

SPECIALIZING TREES AND ANSWER TO A QUESTION OF WILLIAMS

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ABSTRACT. We show that if $\text{cf}(2^{\aleph_0}) = \aleph_1$, then any non-trivial \aleph_1 -closed forcing notion of size $\leq 2^{\aleph_0}$ is forcing equivalent to $\text{Add}(\aleph_1, 1)$, the Cohen forcing for adding a new Cohen subset of ω_1 . We also produce, relative to the existence of suitable large cardinals, a model of ZFC in which $2^{\aleph_0} = \aleph_2$ and all \aleph_1 -closed forcing notion of size $\leq 2^{\aleph_0}$ collapse \aleph_2 , and hence are forcing equivalent to $\text{Add}(\aleph_1, 1)$. These results answer a question of Scott Williams from 1978. We also extend a result of Todorcevic and Foreman-Magidor-Shelah by showing that it is consistent that every partial order which adds a new subset of \aleph_2 , collapses \aleph_2 or \aleph_3 .

1. INTRODUCTION

For an infinite cardinal κ , let $\text{Add}(\kappa, 1)$ denote the Cohen forcing for adding a new Cohen subset of κ ; thus conditions in $\text{Add}(\kappa, 1)$ are partial functions $p : \kappa \rightarrow \{0, 1\}$ of size less than κ , ordered by reverse inclusion. The forcing is $\text{cf}(\kappa)$ -closed and satisfies $(2^{<\kappa})^+$ -c.c., in particular, if κ is regular and $2^{<\kappa} = \kappa$, then it preserves all cardinals.

It is well-known that if the continuum hypothesis, CH, holds, then any \aleph_1 -closed forcing notion of size continuum is forcing equivalent to $\text{Add}(\aleph_1, 1)$. In [19] (see also [20]), Scott Williams asked if the converse is also true, i.e., if CH follows from the assumption “any \aleph_1 -closed forcing notion of size continuum is forcing equivalent to the Cohen forcing $\text{Add}(\aleph_1, 1)$ ”. We will show that $\text{cf}(2^{\aleph_0}) = \aleph_1$ is sufficient to conclude that all \aleph_1 -closed forcing notions

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of size continuum are forcing equivalent to $\text{Add}(\aleph_1, 1)$. Since $\text{cf}(2^{\aleph_0}) = \aleph_1$ is consistent with $\neg\text{CH}$, this gives a negative answer to Williams question.

Theorem 1.1. *Assume $\text{cf}(2^{\aleph_0}) = \aleph_1$. Then any non-trivial \aleph_1 -closed forcing notion of size $\leq 2^{\aleph_0}$ is forcing equivalent to $\text{Add}(\aleph_1, 1)$.*

Remark 1.2. (1) *We can replace $\aleph_1, 2^{\aleph_0}$ by $\kappa = \mu^+, 2^\mu$ resp., with $\text{cf}(2^\mu) = \kappa$; or by $\kappa, 2^\mu$ resp., if κ is weakly inaccessible, $\mu < \kappa$, $2^\mu = 2^{<\kappa}$ and $\text{cf}(2^\mu) = \kappa$.*

(2) *If $2^{\aleph_0} = 2^{\aleph_1}$, then $\text{Add}(\aleph_2, 1)$ is \aleph_1 -closed of size continuum, but it is not forcing equivalent to $\text{Add}(\aleph_1, 1)$.*

On the other hand, it is not difficult to prove the consistency of “ $2^{\aleph_0} = \aleph_2$ and there exists a non-trivial \aleph_1 -closed (but not \aleph_2 -closed) forcing notion of size \aleph_2 which preserves all cardinals” (see [7]). So it is natural to ask if we can have the same result as in Theorem 1.1 with 2^{\aleph_0} being regular. We show that this is indeed the case, if we assume the existence of large cardinals.

Recall that an uncountable cardinal κ is *supercompact* if for every cardinal $\lambda > \kappa$ there exists a non-trivial elementary embedding $j : V \rightarrow M$ with critical point κ such that $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$. It is *2-Mahlo* if $\{\mu < \kappa : \mu \text{ is a Mahlo cardinal}\}$ is stationary in κ .

Theorem 1.3. *Assume κ is a supercompact cardinal and $\lambda > \kappa$ is a 2-Mahlo cardinal. Then there is a generic extension of the universe in which the following hold:*

(a) $2^{\aleph_0} = \kappa = \aleph_2$,

(b) $2^{\aleph_1} = \lambda = \aleph_3$,

(c) *Any \aleph_1 -closed forcing notion of size $\leq \aleph_2$ collapses \aleph_2 into \aleph_1 , in particular it is forcing equivalent to $\text{Add}(\aleph_1, 1)$.*

Following [4], let *Todorćevic’s maximality principle* be the assertion: “every partial order which adds a fresh subset of \aleph_1 , collapses \aleph_1 or \aleph_2 ”, where by a fresh subset of a cardinal κ we mean a subset of κ which is not in the ground model but all of its proper initial segments are in the ground model.

In [16], Todorćevic showed that if $2^{\aleph_0} = \aleph_2$ and every \aleph_1 -tree of size \aleph_1 is special, then Todorćevic’s maximality principle holds.

By results of Baumgartner [2] and Todorćevic [15], “ $2^{\aleph_0} = \aleph_{2+}$ every \aleph_1 -tree of size \aleph_1 is special” is consistent, and hence Todorćevic’s maximality principle is consistent as well. On the other hand, Foreman-Magidor-Shelah [10] showed that PFA implies the same conclusion. In [18], Viale and Weiss introduced the principle GMP (guessing model principle) and showed that it follows from PFA. Cox and Krueger [3], introduced the stronger principle IGMP (indestructible guessing model principle) and showed that PFA implies IGMP which in turn implies Todorćevic’s maximality principle. On the other hand, in [4], they showed that Todorćevic’s maximality principle does not follow from GMP.

We extend the above result of Todorćevic to higher cardinals, and prove the following theorem.

Theorem 1.4. *Assume κ is a supercompact cardinal and $\lambda > \kappa$ is a 2-Mahlo cardinal. Then there is a generic extension of the universe in which the following hold:*

- (a) $2^{\aleph_0} = \aleph_1$,
- (b) $\kappa = \aleph_2$,
- (c) $2^{\aleph_1} = \lambda = \aleph_3$,
- (d) *Every partial order which adds a fresh subset of \aleph_2 , collapses \aleph_2 or \aleph_3 .*

Remark 1.5. *In theorems 1.3 and 1.4, we can replace the cardinals \aleph_0, \aleph_1 and \aleph_2 by the cardinals η, η^+ and η^{++} respectively, where η is a regular cardinal less than κ .*

The above result is related to Foreman’s maximality principle [9], which asserts that any non-trivial forcing notion either adds a new real or collapses some cardinals.

The paper is organized as follows. In section 2 we prove Theorem 1.1. Sections 3 and 4 are devoted to some preliminary results which are then used in section 5 for the proof of Theorem 1.3. In the last section 6, we prove Theorem 1.4.

To avoid trivialities, by a forcing notion we always mean a non-trivial separative forcing notion. We use \simeq for the equivalence of forcing notions, so

$$\mathbb{P} \simeq \mathbb{Q} \Leftrightarrow RO(\mathbb{P}) \text{ is isomorphic to } RO(\mathbb{Q}),$$

where $RO(\mathbb{P})$ denotes the Boolean completion of \mathbb{P} . Also $\mathbb{P} \triangleleft \mathbb{Q}$ means that \mathbb{P} is a regular sub-forcing of \mathbb{Q} .

2. A NEGATIVE ANSWER TO WILLIAMS QUESTION WHEN THE CONTINUUM IS SINGULAR

In this section we prove Theorem 1.1. In [7] it is shown that if \mathbb{Q} is any \aleph_1 -closed forcing notion¹ of size $\leq 2^{\aleph_0}$ and if λ is the least cardinal such that forcing with \mathbb{Q} adds a fresh λ -sequence of ordinals, then forcing with \mathbb{Q} collapses 2^{\aleph_0} into λ , and hence, if in addition $\lambda = \aleph_1$, then $\mathbb{Q} \simeq \text{Add}(\aleph_1, 1)$. Thus to prove Theorem 1.1, it suffices to show that if $\text{cf}(2^{\aleph_0}) = \aleph_1$, then any \aleph_1 -closed forcing notion \mathbb{Q} of size at most 2^{\aleph_0} adds a fresh set of ordinals of size \aleph_1 . We give a direct proof of this fact which is of its own interest, and avoids the use of the results of [7].

If $2^{\aleph_0} = \aleph_1$, then the result is known to hold, so assume that $\aleph_1 < 2^{\aleph_0}$ and $\text{cf}(2^{\aleph_0}) = \aleph_1$. Let \mathbb{Q} be a non-trivial \aleph_1 -closed forcing notion of size $\leq 2^{\aleph_0}$. We are going to show that \mathbb{Q} is forcing equivalent to $\text{Add}(\aleph_1, 1)$.

Notation 2.1. For a forcing notion \mathbb{P} and a condition $p \in \mathbb{P}$, let $\mathbb{P} \downarrow p$ denote the set of all conditions in \mathbb{P} which extend p ; i.e., $\mathbb{P} \downarrow p = \{q \in \mathbb{P} : q \leq_{\mathbb{P}} p\}$.

Let $\langle \mathbb{Q}_i : i < \omega_1 \rangle$ be a \subseteq -increasing and continuous sequence of subsets of \mathbb{Q} such that $\mathbb{Q}_0 = \emptyset$, for all $i < \omega_1$, $|\mathbb{Q}_i| < 2^{\aleph_0}$, and $\mathbb{Q} = \bigcup_{i < \omega_1} \mathbb{Q}_i$.

Lemma 2.2. For every $i < \omega_1$ and every $p \in \mathbb{Q}$, there exists $q \leq_{\mathbb{Q}} p$ such that there is no $r \in \mathbb{Q}_i$ with $r \leq q$.

Proof. Let A be a maximal antichain in \mathbb{Q} below p of size 2^{\aleph_0} , which exists as \mathbb{Q} is non-trivial and \aleph_1 -closed. As $|\mathbb{Q}_i| < 2^{\aleph_0}$, we can find $q \in A$ such that $(\mathbb{Q} \downarrow q) \cap \mathbb{Q}_i = \emptyset$. Then q is as required. \square

We now define by induction on $i < \omega_1$ a sequence \bar{p}_i such that:

- (1) $\bar{p}_i = \langle \bar{p}_i(\eta) : \eta \in ({}^{i+1}(2^{\aleph_0})) \rangle$ is a maximal antichain in \mathbb{Q} ,
- (2) If $j < i$ and $\eta \in ({}^{i+1}(2^{\aleph_0}))$, then $\bar{p}_i(\eta) \leq_{\mathbb{Q}} \bar{p}_j(\eta \upharpoonright (j+1))$,
- (3) If $\eta \in ({}^{i+1}(2^{\aleph_0}))$, then there is no member of \mathbb{Q}_i which is below $\bar{p}_i(\eta)$.

$i = 0$: Let $\bar{p}_0 = \langle \bar{p}_0(\eta) : \eta \in ({}^1(2^{\aleph_0})) \rangle$ be any maximal antichain in \mathbb{Q} . Note that clauses

(2) and (3) above are vacuous as \mathbb{Q}_0 is empty.

¹In fact being $\omega + 1$ -strategically closed is sufficient

$i > 0$: For every $\nu \in ({}^i 2^{\aleph_0})$ set $\bar{p}_{i,\nu}^1 = \langle \bar{p}_j(\nu \upharpoonright (j+1)) : j < i \rangle$. Then, by the induction hypothesis, $\bar{p}_{i,\nu}^1$ is a countable decreasing sequence of conditions in \mathbb{Q} , and so the set

$$\mathbb{P}_{i,\nu}^2 = \{q \in \mathbb{Q} : j < i \Rightarrow q \leq \bar{p}_j(\nu \upharpoonright (j+1))\}$$

is non-empty. Let

$$\mathbb{P}_{i,\nu}^3 = \{q \in \mathbb{P}_{i,\nu}^2 : \forall z \in \mathbb{Q}[z \not\leq_{\mathbb{Q}} q \text{ and moreover } z \Vdash \text{“}q \in \dot{G}_{\mathbb{Q}}\text{”}]\}.$$

$\mathbb{P}_{i,\nu}^3$ is easily seen to be a dense subset of $\mathbb{P}_{i,\nu}^2$, hence we can find a maximal antichain, say $\bar{P}_{i,\nu} = \{p_{i,\nu}(\alpha) : \alpha < 2^{\aleph_0}\}$, in it. For $\eta \in ({}^{i+1} 2^{\aleph_0})$ set $\bar{p}_i(\eta) = p_{i,\eta \upharpoonright i}(\eta(i))$. Then it is easily seen that $\bar{p}_i = \langle \bar{p}_i(\eta) : \eta \in ({}^{i+1} 2^{\aleph_0}) \rangle$ is as required.

Let $\check{\nu} \in V^{\mathbb{Q}}$ be the \mathbb{Q} -name

$$\check{\nu} = \{\langle \langle \check{j}, \check{\eta}(\check{j}) \rangle, \bar{p}_i(\eta) \rangle : j \leq i < \omega_1 \text{ and } \eta \in ({}^{i+1} 2^{\aleph_0})\}.$$

Claim 2.3. (a) For every $i < \omega_1$ and $\eta \in ({}^{i+1} 2^{\aleph_0})$, $\bar{p}_i(\eta) \Vdash \text{“}\eta = \check{\nu} \upharpoonright i + 1\text{”}$.

(b) $1_{\mathbb{Q}} \Vdash \text{“}\check{\nu} \in ({}^{\omega_1} 2^{\aleph_0})\text{”}$.

Proof. (a) is clear from the definition of $\check{\nu}$.

(b) Let G be \mathbb{Q} -generic over V . Then for each $i < \omega_1$ we can find a unique $\eta_i \in ({}^{i+1} 2^{\aleph_0})$ such that $\bar{p}_i(\eta_i) \in G$. If $j < i$, then $\bar{p}_i(\eta_i) \leq_{\mathbb{Q}} \bar{p}_j(\eta_i \upharpoonright (j+1))$, and so $\eta_j = \eta_i \upharpoonright j + 1$. It then immediately follows that

$$\check{\nu}[G] = \{(i, \eta_i(j)) : j \leq i < \omega_1\} = \bigcup_{i < \omega_1} \{\eta \in ({}^{i+1} 2^{\aleph_0}) : \bar{p}_i(\eta) \in G\}$$

is a function from ω_1 into 2^{\aleph_0} . □

We now define a \mathbb{Q} -name $\check{\tau}$ for a function from ω_1 into 2 as follows: let $\langle \rho_\alpha : \alpha < 2^{\aleph_0} \rangle$ be an enumeration of ${}^\omega 2$ with no repetitions. Then let $\check{\tau}$ be such that

$$\Vdash_{\mathbb{Q}} \check{\tau}(\omega \cdot i + n) = \rho_{\check{\nu}(i)}(n).$$

Lemma 2.4. $\Vdash_{\mathbb{Q}} \text{“}\check{\tau} \in ({}^{\omega_1} 2) \text{ and } \check{\tau} \notin \check{V}\text{”}$.

Proof. Let $q_1 \in \mathbb{Q}$. Then for some $i < \omega_1$, $q_1 \in \mathbb{Q}_i$. Since $\langle \bar{p}_i(\eta) : \eta \in ({}^{i+1} 2^{\aleph_0}) \rangle$ is a maximal antichain, we can find $\eta \in ({}^{i+1} 2^{\aleph_0})$ such that q_1 is compatible with $\bar{p}_i(\eta)$. But $q_1 \Vdash \text{“}\bar{p}_i(\eta) \in \dot{G}_{\mathbb{Q}}\text{”}$, so there is $q_2 \leq q_1$ such that q_2 is incompatible with $\bar{p}_i(\eta)$. But again as

$\langle \bar{p}_i(\eta) : \eta \in ({}^{i+1}2^{\aleph_0}) \rangle$ is a maximal antichain, there exists $\rho \in ({}^{i+1}2^{\aleph_0})$ such that q_2 and $\bar{p}_i(\rho)$ are compatible. Let $q_3 \leq q_2, \bar{p}_i(\rho)$, and let $j \leq i$ be maximal such that $\eta \upharpoonright j = \rho \upharpoonright j$ and $\eta \upharpoonright (j+1) \neq \rho \upharpoonright (j+1)$. Then $q_3, \bar{p}_i(\eta)$ are compatible with q_1 , but they force contradictory information about $\mathcal{T} \upharpoonright [\omega \cdot j, \omega \cdot j + \omega)$. The result follows immediately. \square

Lemma 2.5. *There exists a dense subset \mathbb{Q}' of \mathbb{Q} which is the union of \aleph_1 -many maximal antichains $\langle I_i^* : i < \omega_1 \rangle$ of \mathbb{Q} .*

Proof. For any $p \in \mathbb{Q}$, by the previous lemma, p does not force any value for \mathcal{T} , hence there are ordinal $i < \omega_1$ and conditions $p_0, p_1 \leq p$ such that $p_l \Vdash \mathcal{T}(i) = l$, $l = 0, 1$. Hence we can define by recursion a sequence

$$\langle \langle q_{p,\eta}, i_{p,\eta}, \sigma_{p,\eta} \rangle : \eta \in ({}^{<\omega}2) \rangle$$

such that:

- (4) $q_{p, \langle \rangle} = p$,
- (5) $\nu \triangleleft \eta \Rightarrow q_{p,\eta} \leq q_{p,\nu}$,
- (6) $i_{p,\eta}$ is the least ordinal i less than ω_1 such that $q_{p,\eta}$ does not decide $\mathcal{T}(i)$,
- (7) $q_{p,\eta} \Vdash \forall j < i_{p,\eta}, \mathcal{T}(i_{p,\eta} \upharpoonright j) = \sigma_{p,\eta}(j)$.

It is evident that if $\nu \triangleleft \eta$, then $i_{p,\nu} < i_{p,\eta}$.

Claim 2.6. *For any $p \in \mathbb{Q}$, there exists a perfect subtree T_p of ${}^\omega 2$ such that for some limit ordinal δ_p and every $\rho \in \text{Lim}(T_p), \bigcup_n i_{p,\rho \upharpoonright n} = \delta_p$, where $\text{Lim}(T_p)$ is the set of all branches through T_p .*

Proof. For any $\eta \in ({}^{<\omega}2)$ set

$$\delta_{p,\eta} = \sup\{i_{p,\nu} : \eta \triangleleft \nu \in ({}^{<\omega}2)\}.$$

For some η_* , the ordinal δ_{p,η_*} is minimal. δ_{p,η_*} is a limit ordinal of cofinality \aleph_0 , so let $\langle \eta_{p,m} : m < \omega \rangle$ be an increasing sequence cofinal in δ_{p,η_*} such that $\eta_{p,0} = \text{lh}(\eta_*)$. We define $h_m : {}^m 2 \rightarrow \eta_{p,m} 2$, by induction on $m < \omega$, such that:

- (1) h_m is 1-1,
- (2) $h_0(\langle \rangle) = \eta_*$,
- (3) If $n < m$ and $\eta \in ({}^m 2)$, then $h_n(\eta \upharpoonright n) \triangleleft h_m(\eta)$,

(4) If $\eta \in ({}^m 2)$, then $i_{p, h_m(\eta)} > \eta_{p, m}$.

Then $T_p = \{h_m(\eta) : m < \omega \text{ and } \eta \in ({}^m 2)\}$ and $\delta_p = \delta_{p, \eta^*}$ are as required. \square

For each limit ordinal $\delta < \omega_1$ set

$$I_\delta^1 = \{p \in \mathbb{Q} : \delta_p = \delta\}.$$

Then clearly $\mathbb{Q} = \bigcup \{I_\delta^1 : \delta \text{ is a limit ordinal less than } \omega_1\}$.

Claim 2.7. *Let δ be a countable limit ordinal. Then there exists an antichain $\bar{q}^\delta = \langle q_p^\delta : p \in I_\delta^1 \rangle$ such that for each $p \in I_\delta^1$, $q_p^\delta \leq p$.*

Proof. Let $\langle p_\alpha : \alpha < \alpha_\delta \leq 2^{\aleph_0} \rangle$ enumerate I_δ^1 . We choose, by induction on α , a pair $\langle r_\alpha, v_\alpha \rangle$ such that

$$(8) \quad r_\alpha \leq p_\alpha, \text{ and } v_\alpha \in ({}^\delta 2),$$

$$(9) \quad r_\alpha \Vdash \mathcal{T} \upharpoonright \delta_{p_\alpha} = v_\alpha,$$

$$(10) \quad \alpha \neq \beta \Rightarrow v_\alpha \neq v_\beta.$$

Suppose $\alpha < \alpha_\delta$ and we have defined $\langle r_\beta, v_\beta \rangle$ for all $\beta < \alpha$ as above. We define $\langle r_\alpha, v_\alpha \rangle$.

For every $\rho \in \text{Lim}(T_{p_\alpha})$, the sequence $\langle q_{p, \rho \upharpoonright n} : n < \omega \rangle$ is a decreasing chain of conditions in \mathbb{Q} , and hence there is a condition $q_{\rho, \alpha}^*$ which extends all of them. We may further suppose that it forces a value $v_{\rho, \alpha}$ for $\mathcal{T} \upharpoonright \delta$, where $\delta = \delta_{p_\alpha}$. Also note that by the choice of $\langle q_{\rho, \alpha} : \rho \in ({}^{<\omega} 2) \rangle$, for $\rho_1 \neq \rho_2$ in $\text{Lim}(T_{p_\alpha})$, we have $v_{\rho_1, \alpha} \neq v_{\rho_2, \alpha}$. Now $\{v_\beta : \beta < \alpha\} \subseteq {}^\delta 2$, hence for some $\rho = \rho_\alpha \in \text{Lim}(p_\alpha)$ we have that $v_{\rho, \alpha} \notin \{v_\beta : \beta < \alpha\}$. Let $r_\alpha = q_{\rho_\alpha, \alpha}$ and $v_\alpha = v_{\rho_\alpha, \alpha}$. \square

Now for each limit ordinal $\delta < \omega_1$ let I_δ be a maximal antichain of \mathbb{Q} , such that $I_\delta \supseteq \{q_{p, \delta} : p \in I_\delta^1\}$, and let $\mathbb{Q}' = \bigcup \{I_\delta : \delta \text{ is a countable limit ordinal}\}$. Then \mathbb{Q}' is as required which completes the proof of Lemma 2.5. \square

As each I_i^* is a maximal antichain in \mathbb{Q}' and hence also in \mathbb{Q} , it can easily be seen that there are $\bar{p}_i^*, i < \omega_1$, such that

$$(11) \quad \bar{p}_i^* = \langle p_\eta^* : \eta \in ({}^{i+1} 2^{\aleph_0}) \rangle \text{ is a maximal antichain of } \mathbb{Q}' \text{ (and hence of } \mathbb{Q}),$$

$$(12) \quad \text{If } j < i \text{ and } \eta \in ({}^{i+1} 2^{\aleph_0}), \text{ then } p_\eta^* \leq p_{\eta \upharpoonright (j+1)}^*,$$

$$(13) \quad \text{If } i = j + 1 \text{ and } \eta \in ({}^{i+1} 2^{\aleph_0}), \text{ then } p_\eta^* \text{ is stronger than some condition in } I_i^*.$$

Let

$$\mathbb{Q}'' = \{p_\eta^* : \exists i < \omega_1, \eta \in ({}^{i+1}(2^{\aleph_0}))\}.$$

Lemma 2.8. \mathbb{Q}'' is a dense subset of \mathbb{Q} .

Proof. Let $p \in \mathbb{Q}$. By Lemma 2.5, we can find some $i < \omega_1$ and some $p_1 \in I_i^* \subseteq \mathbb{Q}'$ such that $p_1 \leq p$. By (13), each $p_\eta^*, \eta \in ({}^{i+1}(2^{\aleph_0}))$, is stronger than some condition in I_i^* . If there is no η with $p_\eta^* \leq p_1$, then we contradict with (11). The result follows immediately. \square

Finally note that the map

$$\eta \mapsto p_\eta^*$$

defines an isomorphism between a dense subset of $\text{Col}(\aleph_1, 2^{\aleph_0})$ and \mathbb{Q}'' . It follows that

$$\mathbb{Q} \simeq \mathbb{Q}'' \simeq \text{Col}(\aleph_1, 2^{\aleph_0}) \simeq \text{Add}(\aleph_1, 1).$$

The theorem follows. \square

3. A NOTE ON \aleph_1 -CLOSED FORCING NOTIONS OF SIZE CONTINUUM

In this section we present a result about \aleph_1 -closed forcing notions of size continuum which will be used in section 5 for the proof of Theorem 1.3.

Assume that $2^{\aleph_0} = \aleph_2$ and that \mathbb{R} is an \aleph_1 -closed forcing notions of size continuum which does not collapse \aleph_2 . It then follows from [7] that the forcing notion \mathbb{R} does not add a fresh sequence of ordinals of size \aleph_1 and hence it is \aleph_2 -distributive. The following result is proved in [1] Theorem 2.1.

Lemma 3.1. *There exists a sequence $\langle T_\alpha : \alpha < \aleph_2 \rangle$ of subsets of \mathbb{R} such that:*

- (1) *Each T_α is a maximal antichain in \mathbb{R} ,*
- (2) *If $T = \bigcup \{T_\alpha : \alpha < \aleph_2\}$, then $(T, \geq_{\mathbb{R}})$ is a tree of height \aleph_2 , where T_α is the α -th level of T ,*
- (3) *Each $t \in T$ has \aleph_2 -many immediate successors,*
- (4) *T is dense in \mathbb{R} .*

We denote the above tree T by $T(\mathbb{R})$, and call it a base tree of \mathbb{R} . Note that by clause (4), $\mathbb{R} \simeq T(\mathbb{R})$.

4. SPECIALIZING \aleph_2 -TREES WHICH HAVE FEW BRANCHES

In this section we consider trees of size and height κ which have $\leq \kappa$ -many branches, and define a suitable forcing notion for specializing them. As we allow our trees to have cofinal branches, we need a slightly different definition of the concept of a special tree than the usual ones.

Definition 4.1. *Let $\kappa = \varrho^+$, where ϱ is a regular cardinal.*

- (1) *A κ -tree is a tree of height and size κ (so we allow the levels of the tree to have size κ).*
- (2) *([2]) Let T be a κ -tree. T is special if there exists a function $F : T \rightarrow \varrho$ such that for all $x, y, z \in T$ if $x \leq_T y, z$ and $F(x) = F(y) = F(z)$, then either $y \leq_T z$ or $z \leq_T y$.*

By [2, Theorem 8.1], a κ -special tree has at most κ -many cofinal branches.

Let $\kappa = \varrho^+$, where ϱ is a regular cardinal. Let also $\theta > \kappa$ be large enough regular and let $<$ be a well-ordering of $H(\theta)$. Let Λ denote the set of all κ -trees T with at most κ -many cofinal branches, such that for all $t \in T$, $Suc_T(t)$, the set of successors of t in T , has size κ .

We define a map $\star : \Lambda \rightarrow \Lambda$, where to each $T \in \Lambda$, assigns a subtree $T^\star = \star(T)$ of T , such that T^\star is dense in T and it has no cofinal branches. Thus let $T \in \Lambda$. Let $\langle b_\alpha : \alpha < \kappa \rangle$ be the $<$ -least enumeration of the cofinal branches through T , and for each $\alpha < \kappa$ set

$$s_\alpha = \text{the } \leq_T\text{-least element of } b_\alpha \setminus \bigcup_{\beta < \alpha} b_\beta.$$

Finally set

$$T^\star = \{t \in T : \neg(\exists \alpha, s_\alpha <_T t \in b_\alpha)\}.$$

Lemma 4.2. (a) *T^\star has no cofinal branches.*

(b) *(T^\star, \geq_T) is dense in (T, \geq_T) (when considered as forcing notions), in particular $(T^\star, \geq_T) \simeq (T, \geq_T)$.*

Proof. (a) Assume not, and let b be a branch through T^\star . then for some $\alpha, b \subseteq b_\alpha$, and then clearly $b \cap (T \setminus T^\star) \neq \emptyset$, which is a contradiction.

(b) Let $t \in T$. If $t \in T^\star$, then we are done; so assume that $t \notin T^\star$. Then for some $\alpha < \kappa, s_\alpha <_T t \in b_\alpha$. Let $t' \in Suc_T(t) \setminus \bigcup_{\beta \leq \alpha} b_\beta$. Then $t' \in T^\star$ and $t' \geq_T t$. \square

Lemma 4.3. *Assume there exists $F : T^* \rightarrow \varrho$ such that if $F(x) = F(y)$, then x and y are incomparable in T . Then there exists $F' : T \rightarrow \varrho$ such that $F' \supseteq F$ and F' specializes T .*

Proof. Define $G : (T \setminus T^*) \rightarrow \varrho$ as follows: Let $t \in (T \setminus T^*)$. Then for some $\alpha < \kappa$, $s_\alpha <_T t \in b_\alpha$. Set $G(t) = F(s_\alpha)$. It is now easily seen that $F' = F \cup G$ is as required. \square

Thus, in order to define a forcing notion which specializes T , it suffices to define a forcing notion which adds a function $F : T^* \rightarrow \varrho$ as in Lemma 4.3.

Definition 4.4. *The forcing notion $\mathbb{Q}(T^*)$, for specializing T^* , is defined as follows:*

(a) *A condition in $\mathbb{Q}(T^*)$ is a partial function $f : T^* \rightarrow \varrho$ such that:*

(1) *$\text{dom}(f)$ has size $< \varrho$,*

(2) *If $x <_T y$ and $x, y \in \text{dom}(f)$ then $f(x) \neq f(y)$.*

(b) *$f \leq_{\mathbb{Q}(T^*)} g$ iff $f \supseteq g$.*

It is clear that the forcing notion $\mathbb{Q}(T^*)$ is ϱ -directed closed. But in general, there is no guarantee that the forcing $\mathbb{Q}(T^*)$ satisfies the κ -c.c., or preserves all cardinals, even if we assume GCH (see [5] and [14]).

Lemma 4.5. *Forcing with $\mathbb{Q}(T^*) * T$ collapses κ into ϱ .*

Proof. By Lemma 4.2(b), $\mathbb{Q}(T^*) * T \simeq \mathbb{Q}(T^*) * T^*$. Let G be $\mathbb{Q}(T^*)$ -generic over V and H be T^* -generic over $V[G]$. Let also $F = \bigcup \{f : f \in G\}$. Then $F : T^* \rightarrow \varrho$ and for all $x <_T y$ in T^* we have $F(x) \neq F(y)$. Let $b \in V[G * H]$ be a cofinal branch in T^* . Then $F \upharpoonright b : b \rightarrow \varrho$ is an injection, and $|b| = \kappa$. Hence κ is collapsed into ϱ . \square

Given an infinite cardinal κ , let $\text{Add}(\aleph_0, \kappa)$ denote the Cohen forcing for adding κ -many new Cohen reals; thus conditions are finite partial functions $p : \kappa \times \omega \rightarrow \{0, 1\}$ ordered by reverse inclusion. The forcing is c.c.c., and hence it preserves all cardinals and cofinalities.

For our purpose in the next section, we will work with $\text{Add}(\aleph_0, \kappa)$ -names of trees, and we now modify the above results to cover this case.

Assume $\kappa = \varrho^+$, where ϱ is a regular cardinal and let \mathcal{T} be an $\text{Add}(\aleph_0, \kappa)$ -name for a κ -tree which is forced to have $\leq \kappa$ -many cofinal branches. Let \mathcal{T}^* be an $\text{Add}(\aleph_0, \kappa)$ -name such that it is forced by $\text{Add}(\aleph_0, \kappa)$ that “ \mathcal{T}^* is the subtree of \mathcal{T} defined using the function

\star ". We assume, without loss of generality, that it is forced by $\text{Add}(\aleph_0, \kappa)$ that "the set of nodes of \mathcal{T}^\star is $\kappa \times \kappa$ and for each $\alpha < \kappa$, the α -th level of \mathcal{T}^\star is $\{\alpha\} \times \kappa$ ". We now define $\mathbb{Q}_A(\mathcal{T}^\star) \in V$ as follows:

Definition 4.6. (a) *A condition in $\mathbb{Q}_A(\mathcal{T}^\star)$ is a partial function $f : \kappa \times \kappa \rightarrow \varrho$ such that:*

- (1) *$\text{dom}(f)$ is a subset of $\kappa \times \kappa$ of size $< \varrho$.*
- (2) *If $x, y \in \text{dom}(f)$ and $f(x) = f(y)$, then $\Vdash_{\text{Add}(\aleph_0, \kappa)}$ "x and y are incompatible in the tree ordering, $x \perp y$ ".*

- (b) *$f \leq_{\mathbb{Q}_A(\mathcal{T}^\star)} g$ iff $f \supseteq g$.*

Note that we defined the forcing notion $\mathbb{Q}_A(\mathcal{T}^\star)$ in V and not in the generic extension by $\text{Add}(\aleph_0, \kappa)$.

Lemma 4.7. (a) *$\mathbb{Q}_A(\mathcal{T}^\star)$ is ϱ -directed closed.*

- (b) *Let G be $\mathbb{Q}_A(\mathcal{T}^\star)$ -generic over V . Then in $V[G]$, there exists a function $F : \kappa \times \kappa \rightarrow \varrho$, such that for any H which is $\text{Add}(\aleph_0, \kappa)$ -generic over $V[G]$, F is a specializing function for $\mathcal{T}^\star[H]$.*

The next lemma can be proved as in Lemma 4.5.

Lemma 4.8. *Let \mathcal{T} be an $\text{Add}(\aleph_0, \kappa)$ -name for a κ -tree which has $\leq \kappa$ -many cofinal branches. Then*

$$\Vdash_{\mathbb{Q}_A(\mathcal{T}^\star) * \text{Add}(\aleph_0, \kappa)} \text{"forcing with } \mathcal{T} \text{ collapses } \kappa\text{"}.$$

Proof. By Lemma 4.2(b),

$$(\mathbb{Q}_A(\mathcal{T}^\star) * \text{Add}(\aleph_0, \kappa)) * \mathcal{T} \simeq (\mathbb{Q}_A(\mathcal{T}^\star) * \text{Add}(\aleph_0, \kappa)) * \mathcal{T}^\star,$$

and hence it suffices to show that

$$\Vdash_{\mathbb{Q}_A(\mathcal{T}^\star) * \text{Add}(\aleph_0, \kappa)} \text{"forcing with } \mathcal{T}^\star \text{ collapses } \kappa\text{"}.$$

Let $(G_1 * G_2) * H$ be $(\mathbb{Q}_A(\mathcal{T}^\star) * \text{Add}(\aleph_0, \kappa)) * \mathcal{T}^\star$ -generic over V and $F = \bigcup \{f : f \in G_1\}$.

By Lemma 4.7, $F : \kappa \times \kappa \rightarrow \varrho$, and if $T^\star = \mathcal{T}^\star[G_2]$, then for $x <_{T^\star} y$, $F(x) \neq F(y)$.

Let $b \in V[(G_1 * G_2) * H]$ be a cofinal branch of T^\star . Then $F \upharpoonright b : b \rightarrow \varrho$ is an injection, and hence κ is collapsed into ϱ . □

5. A NEGATIVE ANSWER TO WILLIAMS QUESTION WHEN THE CONTINUUM IS REGULAR

In this section we prove Theorem 1.3. In Subsection 5.1 we define the main forcing construction \mathbb{P} and prove some of its basic properties. In Subsection 5.2 we show that forcing with \mathbb{P} preserves κ . Then in Subsection 5.3 more properties of the forcing notion \mathbb{P} are proved and finally in subsection 5.4 we complete the proof of Theorem 1.3.

5.1. The main forcing construction and its basic properties. In this subsection we define the main forcing notion that will be used in the proof of Theorem 1.3. Let κ be a supercompact cardinal, and let $\lambda > \kappa$ be a 2-Mahlo cardinal. By [12], We may assume that κ is Laver indestructible, i.e., the supercompactness of κ is preserved under κ -directed closed forcing notions, and that GCH holds at and above κ .

Let $\Phi : \lambda \rightarrow H(\lambda)$ be such that for each $x \in H(\lambda)$, $\Phi^{-1}(x) \cap \{\beta + 2 : \beta \text{ is Mahlo}\}$ is unbounded in λ . Such a Φ exists as $|H(\lambda)| = 2^{<\lambda} = \lambda$ and λ is a 2-Mahlo cardinal.

We define, by induction on $\alpha \leq \lambda$, an iteration

$$\mathbb{P} = \langle \langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle, \langle \mathbb{Q}_\alpha : \alpha < \lambda \rangle \rangle$$

of forcing notions of length λ . Suppose $\alpha \leq \lambda$ and we have defined \mathbb{P}_β , for all $\beta < \alpha$. We define \mathbb{P}_α as follows.

Definition 5.1. *A condition p is in \mathbb{P}_α , if and only if p is a function with domain α such that for every $\beta < \alpha$, $\Vdash_\beta "p(\beta) \in \mathbb{Q}_\beta"$, where:*

- (1) $\text{supp}(p)$ has size less than κ , where $\text{supp}(p)$ denotes the the support of p .
- (2) $\{\beta \in \text{supp}(p) : \beta \equiv 0 \pmod{3} \text{ or } \beta \equiv 2 \pmod{3}\}$ has size less than \aleph_1 .
- (3) If $\beta < \kappa$ and $\beta \equiv 0 \pmod{3}$ or $\beta \equiv 2 \pmod{3}$, then $\Vdash_\beta " \mathbb{Q}_\beta = \text{Col}(\aleph_1, \aleph_2 + |\beta|) "$.
- (4) If $\beta \geq \kappa$, $\beta \equiv 0 \pmod{3}$ and β is inaccessible, then $\Vdash_\beta " \mathbb{Q}_\beta = \text{Add}(\aleph_1, \kappa) "$.
- (5) If $\beta \geq \kappa$, $\beta \equiv 1 \pmod{3}$ and $\beta-1$ is inaccessible, then $\Vdash_\beta " \mathbb{Q}_\beta = \text{Col}(\kappa, 2^{|\mathbb{P}_\beta|}) = \text{Col}(\kappa, 2^{|\beta|}) "$ (as $|\mathbb{P}_\beta| = |\beta|$).

- (6) If $\beta \geq \kappa$, $\beta \equiv 2 \pmod{3}$, $\beta-2$ is inaccessible and if $\Vdash_\beta \text{“}\kappa = \aleph_2\text{”}$ and $\Phi(\beta)$ is a $\mathbb{P}_\beta * \text{Add}(\aleph_0, \kappa)$ -name for a κ -tree which has $\leq \kappa$ -many cofinal branches, then $\Vdash_\beta \text{“}\mathbb{Q}_\beta = \mathbb{Q}_A(\Phi(\beta)^*)\text{”}$.²
- (7) Otherwise, $\Vdash_\beta \text{“}\mathbb{Q}_\beta \text{ is the trivial forcing notion”}$.

Also set $\mathbb{P} = \mathbb{P}_\lambda$.

The next lemma gives some basic properties of the forcing notion \mathbb{P} .

Lemma 5.2. (a) \mathbb{P} is \aleph_1 -directed closed, and hence it preserves CH.

- (b) If $\mu \in (\kappa, \lambda]$ is Mahlo, then \mathbb{P}_μ satisfies the μ -c.c.
- (c) \mathbb{P}_λ collapses all cardinals in (\aleph_1, κ) into \aleph_1 , so if κ is not collapsed, then $\Vdash_{\mathbb{P}} \text{“}\kappa = \aleph_2\text{”}$.
- (d) In $V^{\mathbb{P}}$, λ is preserved, but all $\mu \in (\kappa, \lambda)$ are collapsed, so if κ is not collapsed, then $\Vdash_{\mathbb{P}} \text{“}\lambda = \kappa^+ = \aleph_3\text{”}$.
- (e) $\Vdash_{\mathbb{P}} \text{“}2^{\aleph_1} = \lambda\text{”}$.

Proof. (a) is clear as all forcing notions considered in the iteration are \aleph_1 -directed closed and the support of the iteration is at least countable.

(b) Assume $A \subseteq \mathbb{P}_\mu$ is a maximal antichain of size μ and let $\langle p^\xi : \xi < \mu \rangle$ be an enumeration of A . Define $F : \mu \rightarrow \mu$ by $F(\xi) =$ the least η such that $\text{supp}(p^\xi) \upharpoonright \xi \subseteq \eta$. F is a regressive function on the stationary set $X = \{\xi < \mu : \xi \text{ is inaccessible}\}$, and hence F is constant on some stationary subset Y of X . Let η be the resulting fixed value. So for all $\xi \in Y$, $\text{supp}(p^\xi) \upharpoonright \xi \subseteq \eta$. We may further suppose that if $\xi_1 < \xi_2$ are in Y , then $\text{supp}(p^{\xi_1}) \subseteq \xi_2$.

As \mathbb{P}_η has size less than μ , there are $\xi_1 < \xi_2$ in Y such that $p^{\xi_1} \upharpoonright \eta$ is compatible with $p^{\xi_2} \upharpoonright \eta$. But then in fact p^{ξ_1} is compatible with p^{ξ_2} and we get a contradiction.

(c), (e) and the fact that forcing with \mathbb{P}_λ collapses all cardinals in (\aleph_1, κ) into \aleph_1 are clear and the rest of (d) follows from (b). The lemma follows. \square

5.2. Preservation of κ . In this subsection we show that forcing with \mathbb{P} preserves κ . Let

$$G = \langle \langle G_\alpha : \alpha \leq \lambda \rangle, \langle H_\alpha : \alpha < \lambda \rangle \rangle$$

² $\Phi(\beta)^*$ is defined from $\Phi(\beta)$ as in Section 4 using a fixed well-ordering of a large initial segment of the universe.

be \mathbb{P} -generic over V , i.e., each G_α is \mathbb{P}_α -generic over V and H_α is $\mathbb{Q}_\alpha[G_\alpha]$ -generic over $V[G_\alpha]$.

For each $\alpha \leq \lambda$, we define the forcing notion $\mathbb{P}_\alpha^U \in V$, the \mathbb{P}_α^U -name \mathbb{P}_α^C for a forcing notion, in such a way that:

- (a) There are projections $\chi_\alpha : \mathbb{P}_\alpha \rightarrow \mathbb{P}_\alpha^U$ and $\pi_\alpha : \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C \rightarrow \mathbb{P}_\alpha$.
- (b) If $G_\alpha^U = \chi_\alpha[G_\alpha]$, for $\alpha \leq \lambda$, then there exists a function $\Psi \in V[G_\lambda^U]$ such that for each ordinal $\alpha = \beta + 2 > \kappa$, where β is inaccessible, $\Psi(\alpha) \in V[G_\alpha^U]$ is a $\mathbb{P}_\alpha^C * \text{Add}(\aleph_0, \kappa)$ -name such that

$$\Phi(\alpha)[G_\alpha^U * H] = \Psi(\alpha)[H],$$

for any H which is $\mathbb{P}_\alpha^C * \text{Add}(\aleph_0, \kappa)$ -generic over $V[G_\alpha^U]$. Further, if $\Phi(\alpha)$ is a $\mathbb{P}_\alpha * \text{Add}(\aleph_0, \kappa)$ -name for a κ -tree with $\leq \kappa$ -many cofinal branches, then in $V[G_\lambda^U]$, $\Psi(\alpha)$ is a $\mathbb{P}_\alpha^C * \text{Add}(\aleph_0, \kappa)$ -name for a κ -tree with $\leq \kappa$ -many cofinal branches.

- (c) \mathbb{P}_α^U is κ -directed closed.
- (d) For every $\gamma \in [\alpha, \lambda]$, $\mathbb{P}_\gamma^U \Vdash \mathbb{P}_\alpha^C$ is \aleph_1 -directed closed".³

Let us first define the forcing notions \mathbb{P}_α^U and the corresponding projections χ_α . Let

$$U = \{\beta < \lambda : \beta \equiv 1 \pmod{3} \text{ and } \beta - 1 \text{ is inaccessible}\}.$$

For $\beta \in U$, let \mathbb{Q}_β^U be the term forcing, whose conditions are \mathbb{P}_β -names \underline{p} such that $\Vdash_{\mathbb{P}_\beta} \text{“}\underline{p} \in \text{Col}(\kappa, 2^{|\beta|})\text{”}$, ordered by $\underline{p} \leq_{\mathbb{Q}_\beta^U} \underline{q}$ iff $\Vdash_{\mathbb{P}_\beta} \text{“}\underline{p} \leq_{\text{Col}(\kappa, 2^{|\beta|})} \underline{q}\text{”}$. Then set \mathbb{P}_α^U be the $< \kappa$ -support product of the forcing notions \mathbb{Q}_β^U , where $\beta \in U \cap \alpha$. Then there is a natural projection $\chi_\alpha : \mathbb{P}_\alpha \rightarrow \mathbb{P}_\alpha^U$.

We now define, by induction on $\alpha \leq \lambda$, the \mathbb{P}_α^U -name \mathbb{P}_α^C and the corresponding projection $\pi_\alpha : \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C \rightarrow \mathbb{P}_\alpha$. We also inductively verify (d) along the way.

³It is also possible to define the forcing notions $\mathbb{P}_\alpha^U \in V$ and $\mathbb{P}_\alpha^C \in V[G_\alpha^U]$ directly, by setting

$$\mathbb{P}_\alpha^U = \{p \in \mathbb{P}_\alpha : \text{supp}(p) \subseteq \{\beta < \lambda : \beta \equiv 1 \pmod{3}\}\}$$

and

$$\mathbb{P}_\alpha^C = \{p \in \mathbb{P}_\alpha : \text{supp}(p) \subseteq \{\beta < \lambda : \beta \equiv 0 \pmod{3} \text{ or } \beta \equiv 2 \pmod{3}\} \text{ and } p \text{ is compatible with } G_\alpha^U\},$$

where $G_\alpha^U = G_\alpha \cap \mathbb{P}_\alpha^U$. We will give a more explicit definition for \mathbb{P}_α^C , that will be useful in the proof of Lemma 5.3.

- (1) $\alpha \leq \kappa$: Let \mathbb{P}_α^C to be $\mathbb{P}_\alpha / \chi_\alpha^{-1}[\dot{G}_\alpha^U]$. It is then clear that $\mathbb{P}_\alpha^C \simeq \mathbb{P}_\alpha$ and there exist a forcing isomorphism $\pi_\alpha : \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C \simeq \mathbb{P}_\alpha$. It is also clear that clause (d) holds.
- (2) $\alpha = \beta + 1 > \kappa$, $\beta \equiv 0 \pmod{3}$ and β is inaccessible: Then set

$$\Vdash_{\mathbb{P}_\alpha^U} \text{“} \mathbb{P}_\alpha^C \simeq \mathbb{P}_\beta^C * \text{Add}(\aleph_1, \kappa) \text{”}.$$

By the induction, there is a projection $\pi_\beta : \mathbb{P}_\beta^U * \mathbb{P}_\beta^C \rightarrow \mathbb{P}_\beta$. Since $\mathbb{P}_\alpha^U = \mathbb{P}_\beta^U$, and since by clause (d) we can regard each $\mathbb{P}_\beta^U * \mathbb{P}_\beta^C$ -name for an element of $\text{Add}(\aleph_1, \kappa)$ as a \mathbb{P}_β -name, this induces the projection $\pi_\alpha : \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C \rightarrow \mathbb{P}_\alpha$, which is defined by $\pi_\alpha(p, (q, \mathcal{L})) = (\pi_\beta(p, q), \mathcal{L})$. It is also clear that clause (d) holds in this case.

- (3) $\alpha = \beta + 1 > \kappa$, $\beta \equiv 1 \pmod{3}$ and $\beta-1$ is inaccessible: Let $\Vdash_{\mathbb{P}_\beta^U} \text{“} \mathbb{P}_\alpha^C = \mathbb{P}_\beta^C \text{”}$ ⁴. In this case $\mathbb{P}_\alpha^U = \mathbb{P}_\beta^U \times \mathbb{Q}_\beta^U$, and by the induction hypothesis, there is a projection $\pi_\beta : \mathbb{P}_\beta^U * \mathbb{P}_\beta^C \rightarrow \mathbb{P}_\beta$. This induces the projection $\pi_\alpha : \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C \rightarrow \mathbb{P}_\alpha$, which is defined by $\pi_\alpha((p, q), \mathcal{L}) = (\pi_\beta(p, \mathcal{L}), q)$. Again clause (d) is easily verified.

- (4) $\alpha = \beta + 1 > \kappa$, $\beta \geq \kappa$, $\beta \equiv 2 \pmod{3}$ and $\beta-2$ is inaccessible: Then we may assume that $\Phi(\beta)$ is a $\mathbb{P}_\beta * \text{Add}(\aleph_0, \kappa)$ -name for a κ -tree which has $\leq \kappa$ -many cofinal branches, as otherwise the forcing at stage β is the trivial forcing and the result follows from the induction hypothesis. By the induction hypothesis, we have projections χ_β , π_β , and so there exists $\Psi(\beta) \in V[G_\beta^U]$ as in clause (b) above. Set

$$\Vdash_{\mathbb{P}_\alpha^U} \text{“} \mathbb{P}_\alpha^C \simeq \mathbb{P}_\beta^C * \mathbb{Q}_A(\Psi(\beta)^*) \text{”}.$$

By the choice of $\Psi(\beta)$, it is clear that we have a natural projection $\pi_\alpha : \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C \rightarrow \mathbb{P}_\alpha$, which extends π_β . Clause (d) is easily verified in this case as well.

- (5) $\alpha = \beta + 1$ is not as above: Then set

$$\Vdash_{\mathbb{P}_\alpha^U} \text{“} \mathbb{P}_\alpha^C = \mathbb{P}_\beta^C \text{”}.$$

Set also $\pi_\alpha = \pi_\beta$. It is also clear that clause (d) holds.

- (6) α is a limit ordinal: Then set

$$\Vdash_{\mathbb{P}_\alpha^U} \text{“} \mathbb{P}_\alpha^C \text{ is the countable support iteration of } \langle \mathbb{P}_\beta^C : \beta < \alpha \rangle \text{”}.$$

Note that π_α can be defined in a uniform way from π_β 's, $\beta < \alpha$. To be more precise,

let $\pi_\alpha : \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C \rightarrow \mathbb{P}_\alpha$ be defined by

$$\pi_\alpha(p, \langle q_\beta : \beta < \alpha \rangle) = \langle \pi_\beta(p, q_\beta) : \beta < \alpha \rangle.$$

⁴Note that we can consider \mathbb{P}_β^C as a \mathbb{P}_α^U -name as well.

It is evident that $\pi_\alpha(1_{\mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C}) = 1_{\mathbb{P}_\alpha}$ and that π_α is order preserving. Now suppose $\langle p, \langle q_\beta : \beta < \alpha \rangle \rangle \in \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C$, $r \in \mathbb{P}_\alpha$ and suppose that $r \leq \pi_\alpha(p, \langle q_\beta : \beta < \alpha \rangle) = \langle \pi_\beta(p, q_\beta) : \beta < \alpha \rangle$. By 5.1(2), the set

$$S = \{\beta \in \text{supp}(r) : \beta \equiv 0 \pmod{3} \text{ or } \beta \equiv 2 \pmod{3}\}$$

is at most countable. By induction on $\beta \in S$, we can find $(\bar{p}_\beta, \bar{q}_\beta) \in \mathbb{P}_\beta^U * \mathbb{P}_\beta^C$ such that

$$(6-1) \quad \pi_\beta(\bar{p}_\beta, \bar{q}_\beta) \leq r \upharpoonright \beta.$$

$$(6-2) \quad \text{If } \beta_0 < \beta_1 \text{ are in } S, \text{ then } \bar{p}_{\beta_1} \leq \bar{p}_{\beta_0}.$$

Each \bar{p}_β , for $\beta \in S$, is in \mathbb{P}_α^U , and since it is κ -closed, we can find $\bar{p} \in \mathbb{P}_\alpha^U$ which extends all \bar{p}_β 's, $\beta \in S$. Then $(\bar{p}, \langle \bar{q}_\beta : \beta < \alpha \rangle) \in \mathbb{P}_\alpha^U * \mathbb{P}_\alpha^C$, where for $\beta \in \alpha \setminus S$, $\Vdash_{\mathbb{P}_\beta^U} \text{“}\bar{q}_\beta = 1_{\mathbb{P}_\beta^C}\text{”}$, and it satisfies

$$\pi_\alpha(\bar{p}, \langle \bar{q}_\beta : \beta \in S \rangle) = \langle \pi_\beta(\bar{p}, \bar{q}_\beta) : \beta < \alpha \rangle \leq r.$$

Clause (d) follows easily from the induction hypothesis and the fact that \mathbb{P}_β^U 's are κ -directed closed and hence κ -distributive.

Lemma 5.3. *For every $\alpha \leq \lambda$, $\mathbb{P}_\lambda^U \Vdash \text{“}\mathbb{P}_\alpha^C \text{ is } \kappa\text{-c.c.}\text{”}$.*

Proof. Let G_λ^U be \mathbb{P}_λ^U -generic over V and for each $\beta < \lambda$ let $G_\beta^U = G_\lambda^U \cap \mathbb{P}_\beta^U$. Then G_β^U is \mathbb{P}_β^U -generic over V . It follows that for any ordinal $\alpha = \beta + 1 > \kappa$, where $\beta \equiv 2 \pmod{3}$ and $\beta-2$ is inaccessible, we have

$$V[G_\lambda^U] \models \text{“}\mathbb{P}_{\alpha+1}^C \simeq \mathbb{P}_\alpha^C * \mathbb{Q}_A(\Psi(\alpha)^*)\text{”}.$$

As \mathbb{P}_λ^U is κ -directed closed and κ is assumed to be Laver indestructible, κ remains supercompact, and hence weakly compact, in $V[G_\lambda^U]$.

Working in $V[G_\lambda^U]$, let \mathcal{F} be the weakly compact filter on κ , i.e., the filter on κ generated by the sets $\{\lambda < \kappa : (V_\lambda, \in, B \cap V_\lambda) \models \psi\}$, where $B \subseteq V_\kappa$ and ψ is a Π_1^1 sentence for the structure (V_κ, \in, B) . Let also \mathcal{S} be the collection of \mathcal{F} -positive sets, i.e., $\mathcal{S} = \{X \subseteq \kappa : \forall B \in \mathcal{F}, X \cap B \neq \emptyset\}$.

The proof of the next claim is as in the proof of [8, Lemma 2.13], where the forcing notions \mathbb{P}_α^2 and \mathbb{P}_α^1 there, are replaced with \mathbb{P}_α^C and $\text{Add}(\aleph_0, \kappa)$.⁵

Claim 5.4. *Work in $V[G_\lambda^U]$. Let $\alpha \leq \lambda$. For any sequence $\langle q_i : i < \kappa \rangle$ of conditions in \mathbb{P}_α^C , there exist a set $X \in \mathcal{S}$ and two sequences $\langle q_i^1 : i \in X \rangle$ and $\langle q_i^2 : i \in X \rangle$ of conditions in \mathbb{P}_α^C , such that*

- For all $i \in X$, $q_i^1, q_i^2 \leq q_i$.
- For all $i < j$ in X , q_i^1 is compatible with q_j^2 , and this is witnessed by a condition q , such that for every $\xi < \alpha$, $q \restriction \xi \Vdash "q(\xi) = q_i^1(\xi) \cup q_j^2(\xi)"$.

By Claim 5.4, for every $\alpha \leq \lambda$,

$$V[G_\lambda^U] \models " \mathbb{P}_\alpha^C \text{ is } \kappa\text{-c.c.} ",$$

and the lemma follows. □

The following is immediate.

Lemma 5.5. $V^{\mathbb{P}} \models " CH + \kappa = \aleph_2 + \lambda = \aleph_3 = 2^{\aleph_1} "$.

5.3. More on the forcing notion \mathbb{P} . In this subsection we prove a few more properties of the forcing notions \mathbb{P} that will be used in the proof of Theorem 1.3.

Lemma 5.6. *Assume that $\mu \in (\kappa, \lambda)$ is a Mahlo cardinal. Let \mathcal{T} be a $\mathbb{P}_\mu * \text{Add}(\aleph_0, \kappa)$ -name of a κ -tree. Then*

$$\Vdash_{\mathbb{P}_{\mu+2} * \text{Add}(\aleph_0, \kappa)} " \mathcal{T} \text{ has } \leq \kappa\text{-many } \kappa\text{-branches} .$$

Proof. Let G_μ be \mathbb{P}_μ -generic over V and $V_1 = V[G_\mu]$. In V_1 , \mathcal{T} can be considered as an $\text{Add}(\aleph_0, \kappa)$ -name. Note that in V_1 , $\kappa = \aleph_2$, $\mu = \aleph_3$ and $2^{\aleph_1} = \aleph_3$. Further we have

$$\mathbb{P}_{\mu+2}/\mathbb{P}_\mu \simeq \mathbb{Q}_\mu * \mathbb{Q}_{\mu+1} = \text{Add}(\aleph_1, \kappa) * \text{Cof}(\kappa, 2^\mu),$$

and

$$\Vdash_{\mathbb{P}_{\mu+2} * \text{Add}(\aleph_0, \kappa)} " |\{b \in V_1 : b \text{ is a branch of } \mathcal{T}\}| \leq |(2^\kappa)^{V_1}| = \kappa " .$$

⁵In [8], only forcing notions for specializing \mathbb{P}_α^1 -names for trees are considered, while in our forcing, we also consider the Cohen forcing $\text{Add}(\aleph_1, \kappa)$, but this does not produce any problems, as the forcing $\text{Add}(\aleph_1, \kappa)$ is well-behaved and is κ -c.c.

So it suffices to show that forcing with $\mathbb{Q}_\mu * \mathbb{Q}_{\mu+1} * \text{Add}(\aleph_0, \kappa)$ adds no new cofinal branches. Assume by contradiction that η is a $\mathbb{Q}_\mu * \mathbb{Q}_{\mu+1} * \text{Add}(\aleph_0, \kappa)$ -name which is forced to be a new κ -branch of \mathcal{T} . The next claim follows easily from the assumption that η is forced to be a new branch.

Claim 5.7. *For every $\langle p^0, p^1, p^2 \rangle \in \mathbb{Q}_\mu * \mathbb{Q}_{\mu+1} * \text{Add}(\aleph_0, \kappa)$, there are conditions $\langle q_i^0, q_i^1, q_i^2 \rangle$, for $i = 0, 1$, $\delta < \kappa$ and x_0, x_1 such that:*

- (a) $\langle q_0^0, q_0^1, q_0^2 \rangle, \langle q_1^0, q_1^1, q_1^2 \rangle \leq \langle p^0, p^1, p^2 \rangle$,
- (b) $x_0 \neq x_1$,
- (c) $\Vdash "x_0, x_1 \in \mathcal{T}_\delta, \text{ the } \delta\text{-th level of } \mathcal{T}"$,
- (d) $\langle q_i^0, q_i^1, q_i^2 \rangle \Vdash "x_i \in \eta"$ ($i=0,1$). □

In fact, as the forcing notions $\text{Add}(\aleph_1, \kappa)$ and $\text{Add}(\aleph_0, \kappa)$ are κ -c.c. and $\text{Col}(\kappa, 2^\mu)$ is forced to be κ -closed, we can show that the conditions $\langle q_0^0, q_0^1, q_0^2 \rangle$ and $\langle q_1^0, q_1^1, q_1^2 \rangle$ in the claim can be chosen so that $q_0^0 = q_1^0 = p^0$ and $q_0^2 = q_1^2 = p^2$ (see [11] for similar arguments).

Let us assume that the empty condition forces η is a new branch. By repeated application of Claim 5.7, we can build a sequence $\langle q_\nu^1 : \nu \in \langle \omega_1, 2 \rangle \rangle$ of \mathbb{Q}_μ -names of elements of $\mathbb{Q}_{\mu+1}$, an increasing continuous sequence $\langle \delta_i : i < \omega_1 \rangle$ of ordinals less than κ and a sequence $\langle x_\nu : \nu \in \langle \omega_1, 2 \rangle \rangle$ such that:

- (1) $\nu_1 \trianglelefteq \nu_2 \Rightarrow \Vdash_{\mathbb{Q}_\mu} "q_{\nu_2}^1 \leq q_{\nu_1}^1"$,
- (2) $\langle \emptyset, q_\nu^1, \emptyset \rangle \Vdash "x_\nu \in \mathcal{T}_{\delta_i}"$ where $i = \text{lh}(\nu)$,
- (3) $x_{\nu \smallfrown \langle 0 \rangle} \neq x_{\nu \smallfrown \langle 1 \rangle}$,
- (4) $\langle \emptyset, q_\nu^1, \emptyset \rangle \Vdash "x_\nu \in \eta"$,
- (5) $\nu_1 \trianglelefteq \nu_2 \Rightarrow \langle \emptyset, q_{\nu_2}^1, \emptyset \rangle \Vdash "x_{\nu_1} <_{\mathcal{X}} x_{\nu_2}"$.

For some $\xi < \kappa$, $\langle q_\nu^1 : \nu \in \langle \omega_1, 2 \rangle \rangle$ is in fact an $\text{Add}(\aleph_1, \xi)$ -name. Now we have

$$\mathbb{Q}_\mu * \mathbb{Q}_{\mu+1} * \text{Add}(\aleph_0, \kappa) \simeq \text{Add}(\aleph_1, \xi) * \text{Add}(\aleph_1, [\xi, \kappa]) * \mathbb{Q}_{\mu+1} * \text{Add}(\aleph_0, \kappa),$$

and in the generic extension $V^{\mathbb{P}_\mu * \text{Add}(\aleph_1, \xi)}$, we have an interpretation q_ν^1 of the name q_ν^1 , where $\nu \in \langle \omega_1, 2 \rangle$.

Work in $V^{\mathbb{P}_\mu * \mathbb{Q}_\mu}$. For each $\tau \in (\omega_1^2)$, let $q_\tau^1 \leq q_{\tau \upharpoonright i}^1$, $i < \omega_1$ and let $\delta = \sup\{\delta_i : i < \omega_1\} < \kappa$.

By extending q_τ^1 if necessary, we can assume that for some x_τ ,

$$\langle \emptyset, q_\tau^1, \emptyset \rangle \Vdash "x_\tau \in \mathcal{T}_\delta \cap \eta".$$

But then for all $\tau_1 \neq \tau_2$ in ω_1^2 we have $x_{\tau_1} \neq x_{\tau_2}$, and so

$$\Vdash_{\mathbb{P}_\mu * \text{Add}(\aleph_0, \kappa)} " \text{the } \delta\text{-th level of the tree has at least } 2^{\aleph_1} = \mu = \aleph_3\text{-many nodes} ".$$

But $\Vdash_{\mathbb{P}_\mu * \text{Add}(\aleph_0, \kappa)} " |\mathcal{T}_\delta| \leq \kappa < \mu "$, and we get a contradiction. \square

The next lemma follows from Lemma 5.6 and the fact that $\mathbb{P}_\mu * \text{Add}(\aleph_0, \kappa) \leq \mathbb{P}_{\mu+2} * \text{Add}(\aleph_0, \kappa)$

Lemma 5.8. *With the same hypotheses as in Lemma 5.6, we have the following: In $V^{\mathbb{P}_{\mu+2}}$, \mathcal{T} is isomorphic to some \mathcal{T}' , which is an $\text{Add}(\aleph_0, \kappa)$ -name of a κ -tree with $\leq \kappa$ -many cofinal branches.*

5.4. Completing the proof of Theorem 1.3. Finally in this subsection we complete the proof of Theorem 1.3. We first prove the following lemma.

Lemma 5.9. $\Vdash_{\mathbb{P} * \text{Add}(\aleph_0, \kappa)}$ “Any \aleph_1 -closed forcing notion of size $\leq \kappa$ collapses κ ”.

Proof. Let $G * H$ be $\mathbb{P} * \text{Add}(\aleph_0, \kappa)$ -generic over V and assume $\mathbb{R} \in V[G * H]$ is an \aleph_1 -closed forcing notion of size $\leq \kappa = \aleph_2$.

Assume towards a contradiction that forcing with \mathbb{R} over $V[G * H]$ does not collapse \aleph_2 . It then follows from [8] that \mathbb{R} is \aleph_2 -distributive, and hence by Lemma 3.1, there exists a κ -tree $T = T(\mathbb{R})$, the base tree of \mathbb{R} , which is dense in \mathbb{R} .

Let \mathcal{T} be a $\mathbb{P} * \text{Add}(\aleph_0, \kappa)$ -name for T . By Lemma 5.2, we may assume that $\mathcal{T} \in H(\lambda)$, and hence there exists some Mahlo cardinal $\beta \in (\kappa, \lambda)$ such that \mathcal{T} is a $\mathbb{P}_\beta * \text{Add}(\aleph_0, \kappa)$ -name. By Lemma 5.6, \mathcal{T} is isomorphic to some $\mathcal{T}' \in H(\lambda)$ which is a $\mathbb{P}_{\beta+2} * \text{Add}(\aleph_0, \kappa)$ -name for a κ -tree which has $\leq \kappa$ -many cofinal branches. On the other hand $\{\beta + 2 : \beta \in (\kappa, \lambda)\}$ is a Mahlo cardinal and $\Phi(\beta + 2) = \mathcal{T}'$ is unbounded in λ , and hence we can choose β as above such that $\mathcal{T}' = \Phi(\beta + 2)$. Then $\mathbb{P}_{\beta+3} \simeq \mathbb{P}_{\beta+2} * \mathbb{Q}_A(\Phi(\beta + 2)^*)$, and by Lemma 4.8,

$$\Vdash_{\mathbb{P}_{\beta+3} * \text{Add}(\aleph_0, \kappa)} " \text{Forcing with } \mathcal{T}' \text{ collapses } \kappa \text{ into } \aleph_1 ".$$

As $\mathbb{P}_{\beta+3} * \text{Add}(\aleph_0, \kappa) \triangleleft \mathbb{P} * \text{Add}(\aleph_0, \kappa)$,

$$\Vdash_{\mathbb{P} * \text{Add}(\aleph_0, \kappa)} \text{“ Forcing with } T' \text{ collapses } \kappa \text{ into } \aleph_1 \text{”}.$$

This implies

$$\Vdash_{\mathbb{P} * \text{Add}(\aleph_0, \kappa)} \text{“ Forcing with } \mathbb{R} \text{ collapses } \kappa \text{ into } \aleph_1 \text{”},$$

The lemma follows. \square

Now let $G * H$ be $\mathbb{P} * \text{Add}(\aleph_0, \kappa)$ -generic over V . Let also $\mathbb{R} \in V[G * H]$ be an \aleph_1 -closed forcing notion of size $\leq \kappa = \aleph_2$. By Lemma 5.9,

$$V[G * H] \models \text{“ } \mathbb{R} \simeq \text{Col}(\aleph_1, \kappa) \simeq \text{Add}(\aleph_1, 1) \text{”}.$$

6. CONSISTENCY, EVERY FORCING WHICH ADDS A FRESH SUBSET OF \aleph_2 COLLAPSES A CARDINAL

In this section we prove Theorem 1.4. Thus assume that GCH holds and $\lambda > \kappa$ are such that κ is supercompact and Laver indestructible, and λ is a 2-Mahlo cardinal. Let also $\Phi : \lambda \rightarrow H(\lambda)$ be such that for each $x \in H(\lambda)$, $\Phi^{-1}(x) \cap \{\beta + 2 : \beta \text{ is Mahlo}\}$ is unbounded in λ . The forcing notion we define is very similar the forcing notion of Section 5.

Definition 6.1. *Let*

$$\langle\langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle\rangle, \langle\langle \mathbb{Q}_\alpha : \alpha < \lambda \rangle\rangle$$

be an iteration of forcing notions such that for each $\alpha \leq \lambda$, $p \in \mathbb{P}_\alpha$ if and only if p is a function with domain α such that:

- (1) p has support of size less than κ .
- (2) $\{\beta \in \text{supp}(p) : \beta \equiv 0 \pmod{3} \text{ or } \beta \equiv 2 \pmod{3}\}$ has size less than \aleph_1 ,
- (3) If $\beta < \kappa$ and $\beta \equiv 0 \pmod{3}$ or $\beta \equiv 2 \pmod{3}$, then $\Vdash_\beta \text{“ } \mathbb{Q}_\beta = \text{Col}(\aleph_1, \aleph_2 + |\beta|) \text{”}$,
- (4) If $\beta \geq \kappa$, $\beta \equiv 0 \pmod{3}$ and β is inaccessible, then $\Vdash_\beta \text{“ } \mathbb{Q}_\beta = \text{Add}(\aleph_1, \kappa) \text{”}$,
- (5) If $\beta \geq \kappa$, $\beta \equiv 1 \pmod{3}$ and $\beta-1$ is inaccessible, then $\Vdash_\beta \text{“ } \mathbb{Q}_\beta = \text{Col}(\kappa, 2^{\mathbb{P}_\beta}) \text{”}$,
- (6) If $\beta \geq \kappa$, $\beta \equiv 2 \pmod{3}$ $\beta-2$ is inaccessible, and $\Phi(\beta)$ is a \mathbb{P}_β -name for κ -tree with $\leq \kappa$ -many cofinal branches, then $\Vdash_\beta \text{“ } \mathbb{Q}_\beta = \mathbb{Q}_A(\Phi(\beta)^*) \text{”}$,
- (7) Otherwise, $\Vdash_\beta \text{“ } \mathbb{Q}_\beta \text{ is the trivial forcing notion”}$,

Finally set $\mathbb{P} = \mathbb{P}_\lambda$.

The next lemma can be proved as in Section 5.

Lemma 6.2. *Let G be \mathbb{P} -generic over V . Then the following hold in $V[G]$:*

- (a) $2^{\aleph_0} = \aleph_1 < \kappa = \aleph_2 < 2^{\aleph_1} = \lambda = \aleph_3$,
- (b) *Every Tree of size and height \aleph_2 is specialized.*

Thus (a)-(c) of Theorem 1.4 are satisfied. Let's prove Theorem 1.4.(d). The proof is similar to Todorcevic's proof in [16], and we present it here for completeness.

Work in $V[G]$. Let \mathbb{P} be any forcing notion, and suppose that forcing with \mathbb{P} adds a fresh subset of \aleph_2 without collapsing it. We show that forcing with \mathbb{P} collapses \aleph_3 . Let $\mathbb{B} = RO(\mathbb{P})$.

Let also \mathcal{T} be a name for a fresh subset of \aleph_2 such that

$$\|(\mathcal{T} \subseteq \aleph_2) \wedge (\mathcal{T} \notin \check{V}) \wedge (\forall \alpha < \aleph_2, \mathcal{T} \cap \alpha \in \check{V})\|_{\mathbb{B}} = 1$$

For $\alpha < \aleph_2$, set $a_{\alpha,0} = \|\alpha \in \mathcal{T}\|_{\mathbb{B}}$ and $a_{\alpha,1} = \|\alpha \notin \mathcal{T}\|_{\mathbb{B}}$. Let $T_0 = \{1_{\mathbb{B}}\}$, and for $0 < \alpha < \aleph_2$ set

$$T_{\alpha} = \left\{ \bigwedge \{a_{\beta, f(\beta)} : \beta < \alpha\} : f \in {}^{\alpha}2, \bigwedge \{a_{\beta, f(\beta)} : \beta < \alpha\} \neq 0_{\mathbb{B}} \right\}.$$

By the assumption on \mathcal{T} , each T_{α} is a partition of $1_{\mathbb{B}}$, for $\beta < \alpha$, T_{α} refines T_{β} and so $T = \bigcup \{T_{\alpha} : \alpha < \aleph_2\}$ is a tree of height \aleph_2 , whose α -th level is T_{α} . Also clearly $|T| = 2^{\aleph_1} = \aleph_3$.

Claim 6.3. *For every $0_{\mathbb{B}} \neq b \in \mathbb{B}$, there exists $\alpha < \aleph_2$ such that*

$$|\{a \in T_{\alpha} : a \wedge b \neq 0_{\mathbb{B}}\}| > \aleph_2.$$

Proof. Suppose not. So we can find $0_{\mathbb{B}} \neq b \in \mathbb{B}$ such that for each $\alpha < \aleph_2$, $|\{a \in T_{\alpha} : a \wedge b \neq 0_{\mathbb{B}}\}| \leq \aleph_2$. Define a new tree $T^* = \bigcup \{T_{\alpha}^* : \alpha < \aleph_2\}$, where for each α ,

$$T_{\alpha}^* = \{a \wedge b : a \in T_{\alpha}, a \wedge b > 0_{\mathbb{B}}\}.$$

Then T^* is an \aleph_2 -tree of size \aleph_2 and hence it is specialized. But then

$$\|\dot{G}_{\mathbb{B}} \cap T^* \text{ is a new cofinal branch of } T^*\|_{\mathbb{B}} \geq b,$$

where $\dot{G}_{\mathbb{B}}$ is the canonical name for a generic ultrafilter over \mathbb{B} . This is impossible as T^* is specialized and forcing with \mathbb{B} preserves \aleph_2 . \square

For each $\alpha < \aleph_2$, let $\langle a_\alpha(\xi) : \xi < \lambda_\alpha \leq \aleph_3 \rangle$ be an enumeration of T_α , and let $\underset{\sim}{f}$ be a name for a function from \aleph_2 into \aleph_3 defined by

$$\|\underset{\sim}{f}(\alpha) = \xi\|_{\mathbb{B}} = \begin{cases} a_\alpha(\xi) & \text{if } \xi < \lambda_\alpha, \\ 0_{\mathbb{B}} & \text{Otherwise.} \end{cases}$$

Claim 6.4. $\|\text{range}(\underset{\sim}{f}) \text{ is unbounded in } \aleph_3\|_{\mathbb{B}} = 1_{\mathbb{B}}$.

Proof. Assume not. Then for some $\delta < \aleph_3$, $b = \|\text{range}(\underset{\sim}{f}) \subseteq \delta\|_{\mathbb{B}} > 0_{\mathbb{B}}$. By Claim 6.3, we can find $\alpha < \aleph_2$ such that $|\{a \in T_\alpha : a \wedge b \neq 0_{\mathbb{B}}\}| = \aleph_3$, so $\lambda_\alpha = \aleph_3$. Pick some $\xi > \delta$ so that $a_\alpha(\xi) \wedge b \neq 0_{\mathbb{B}}$. This implies

$$\|\underset{\sim}{f}(\alpha) = \xi\|_{\mathbb{B}} \wedge \|\text{range}(\underset{\sim}{f}) \subseteq \delta\|_{\mathbb{B}} \neq 0_{\mathbb{B}},$$

and we get a contradiction (as $\xi > \delta$). \square

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