \cup \{z_h : h \in H'\} \cup \{x_{h,i} : h \in H, i \in \mathrm{Dom}(h)\}$.
Let $g : \lambda \to \theta$ be such that for every $h \in H'$, for some $\zeta[h] < \theta$ we have

$$i \in \mathrm{Dom}(h) \Rightarrow (g(i), \zeta[h]) \in C\ell(h(i)).$$

Let us choose for $t \in X[H']$ a neighborhood v_t:

$$\begin{array}{llll} \underline{\text{if}} & t = x_{h,i} & \underline{\text{then}} & v_t = \{x_{h,i}\} \\ \underline{\text{if}} & t = y_i & \underline{\text{then}} & v_t = u_{g(i)}[y_i] \\ \underline{\text{if}} & t = z_h & \underline{\text{then}} & v_t = u_{\zeta[h]}[z_h]. \end{array}$$

Now

$$\langle v_{y_i} : i < \lambda\rangle\hat{\ }\langle v_{z_h} : h \in H'\rangle\hat{\ }\langle v_{x_{h,i}} : i < \lambda, h \in H \text{ and } x_{h,i} \notin \bigcup_{j<\lambda} v_{y_j} \cup \bigcup_{h \in H'} v_{z_h}\rangle$$

is a partition of $X[H']$ to pairwise disjoint open sets. In each basic open set there is at most one point which is not isolated, and if so it has a neighborhood base consisting of a decreasing sequence of (open) sets of length θ.
This suffices to show that X is $(< \lambda)$-metrizable when $\theta = \aleph_0$ and as required generally (for 4)).
[Why? Suppose $X' \subseteq X$ and $X = \bigcup_{i<i(*)} U_i$, where each U_i is open, U_i for $i < i(*)$ are pairwise disjoint and for every $i < i(*)$ at most one $x_i \in U_i$ is not isolated, and x_i has a countable neighborhood base. If $i < i(*)$ and $x_i \in U_i$ is non-isolated, let $\{u^i_n : n < \omega\}$ be a neighborhood for x_i, and without los of generality we have $u^i_{n+1} \subseteq u^i_n \subseteq U_i$ for $n < \omega$. For $x \in X'$ let $i(*)$ be the $i < i(*)$ such that $x \in U_i$. Now define

$$d(x,y) = \begin{cases} 1 & \text{if} \quad i(x) \neq i(y) \\ 0 & \text{if} \quad x = y \\ \mathrm{Min}\{1/n : x, y \in u^i_n\} & \text{if} \quad x \neq y \text{ but } i(x) = i(y) \end{cases}$$

and check.]

As for showing that X is not CWH (hence not metrizable and not CWN), note that $\{y_i : i < \lambda\} \cup \{z_h : h \in H\}$ is a discrete subspace.
If it is separated, we have a sequence of pairwise disjoint neighborhoods:

$$\langle u_{g(i)}[y_i] : i < \lambda\rangle{}^\frown\langle u_{\zeta(h)}[z_h] : h \in H\rangle.$$

But H is not free (in the sense of Definition 4.2(1)) and we get a contradiction.
<u>$(b)^+ \Rightarrow (b)$</u>.
Trivial.

<u>$(b) \Rightarrow (b)^+$</u>.
Let X exemplify clause (b), so without loss of generality $|X| = \lambda$. Let Y be a discrete subspace of cardinality λ which cannot be separated. Let X^+ be the topology on the set of points of X generated by basic open sets of X and $\{\{x\} : x \in X \setminus Y\}$.

Now X^+ is not $\lambda - CWH$ (Y still exemplifies it). But X^+ is $(< \lambda)$-metrizable as:

If $Z \subseteq X, |Z| < \lambda$, then we can find a sequence $\langle u_z : z \in Z \cap Y\rangle$ of pairwise disjoint open sets, and in $X \restriction u_z$, every point is isolated except z, which has a neighborhood basis of cardinality $\aleph_0$, and every $x \in Z \setminus \bigcup_{z \in Z\cap Y} u_z$ is isolated.

As noted above, this is enough.
<u>$(b)^+ \Rightarrow (c)^+$</u>
Trivial (as $(< \lambda)$-metrizable $\Rightarrow (< \lambda) -^* CWN$).
<u>$(c)^+ \Rightarrow (c)$</u>
Trivial.
<u>$(c) \Rightarrow (b)^+$</u>

If $X, \langle Y_i : i < \alpha\rangle$ exemplify clause (c) in (1) with $\langle u_\zeta(y) : \zeta < \theta\rangle$ a decreasing neighborhood basis of y, we can get another example X' to the third clause, as follows.
We are, without loss of generality, assuming that $|X| = \lambda$. Now we define a topological space X':

$$X' = \bigcup_{i<\alpha} Y_i \cup \Big\{ x_{y,z,\zeta,\xi} : \text{for some } i \neq j < \alpha, y \in Y_i, \\ z \in Y_j, u_\zeta[y] \cap u_\xi[z] \neq \emptyset \Big\}$$

with the neighborhood bases for $y, z \in \bigcup_{i<\alpha} Y_i$ given by

$$u'_\varepsilon[t] = \{t\} \cup \Big\{ x_{y,z,\zeta,\xi} : x_{y,z,\zeta,\xi} \in X', \text{ and: } \\ t = y \wedge \varepsilon \leq \zeta \text{ or } t = z \wedge \varepsilon \leq \xi \Big\}$$

and each $x_{y,z,\zeta,\xi}$ is isolated; note that for $t \in \bigcup_{i<\alpha} Y_i, \langle u'_\varepsilon[t] : \varepsilon < \theta\rangle$ is decreasing with intersection $\{t\}$, so $X' \in \mathcal{T}_\theta$ is a Hausdorff space.
Clearly $Y =: \bigcup_{i<\alpha} Y_i$ is discrete. Assume that $\langle u'_{\varepsilon(y)}[y] : y \in Y\rangle$ is a sequence of pairwise disjoint open sets. Then let

$$U_i = \bigcup\{u_{\varepsilon(y)}[y] : y \in Y_i\}.$$

So in X, U_i is an open set (as a union of open sets),

$$Y_i \subseteq U_i \text{ as } y \in u_{\varepsilon(y)}[y].$$

Therefore, there are i, j such that

$$U_i \cap U_j \neq \emptyset \text{ and clearly}$$

$$\begin{aligned} i \neq j \ \& \ U_i \cap U_j \neq \emptyset &\Rightarrow \exists y \in Y_i \exists z \in Y_j (u_{\varepsilon(y)}[y] \cap u_{\varepsilon(z)}[z] \neq \emptyset) \\ &\Rightarrow x_{y,z,\varepsilon(y),\varepsilon(z)} \text{ is well defined} \\ &\Rightarrow \text{ in } X' \text{ we have that } u'_{\varepsilon(y)}[y] \cap u'_{\varepsilon(z)}[z] \neq \emptyset. \end{aligned}$$

This is a contradiction.

So we conclude that Y cannot be separated in X', so X' is not $\lambda - CWH$.

Next, assume that $Z \subseteq X', |Z| < \lambda$, so in $X, \langle Y_i \cap Z : i < \alpha, Y_i \cap Z \neq \emptyset\rangle$ can be separated, say by $\langle U_i : i < \alpha, Y_i \cap Z \neq \emptyset\rangle$. So for $y \in Y \cap Z$, there is an $\varepsilon(y)$, such that $u'_{\varepsilon(y)}[y] \subseteq U_i$ (the isolated points in $X' \cap Z \backslash Y$ can be taken care of easily so we ignore them).

Now, if $y \neq z \in Y \cap Z$ then:

(*i*) if $(\exists i)(y, z \in Y_i)$ then $u'_{\varepsilon(y)}[y] \cap u'_{\varepsilon(z)}[z] = \emptyset$.

(*ii*) If $y \in Y_i, z \in Y_j, i \neq j$, and $u'_{\varepsilon(y)}[y] \cap u'_{\varepsilon(z)}[z] \neq \emptyset$,
then $x_{y,z,\varepsilon(y),\varepsilon(z)}$ exists, so
$\emptyset \neq u_{\varepsilon(y)}[y] \cap u_{\varepsilon(z)}[z] \subseteq U_i \cap U_j$ which is a contradiction.

That X' is $(< \lambda)$-metrizable now follows as in $(b) \Rightarrow (b)^+$.

$(c) \Rightarrow (d)$.
Assume that X is a first countable $(< \lambda) - {}^*CWN$ not $\lambda - {}^*CWN$-space, without loss of generality with the set of points λ, so there is a sequence $\langle Y_i : i < \lambda\rangle$ of pairwise disjoint subsets of X such that $Y_i \neq \emptyset$, Y_i is clopen in $X \restriction (\bigcup_{j<\lambda} Y_j)$ and $\langle Y_i : i < \lambda\rangle$ cannot be separated. For $y \in Y =: \bigcup_{i<\lambda} Y_i$ let $\bar{u}[y] = \langle u_\zeta[y] : \zeta < \theta\rangle$ be a neighborhood basis of the topology for y, and without loss of generality $\varepsilon < \zeta < \theta \Rightarrow u_\zeta[y] \subseteq u_\varepsilon[y]$. Let

$$H = \Big\{ (h,\bar u) : \text{for some } i<\lambda \text{ and for some } y\in Y_i,$$
$$(h,\bar u) = (h_y,\bar u_y), \text{ which means :}$$
$$\mathrm{Dom}(h) = \bigcup_{j\neq i} Y_j, C\ell(h(z)) = \{(\zeta,\xi)\in\theta\times\theta : u_\zeta[y]\cap u_\xi[z]=\emptyset\}$$
$$\text{and } \bar u \text{ is } \langle u_\zeta(y)\cap \mathrm{Dom}(h) : \zeta<\theta\rangle \Big\}.$$

Note that $h(z)$ is uniquely determined by $C\ell(h(z))$ (since we know that $u(z)$ is a pie). Now we check that H exemplifies SQw_θ, i.e. the clauses in 4.2(3). Clauses (a), (b) are immediate.
As for clause (c), let $H'\subseteq H, |H'|<\lambda$, and, without loss of generality, let $Z'\subseteq \bigcup\{\mathrm{Dom}(h) : (h,\bar u)\in H\}$ with $|Z'|<\lambda$. Let

$$Y' =: \{y : y\in \bigcup_{i<\lambda} Y_i, \text{ and } y\in Z' \text{ or } (h_y,\bar u_y)\in H'\},$$

so $|Y'|<\lambda$; we can find
$X'\subseteq X, |X'|\leq |Y'|+\theta<\lambda$ such that $Y'\subseteq X'$, and for every $y,z\in Y'$ and $\zeta<\theta, \xi<\theta$, we have $u_\zeta[y]\cap u_\xi[z]\neq\emptyset \Rightarrow u_\zeta[y]\cap u_\xi[z]\cap X'\neq\emptyset$. As $|X'|<\lambda$ we know that X' (i.e. $X\restriction X'$) is CWN, and $\langle Y_i\cap X' : i<\alpha\rangle$ is a discrete sequence of closed sets in X' hence there is a function $g : Y'\to\theta$ such that

(*) if $i<j<\lambda, y\in Y'\cap Y_i, z\in Y'\cap Y_j$, then
$u_{g(y)}[y]\cap u_{g(z)}[z]=\emptyset$ (intersecting with X' is immaterial).

Hence by the choice of g

(**) if $i\neq j$ $(i<\lambda,\ j<\lambda)$, $y\in Y'\cap Y_i$, $z\in Y'\cap Y_j$
then $(g(y),g(z))\in C\ell(h_y(z))$.

This is enough.

We are left with proving that H is not free, so suppose $f,g : Y\to\theta$ satisfies

$\otimes$ for every $y\in Y$,
for every $z\in\ \mathrm{Dom}(h_y)$ we have $(g(y),f(z))\in C\ell(h_y(z))$,

so without loss of generality $f=g$.
For $i<\lambda$ let

$$U_i = \bigcup\{u_{g(y)}[y] : y\in Y_i\}.$$

So U_i, being the union of open sets is open.
If $i<j, y\in Y_i, z\in Y_j$ then

$$u_{g(y)}[y]\cap u_{g(z)}[z]\neq\emptyset \Rightarrow (g(y),g(z))\in C\ell(h_y(z))$$
$$\Rightarrow (g(y),f(z)) = (g(y),g(z))\in C\ell(h_y(z)).$$

Contradiction, by the choice of f and g.
So $u_{g(y)}[y] \cap u_{g(z)}[z] = \emptyset$, as $y \in Y_i, z \in Y_j$ were arbitrary, $U_i \cap U_j = \emptyset$.
We conclude that $\langle Y_i : i < \lambda \rangle$ can be separated, which is a contradiction.
2) We prove each implication
(A) $\lambda \in SQd_{\theta,\sigma} \Rightarrow \lambda \in SQ_{\theta,\sigma} \Rightarrow \lambda \in SQw_{\theta,\sigma}$. Obvious.
(B) $\lambda \in SQw_{\theta,\sigma} \Rightarrow SQd_{\theta,\sigma} \cap [\lambda, \lambda^\theta] \neq \emptyset$ when $\sigma \leq \theta^+$.

Now without loss of generality $\sigma \in \{1, \theta\}$ and the reader can think of $\sigma = 1$ only. Assume that H exemplifies $\lambda \in SQw_{\theta,\sigma}$. By the definition, $(h, \bar{u}) \in H \Rightarrow u_\zeta \subseteq \text{Dom}(h)$ & $\bigcap_{\xi<\theta} u_\xi = \emptyset$. Let for each $(h, \bar{u}) \in H$,

$$H^*_{(h,\bar{u})} = \Big\{ f : f \text{ is a function from ordinals to Pie}(\theta \times \theta) \text{ and for some set } v,\ \text{Dom}(f) = v \subseteq \text{Dom}(h), |v| = \theta, \text{ but } \zeta < \theta \Rightarrow |v \backslash u_\zeta| < \theta, \text{ and } (\forall \alpha \in v)[C\ell(f(\alpha)) \supseteq C\ell(h(\alpha))], \text{ and } f \text{ is simple} \Big\}$$

and $H^* = \bigcup \{H^*_{(h,\bar{u})} : (h, \bar{u}) \in H\}$.
It is easy to check that H^* satisfies clauses (a) and (b) from 4.2(1) and (e) of 4.2(2) and $|H^*| \in [\lambda, \lambda^\theta]$.

As for clause (c) of 4.2(1), let $H' \subseteq H^*$, $|H'| < \lambda$, let $H' = \{f_j : j < j(*)\}$, $j(*) < \lambda$, and for every $j < j(*)$,$f_j \in H^*_{(h_j,\bar{u}_j)}$ for some $(h_j, \bar{u}_j) \in H$. So $v_j =:$ $\text{Dom}(f_j)$ is as in the definition of $H^*_{(h_j,\bar{u}_j)}$. Define $H'' = \{(h_j, \bar{u}_j) : j < j(*)\}, Y = \bigcup_{j<j(*)} v_j$. Now H'' is a subset of H of cardinality $< \lambda$, $Y \subseteq Ord$ and $|Y| < \lambda$. As H exemplifies $\lambda \in SQw_{\theta,\sigma}$, we can find a $\langle g_i : i < i(*) \rangle$, $i(*) < 1 + \sigma$, $g_i \in {}^\lambda\theta$ and $\langle (H''_i, Y_i) : i < i(*) \rangle$ such that $H'' \times Y = \bigcup_{i<i(*)} H''_i \times Y_i$ and for every $(h_j, \bar{u}_j) \in H''$ for some $i = i(j) < i(*)$ we have

$$(\exists \zeta < \theta)(\exists \xi < \theta)(\forall \alpha \in u_{j,\zeta} \cap Y_i)[(g_i(\alpha), \xi) \in C\ell(h_j(\alpha))].$$

Here $\bar{u}_j = \langle u_{j,\zeta} : \zeta < \theta \rangle$. Let $H_i =: \{f_j : i(j) = i\}$.

Now $\langle g_i : i < i(*) \rangle$, $\langle (H_i, Y_i) : i < i(*) \rangle$ are O.K. for H', too, as $C\ell(f_j(\alpha)) \supseteq C\ell(h_j(\alpha))$ and $|v_j \backslash u_{i,\zeta}| < \theta$.

We are left with clause (d) of 4.2(1), so assume $i(*) < 1 + \sigma$ and $g_i \in {}^\lambda\theta$, H^*_i, Y_i for $i < i(*)$ exemplify H^* is σ-free. Let

$$H_i =: \{(h, \bar{u}) : \neg(\exists \zeta < \theta)(\exists \xi < \theta)(\forall \alpha \in u_\zeta \cap Y_i)[(g_i(\alpha), \xi) \in C\ell(h(\alpha))]\}.$$

By the choice of H we have $H \times \lambda \neq \bigcup_{i<i(*)} H_i \times Y$, but the inclusion $\supseteq$ is obviously true. So there is a pair $((h, \bar{u}), \alpha^*) \in H \times \lambda \setminus \bigcup_{i<i(*)} H_i \times Y_i$. Let $\langle a_i : i < i(*) \rangle$ be a partition of θ to unbounded subsets, and we choose by induction on $\zeta < \theta$, an ordinal $\alpha_\zeta \in u_\zeta$ and $\Upsilon_\zeta < \theta$ such that if $\zeta \in a_i$ then $\alpha_\zeta \in Y_i$ if possible and

$$\Upsilon_\zeta \in \theta \backslash \left[\bigcup_{\xi<\zeta}(g_i(\Upsilon_\xi) \cup \Upsilon_\xi) + 1\right]$$

$$\text{and } (g_i(\alpha_\zeta), \Upsilon_\zeta) \notin C\ell(h(\alpha_\zeta)).$$

Lastly we define a function f, $\operatorname{Dom}(f) = \{\alpha_\zeta : \zeta < \theta\}$ and $f(\alpha_\zeta)$ is such that

$$C\ell(f(\alpha_\zeta)) = \{(\gamma_1, \gamma_2) : \gamma_1 < \theta, \gamma_2 < \theta, \text{ and } (\gamma_1, \gamma_2) \nleq (g_i(\alpha_\zeta), \Upsilon_\zeta)\}.$$

Let $v =: \{\alpha_\zeta : \zeta < \theta\}$, so $f \in H^*_{(h,\bar{u})} \subseteq H^*$. Hence for some $j < i$ the pair (f, α^*) belongs to (H_i, Y_i). This contradicts $((h, \bar{u}), \alpha^*) \notin (H^*_i, Y_i)$.
(3) As in (2), $\lambda \in SPd_{\theta,\sigma} \Rightarrow \lambda \in SP_{\theta,\sigma} \Rightarrow \lambda \in SPw_{\theta,\sigma}$ is obvious.
We need to prove that $\lambda \in SPw_{\theta,\sigma} \Rightarrow SPd_{\theta,\sigma} \cap [\lambda, \lambda^\theta] \neq \emptyset$ when $\sigma \leq \theta^+$.
The proof if similar to that of (2). We start with H exemplifying that $\lambda \in SPw_{\theta,\sigma}$. We assume that for each $(h, \bar{u}) \in H, \bar{u}$ is standard. So for $(h, \bar{u}) \in H$, we define

$$\begin{aligned} H^*_{(h,\bar{u})} = \Big\{ f : \; & f \text{ is a function from ordinals (i.e. from } \lambda) \text{ to } \theta \text{ and } f \text{ is } 1-1, \\ & \text{and for some set } v,\ \operatorname{Dom}(f) = v \subseteq \operatorname{Dom}(h), \\ & \text{we have that } |v| = \theta, \text{ but} \\ & \zeta < \theta \Rightarrow |v \backslash u_\zeta| < \theta, \text{ while } (\forall \alpha \in v) f(\alpha) \leq h(\alpha) \Big\}. \end{aligned}$$

Let $H^* = \bigcup\{H^*_{(h,\bar{u})} : (h, \bar{u}) \in H\}$.
Checking that this H^* is as required is similar to (2). For example, to see 4.1(1)d), with $\sigma = 1$ suppose that g exemplifies that H^* is free. By the choice of H, there is an $(h, \bar{u}) \in H$ such that

$$\neg(\exists \zeta < \theta)(\exists \zeta < \theta)(\forall \alpha \in u_\zeta)[h(\alpha) \leq \operatorname{Max}\{g(\alpha), \xi\}].$$

We choose by induction on $\zeta < \theta$, an ordinal $\alpha_\zeta \in u_\zeta \cap Y$ and $\Upsilon_\zeta \in \theta$ such that

$$\Upsilon_\zeta \in \theta \backslash \left[\bigcup_{\xi<\zeta}(g(\Upsilon_\xi) \cup \Upsilon_\xi) + 1\right]$$

and

$$h(\alpha_\zeta) > \operatorname{Max}\{g(\alpha_\zeta), \Upsilon_\zeta\}.$$

Then we let $f(\alpha_\zeta)$ be such that

$$f(\alpha_\zeta) \leq \operatorname{Max}\{g(\alpha_\zeta), \Upsilon_\zeta\}$$

but

$$f(\alpha_\zeta) \notin \{f(\alpha_\xi) : \xi < \zeta\}.$$

(4) Included in the proof of (1). $\square_{4.4}$

4.5 Claim. *Assume $\lambda \in SP_{\theta,\sigma}$, μ is a strong limit with $cf(\mu) > \theta$, and $2^\mu = \mu^+ > \lambda$.*

Then there is a $\kappa \in [\lambda, \mu^+]$, a regular cardinal such that $\kappa \in SP^+_{\theta,\sigma}$ where

4.6 Definition. 1) $\kappa \in SP^+_{\theta,\sigma}$ means that κ is regular $> \theta$ and we can find an $S \subseteq \{\delta < \kappa : cf(\delta) = \theta\}$ stationary, $\bar{\eta} = \langle \eta_\delta : \delta \in S\rangle$, $\bar{h} = \langle h_\delta : \delta \in S\rangle$, such that

(a) η_δ is a strictly increasing sequence of ordinals of length θ with limit δ

(b) $h_\delta : \text{Rang}(\eta_\delta) \to \theta$ is strictly increasing

(c) $H = \{h_\delta : \delta \in S\}$ is $(< \kappa)$-σ- free not σ-free (in 4.1's sense).

2) $\kappa \in SP^*_{\theta,\sigma}$ if in the above we add:

(d) $\bar{\eta}$ is tree like, i.e. $\eta_{\delta_1}(\varepsilon_1) = \eta_{\delta_2}(\varepsilon_2) \Rightarrow \varepsilon_1 = \varepsilon_2$ & $\eta_{\delta_1} \restriction \varepsilon_1 = \eta_{\delta_2} \restriction \varepsilon_2$.

3) $\kappa \in SP^{\otimes}_{\theta,\sigma}$ if in part (1) we replace (b) by

(b)$'$ $h_\delta : \text{Rang}(\eta_\delta) \to \theta$ is constant.

Remark. 1) The assumption "$\mu^+ = 2^\mu$" (in 4.5) is very reasonable because of 4.3(2) (and 4.3A(3) from the topological point of view).
2) Basically the proof of 3.3(2) is a way of getting nicer examples of incompactness.

Proof of 4.5. Use $\diamondsuit_{\{\delta<\mu^+ : cf(\delta)=\theta\}}$ and imitate the proof of 3.3(2) (noting that if h fails g then for some $a \subsetneq \text{Dom}(h)$, $|a| = h$, $h \restriction a$ is strictly increasing and $h \restriction a$ fails g). $\square_{4.5}$

4.6A Observation. 1) $SP^*_{\theta,\sigma} \subseteq SP^+_{\theta,\sigma} \subseteq SP_{\theta,\sigma}$.
2) If $\langle h_\delta, \eta_\delta : \delta \in S\rangle$, κ satisfies the preliminary requirements and clauses (a), (b) of Definition 4.6 and H is $(< \kappa_1)$-free, $\kappa_1 > \theta$ then for some $\mu \in [\kappa_1, \kappa]$, $\mu \in SP^+_\theta$.
3) Similarly for $SP^*_{\theta,\sigma}$.

Proof. (1) is trivial. For (2) and (3), do like in 2.3. $\square_{4.6A}$

4.6.B Conclusion. For $\lambda > \theta = cf(\theta), \chi = \beth_\chi > \lambda$, the following are equivalent:

(A) for some $\mu \in [\lambda, \chi)$, $\mu \in SP_\theta$

(B) for some $\mu \in [\lambda, \chi)$, $\mu \in SP^+_\theta$.

Proof. By 4.6A(1), (B) $\Rightarrow$ (A), as for (A) $\Rightarrow$ (B), let $\mu = \beth_{\lambda+(\theta^+)}$, if $pp(\mu) > \mu^+$ use 4.3(2) and if $pp(\mu) = \mu^+$ use 4.5. $\square_{4.6.B}$

4.7 Claim. *Assume $\theta = \theta^{<\theta}$.*
Let $\langle h_\delta, \eta_\delta : \delta \in S\rangle$ exemplify $\lambda \in SP^*_\theta$ (even omitting "η_δ converge to δ, η_δ strictly increasing"). Then any θ^+-complete forcing preserves the non-freeness of $\{h_\delta : \delta \in S\}$.

Proof. Instead of the domain of the functions h_δ being a subset of λ, we can assume that it is $T =: \{\eta_\delta \restriction \zeta : \delta \in S$ and $\zeta < \theta$ is a successor ordinal$\}$ we can by (d) of 4.6(2) (identify $\eta_\delta(\zeta)$ with $\eta_\delta \restriction (\zeta+1)$, so $\mathrm{Dom}(h_\delta) = \{\eta_\delta \restriction \zeta : \zeta < \theta$ is a successor ordinal $\}$). Suppose Q is a θ^+-complete forcing notion, $p \in Q$ and $p \Vdash$ "$\underset{\sim}{g} : T \to \theta$ exemplifies $\{h_\delta : \delta \in S\}$ is free ". We now define by induction on $\ell g(\eta) < \theta$ a sequence $\langle p_{\eta,t}, \varepsilon_{\eta,t} : t \in T_\eta\rangle$ for $\eta \in T$ such that:

- (α) $T_\eta \subseteq {}^{\ell g(\eta) \geq}\theta$, is closed under initial segments
- (β) $t \triangleleft s \in T_\eta \Rightarrow p_{\eta,t} \leq_Q p_{\eta,s}$
- (γ) if $t \in T_\eta$, either $\bigwedge\limits_{\zeta<\theta} t\hat{\ } < \zeta > \in T_\eta$ or $\bigwedge\limits_{\zeta<\theta} t\hat{\ } < \zeta > \notin T_\eta$
- (δ) if $t \in {}^{\ell g(\eta) \geq}\theta$, $\ell g(t)$ is a limit ordinal and $(\forall \zeta < \ell g(t))(t \restriction \zeta \in T_\eta)$, then $t \in T_\eta$,
- (ε) if $\nu \triangleleft \eta$ then $T_\nu \subseteq T_\eta$ and $t \in T_\nu \Rightarrow (p_{\eta,t}, \varepsilon_{\eta,t}) = (p_{\nu,t}, \varepsilon_{\nu,t})$
- (ζ) if $\ell g(\eta)$ is a limit ordinal then
  $T_\eta = \{t : t \in \bigcup\limits_{\nu \triangleleft \eta} T_\nu$ or $\ell g(t)$ is a limit ordinal and $(\forall s)[s \triangleleft t \Rightarrow s \in \bigcup\limits_{\nu \triangleleft \eta} T_\nu]\}$.
- (η) assume $\eta = \nu\hat{\ } < \alpha >$ and s is a $\triangleleft$-maximal element of T_ν, then:
  - (a) if $\{\zeta < \theta : p_{\eta,s} \nVdash_Q$ "$\underset{\sim}{g}(\eta) \neq \zeta$"$\}$ is bounded in θ then s is a $\triangleleft$-maximal element of T_η.
  - (b) if $A = \{\zeta < \theta : p_{\eta,s} \nVdash$ "$\underset{\sim}{g}(\eta) \neq \zeta$"$\}$ is unbounded in θ, then for every $\zeta < \theta$, $s\hat{\ } < \zeta >$ is a maximal member of T_η, and $p_{s\hat{\ }<\zeta>}$ forces a value $\varepsilon_{s\hat{\ }<\zeta>} > \ell g(\eta)$ to $\underset{\sim}{g}(\eta)$.

We can carry this definition.

- ($*$) if $\delta \in S$ then for some $\zeta = \zeta_\delta < \theta$ and $t = t_\delta \in T_{\eta_\delta \restriction \zeta}$ we have: t is a $\triangleleft$-maximal member of $T_{\eta \restriction \xi}$ for every $\xi \in [\zeta, \theta)$.

[why? otherwise we can construct a $t \in {}^\theta\theta$ such that $(\forall s)[s \triangleleft t \Rightarrow s \in \bigcup\limits_{\xi<\theta} T_{\eta_\delta \restriction \xi}]$, $t(\varepsilon) > \varepsilon$ and for unboundedly many $\xi < \theta$, for some $s\hat{\ } < \zeta > \triangleleft t$ we have $s\hat{\ } < \zeta > \in T_{\eta_\delta \restriction (\xi+1)} \backslash T_{\eta_\delta \restriction \xi}$ and $\varepsilon_{s\hat{\ }<\zeta>} > h_\delta(\eta_\delta \restriction (\xi+1))$.
Now $\{p_{\nu,s} : \nu \triangleleft \eta_\delta, s \triangleleft t, s \in T_\nu\}$ has an upper bound in Q, say p^*. Then p^* forces for $\underset{\sim}{g}(\eta \restriction (\xi+1))$ a value $> h_\delta(\eta \restriction (\xi+1)), \xi$; this is a contradiction to $p \Vdash$ "$\underset{\sim}{g}$ exemplifies the freeness of $\{h_\delta : \delta \in S\}$"].
For $t \in {}^{\theta>}\theta$ and $\zeta < \theta$ we define $S_{t,\zeta} = \{\delta \in S : \zeta, t$ can serve as ζ_δ, t_δ from $(*)\}$. Clearly $\{h_\delta : \delta \in S_{t,\zeta}\}$ is free, hence $\{h_\delta : \delta \in S\}$ is $(2^{<\theta})^+$-free. Now as $2^{<\theta} = \theta$,

clearly we can have $S = \bigcup_{\zeta<\theta} S_\zeta$ and $\{h_\delta : \delta \in S_\zeta\}$ is free. Define $h : T \to \theta$ by $h(\eta) = \sup\{h_\zeta(\eta) : \zeta < \ell g(\eta)\}$. This h shows $\{h_\delta : \delta \in S\}$ is free. Contradiction. $\square_{4.7}$

Remark. We now sum up our results; for simplicity we speak on the case $\theta = \aleph_0$, $\sigma = 1$.

4.8 Theorem. *Assume $\lambda < \mu$ and $(\forall \chi < \mu)[\chi^{\aleph_0} < \mu)$ (possibly $\mu = \infty$). Then the following are equivalent:*

(A) *There is a first countable Hausdorff space X such that:*
- *(a) X is $(< \lambda)$-CWH*
- *(b) X is not λ-CWH*
- *(c) X has $< \mu$ points.*

(A)$^+$ *There is a space X like in (B), and in addition*
- *$(a)^+$ X is $(< \lambda)$-metrizable.*

(B) *There is a first countable Hausdorff space X such that:*
- *(a) X is $(< \lambda)-^*$ CWN*
- *(b) X is not $\lambda-^*$ CWN*
- *(c) X has $< \mu$ points.*

(B)$^+$ *There is an X like in (C), and in addition,*
- *$(a)^+$ X is $(< \lambda)$-metrizable.*

(C) *there is a family H of functions, each with domain a countable set of ordinals and range $\subseteq \omega$ such that:*
- *(a) H is $(< \lambda)$-free*
- *(b) H is not free*
- *(c) $|H| < \mu$.*

$(C)^+$ as in (D) and
- *(d) $\bigcup\{Dom(h) : h \in H\} = \lambda' \in [\lambda, \mu)$*
- *(e) each h is one to one.*

$(C)'$ $[\lambda, \mu) \cap SP_{\aleph_0} \neq \emptyset$

$(C)''$ $[\lambda, \mu) \cap SPw_{\aleph_0} \neq \emptyset$

$(C)'''$ $[\lambda, \mu) \cap SPd_{\aleph_0} \neq \emptyset$

(D) *there is a* $H = \langle h_\alpha, < u_{\alpha,n} : n < \omega >: \alpha \in v\rangle$ *with* $u_{\alpha,n+1} \subseteq u_{\alpha,n} \subseteq v$ *and each* h_α *is a function from ordinals to* $Pie(\aleph_0 \times \aleph_0)$ *defined on* $u_{\alpha,0}$ *such that:*

(a) H *is not free in the sense of* SQw

(b) *for* $v' \in [v]^{<\lambda}$, $H \restriction v'$ *is free in the sense of* SQw

(c) $|v| < \mu$.

$(D)'$ $[\lambda, \mu) \cap SQ_{\aleph_0} \neq \emptyset$

$(D)''$ $[\lambda, \mu) \cap SQw_{\aleph_0} \neq \emptyset$

$(D)'''$ $[\lambda, \mu) \cap SQd_{\aleph_0} \neq \emptyset$

4.8A Theorem. *In 4.8 if* $(\forall\kappa < \mu)(\beth_{\theta^+}(\kappa) < \mu)$ *(really* $(\forall\kappa < \mu)(\beth_{\omega_1}(\kappa) < \mu)$ *is O.K.,* <u>*then*</u> *we can add*

(E) *for some regular* $\kappa \in [\lambda, \mu)$ *we have* $INCWH^2(\kappa)$

$(E)'$ $\lambda \in SP^+_{\aleph_0}$

Proof of 4.8 and 4.8A. By 4.4(1) (the $(b) \Leftrightarrow (b)^+ \Leftrightarrow (c) \Leftrightarrow (c)^+$ part) we know the equivalence of (A), $(A)^+$, (B) $(B)^+$.
By 4.3(1) and 4.1 $(C) \Leftrightarrow (C)'$.
By 4.4(3) we have $(C)' \Rightarrow (C)'' \Rightarrow (C)'''$.
By 4.3A(1) and 4.2 $(D) \Leftrightarrow (D)'$.
By 4.4(2) $(D)' \Rightarrow (D)'' \Rightarrow (D)'''$.
By 4.2B(2) $(D)''' \Rightarrow (C)'$.
By 4.2B(1) $(C)' \Rightarrow (D)'$, $(C)'' \Rightarrow (D)''$, $(C)''' \Rightarrow (D)''$.

Together we get the equivalence of $(C), (D), (C)', (D)', (C)'', (D)'', (C)'', (D)''$.

By 4.4(1), i.e. $(a) \Rightarrow (b) \Rightarrow (d)$ we have $(D)' \Rightarrow (A) \Rightarrow (D)''$, so by the last sentence and the first paragraph we have finished the proof of 4.8. For 4.8A use 4.6B. $\square_{4.8}$

4.9 Fact. Let $\lambda = cf(\lambda) > \theta = cf(\theta)$. The following statements, (A) and (B), are equivalent:

$(A) = (A)_{\lambda,\theta}$ There is $H = \bigcup\limits_{i<\lambda} H_i$ such that:

(α) H_i is increasing continuous

(β) H_i is a family of functions h to θ, $\mathrm{Dom}(h)$ is a set of θ ordinals, h is one to one

(γ) each H_i is free, but H is not free, in the sense of SP.

$(B) = (B)_{\lambda,\theta}$. Let $X = X_{\lambda,\theta} =: {}^{\lambda}\theta$

$$F = F_{\lambda,\theta} =: \Big\{ f : f \text{ a partial function from } X \text{ to } \theta, |\text{Dom } f| = \theta \text{ and } (\forall^* i < \lambda)(\forall^* \eta \in \text{Dom}(f))[f(\eta) \leq \eta(i)] \text{ and } f \text{ is one to one} \Big\}.$$

Then there is no $G : X \to \omega$ such that

$$f \in F \Rightarrow (\forall^* \eta \in \text{Dom}(f))[f(\eta) \leq G(\eta)].$$

Proof. $\underline{(A) \Rightarrow (B)}$.

Let H, H_i $(i < \lambda)$ exemplify (A), let $A = \bigcup\{\text{Dom}(h) : h \in H\}$, and let $g_i : A \to \theta$ exemplify "H_i is free".

We define an equivalence relation E on $A : \alpha E \beta \Leftrightarrow \bigwedge\limits_{i<\lambda} g_i(\alpha) = g_i(\beta)$. If for some $h \in H$ and α, the set $(\alpha/E) \cap \text{Dom}(h)$ has cardinality θ, choose $i < \lambda$ such that $h \in H_i$, and g_i cannot satisfy the requirement. Hence $|(\alpha/E) \cap \text{Dom}(h)| < \theta$ for all α and h.

Let $h^{\otimes}$ be a function with domain $\text{Dom}(h)$, $h^{\otimes}(\alpha) = \sup\{h(\beta) : \beta \in \alpha/E\}$, for all $h \in H$. Now $H' =: \{h^{\otimes} : h \in H\}$, $H_i'' =: \{h^{\otimes} : h \in H_i'\}$ exemplify (A) too. So without loss of generality E is the equality on A.

Next for each $\alpha \in A$ let $\eta_\alpha \in {}^{\lambda}\theta (= X)$ be defined by $\eta_\alpha(i) = g_i(\alpha)$, so $\alpha \neq \beta \Rightarrow \eta_\alpha \neq \eta_\beta$. For $h \in H$ let $\text{Dom}(h) = \{\alpha_{h,\zeta} : \zeta < \theta\}$ be an enumeration such that $\langle h(\alpha^*_{h,\zeta}) : \zeta < \theta\rangle$ is increasing and not eventually constant. For $h \in H$ let

$$\mathcal{P}_h = \{a \subseteq \theta : \langle h(\alpha^*_{h,\zeta}) : \zeta \in a\rangle \text{ is strictly increasing}\}$$

and let the function f_h be defined by:

$$\text{Dom}(f_h) = \{\eta_{\alpha_{h,\zeta}} : \zeta < \theta\} \text{ and } f_h(\eta_{\alpha_{h,\zeta}}) = h(\alpha_{h,\zeta}).$$

Now

$(*)$ $h \in H \wedge a \in \mathcal{P}_h \Rightarrow f_h \restriction a \in F$.

[Why? Let $i(*) = \text{Min}\{i : h \in H_i\}$ (well defined as $H = \bigcup\limits_{i<\lambda} H_i$), so $i \in [i(*), \lambda)$ implies $h \leq^* (g_i \restriction \text{Dom}(h))$. So for some $\zeta(*) < \theta$, for every $\zeta \in [\zeta(*), \theta)$ we have $h(\alpha_{h,\zeta}) \leq g_i(\alpha_{h,\zeta})$, but $f_h(\eta_{\alpha_{h,\zeta}}) = h(\alpha_{h,\zeta})$ and $g_i(\alpha_{h,\zeta}) = \eta_{\alpha_{h,\zeta}}(i)$ so: for every $i < \lambda$ large enough for all but $< \theta$ members $\eta = \eta_{\alpha_{h,\zeta}}$ of $\text{Dom}(f_h)$, $f_h(\eta) = h(\alpha_{h,\zeta}) \leq g_i(\alpha_{h,\zeta}) = \eta_{\alpha_{h,\zeta}}(i) = \eta(i)$ as required].

So assume G is a function from X to ω such that

$(**)$ $f \in F \Rightarrow (\forall^* \eta \in \text{Dom}(f))[f(\eta) \leq G(\eta)]$

and we should get a contradiction. Let us define $g \in {}^{A}\theta$ by $g(\alpha) = G(\eta_\alpha)$. So for $h \in H$ assume $b = \{\zeta < \theta : h(\alpha_{h,\zeta}) > G(\eta_{\alpha_{h,\zeta}})\}$ is unbounded, so there is $a \subseteq b$, $a \in \mathcal{P}_h$. So we have $f_h \restriction a \in F$ hence by $(*) + (**)$ for some $\zeta(*) < \theta$, $\zeta \in [\zeta(*), \theta) \cap a \Rightarrow f_h(\eta_{\alpha_{h,\zeta}}) \leq G(\eta_{\alpha_{h,\zeta}})$. But $f_h(\eta_{\alpha_{h,\zeta}}) = h(\alpha_{h,\zeta})$, and $g(\alpha_{h,\zeta}) = G(\eta_{\alpha_{h,\zeta}})$ so

$$\zeta \in [\zeta(*), \theta) \cap a \Rightarrow h(\alpha_{h,\zeta}) \leq g(\alpha_{h,\zeta}).$$

So g shows that H is free, contradiction. We have proved (B).

<u>$(B) \Rightarrow (A)$</u>

The demand $A = \bigcup_{h \in H} \mathrm{Dom}(h) \subseteq$ Ord is immaterial, so let $A = X$ and $H = F_{\lambda,\theta}$. Lastly for $i < \lambda$ let $g_i : A \to \theta$ be given by $g_i(\eta) = \eta(i)$, and

$$H_i = \{f \in F : \text{ for every } j \in [i, \lambda) \text{ we have } (\forall^* \eta \in \mathrm{Dom}(f))[f(\eta) \leq \eta(j)\}.$$

$\square_{4.9}$

4.10 Conclusion. $INCWH(\lambda)$ implies $(B)_{\lambda,\theta}$ of Fact 4.9 implies

$$(\exists \mu)[\lambda \leq \mu \leq 2^\lambda \ \& \ INCWH(\mu)].$$

4.11 Remark. It is well known that

$(*)$ if there is a real valued measure m on $P(\lambda)$
$G(f) = \mathrm{Min}\{n : m(f^{-1}(\{n\}) > 0\}$,

then G contradicts $(B)_{\lambda,\aleph_0}$.

Also, it is consistent that $SP_{\aleph_0} \subseteq (2^{\aleph_0})^+$. This follows from the consistency of the PMEA (Product Measure Extension Axiom) and Fact 4.9.
The consistency of PMEA is due to Kunen. See [F184] for an exposition.

§5 More on freeness

5.1 Definition. For $\ell \in \{0, 1, 2, 3, 4\}$ and regular cardinal θ we define

$$SP^\ell_{\theta,\sigma} = \Big\{\lambda : \text{ there is a family } H \text{ such that:}$$

(a) every $h \in H$ is a partial function from ordinals to θ

(b) $h \in H \Rightarrow |\mathrm{Dom}(h)| = \theta$

(c) every $H' \subseteq H$ of cardinality $< \lambda$ is free in the sense of $P^\ell_{\theta,\sigma}$

(d) H is not free in the sense of $P^\ell_{\theta,\sigma}\Big\}$, where H' is σ-free in the sense of $P^\ell_{\theta,\sigma}$ if $H' = \bigcup_{i<i(*)} H'_i$ for some $i(*) < 1+\sigma$, and each H'_i is free in the sense of P^ℓ_θ, which means:

(0) if $\ell = 0$, there is a function g from ordinals to θ such that

$$(\forall h)(\exists \xi < \theta)[h \in H'_i \Rightarrow (\forall \alpha \in \text{ Dom}(h))[h(\alpha) \leq g(\alpha) \vee h(\alpha) \leq \xi]].$$

(So $SP^0_{\theta,\sigma} = SP_{\theta,\sigma}$).

(1) if $\ell = 1$, there is a function g from ordinals to θ, such that

$$(\forall h)[h \in H'_i \Rightarrow |\{\alpha : h(\alpha) > g(\alpha)\}| < \theta].$$

(2) if $\ell = 2$, there is a g like in (1), and

$$(\forall h)[h \in H'_i \Rightarrow h \text{ one to one}].$$

(3) if $\ell = 3$, there is a g like in (1), and

$$(\forall h)[h \in H'_i \Rightarrow h \text{ is constant function}].$$

(4) if $\ell = 4$, there is a g like in (1), and

$$(\forall h)[h \in H'_i \Rightarrow h \text{ is constant or } h \text{ is one to one}].$$

If $\sigma = 1$, we omit it from the notation.

5.2 Claim. *If $\lambda \in SP^1_{\theta,\sigma}$, <u>then</u> $[\lambda, \lambda + 2^\theta] \cap SP^4_{\theta,\sigma} \neq \emptyset$. (Here $\sigma \leq \theta^+$).*

Proof. We divide the proof into two cases.

Case 1 $\sigma = 1$. Let H exemplify $\lambda \in SP^1_\theta$. Let

$$H^\oplus \underset{\text{def}}{=} \Big\{ h \restriction A : h \in H \,\&\, A \in [\text{Dom}(h)]^\theta \,\&\, (h \restriction A \text{ is constant } \underline{\text{or}}\ h \restriction A \text{ is one to one}) \Big\}.$$

So

(α) $|H^\oplus| \leq \lambda + 2^\theta$.

(β) Let $H' \subseteq H^\oplus$ with $|H'| < \lambda$. Hence

$$H' = \{h_j \restriction A_j : j < j^* < \lambda\}$$

for some $H'' = \{h_j : j < j^*\} \subseteq H$ and $A_j \in [\text{Dom}(h_j)]^\theta$, for $j < j^*$.

Hence H'' is free in the sense of P^1_θ. Let g exemplify this. Hence for every $h_j \in H''$, we have

$$|\{\alpha : h_j(\alpha) > g(\alpha) \,\&\, \alpha \in \text{Dom}(h_j)\}| < \theta.$$

In particular, g exemplifies that H' is free in the sense of P^4_θ.

(γ) Assume that $H^\oplus$ is free in the sense of P^4_θ.

Before we proceed, an easy observation.

Subclaim (A). If h is a function from ordinals to θ, then at least one of the following holds:

(i) there is a subset A of $\mathrm{Dom}(h)$ such that $h \restriction A$ is a constant.
(ii) there is a subset A of $\mathrm{Dom}(h)$ such that $h \restriction A$ is one to one.

[Why? If not (i), then $|\mathrm{Rang}(h)| = \theta$, by the regularity of θ.]

Assume that g exemplifies that $H^{\otimes}$ is free in the sense of P^4_θ. For every $h \in H^{\otimes}$ let

$$B_h = \{\alpha \in A : h(\alpha) > g(\alpha)\}.$$

If $|B_h| = \theta$ let $A \subseteq B_h$ be such that $|A| = \theta$, $h \restriction A$ is one to one or constant, but $h \restriction A \in H^{\oplus}$ and $\bigwedge_{\alpha \in A} h \restriction A(\alpha) > g(\alpha)$, contradiction. So $h \in H \Rightarrow |B_h| < \theta$ as required.

Case 2 $\sigma = \theta^+$.

Let H exemplify $\lambda \in SP^1_{\theta,\theta^+}$. For $h \in H$, let $\mathrm{Dom}(h) = \{\alpha^h_\zeta : \zeta < \theta\}$ be an increasing enumeration.

As before we define

$$H^{\oplus} \underset{\mathrm{def}}{=} \Big\{ h \restriction A : h \in H \,\&\, A \in [\mathrm{Dom}(h)]^\theta \;\&\, (h \restriction A \text{ is constant } \underline{\text{or}} \; h \restriction A \text{ is one to one}) \Big\}.$$

It is easily seen that

(a) $|H^{\oplus}| \leq 2^\theta + \lambda$
(b) $H^{\oplus}$ is $(< \lambda) - \theta^+$-free in the sense of P^4_θ.

Fact (B). $H^{\oplus}$ is not θ^+-free in the sense of P^4_θ

Proof of Fact (B). Suppose otherwise, so without loss of generality $H^{\oplus} = \bigcup_{i<\theta} H^{\oplus}_i$, and each $H^{\oplus}_i$ is free in the sense of P^4_θ. Let this be witnessed by g_i.

without loss of generality, $\mathrm{Dom}(g_i) = \bigcup_{h \in H} \mathrm{Dom}(h)$, as $|H| = \lambda$.

Also without loss of generality,

$$i < j < \theta \Rightarrow \bigwedge_{\beta \in \bigcup_{h \in H} \mathrm{Dom}(h)} g_i(\beta) < g_i(j).$$

[Why? Since we can replace $\langle g_i : i < \theta\rangle$ by $\langle g'_i : i < \theta \rangle$ defined by

$$g'_i(\beta) = \sup[\{g_j(\beta) + 1 : j < i\} \cup \{g_i(\beta)\}].]$$

Let

$$H_i \overset{\mathrm{def}}{=} \{h \in H : (\exists \xi < \theta)[\zeta \geq \xi \Rightarrow h(\alpha^h_\zeta) \leq g_i(\alpha^h_\zeta)]\},$$

for $i < \theta$.

Hence, each H_i is free in the sense of P^1_θ, so it suffices to show that $H = \bigcup_{i<\theta} H_i$. So suppose $h \in H \setminus \bigcup_{i<\theta} H_i$. Let

$$A_i \stackrel{\text{def}}{=} \{\alpha^h_\zeta : \zeta < \theta \,\&\, h(\alpha^h_\zeta) > g_i(\alpha^h_\zeta)\},$$

for $i < \theta$. So, as $h \notin \bigcup_{i<\theta} H_i$, for every $i < \theta$ we have $|A_i| = \theta$. Also, by the choice of $\langle g_i : i < \theta\rangle$, we have

$$i < j < \theta \Rightarrow A_i \supseteq A_j.$$

Hence, we can find a set $A \in [\mathrm{Dom}(h)]^\theta$ such that for all $i < \theta$ we have $|A \setminus A_i| < \theta$. There is a subset $B \subseteq A$ such that $|B| = \theta$ and $h \restriction B \in H^\oplus$. Hence $h \restriction B \in H_i$ for some $i < \theta$. But $|\{\alpha \in B : h(\alpha) < g_i(\alpha)\}| < \theta$, in contradiction with the choice of g_i.

This finishes the proof of the second case. Since, as remarked in 4.1A, the σ-freeness for $\sigma < \theta$ is equivalent to 1-freeness, we have finished the proof. $\square_{5.2}$

5.3 Observation. 1) For $\sigma > \theta^+$, we have $SP^3_{\theta,\sigma} = \emptyset$.

[Why? Suppose $\lambda \in SP^3_{\theta,\sigma}$ for some $\sigma > \theta^+$, and this is exemplified by a family H. Let

$$H_i \stackrel{\text{def}}{=} \{h \in H : h \text{ is constantly } i \text{ on its domain}\}.$$

Hence $H = \bigcup_{i<\theta} H_i$.

But each H_i is free in the sense of P^3_θ, as exemplified by $g_i \equiv i+1$.]

2) For $\sigma > \theta$, for every λ we have $\lambda \in SP^2_{\theta,\sigma}$ iff $\lambda \in SP^4_{\theta,\sigma}$.

[Why? Certainly $SP^2_{\theta,\sigma} \subseteq SP^4_{\theta,\sigma}$. Suppose H exemplifies that $\lambda \in SP^4_{\theta,\sigma}$. Let

$$H^\oplus \stackrel{\text{def}}{=} \{h \in H : h \text{ is one to one}\}.$$

Then by (1) we know $|H^\oplus| = \lambda$ and $H^\oplus$ is not σ-free in the sense of P^4_θ. However, each $H' \subsetneq H^\oplus$ is σ-free in the sense of P^4_θ, so $H^\oplus$ exemplifies that $\lambda \in SP^2_{\theta,\sigma}$.]

3) Observation 4.1A(0) now means that $SP^3_{\theta,\sigma}$ is the same as $SPd_{\theta,\sigma}$.

5.4 Claim.

(1) $\lambda \in SP^3_{\theta,\sigma} \Rightarrow \lambda \in SP^1_{\theta,\sigma}$

(2) $\lambda \in SP^2_{\theta,\sigma} \Rightarrow \lambda \in SP^1_{\theta,\sigma}$

(3) $\lambda \in SP^4_{\theta,\sigma} \Rightarrow \lambda \in SP^1_{\theta,\sigma}$

Proof. (1)-(3) Obvious from the definition. $\square_{5.4}$

5.5 Claim. $\lambda \in SP^4_{\theta,\sigma} \Leftrightarrow \lambda \in SP^2_{\theta,\sigma} \vee \lambda \in SP^3_{\theta,\sigma}$.

Proof. $\Leftarrow$ Follows from the definition.
$\Rightarrow$ Let H exemplify $\lambda \in SP^4_{\theta,\sigma}$, and let

$$H^1 \stackrel{\text{def}}{=} \{h \in H : h \text{ is one to one}\} \text{ and}$$

$$H^2 \stackrel{\text{def}}{=} \{h \in H\, h \text{ is a constant}\}.$$

So both H^1, H^2 are $(< \lambda) - \sigma$-free in the sense of P_θ^1.

If one of H^1, H^2 is not $\lambda - \sigma$-free in the sense of P_θ^1, we are done. Otherwise let $H^\ell = \bigcup_{i < i_\ell(*)} H_i^\ell$ for $\ell \in \{1, 2\}$ and $i_\ell(*) < 1 + \sigma$ such that H_i^ℓ is free in the sense of P_θ^1, as exemplified by g_i^ℓ.

If $\sigma = 1$ let g be

$$g_0(\alpha) \stackrel{\text{def}}{=} \text{Max}\{g_0^1(\alpha), g_0^2(\alpha)\}.$$

It exemplifies H is free.

If $\sigma > 1$ without loss of generality $\sigma = \theta^+$ and let $i(*) = i_1(*) + i_2(*)$

$$H_i = \begin{cases} H_i^1 & \underline{\text{if}} \quad i < i_1(*) \\ H_{i-i_1(*)}^2 & \underline{\text{if}} \quad i \in [i_1(*), i_1(*) + i_2(*)) \end{cases}$$

$$g_i = \begin{cases} g_i^1 & \underline{\text{if}} \quad i < i_1(*) \\ g_i^2 & \underline{\text{if}} \quad i \in [i_1(*), i_1(*) + i_2(*)). \end{cases}$$

Then $H = \bigcup_{i < i(*)} H_i$ and each H_i is free, as exemplified by g_i. $\square_{5.5}$

5.6 Claim. $\lambda \in SP_\theta^3 \Rightarrow [\lambda, \lambda^\theta] \cap SP_\theta^2 \neq \emptyset$.

Proof.

Let H exemplify $\lambda \in SP_\theta^3$, and let for $\varepsilon < \theta$,

$$H_\varepsilon \stackrel{\text{def}}{=} \{h \in H : h \text{ is constantly } \varepsilon\}.$$

Let $\text{Dom}(h) = \{\alpha_\zeta^h : \zeta < \theta\}$ be an enumeration with no repetitions, for $h \in H$, and let

$$A \stackrel{\text{def}}{=} \{\alpha_\zeta^h : h \in H \;\&\; \zeta < \theta\}.$$

We define

$$G \stackrel{\text{def}}{=} \left\{ (\beta, \bar{h}) : \bar{h} = \langle h_\zeta ; \zeta \in W \rangle \text{ for some } W = W_{(\beta,\bar{h})} \in [\theta]^\theta \text{ we have} \right.$$
$$\text{(i)} \;\; h_\zeta \in H_\zeta \text{ for } \zeta \in W$$
$$\text{(ii)} \;\; \forall \zeta \in W \; [\beta \in \{\alpha_\varepsilon^{h_\zeta} : \varepsilon \in (\zeta, \theta)\}]$$
$$\left. \text{(iii)} \;\; \langle \text{Min}\{\varepsilon : \alpha_\varepsilon^{h_\zeta} = \beta\} : \zeta \in W \rangle \text{ is strictly increasing} \right\}.$$

For each $(\beta, \bar{h}) \in G$, we define a function $f_{(\beta,\bar{h})}$ such that

$$\text{Dom}(f_{(\beta,\bar{h})}) = \text{Rang}(\bar{h}) = \{h_\zeta : \zeta \in W_{(\beta,\bar{h})}\}, \text{ and}$$

$$f_{(\beta,\bar{h})}(h_\zeta) = \text{ the unique } \varepsilon \text{ such that } \beta = \alpha_\varepsilon^{h_\zeta}.$$

Let $F \stackrel{\text{def}}{=} \{f_{(\beta,\bar{h})} : (\beta, \bar{h}) \in G\}$.

Remark. Our aim is to use F to exemplify that $[\lambda, \lambda^\theta] \cap SP^2_\theta \neq \emptyset$. However, if $f \in F$, then the domain of f is not a set of ordinals, but a subset of H. This does not matter, as $|H| = \lambda$.

Fact (a). Each $f_{(\beta,\bar{h})}$ is one to one, in fact $f_{(\beta,\bar{h})}(h_\zeta)$ is strictly increasing in ζ.

[Why? Suppose $\zeta_1 < \zeta_2 \in W_{(\beta,\bar{h})}$. By the last clause in the definition of G, we know that $f_{(\beta,\bar{h})}(h_{\zeta_1}) < f_{(\beta,\bar{h})}(h_{\zeta_2})$.]

Fact (b). F is not free in the sense of P^1_θ.

Proof of Fact (b). Suppose otherwise, and let g witness this. We define $g^\oplus$ on $\bigcup\{\mathrm{Dom}(h) : h \in H\}$ by

$$g^\oplus(\beta) \stackrel{\text{def}}{=} \sup\{\varepsilon(h) : h \in H \,\&\, \beta \in \{\alpha^h_\zeta : \zeta \in (g(h), \theta)\}\}.$$

Subfact 1. $g^\oplus(\beta) < \theta$.

[Why? Otherwise we can find for some $\beta \in \mathrm{Dom}(g^\oplus)$, a sequence $\langle h'_\zeta : \zeta < \theta\rangle$ in H such that

$$\varepsilon(h'_\zeta) > \zeta \,\&\, \beta \in \{\alpha^{h'_\zeta}_\xi : \xi \in (g(h'_\zeta), \theta)\}.$$

By thinning out, we can find a sequence $\langle h_\zeta : \zeta \in W\rangle$ for some $W \in [\theta]^\theta$, such that for $\zeta \in W$ we have

$$\varepsilon(h_\zeta) = \zeta \,\&\, \beta \in \{\alpha^{h_\zeta}_\xi : \xi \in (g(h_\zeta), \theta)\}.$$

Hence $(\beta, \bar{h}) \stackrel{\text{def}}{=} \langle h_\zeta : \zeta \in W\rangle) \in G$, and so $f_{(\beta,\bar{h})} \in F$. However, for every $\zeta \in W$ we have $f_{(\beta,\bar{h})}(h_\zeta) > g(h_\zeta)$, in contradiction with the choice of g.]

Subfact 2. For $h \in H$, for every $\zeta < \theta$ large enough, we have $g^\oplus(\alpha^h_\zeta) \geq h(\alpha^h_\zeta)$ $(= \varepsilon(h))$.

[Why? Suppose $\zeta > g(h)$, hence

$$\alpha^h_\zeta \in \{\alpha^h_\xi : \xi \in (g(h), \theta)\},$$

so

$$g^\oplus(\alpha^h_\zeta) \geq \varepsilon(h) = h(\alpha^h_\zeta),$$

by the definition of $g^\oplus$.]

Hence we proved Fact (b). $\square_{(b)}$

Fact (c). F is $(< \lambda)$-free in the sense of P^1_θ.

Proof of Fact (c). Let $F' \subseteq F$ with $|F'| < \lambda$. Let $F' = \{f_{(\beta_i,\bar{h}_i)} : i < i(*) < \lambda\}$.

Now let

$$H' \stackrel{\text{def}}{=} \{h_\zeta : \zeta \in W(\beta_i, h_i) \,\&\, i < i(*)\}.$$

Hence $H' \in [H]^{<\lambda}$ (as $\lambda > \theta$). Let $g^\oplus$ be a function which exemplifies that H' is free in the sense of P^3_θ. We claim that g below shows that F' is free.

For $h \in H$ we let $\mathrm{Dom}(h) = \{\alpha^h_\zeta : \zeta < \theta\}$ be an increasing enumeration. Then we let

$$g(h) \stackrel{\text{def}}{=} \mathrm{Min}\{\xi < \theta : \varepsilon \geq \xi \Rightarrow g^\oplus(\alpha^h_\varepsilon) \geq h(\alpha^h_\varepsilon)\}.$$

Note: $g(h)$ is well defined by the choice of $g^\oplus$.
So, let $f_{(\beta,\bar{h})} \in F'$, and let $W = W(\beta, \bar{h})$. Let

$$A \stackrel{\text{def}}{=} \{\zeta < \theta : f_{(\beta,\bar{h})}(h_\zeta) > g(h_\zeta)\}.$$

If $\zeta \in A$, then $f_{(\beta,\bar{h})}(h_\zeta) > g(h_\zeta)$, so $g^\oplus(\alpha^{h_\zeta}_{f_{(\beta,\bar{h})}(h_\zeta)}) \geq h_\zeta(\alpha^{h_\zeta}_{f_{(\beta,\bar{h})}(h_\zeta)})$ by the definition of $g(h_\zeta)$. In other words, $g^\oplus(\beta) \geq \zeta$. Hence $A \subseteq g^\oplus(\beta) + 1$, and so $|A| < \theta$. $\square_{5.6}$

5.7 Claim. $\lambda \in SQd_{\theta,\sigma} \Rightarrow [\lambda, \lambda^\theta) \cap SP^1_{\theta,\sigma} \neq \emptyset$.

Proof. Let $H = \{h_j : j < \lambda\}$ exemplify that $\lambda \in SQd_{\theta,\sigma}$. Let us enumerate $\mathrm{Dom}(h_j) = \{\alpha^j_\zeta : \zeta < \theta\}$, as in clause (e) of 4.2(2). Hence

$$C\ell(h_j(\alpha^j_\zeta)) = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_1 < \theta \,\&\, \varepsilon_2 < \theta \,\&\, \neg[(\varepsilon_1, \varepsilon_2) \leq (\beta^j_\zeta, \gamma^j_\zeta)]\}$$

for some $\langle \gamma^j_\zeta : \zeta < \theta\rangle$ which is strictly increasing and $\gamma^j_\zeta > \bigcup_{\xi<\zeta} \beta^j_\xi$. Let $h^\oplus_j$ be the function with $\mathrm{Dom}(h^\oplus_j) = \{\alpha^j_\zeta : \zeta < \theta\}$ and defined by $h^\oplus_j(\alpha^j_\zeta) = \beta^j_\zeta + 1$. Let $H^\oplus = \{h^\oplus_j : j < \lambda\}$.

Suppose that $H^\oplus = \bigcup_{i<i(*)} H^\oplus_i$ for some $i(*) < 1 + \sigma$, and each $H^\oplus_i$ is free in the sense of P^1_θ, and let g_i $(i < i(*))$ exemplify this.

Hence for every $j < \lambda$ we have that $h_j \in H^\oplus_i \Rightarrow \{\zeta : h^\oplus_j(\alpha^j_\zeta) = \beta^j_\zeta + 1 > g_i(\alpha^j_\zeta)\}$ is bounded in θ. In particular, there is an ordinal $\xi < \theta$ such that

$$\zeta \in [\xi, \theta) \Rightarrow (g_i(\alpha^j_\zeta), g_i(\alpha^j_\zeta)) \in C\ell(h_j(\alpha^j_\zeta)).$$

This contradicts the assumption that H is not λ-free in the sense of $Qd_{\theta,\sigma}$.

Now suppose that $H' \subseteq H^\oplus$ with $|H'| < \lambda$. Let

$$H'' \stackrel{\text{def}}{=} \{h_j : h^\oplus_j \in H' \text{ (and } j < \lambda \text{ of course)}\},$$

hence $H'' \subseteq H$ with $|H''| < \lambda$. So $H'' = \bigcup_{i<i(*)} H''_i$ for some $i(*) < 1 + \sigma$, and each H''_i is free in the sense of Qd_θ.
Let this be exemplified by g_i, for $i < i(*)$.
For $i < i(*)$ let $H'_i \stackrel{\text{def}}{=} \{h^\oplus_j : h_j \in H''_i\}$, so $H' = \bigcup_{i<i(*)} H'_i$. Suppose $h^\oplus_j \in H'_i$. If $h^\oplus_j(\alpha^j_\zeta) > g_i(\alpha^j_\zeta)$, then $\beta^j_\zeta + 1 > g_i(\alpha^j_\zeta)$.
Let $\xi_j < \theta$ be such that

$$(\forall \alpha \in \mathrm{Dom}(h_j))[(g_i(\alpha), \xi_j) \in C\ell(h_j(\alpha))]$$

Let $\xi < \theta$ be such that $\zeta \geq \xi \Rightarrow \gamma^j_\zeta > \xi_j$. Hence $g_i(\alpha^j_\zeta) > \beta^j_\zeta$, so $g_i(\alpha^j_\zeta) \geq \beta^j_\zeta + 1 = h_j(j_\zeta)$. Hence H'_i is free in the sense of P^1_θ. $\square_{5.7}$

5.8 Remark.

We have now finally proved 4.2B(2) (i.e. $\lambda \in SQd_{\theta,\sigma} \Rightarrow [\lambda, \lambda^\theta] \cap SP_{\theta,\sigma} \neq \emptyset$), as:
By 5.7 $\lambda \in SQd_{\theta,\sigma} \Rightarrow [\lambda, \lambda^\theta] \cap SP^1_{\theta,\sigma} \neq \emptyset$.
By 5.2 $[\lambda, \lambda^\theta] \cap SP^1_{\theta,\sigma} \Rightarrow [\lambda, \lambda^\theta] \cap SP^4_{\theta,\sigma} \neq \emptyset$.
By 5.5 and 5.6, $[\lambda, \lambda^\theta] \cap SP^4_{\theta,\sigma} \neq \emptyset \Rightarrow [\lambda, \lambda^\theta] \cap SP^2_{\theta,\sigma} \neq \emptyset$.
By 4.1A(0), $[\lambda, \lambda^\theta] \cap SP_{\theta,\sigma} \neq \emptyset$.

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