Saccharinity with ccc

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Abstract

Using creature technology, we construct families of Suslin ccc non-sweet forcing notions $\mathbb{Q}$ such that ZFC is equiconsistent with $ZF+$"Every set of reals equals a Borel set modulo the $(\leq \aleph_1)$-closure of the null ideal associated with $\mathbb{Q}$"+"There is an ω_1-sequence of distinct reals".[1]

1. Introduction

Some history

The study of the consistency strength of regularity properties originated in Solovay's celebrated work [So2], where he proved the following result:

Theorem ([So2]): Suppose there is an inaccessible cardinal, then after forcing (by Levy collapse) there is an inner model of $ZF + DC$ where all sets of reals are Lebesgue measurable and have the Baire property.

Following Solovay's result, it was natural to ask whether the existence of an inaccessible cardinal is necessary for the above theorem. This problem was settled by Shelah ([Sh176]) who proved the following theorems:

Theorem ([Sh176]): 1. If every Σ^1_3 set of reals is Lebesgue measurable, then $\aleph_1$ is inaccessible in L.

2. $ZF + DC +$ "all sets of reals have the Baire property" is equiconsistent with ZFC.

A central concept in the proof of the second theorem is the amalgamation of forcing notions, which allows the construction of a suitably homogeneous forcing notion, thus allowing the use of an argument similar to the one used by Solovay, in which we have "universal amalgamation" (for years it was a quite well known problem). As the problem was that the countable chain condition is not necessarily preserved by amalgamation, Shelah isolated a property known as "sweetness", which implies *ccc* and is preserved under amalgamation. See more on the history of the subject in [RoSh672].

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2. General regularity properties

Given an ideal I on the reals, we say that a set of reals X is $I-$measurable if $X \Delta B \in I$ for some Borel set B, this is a straightforward generalization of Lebesgue measurability and the Baire property.

Given a definable forcing notion $\mathbb{Q}$ adding a generic real $\underset{\sim}{\eta}$ (we may write $\mathbb{Q}$ instead of $(\mathbb{Q}, \underset{\sim}{\eta})$) and a cardinal $\aleph_0 \leq \kappa$, there is a natural ideal on the reals $I_{\mathbb{Q},\kappa}$ associated to $(\mathbb{Q}, \kappa)$ (see definition 18), such that, for example, $I_{Cohen,\aleph_0}$ and $I_{Random,\aleph_0}$ are the meagre and null ideals, respectively. Hence in many cases the study of ideals on the reals corresponds to the study of definable forcing notions adding a generic real. On the study of ideals from the point of view of classical descriptive set theory, see [KeSo] and [So1]. For a forcing theoretic point of view, see [RoSh672]. Another approach to the subject can be found in [Za].

We are now ready to formulate the first approximation for our general problem:

Problem: Classify the definable *ccc* forcing notions according to the consistency strength of $ZF + DC +$ "all sets of reals are $I_{\mathbb{Q},\kappa}-$measurable".

Towards this we may ask: Given a definable *ccc* forcing notion $\mathbb{Q}$, is it possible to get a model where all sets of reals are $I_{\mathbb{Q},\kappa}-$measurable without using an inaccessible cardinal and for non-sweet forcing notions?

3. Saccharinity

A positive answer to the last question was given by Kellner and Shelah in [KrSh859] for a proper **non-ccc** (very non-homogeneous) forcing notion $\mathbb{Q}$, where the ideal is $I_{\mathbb{Q},\aleph_1}$.

In this paper we shall prove a similar result for a **ccc** forcing notion, omitting the DC but getting an ω_1-sequence of distinct reals. By [Sh176], the existence of such sequence is inconsistent with the Lebesgue measurability of all sets of reals, hence our forcing notions are, in a sense, closer to Cohen forcing than to Random real forcing.

Our construction will involve the creature forcing techniques of [RoSh470] and [RoSh628], and will result in definable forcing notions $\mathbb{Q}^i_{\mathbf{n}}$ which are non-homogeneous in a strong sense: Given a finite-length iteration of the forcing, the only generic reals are those given explicitly by the union of trunks of the conditions that belong to the generic set.

The homogeneity will be achieved by iterating along a very homogeneous (thus non-wellfounded) linear order. By moving to a model where all sets of reals are definable from a finite sequence of generic reals, we shall obtain the consistency of $ZF +$ "all sets of reals are $I_{\mathbb{Q}^i_{\mathbf{n}},\aleph_1}-$measurable"+"There exists an ω_1-sequence of distinct reals".

It's interesting to note that our model doesn't satisfy $AC_{\aleph_0}$, thus leading to a finer version of the problem presented earlier:

Problem: Classify the definable *ccc* forcing notions according to the consistency strength of T + "all sets of reals are $I_{\mathbb{Q},\kappa}-$measurable" where $T \in \{ZF, ZF + AC_{\aleph_0}, ZF + DC, ZF + DC(\aleph_1), ZFC\}$, and similarly for $T' = T + WO_{\omega_1}$ where T is as above and WO_{ω_1} is the statement "There is an ω_1-sequence of distinct reals".

Remark: Note that for some choices of T, $\mathbb{Q}$ and κ, the above statement might be inconsistent.

We intend to address this problem in [F1424] and other continuations.

A remark on notation: 1. Given a tree $T \subseteq \omega^{<\omega}$ and a node $\eta \in T$, we shall denote by $T^{[\eta\leq]}$ the subtree of T consisting of the nodes $\{\nu : \nu \leq \eta \vee \eta \leq \nu\}$.

2. For T as above, if $\eta \in T$ is the trunk of T, let $T^+ := \{\nu \in T : \eta \leq \nu\}$.

2. Norms, $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$

In this section we shall define a collection $\mathbf{N}$ of parameters. Each parameter $\mathbf{n} \in \mathbf{N}$ consists of a subtree with finite branching of $\omega^{<\omega}$ with a rapid growth of splitting and a norm on the set of successors of each node in the tree.

From each parameter $\mathbf{n} \in \mathbf{N}$ we shall define two forcing notions, $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$. We shall prove that they're nicely definable ccc. We will show additional nice properties in the case of $\mathbb{Q}^2_{\mathbf{n}}$, such as a certain compactness property and the fact that being a maximal antichain is a Borel property. We refer the reader to [RoSh470] and [RoSh628] for more information on creature forcing.

Definition 1: 1. A norm on a set A is a function assigning to each $X \in P(A) \setminus \{\emptyset\}$ a non-negative real number such that $X_1 \subseteq X_2 \to nor(X_1) \leq nor(X_2)$.

2. Let $\mathbf{M}$ be the collection of pairs $(\mathbb{Q}, \underset{\sim}{\eta})$ such that $\mathbb{Q}$ is a Suslin ccc forcing notion and $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real.

Definition 2: Let $\mathbf{N}$ be the set of tuples $\mathbf{n} = (T, nor, \bar{\lambda}, \bar{\mu}) = (T_{\mathbf{n}}, nor_{\mathbf{n}}, \bar{\lambda}_{\mathbf{n}}, \bar{\mu}_{\mathbf{n}})$ such that:

a. T is a subtree of $\omega^{<\omega}$.

b. $\bar{\mu} = (\mu_\eta : \eta \in T)$ is a sequence of non-negative real numbers.

c. $\bar{\lambda} = (\lambda_\eta : \eta \in T)$ is a sequence of pairwise distinct non-zero natural numbers such that:

1. $\lambda_\eta = \{m : \hat{\eta m} \in T\}$, so $T \cap \omega^n$ is finite and non-empty for every n.

2. If $lg(\eta) = lg(\nu)$ and $\eta <_{lex} \nu$ then $\lambda_\eta \ll \lambda_\nu$.

3. If $lg(\eta) < lg(\nu)$ then $lg(\eta) \ll \lambda_\eta \ll \lambda_\nu$.

4. $lg(\eta) \ll \mu_\eta \ll \lambda_\eta$ for $\eta \in T$.

d. For $\eta \in T$, nor_η is a function with domain $\mathcal{P}^-(suc_T(\eta)) = \mathcal{P}(suc_T(\eta)) \setminus \emptyset$ and range $\subseteq \mathbb{R}^+$ such that:

1. nor_η is a norm on $suc_T(\eta)$ (see definition 1).

2. $(lg(\eta)+1)^2 \leq \mu_\eta \leq nor_\eta(suc_T(\eta))$.

e. $\lambda_{<\eta} := \Pi\{\lambda_\nu : \lambda_\nu < \lambda_\eta\} \ll \mu_\eta$.

f. (Co-Bigness) If $k \in \mathbb{R}^+$, $a_i \subseteq suc_{T_{\mathbf{n}}}(\eta)$ for $i < i(*) \leq \mu_\eta$ and $k + \frac{1}{\mu_\eta} \leq nor_\eta(a_i)$ for every $i < i(*)$, then $k \leq nor_\eta(\bigcap\limits_{i<i(*)} a_i)$.

g. If $1 \leq nor_\eta(a)$ then $\frac{1}{2} < \frac{|a|}{|suc_{T_{\mathbf{n}}}(\eta)|}$.

h. If $k + \mu_\eta \leq nor_\eta(a)$ and $\rho \in a$, then $k \leq nor_\eta(a \setminus \{\rho\})$.

Definition 3: For $\mathbf{n} \in \mathbf{N}$ we shall define the forcing notions $\mathbb{Q}^1_{\mathbf{n}} \subseteq \mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}} \subseteq \mathbb{Q}^0_{\mathbf{n}}$ as follows:

1. $p \in \mathbb{Q}^0_{\mathbf{n}}$ iff for some $tr(p) \in T_{\mathbf{n}}$ we have:

a. p or T_p is a subtree of $T^{[tr(p)\leq]}_{\mathbf{n}}$ (so it's closed under initial segments) with no maximal node.

b. For $\eta \in lim(T_p)$, $lim(nor_{\eta\restriction l}(suc_{T_p}(\eta \restriction l)) : lg(tr(p)) \leq l < \omega) = \infty$.

c. $2 - \frac{1}{\mu_{tr(p)}} \leq nor(p)$ (where $nor(p)$ is defined in C(b) below).

2. $p \in \mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}}$ if $p \in \mathbb{Q}^0_{\mathbf{n}}$ and $nor_\eta(Suc_p(\eta)) > 2$ for every $tr(p) \leq \eta \in T_p$.

We shall prove later that $\mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}}$ is dense in $\mathbb{Q}^0_{\mathbf{n}}$.

3. $p \in \mathbb{Q}^1_{\mathbf{n}}$ if $p \in \mathbb{Q}^0_{\mathbf{n}}$ and for every $n < \omega$, there exists $k^p(n) = k(n) > lg(tr(p))$ such that for every $\eta \in T_p$, if $k(n) \leq lg(\eta)$ then $n \leq nor_\eta(Suc_p(\eta))$.

B. $\mathbb{Q}^i_{\mathbf{n}} \models p \leq q$ $(i \in \{0, \frac{1}{2}, 1\})$ iff $T_q \subseteq T_p$.

C. a. For $i \in \{0, \frac{1}{2}, 1\}$, $\underset{\sim}{\eta}^i_{\mathbf{n}}$ is the $\mathbb{Q}^i_{\mathbf{n}}-$name for $\cup\{tr(p) : p \in \underset{\sim}{G}_{\mathbb{Q}^i_{\mathbf{n}}}\}$.

b. For $i \in \{0, \frac{1}{2}, 1\}$ and $p \in \mathbb{Q}$ let $nor(p) := sup\{a \in \mathbb{R}_{>0} : \eta \in T^+_p \rightarrow a \leq nor_\eta(suc_{T_p}(\eta))\} = inf\{nor_\eta(suc_{T_p}(\eta)) : \eta \in T_p\}$.

D. For $i \in \{0, \frac{1}{2}, 1\}$ let $\mathbf{m}^i_{\mathbf{n}} = \mathbf{m}_{i,\mathbf{n}} = (\mathbb{Q}^i_{\mathbf{n}}, \underset{\sim}{\eta}^i_{\mathbf{n}}) \in \mathbf{M}$ where $\mathbf{M}$ denotes the set of pairs of the form $(\mathbb{Q}^i_{\mathbf{n}}, \underset{\sim}{\eta}^i_{\mathbf{n}})$.

We shall now describe a concrete construction of some $\mathbf{n} \in \mathbf{N}$:

Definition 4: We say $\mathbf{n} \in \mathbf{N}$ is special when:

a. For each $\eta \in T_{\mathbf{n}}$ the norm nor_η is defined as follows: for $\emptyset \neq a \subseteq suc_T(\eta)$, $nor_\eta(a) = \frac{log_*(|suc_T(\eta)|)}{\mu^2_\eta} - \frac{log_*|suc_T(\eta)\setminus a|}{\mu^2_\eta}$ where $log_*(x) = max\{n : \beth_n \leq x\}$ $(\beth_0 = 0)$.

a'. For each $\eta \in T_{\mathbf{n}}$, the dual norm nor^1_η is defined by $nor^1_\eta(a) = \frac{log_*(|a|)}{\mu_n}$.

b. $\mu_\eta = nor_\eta(suc_{t_{\mathbf{n}}}(\eta))$.

Observation 4A: There are $T_{\mathbf{n}}$, $(\lambda_\eta, \mu_\eta : \eta \in T_{\mathbf{n}})$ and $(nor_\eta : \eta \in T_{\mathbf{n}})$ satisfying the requirements of definition 2, where the norm is defined as in definition 4 (hence $\mathbf{n} \in \mathbf{N}$ is special).

Proof: It's easy to check that the following $(T_{\mathbf{n}}, (\mu_\eta, \lambda_\eta : \eta \in T_{\mathbf{n}}))$ together with the norm from deifnition 4 form a special $\mathbf{n} \in \mathbf{N}$ where $T_{\mathbf{n}} \cap \omega^n$, $(\mu_\eta, \lambda_\eta : \eta \in T_{\mathbf{n}} \cap \omega^n)$ are defined by induction on $n < \omega$ as follows:

a. $T_{\mathbf{n}} \cap \omega^0 = \{<>\}$.

b. At stage $n+1$, for $\eta \in T_{\mathbf{n}} \cap \omega^n$, by induction according to $<_{lex}$, define $\mu_\eta = \beth_{\lambda_{<\eta}}$, $\lambda_\eta = \beth_{\mu_\eta^2}$ and the set of succesors of η in $T_{\mathbf{n}}$ is defined as $\{\eta\hat{(}l) : l < \lambda_\eta\}$.

For example, we shall prove the co-bigness property:

Suppose that $\eta \in T_{\mathbf{n}}$ $(a_i : i < i(*))$ are as in definition 2(f). Denote $k_1 = |suc_{T_{\mathbf{n}}}(\eta)|$ and $k_2 = max\{|suc_{T_{\mathbf{n}}}(\eta) \setminus (a_i)| : i < i(*)\}$. Therefore, $\frac{log_*(k_1)}{\mu_n^2} - \frac{log_*(k_2)}{\mu_n^2} \leq nor_\eta(a_i)$ (so necessarily $k + \frac{1}{\mu_\eta} \leq \frac{log_*(k_1)}{\mu_n^2} - \frac{log_*(k_2)}{\mu_n^2}$). Let $a = \bigcup\limits_{i<i(*)} a_i$ and $k_3 = |suc_{T_{\mathbf{n}}}(\eta) \setminus a| \leq i(*)k_2 \leq \mu_\eta k_2$. Therefore $\frac{log_*(k_1)}{\mu_\eta^2} - \frac{log_*(\mu_\eta k_2)}{\mu_n^2} \leq \frac{log_*(k_1)}{\mu_\eta^2} - \frac{log_*(k_3)}{\mu_\eta^2} = nor_\eta(a)$. We have to show that $k \leq nor_\eta(a)$, so it's enough to show that $k \leq \frac{log_*(k_1)}{\mu_\eta^2} - \frac{log_*(\mu_\eta k_2)}{\mu_n^2}$. Recalling that $k + \frac{1}{\mu_\eta} \leq \frac{log_*(k_1)}{\mu_\eta^2} - \frac{log_*(k_2)}{\mu_\eta^2}$, it's enough to show that $\frac{log_*(\mu_\eta k_2)}{\mu_\eta^2} - \frac{log_*(k_2)}{\mu_\eta^2} \leq \frac{1}{\mu_\eta}$.

Case 1: $k_2 \leq \mu_\eta$. In this case, it's enough to show that $log_*(\mu_\eta k_2) - log_*(k_2) \leq \mu_\eta$, and indeed, $log_*(\mu_\eta k_2) - log_*(k_2) \leq log_*(\mu_\eta^2) \leq \mu_\eta$.

Case 2: $\mu_\eta < k_2$. By the properties of log_*, $log_*(k_2) \leq log_*(\mu_\eta k_2) \leq log_*(k_2^2) = log_*(k_2) + 1$, therefore $\frac{log_*(\mu_\eta k_2)}{\mu_\eta^2} - \frac{log_*(k_2)}{\mu_\eta^2} \leq \frac{1}{\mu_\eta}$.

$\square$

Definition 5: For $\mathbf{n} \in \mathbf{N}$ we define $\mathbf{m} = \mathbf{m}_{\mathbf{n}}^2 = (\mathbb{Q}_{\mathbf{n}}^2, \underset{\sim}{\eta}_{\mathbf{n}}^2)$ by:

A) $p \in \mathbb{Q}_{\mathbf{n}}^2$ iff p consists of a trunk $tr(p) \in T_{\mathbf{n}}$, a perfect subtree $T_p \subseteq T_{\mathbf{n}}^{[tr(p)\leq]}$ and a natural number $n \in [1, lg(tr(p)) + 1]$ such that $1 + \frac{1}{n} \leq nor_\eta(suc_{T_p}(\eta))$ for every $\eta \in T_p^+$.

B) Order: reverse inclusion.

C) $\underset{\sim}{\eta}_{\mathbf{n}}^2 = \cup\{tr(p) : p \in \underset{\sim}{G}_{\mathbb{Q}_{\mathbf{n}}^2}\}$.

D) If $p \in \mathbb{Q}_{\mathbf{n}}^2$ we let $nor(p) = min\{n : \eta \in T_p \to 1 + \frac{1}{n} \leq nor_\eta(suc_p(\eta))\}$.

Claim 6: $\mathbb{Q}_{\mathbf{n}}^i \models ccc$ for $i \in \{0, \frac{1}{2}, 1, 2\}$.

Proof: First we shall prove the claim for $\mathbb{Q}_{\mathbf{n}}^i$ where $i \in \{0, \frac{1}{2}, 1\}$. Observe that if $p \in \mathbb{Q}_{\mathbf{n}}^i$ and $0 < k < \omega$, then there is $p \leq q \in \mathbb{Q}_{\mathbf{n}}^i$ such that $nor_\eta(Suc_q(\eta)) > k$ for every $\eta \in T_q$. The claim is trivial for $i = 1$, so suppose that $i \in \{0, \frac{1}{2}\}$. In order to prove this fact, let $Y = \{\eta \in T_p$:for every $\eta \leq \nu \in T_p$, $nor_\nu(Suc_{T_p}(\nu)) > k\}$, then Y is dense in T_p (suppose otherwise, then we can construct a strictly increasing sequence of memebrs $\eta_i \in T_p$ such that $nor_{\eta_i}(Suc_{T_p}(\eta_i)) \leq k$, so $\bigcup\limits_{i<\omega} \eta_i \in lim(T_p)$ contradicts the definition of $\mathbb{Q}^i_{\mathbf{n}}$). Now pick $tr(p) \leq \eta \in Y$, then $q = p^{[\eta \leq]}$ is as required. It also follows that from this claim that $\mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}}$ is dense in $\mathbb{Q}^0_{\mathbf{n}}$.

Now suppose towards contradiction that $\{p_\alpha : \alpha < \aleph_1\} \subseteq \mathbb{Q}^i_{\mathbf{n}}$ is an antichain, for every α, there is $p_\alpha \leq q_\alpha$ such that $nor_\eta(Suc_{q_\alpha}(\eta)) > 2$ for every $\eta \in q_\alpha$. For some uncountable $S \subseteq \aleph_1$, $tr(q_\alpha) = \eta_*$ for every $\alpha \in S$. By the claim below, q_α, q_β are compatible for $\alpha, \beta \in S$, contradicting our assumption.

As for $\mathbb{Q}^2_{\mathbf{n}}$, given $I = \{p_i : i < \aleph_1\} \subseteq \mathbb{Q}^2_{\mathbf{n}}$ ($\mathbb{Q}^1_{\mathbf{n}}$), the set $\{(tr(p), nor(p)) : p \in I\}$ is countable, hence there is $p_* \in I$ such that for uncountably many $p_i \in I$ we have $(tr(p_i), nor(p_i)) = (tr(p_*), nor(p_*))$. By the claim below, those p_i are pairwise compatible.

$\square$

Claim 7: 1) $p, q \in \mathbb{Q}^2_{\mathbf{n}}$ are compatible in $\mathbb{Q}^2_{\mathbf{n}}$ iff $tr(p) \leq tr(q) \in T_p$ or $tr(q) \leq tr(p) \in T_q$.

2) Similarly, $p, q \in \mathbb{Q}^i_{\mathbf{n}}$ are compatible in $\mathbb{Q}^i_{\mathbf{n}}$ for $i \in \{0, \frac{1}{2}, 1\}$ iff $tr(p) \leq tr(q) \in T_p \vee tr(q) \leq tr(p) \in T_q$.

Proof: In both clauses, the implication $\rightarrow$ is obvious, we shall prove thee other direction.

1) First observe that if $p \in \mathbb{Q}^2_{\mathbf{n}}$ and $\nu \in T_p$, then $p^{[\nu]} \in \mathbb{Q}^2_{\mathbf{n}}$ and $p \leq p^{[\nu]}$ (where $p^{[\nu]}$ is the set of nodes in p comparable with ν).

$\square_1$ If $tr(p) \leq tr(q) \in T_p$ then $T_p \cap T_q$ has arbitrarily long sequences.

Proof: Let $\eta = tr(q)$, then by the definition of the norm and $\mathbb{Q}^2_{\mathbf{n}}$, $\frac{1}{2} < \frac{|suc_{T_p}(\eta)|}{|suc_{T_{\mathbf{n}}}(\eta)|}, \frac{|suc_{T_q}\eta|}{|suc_{T_{\mathbf{n}}}(\eta)|}$. Hence there is $\nu \in suc_{T_p}(\eta) \cap suc_{T_q}(\eta)$. Repeating the same argument, we get sequences in $T_p \cap T_q$ of length n for every n large enough.

$\square_2$ Claim: If $tr(p_1) = tr(p_2) = \eta$, $p_1, p_2 \in \mathbb{Q}^2_{\mathbf{n}}$, $1 + \frac{1}{h} \leq nor(p_1), nor(p_2)$ and $h < lg(\eta)$, then p_1 and p_2 are compatible.

Proof: For every $\nu \in T_{p_1} \cap T_{p_2}$, by the co-bigness property, $min\{nor_\nu(suc_{p_1}(\nu)), suc_{p_2}(\nu)\} - \frac{1}{\mu_\nu} \leq nor(suc_{p_1}(\nu) \cap suc_{p_2}(\nu))$. By the definition of $nor(p_i)$ (recalling that $lg(\eta)^2 \leq \mu_\eta$), $1 + \frac{1}{h+1} \leq (1 + \frac{1}{h+1}) + (\frac{1}{(h+1)^2} - \frac{1}{\mu_\eta}) \leq (1 + \frac{1}{h+1}) + (\frac{1}{h} - \frac{1}{h+1} - \frac{1}{\mu_\nu}) = 1 + \frac{1}{h} - \frac{1}{\mu_\nu} \leq min\{nor(p_1), nor(p_2)\} - \frac{1}{\mu_\nu} \leq min\{nor_\nu(suc_{p_1}(\nu)), suc_{p_2}(\nu)\} - \frac{1}{\mu_\nu}$. Therefore $1 + \frac{1}{h+1} \leq nor(suc_{p_1}(\nu) \cap suc_{p_2}(\nu))$, so $p_1 \cap p_2$ is as required. Hence:

$\square_3$ p and q are compatible.

Proof: Suppose WLOG that $tr(p) \leq tr(q) \in T_p$ and pick h such that $1 + \frac{1}{h} \leq nor(p), nor(q)$. By $\square_1$, there is $\eta \in T_p \cap T_q$ such that $h < \lg(\eta)$. Now $p \leq p^{[\eta]}, q \leq q^{[\eta]}$ and $(p^{[\eta]}, q^{[\eta]})$ satisfy the assumptions of $\square_2$, therefore they're compatible and so are p and q.

The proof is similar if $tr(q) \leq tr(p) \in T_q$. The implication in the other direction is easy.

2) The proof is similar. First observe that if $\eta \in lim(T_p) \cap lim(T_q)$, then $lim(nor_{\eta \restriction l}(suc_{T_p}(\eta \restriction l)) : l < \omega) = \infty = lim(nor_{\eta \restriction l}(suc_{T_q}(\eta \restriction l)) : l < \omega)$, so by the co-bigness property (definition 2(f)), $lim(nor_{\eta \restriction l}(suc_{T_{p \cap q}}(\eta \restriction l)) : l < \omega) = \infty$. Now let $\nu = tr(q) \in T_p \cap T_q$, as $2 - \frac{1}{\mu_{tr(p)}} \leq nor(p), nor(q)$, it follows from the co-bigness property and definition 2(g) that $\nu \leq \eta \in T_p \cap T_q \to 2 < |Suc_{p \cap q}(\eta)|$, so $p \cap q$ is a perfect tree. It's easy to see that there exists $\eta \in p \cap q$ such that $nor_\nu(Suc_{p \cap q}(\nu)) > 2$ for every $\eta \leq \nu \in p \cap q$ (otherwise, we can repeart the argument in the proof of claim 6, and get a branch through $p \cap q$ along which the norm doesn't tend to infinity). Therefore, $p^{[\leq \eta]} \cap q^{[\leq \eta]} \in \mathbb{Q}^i_{\mathbf{n}}$ $(i \in \{0, \frac{1}{2}\})$ is a common upper bound. Finally, note that if $i = 1$, then for every $n < \omega$ there exist $k^p(n+1), k^q(n+1)$ as in definition 3.3. By the co-bigness property, for every $\eta \in T_p \cap T_q$ of length $> max\{k^p(n+1), k^q(n+1)\}$, $n \leq nor_\eta(Suc_{p \cap q}(\eta))$. Therefore, the common upper bound is in $\mathbb{Q}^1_{\mathbf{n}}$ as well.

$\square$

Claim 8: Let $I \subseteq \mathbb{Q}^2_{\mathbf{n}}$ be an antichain and $A = \cup\{T^+_q : q \in I\} \subseteq T_{\mathbf{n}}$. The following conditions are equivalent:

(a) I is a maximal antichain.

(b) If $\eta \in T_{\mathbf{n}}$ and $0 < n < \omega$ then there is no $p \in \mathbb{Q}^2_{\mathbf{n}}$ such that:

(α) $tr(p) = \eta$.

(β) $nor(p) = n$.

(γ) p is incompatible with every $q \in I$.

(c) Like (b), but replcaing (γ) by

$(\gamma)'$ $T^+_p \cap A = \emptyset$.

(d) Like (b), but replcaing (γ) by

$(\gamma)''$ For every $m > n$ $T^+_p \cap A$ is disjoint to $\{\nu \in T_{\mathbf{n}} : lg(\nu) \leq m\}$.

(e) If $\eta \in T_{\mathbf{n}}$ and $n < \omega$ then for some $m > n$ there is no set T such that:

(α) $T \subseteq T_{\mathbf{n}}$.

(β) $\eta \in T$.

(γ) If $\nu \in T^+$ then $\eta \leq \nu$ and $lg(\nu) \leq m$.

(δ) If $\eta \leq \nu_1 \leq \nu_2$ and $\nu_2 \in T$ then $\nu_1 \in T$.

(ϵ) $T \cap A = \emptyset$.

(ζ) If $\nu \in T$ and $lg(\nu) < m$ then $1 + \frac{1}{n} \leq nor_\nu(suc_T(\nu))$.

Proof: $\neg(a) \to \neg(b)$: If p is incompatible with every $q \in I$ then $(p, tr(p), nor(p))$ is a counterexample to (b).

$\neg(b) \to \neg(c)$: If $(p, tr(p), nor(p))$ is a counterexample to (b), then it is a counterexample to (c) by the characterisation of compatibility in $\mathbb{Q}^2_{\mathbf{n}}$ in claim 7.

$\neg(c) \to \neg(d)$: Obvious.

$\neg(d) \to \neg(e)$: Let $T = T_p$ with p being a counter example to (d) and let $\eta = tr(p), n$ witness $\neg(d)$. We shall check that for every $m > n$, $\{\nu : tr(p) \leq \nu \in T \wedge lg(\nu) \leq m\}$ satisfies $(\alpha) - (\zeta)$ if (e).

$\neg(e) \to \neg(a)$: If (η, n) is a counterexample, then for every m there is T_m satisfying $(\alpha) - (\zeta)$ of clause (e). Let D be a non-principal ultrafilter on ω and define $T := \{\nu \in T_{\mathbf{n}} : \nu \leq \eta$ or $\{m : m > n, \nu \in T_m\} \in D\}$. It remains to show that $T \in \mathbb{Q}^2_{\mathbf{n}}$ (as T^+ is disjoint to A, it follows that I is not a maximal antichain). The proof is similar to claim 12.

□

Claim 9: Let $\mathbf{n} \in \mathbf{N}$.

A) The sets $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$ are Borel, the sets $\mathbb{Q}^0_{\mathbf{n}}$ and $\mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}}$ are Π^1_1.

B) The relation $\leq_{\mathbb{Q}^i_{\mathbf{n}}}$ is Borel for $i \in \{0, \frac{1}{2}, 1, 2\}$.

C) The incompatibility relation in $\mathbb{Q}^i_{\mathbf{n}}$ is Borel for $i \in \{0, \frac{1}{2}, 1, 2\}$.

Proof:

A. The sets $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$ are Borel: We shall first prove the claim for $\mathbb{Q}^1_{\mathbf{n}}$. Consider $T_{\mathbf{n}}$ as a subset of $H(\aleph_0)$. By definition, if $p \in \mathbb{Q}^1_{\mathbf{n}}$ then $T_p \subseteq T_{\mathbf{n}} \subseteq H(\aleph_0)$. Hence $S := \{p \subseteq H(\aleph_0) : p$ is a perfect subtree of $T_{\mathbf{n}}\} \subseteq P(H(\aleph_0))$ is a Borel subset of $P(H(\aleph_0))$. For every $n, k < \omega$ define $S^1_{n,k} = \{p \in S : lg(tr(p)) < k$ and if $\rho \in T_p$ and $k \leq lg(\rho)$ then $n \leq nor_\rho(suc_p(\rho))\}$. Each $S^1_{n,k}$ is closed, hence $S \cap (\bigcap\limits_n \bigcup\limits_k S^1_{n,k})$ is Borel, so it's enough to show that $p \in \mathbb{Q}^1_{\mathbf{n}}$ iff $p \in S \cap (\bigcap\limits_n \bigcup\limits_k S^1_{n,k})$ and $2 - \frac{1}{\mu_{tr(p)}} \leq nor(p)$, which follows directly from the definition of $\mathbb{Q}^1_{\mathbf{n}}$.

In the case of $\mathbb{Q}^2_{\mathbf{n}}$, we replace $\bigcap\limits_n \bigcup\limits_k S^1_{n,k}$ with $\bigcup\limits_{n,k} S^2_{n,k}$ where $S^2_{n,k} = \{p \in S : lg(tr(p)) = n \wedge nor(p) = k\}$. Each $S^2_{n,k}$ is Borel and since "being a perfect subtree" is Borel, $\mathbb{Q}^2_{\mathbf{n}}$ is Borel.

The sets $\mathbb{Q}^0_{\mathbf{n}}$ and $\mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}}$ are Π^1_1: The demand "$\underset{n<\omega}{lim}(nor_{\eta \restriction n}(Suc_p(\eta \restriction n))) = \infty$ for every $\eta \in lim(T_p)$" is Π^1_1,and it's easy to see that $\{p \in S : tr(p) \leq \eta \in T_p \to nor_\eta(Suc_{T_p}(\eta)) > 2\}$ is Borel.

B. The relation $\leq_{\mathbb{Q}^i_{\mathbf{n}}}$ is Borel for $i \in \{0, \frac{1}{2}, 1, 2\}$: For $i \in \{0, \frac{1}{2}, 1, 2\}$, the relation $\leq_{\mathbb{Q}^i_{\mathbf{n}}}$ is simply the reverse inclusion relation restricted to $\mathbb{Q}^i_{\mathbf{n}}$, hence it is Borel.

C. The incompatibility relation in $\mathbb{Q}^i_{\mathbf{n}}$ is Borel for $\{0, \frac{1}{2}, 1, 2\}$: The incompatibility relation is Borel by claim 7.

□

Claim 10: A) Assume that $p_l \in \mathbb{Q}^i_{\mathbf{n}}$ $(l < n)$ where $i \in \{0, 1\}$, $\underset{l<n}{\wedge} tr(p_l) = \rho$, $n \leq lg(\rho)$ and for every $\eta \in p_l$ we have $2 \leq k+1 \leq nor_\eta(suc_{p_l}(\eta))$, then $\{p_l : l < n\}$ have a common upper bound p such that $tr(p) = \rho$ and $k \leq nor_\eta(suc_p(\eta))$ for every $\eta \in T^+_p$.

B) Assume that $p_l \in \mathbb{Q}^2_{\mathbf{n}}$ $(l < n)$, $\underset{l<n}{\wedge} tr(p_l) = \rho$, $n \leq lg(\rho)$ and for every $\eta \in p^+_l$ $(l < n)$ we have $1 + \frac{1}{k} \leq nor_\eta(suc_{p_l}(\eta))$. In addition, assume that $k(k+1) \leq \mu_\eta$ for every $\eta \in p^+_l$ $(l < n)$, then $\{p_l : l < n\}$ have a common upper bound p such that $tr(p) = \rho$ and $1 + \frac{1}{k+1} \leq nor_\eta(suc_p(\eta))$.

Proof: A) Suppose first that $i = 0$. Let $p = \underset{l<n}{\cap} p_l$, then $p \subseteq T^{[\rho\leq]}_{\mathbf{n}}$ is a subtree conatining ρ. If $\nu \in p$ then $\nu \in p_l$ for every $l < n$, hence $Suc_p(\nu) = \underset{l<n}{\cap} Suc_{p_l}(\nu)$. As $n \leq lg(\rho) \leq \mu_\eta$ for every $\rho \leq \eta \in p$, it follows from the properties of the norm in the definition of $\mathbf{n} \in \mathbf{N}$ that $k \leq nor_\eta(Suc_p(\eta))$. Therefore, T_p is a prefect tree, and similarly to the proof of claim 7, it follows that the norm along infinite branches tends to infinity, hence $p \in \mathbb{Q}^0_{\mathbf{n}}$. Suppose now that $i = 1$. The above arguments are still valid, and in addition, similarly to the argument on $\mathbb{Q}^1_{\mathbf{n}}$ in th proof of claim 7(2), it's easy to see that by the co-bigness property, $p \in \mathbb{Q}^1_{\mathbf{n}}$.

Remark: Note that as $2 \leq k+1$, it follows from the above arguments that $2 - \frac{1}{\mu_{tr(p)}} \leq nor_\eta(Suc_{T_p}(\eta))$ for every $tr(p) \leq \eta \in T_p$. In fact, $k + 1 - \frac{1}{\mu_\rho} \leq nor_\eta(Suc_{T_p}(\eta))$, therefore, if $2 < k + 1 - \frac{1}{\mu_\rho}$ then we also get the claim for $i = \frac{1}{2}$.

B) The proof is similar, the only difference is that now we have to prove the following assertion:

$(*)$ If $b_l \subseteq suc_{T_{\mathbf{n}}}(\eta)$ for $l < n \leq \mu_\eta$, $\underset{l<n}{\wedge} 1 + \frac{1}{k} \leq nor_\eta(b_l)$ and $b = \underset{l<n}{\cap} b_l$ then $1 + \frac{1}{k+1} \leq nor_\eta(b)$.

The assertion follows from the co-bigness property (definition 2(f), with b_i and $1 + \frac{1}{k} - \frac{1}{\mu_\eta}$ here standing for a_i and k there).

$\square$

Claim 11: Let $\mathbf{n} \in \mathbf{N}$. "$\{p_n : n < \omega\}$ is a maximal antichain" is Borel for $\{p_n : n < \omega\} \subseteq \mathbb{Q}^2_{\mathbf{n}}$.

Proof: By claim 8.

$\square$

Claim 12: Assume $\{p_n : n < \omega\} \subseteq \mathbb{Q}^2_{\mathbf{n}}$, $\underset{n}{\wedge} tr(p_n) = \eta$ and $\underset{n}{\wedge} nor(p_n) = k$. Then there is $p_* \in \mathbb{Q}^2_{\mathbf{n}}$ such that:

(a) $tr(p_*) = \eta$, $nor(p_*) = k$.

(b) $p_* \Vdash_{\mathbb{Q}^2_{\mathbf{n}}}$ "$(\exists^\infty n)(p_n \in G_{\mathbb{Q}^2_{\mathbf{n}}})$".

Proof: Let D be a uniform ultrafilter on ω and define $T_{p_*} := \{\nu \in T_{\mathbf{n}} : \{n : \nu \in p_n\} \in D\}$. If $\nu \in T_{p_*}$, then for some n, $\nu \in T_{p_n} \subseteq T_{\mathbf{n}}^{[\eta\leq]}$ (recalling that $tr(p_n) = \eta$), hence $T_{p_*} \subseteq T_{\mathbf{n}}^{[\eta\leq]}$. Obviously, $l \leq lg(\eta) \rightarrow \eta \restriction l \in T_{p_*}$ as $\eta = tr(p_n) \in p_n$ for every n.

$(*)_1$ If $\eta \triangleleft \nu \triangleleft \rho$ and $\rho \in T_{p_*}$, then $\nu \in T_{p_*}$.

Why? Define $A_\rho = \{n : \rho \in p_n\}$ and define A_ν similarly. $A_\rho \in D$ by the definition of T_{p_*}. Obviously $A_\rho \subseteq A_\nu$, hence $A_\nu \in D$ and $\nu \in T_{p_*}$.

$(*)_2$ If $\nu \in T_{p_*}$ then $1 + \frac{1}{k} \leq nor_\nu(suc_{p_*}(\nu))$.

Why? Define A_ν as above, so $A_\nu \in D$. Let $(b_l : l < l(*))$ list $\{suc_{p_n}(\nu) : n < \omega\}$. As $\{suc_{p_n}(\nu) : n < \omega\} \subseteq P(suc_{T_{\mathbf{n}}}(\nu))$, we have $l(*) \leq 2^{|suc_{T_{\mathbf{n}}}(\nu)|} = 2^{\lambda_\nu} < \aleph_0$. For $l < l(*)$ let $A_{\nu,l} := \{n \in A_\nu : suc_{p_n}(\nu) = b_l\}$. Obviously this is a finite partition of A_ν, hence there is exactly one $m < l(*)$ such that $A_{\nu,m} \in D$ and therefore $b_m \subseteq suc_{p_*}(\nu)$ and actually $b_m = suc_{p_*}(\nu)$ (if $\eta \in suc_{p_*}(\nu)$ is witnessed by $X \in D$, then $X \cap A_{\nu,m}$ is a witness for $\eta \in b_m$). Therefore $nor_\nu(b_m) = nor_\nu(suc_{p_*}(\nu))$ and for some n we have $1 + \frac{1}{k} = 1 + \frac{1}{nor(p_n)} \leq nor_\nu(suc_{p_n}(\nu)) = nor_\nu(suc_{p_*}(\nu))$.

It follows from the above arguments that $p_* \in \mathbb{Q}^2_{\mathbf{n}}$.

We shall now prove that

$(*)_3 p_* \Vdash_{\mathbb{Q}^2_{\mathbf{n}}}$ "$(\exists^\infty n)(p_n \in G_{\mathbb{Q}^2_{\mathbf{n}}})$".

Why? Suppose that $p_* \leq q$, then $tr(q) \in T_{p_*}$. By the definition of p_*, $\{n : tr(q) \in p_n\} \in D$. For every such p_n, $\eta = tr(p_n) \leq tr(q) \in T_{p_n}$, so p_n is compatible with q and hence with p_*.

$\square$

Claim 12': For $\iota \in \{0, \frac{1}{2}, 1, 2\}$, $\underset{\sim}{\eta}^\iota_{\mathbf{n}}$ is a generic for $\mathbb{Q}^\iota_{\mathbf{n}}$, i.e. $\Vdash_{\mathbb{Q}^\iota_{\mathbf{n}}}$ "$V[\underset{\sim}{G}_{\mathbb{Q}^\iota_{\mathbf{n}}}] = V[\underset{\sim}{\eta}^\iota_{\mathbf{n}}]$".

Proof: Easy.

$\square$

3. The iteration

In this section we shall describe our iteration. Although our definition will be general and will follow the technique of iteration along templates as described in [Sh700], we will eventually use a simple private case of the general construction. In our case, we'll have a non-wellfounded linear order L, and the forcing will be the union of finite-length iterations along subsets of L. Dealing with FS-iterations of Suslin forcing will guarantee that the union is well-behaved.

Iteration parameters

Definition 12: Let $\mathbf{Q}$ be the class of $\mathbf{q}$ (iteration parameters) consisting of:

a. A partial order $L_{\mathbf{q}} = L[\mathbf{q}]$.

b. $\bar{u}_0 = (u^0_t : t \in L_{\mathbf{q}})$ such that $u^0_t \subseteq L_{<t}$ for each $t \in L_{\mathbf{q}}$ (and u^0_t is well-ordered by (d)). In the main case $|u^0_t| \leq \aleph_0$ (in our application, u^0_t is actually empty).

c. $\mathbf{I} = (\mathbf{I}_t : t \in L_{\mathbf{q}})$ such that each $\mathbf{I}_t$ is an ideal on $L_{<t}$ and $u^0_t \in \mathbf{I}_t$. In the main case here, $\mathbf{I}_t = \{u \subseteq L_{<t} : u \text{ is finite}\}$.

d. $\mathbf{L}$ is a directed family of well-founded subsets of $L_{\mathbf{q}}$ closed under initial segments such that $\bigcup\limits_{L \in \mathbf{L}} L = L_{\mathbf{q}}$ and $t \in L \rightarrow u^0_t \subseteq L$ (for $L \in \mathbf{L}$).

e. $(\mathbf{m}_t : t \in L_{\mathbf{q}})$ is a sequence such that each $\mathbf{m}_t$ is a definition of a Suslin ccc forcing notion $\mathbb{Q}^i_{\mathbf{m}_t}$ with a generic $\underset{\sim}{\eta}_{\mathbf{m}_t}$ (depending on a formula using $\mathbf{B}(..., \eta_s, ...)_{s \in u^0_t}$, see f+g and definition 13).

f. Actually, $\mathbf{m}_t = \mathbf{m}_{t, \underset{\sim}{\nu}_t}$ where $\underset{\sim}{\nu}_t = \mathbf{B}_t(\bar{\eta} \restriction u^0_t)$ is a name of a real and $\mathbf{B}_t$ is a Borel function (see definition 13(E) below), i.e. $\mathbf{m}_t$ is computed from the parameter $\underset{\sim}{\nu}_t \in \omega^\omega$.

g. For every $t \in L_{\mathbf{q}}$, $\mathbf{B}_t : \prod\limits_{i \in u^0_t} \omega^\omega \rightarrow \omega^\omega$ is an absolute Borel function.

h. For a linear order L, let $L^+ := L \cup \{\infty\}$ which is obtained by adding an element above all elements of L.

The iteration

Definition and claim 13: For $i \in \{1, 2\}$, $\mathbf{q} \in \mathbf{Q}$ and $L \in \mathbf{L}$ we shall define the FS iteration $\bar{\mathbb{Q}}_L = (\mathbb{P}^L_t, \underset{\sim}{\mathbb{Q}}^L_t : t \in L^+)$ with limit $\mathbb{P}_L$ and the $\mathbb{P}^L_t = \mathbb{P}_{L,<t}$-names $\underset{\sim}{\eta}_t, \underset{\sim}{\nu}_t$ by induction on $dp(L)$ (where $dp(L)$ is the depth of L, recalling that L is well-founded) such that:

A. a) $\mathbb{P}_L$ is a forcing notion.

b) $\underset{\sim}{\eta}_t$ is a $\mathbb{P}_L$ name when $u^0_t \cup \{t\} \subseteq L \in \mathbf{L}$ (so we use a maximal antichain from $\mathbb{P}_L$, moreover, from $\mathbb{P}_{L_1}$ for every $L_1 \in \mathbf{L}$ which is $\subseteq L$).

c) $\underset{\sim}{\nu}_t$ is a $\mathbb{P}_L$ name when $u^0_t \subseteq L \in \mathbf{L}$.

d) If $L_1, L_2 \in \mathbf{L}$ are linearly ordered, $L_1 \subseteq L_2$ and each $\mathbf{I}_t$ has the form $\{L \subseteq L_{<t} : L$ is well-ordered$\}$, then $\mathbb{P}_{L_1} \lessdot \mathbb{P}_{L_2}$.

B. $p \in \mathbb{P}^L_t$ iff

a. $Dom(p) \subseteq L_{<t}$ is finite.

b. If $s \in Dom(p)$ then for some $u \in \mathbf{I}_s \cap \mathcal{P}(L_{<s})$ and a Borel function $\mathbf{B}$, $p(s) = \mathbf{B}(..., \underset{\sim}{\eta}_r, ...)_{r \in u}$ and $\Vdash_{\mathbb{P}^L_s}$ "$p(s) \in \mathbb{Q}^i_{\mathbf{m}_s}$".

C. $\underset{\sim}{\mathbb{Q}}^L_t$ is the $\mathbb{P}^L_t$-name of $\mathbb{Q}^i_{\mathbf{m}_t}$ using the parameter $\underset{\sim}{\nu}_t$.

D. $\bar{\eta} = (\underset{\sim}{\eta}_t : t \in L_{\mathbf{q}})$. Each $\underset{\sim}{\eta}_t$ is defined as the generic of $\mathbb{Q}_t^L$ (by a maximal antichain of $\mathbb{P}_L$ whenever $L \in \mathbf{L}$ and $u_t^0 \subseteq L \subseteq L_{<t}$), meaning: $t \in L \in \mathbf{L} \to \Vdash$ "$\underset{\sim}{\eta}_t$ is a generic for $\underset{\sim}{\mathbb{Q}}_t$" defined as usual.

E. $\bar{\nu} = (\underset{\sim}{\nu}_t : t \in L_{\mathbf{q}})$ such that for each $t \in L_{\mathbf{q}}$, $\mathbf{B}_t$ is a Borel function and $\underset{\sim}{\nu}_t = \mathbf{B}_t(\bar{\eta} \restriction u_t^0)$.

F. The order on $\mathbb{P}_L$ is defined naturally.

Proof: Should be clear.

$\square$

13(A) A special case of the general construction

Of special interest here is the case where $\mathbf{q} \in \mathbf{Q}$ satisfies:

a. $L_{\mathbf{q}}$ is a dense linear order, $\mathbf{I}_t = [L_{<t}]^{<\aleph_0}$ for each $t \in L_{\mathbf{q}}$ and $\mathbf{L} = [L_{\mathbf{q}}]^{<\aleph_0}$.

b. $\mathbf{m}_t$ is a definition of $\mathbb{Q}^i_{\mathbf{n}_t}$ where $i \in \{1, 2\}$ (hence a Suslin c.c.c. forcing), not using a name of the form $\underset{\sim}{\nu}_t$.

c. $\mathbf{m}_t \in V$ and $u_t^0 = \emptyset$ for every $t \in L_{\mathbf{q}}$.

13(B) We shall denote the collection of $\mathbf{q} \in \mathbf{Q}$ as above by $\mathbf{Q}_{sp}$.

13(C) Hypothesis: From now on we assume that $\mathbf{q} \in \mathbf{Q}$ satisfies the requirements of $13(A)$.

Definition/Observation 14: Let $\mathbf{q} \in \mathbf{Q}$.

1. $\{\mathbb{P}_J : J \subseteq L_{\mathbf{q}}$ is finite$\}$ is a $\lessdot$-directed set of forcing notions.

2. For $J \subseteq L_{\mathbf{q}}$, let $\mathbb{P}_J = \cup\{\mathbb{P}_{J'} : J' \subseteq J$ is finite$\}$ and $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{L_{\mathbf{q}}}$.

Proof: (1) follows by [JuSh292].

$\square$

Claim 15: 1) For every $J_1 \subseteq J_2 \subseteq L_{\mathbf{q}}$, $\mathbb{P}_{J_1} \lessdot \mathbb{P}_{J_2}$.

2) If $J \subseteq L_{\mathbf{q}}$ then $\mathbb{P}_J = \mathbb{P}_{\mathbf{q},J} = \cup\{\mathbb{P}_I : I \subseteq J$ is finite$\} \lessdot \mathbb{P}_{\mathbf{q}}$.

Proof: 1) Case 1: $|J_2| < \aleph_0$. Easy by [JuSh292].

Case 2: J_2 is inifinite. Let $q \in \mathbb{P}_{J_2}$, then for some finite $J_2^* \subseteq J_2$, $q \in \mathbb{P}_{J_2^*}$. Let $J_1^* = J_1 \cap J_2^*$. As $\mathbb{P}_{J_1^*} \lessdot \mathbb{P}_{J_2^*}$ by observation 14(1), there is $p \in \mathbb{P}_{J_1^*}$ such that $p \leq p' \in \mathbb{P}_{J_1^*} \to p'$ and q are compatible. It suffices to prove that if $J_1' \subseteq J_1$ is finite and $J_1^* \subseteq J_1'$, then $p \leq p' \in \mathbb{P}_{J_1'} \to p'$ and q are compatible in $\mathbb{P}_{J_2^* \cup J_1'}$ (as if $p \leq p' \in \mathbb{P}_{J_1}$, then $p' \in \mathbb{P}_{J_1'}$ where $J_1' = J_1^* \cup Dom(p')$). We prove this by induction on $sup\{|L_{<t} \cap J_1^*| : t \in J_1' \setminus J_1^*\}$ as in [JuSh292].

2) By (1).

Observation 16: Suppose that $\mathbf{q} \in \mathbf{Q}$, $J \in \mathbf{L}$ is finite and $p_1, p_2 \in \mathbb{P}_J$. If $tr(p_1(t)) = tr(p_2(t))$ for every $t \in Dom(p_1) \cap Dom(p_2)$, then p_1 and p_2 are compatible.

Proof: By induction on $|J|$. The induction step is a corollary of the compatibility condition for $\mathbb{Q}^2_{\mathbf{n}}$ (see claim 7).

$\square$

Claim 17: For $\mathbf{q} \in \mathbf{Q}$, $\mathbb{P}_{\mathbf{q}} \models ccc$.

Proof: Suppose that $\{p_\alpha : \alpha < \aleph_1\} \subseteq \mathbb{P}_{\mathbf{q}}$. For each $\alpha < \aleph_1$ there is a finite $J_\alpha \subseteq L_{\mathbf{q}}$ such that $p_\alpha \in \mathbb{P}_{J_\alpha}$. Hence there is $n_* \in \mathbb{N}$ such that $|\{p_\alpha : |J_\alpha| = n_*\}| = \aleph_1$. For each α denote $J_\alpha = \{t_{\alpha,0} < ... < t_{\alpha,n_{\alpha-1}}\}$, by cardinallity arguments i.e. the Δ-system lemma, WLOG there is $u \subseteq n_*$ such that $t_{\alpha,l} = t_l$ for every $\alpha < \aleph_1$ and $(t_{\alpha,l} : l \in n_* \setminus u, \alpha < \aleph_1)$ is without repetitions. As every condition $p_\alpha \in \mathbb{P}_{J_\alpha}$ belongs to an iteration along J_α in the usual sense, there is $p_\alpha \leq p'_\alpha \in \mathbb{P}_{J_\alpha}$ such that $tr(p'_\alpha(t))$ is an object for every $t \in J_\alpha$ (so $J_\alpha = Dom(p'_\alpha)$). Given $l \in u$ there are countably many possible values for $tr(p_\alpha(t_l))$, hence there is a set $I = \{p_{\alpha_i} : i < i(*)\} \subseteq \{p_\alpha : \alpha < \aleph_1\}$ of cardinality $\aleph_1$ such that $tr(p_{\alpha_i}(t_l))$ is constant for all $i < i(*)$. If $i < j < i(*)$, then $J_{i,j} := J_{\alpha_i} \cup J_{\alpha_j} \subseteq L_{\mathbf{q}}$ is finite, $p_{\alpha_i} \in \mathbb{P}_{J_{\alpha_i}} \lessdot \mathbb{P}_{J_{i,j}}$ and $p_{\alpha_j} \in \mathbb{P}_{J_{\alpha_j}} \lessdot \mathbb{P}_{J_{i,j}}$, so p_{α_i} and p_{α_j} are compatible in $\mathbb{P}_{J_{i,j}}$ (hence in $\mathbb{P}_{\mathbf{q}}$) by observation 16.

$\square$

4. The ideals derived from a forcing notion $\mathbb{Q}$

We shall now define the ideals derived from a Suslin forcing notion $\mathbb{Q}$ and a name $\underset{\sim}{\eta}$ of a real.

Definition 18: 1. Let $\mathbb{Q}$ be a forcing notion such that each $p \in \mathbb{Q}$ is a perfect subtree of $\omega^{<\omega}$, $p \leq_{\mathbb{Q}} q$ iff $q \subseteq p$ and the generic real is given by the union of trunks of conditions that belong to the generic set, that is $\underset{\sim}{\eta} = \bigcup_{p \in \underset{\sim}{G}} tr(p)$ and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in \omega^\omega$".

Let $\aleph_0 \leq \kappa$, the ideal $I^0_{\mathbb{Q},\kappa}$ will be defined as the closure under unions of size $\leq \kappa$ of sets of the form $\{X \subseteq \omega^\omega : (\forall p \in \mathbb{Q})(\exists p \leq q)(lim(q) \cap X = \emptyset)\}$.

2. Let $\mathbf{m} = (\mathbb{Q}, \kappa)$ where $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real, the ideal $I^1_{\mathbf{m},\kappa}$ for $\aleph_0 \leq \kappa$ will be defined as follows:

$A \in I^1_{\mathbf{m},\kappa}$ iff there exists $X \subseteq \kappa$ such that $A \cap \{\underset{\sim}{\eta}[G] : G \subseteq \mathbb{Q}^{L[X]}$ is generic over $L[X]\} = \emptyset$.

3. For $\mathbb{Q}$ and κ as above, we shall denote $I^0_{\mathbb{Q},\kappa}$ by $I_{\mathbb{Q},\kappa}$.

4. Let I be an ideal on the reals, a set of reals X is called I-measurable if there exists a Borel set B such that $X \Delta B \in I$.

5. A set of reals X will be called $(\mathbb{Q}, \kappa)$-measurable if it is $I_{\mathbb{Q},\kappa}$-measurable.

6. Given a model V of ZF, we say that $(\mathbb{Q}, \kappa)$-measurability holds in V if every set of reals in V is $(\mathbb{Q}, \kappa)$-measurable and $I_{\mathbb{Q},\kappa}$ is a non-trivial ideal.

Remark: In [F1424] we shall further investigate the above ideals.

5. Cohen reals

An important feature of $\mathbb{Q}^{\iota}_{\mathbf{n}}$ is the fact that it adds a Cohen real. This fact will be later used to show that $\mathbb{Q}^{\iota}_{\mathbf{n}}$ can turn the ground model reals into a null set with respect to the relevant ideal.

Claim 19: Forcing with $\mathbb{Q}^{\iota}_{\mathbf{n}}$ ($i \in \{0, \frac{1}{2}, 1, 2\}$) adds a Cohen real.

Proof: For every $\eta \in T_{\mathbf{n}}$ let $g_\eta : suc_{T_{\mathbf{n}}}(\eta) \to \{0,1\}$ be a function such that $|g_\eta^{-1}\{l\}| > \frac{\lambda_\eta}{2} - 1$ ($l = 0, 1$) (recall that $\lambda_\eta = |suc_{T_{\mathbf{n}}}(\eta)|$). Define a $\mathbb{Q}^{\iota}_{\mathbf{n}}$-name $\underset{\sim}{\nu}$ by $\underset{\sim}{\nu}(n) = g_{\underset{\sim}{\eta}^{\iota}_{\mathbf{n}} \restriction n}(\underset{\sim}{\eta}^{\iota}_{\mathbf{n}} \restriction (n+1))$ (recalling $\underset{\sim}{\eta}^{\iota}_{\mathbf{n}}$ is the generic). Clearly, $\Vdash_{\mathbb{Q}^{\iota}_{\mathbf{n}}}$ "$\underset{\sim}{\nu} \in 2^{\omega}$". We shall prove that it's forced to be Cohen.

$(*)$ If $p \in \mathbb{Q}^{\iota}_{\mathbf{n}}$ and $i = 1 \to 2 \leq nor_\rho(suc_p(\rho))$ for every $\rho \in T_p$, then for every $\eta \in 2^{\omega}$, for some $\rho \in T_p$, $lg(\rho) = lg(tr(p)) + m$ and if $lg(tr(p)) \leq i < tr(p) + m$ then $p^{[\rho]} \Vdash$ "$\underset{\sim}{\nu}(i) = \eta(i)$".

We prove it by induction on m. For $m = 1$, as $|suc_{T_{\mathbf{n}}}(tr(p)) \setminus suc_p(tr(p))| < \frac{|suc_{T_{\mathbf{n}}}(tr(p))|}{2} - 1$ (by clause (g) of definition 2) and for every $i \in \{0, 1\}$ we have $|g^{-1}_{tr(p)}\{i\}| > \frac{\lambda_{tr(p)}}{2} - 1$, hence there are $\rho_0, \rho_1 \in suc_p(tr(p)) \setminus \{\rho\}$ such that $g_{tr(p)}(\rho_0) = 0, g_{tr(p)}(\rho_1) = 1$ and by the definition of $\underset{\sim}{\nu}$, $p^{[\rho_0]} \Vdash$ "$\underset{\sim}{\nu}(tr(p)+1) = 0$" and $p^{[\rho_1]} \Vdash$ "$\underset{\sim}{\nu}(tr(p)+1) = 1$". Suppose that we proved the theorem for m, then for some $\rho \in T_p$ of length $lg(tr(p)) + m$ the conclusion holds. Now repeat the argument of the first step of the induction for $p^{[\leq \rho]}$ to obtain $\rho \leq \rho'$ of length $lg(tr(p)) + m + 1$ as required.

By $(*)$, $\underset{\sim}{\nu}$ is forced to lie in every open dense set, hence it's Cohen.

□

6. Not adding an unwanted real

A crucial step towards our final goal is to prove that the only generic reals in finite length iterations of $\mathbb{Q}^2_{\mathbf{n}}$ are the η_ts. This will be used later in order to show that $\omega^{\omega} \setminus \{\eta_t : t \in L\}$ is null with resepect to the relevant ideal. We intend to strengthen this result dealing with arbitrary length iterations in [F1424].

Claim 20: If A) then B) where

A) (a) $p_i \in \mathbb{Q}^{\iota}_{\mathbf{n}}$ for $i < m$.

(b) $tr(p_i) = \rho$ for $i < m$.

(c) If $\iota \in \{0, 1\}$ then $2 \leq nor(p_i)$ for every $i < m$.

(d) If $\iota = 2$ then $2 \leq nor(p_i)$ for every $i < m$.

(e) $lg(\rho) < m_* < m$.

(f) There is $\rho < \eta \in T_{\mathbf{n}}$ such that $\lambda_{<\eta} \leq m_* < m \leq \mu_\eta$ (for example, it follows from the assumption $m \leq \mu_\eta \iff m_* \leq \lambda_{\leq\eta}$).

B) There is an equivalence relation E on $\{0, 1, ..., m-1\}$ with $\leq m_*$ equivalence classes such that if $i < m$ then $\{p_j : j \in (i/E)\}$ has a common upper bound.

Proof: Let $\eta \in T_{\mathbf{n}}^{[\rho\leq]}$ be as in clause (f). Let $k_* = lg(\eta)$ and define $\lambda_{\mathbf{n},k} := \Pi\{\lambda_\nu : \nu \in T_{\mathbf{n}}, lg(\nu) < k\}$, $T_{\mathbf{n},\rho,k} := \{\nu \in T_{\mathbf{n}} : \rho \leq \nu \in T_{\mathbf{n}}, lg(\nu) = k\}$. Recall that λ_ν is the size of $suc_{\mathbf{n}}(\nu)$, hence $|T_{\mathbf{n},\rho,k_*}|$ is the product of all λ_ν such that $\rho \leq \nu$ and $lg(\nu) < k_*$, which is $\leq \lambda_{\mathbf{n},k_*}$. For each $i < m$ let $\rho_i \in p_i$ be of length k_*, then $\rho_i \in T_{\mathbf{n},\rho,k_*}$ by the definition of $T_{\mathbf{n},\rho,k_*}$ and the assumptions on p_i. Define ρ_i^+ for $i < m$ as follows: if $\lambda_\eta < \lambda_{\rho_i}$, define $\rho_i^+ := \rho_i$. Otherwise we let $\rho_i^+ \in suc_{p_i}(\rho_i)$.

Define the equivalence relation $E := \{(i, j) : \rho_i^+ = \rho_j^+\}$. Let $j < m$, for every $i \in (j/E)$ define $p'_i = p_i^{[\rho_j^+]}$ (this is well defined, as $\rho_i^+ = \rho_j^+$), then $tr(p'_i) = \rho_j^+$ for every $i \in (j/E)$. By the choice of η, for $j < m$, $|j/E| \leq m \leq \mu_\eta \leq \mu_{\rho_j^+}$ (by the choice of ρ_j^+ and definition 2).

By claim 10, the set $\{p'_i : i \in (j/E)\}$ has a common upper bound, hence $\{p_i : i \in (j/E)\}$ has a common upper bound.

By the choice of p_i^+, the number of E-equivalence classes is bounded by $\lambda_{<\eta}$. As $\lambda_{<\eta} \leq m_*$, we're done.

$\square$

Claim 21: We have $p_* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\rho}$ is not $(\mathbb{Q}_{\mathbf{n}}^\iota, \underset{\sim}{\eta}_{\mathbf{n}}^\iota)$-generic over V" when:

a) $\iota \in \{0, \frac{1}{2}, 1, 2\}$ and $\alpha_* < \omega$.

b) $(\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \alpha_*)$ is a FS iteration with limit $\mathbb{P} = \mathbb{P}_{\alpha_*}$.

c) $\mathbf{n}_\alpha \in \mathbf{N}$ is special (note: $\mathbf{n}_\alpha$ is not a $\mathbb{P}_\alpha$−name).

d) $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = (\mathbb{Q}_{\mathbf{n}_\alpha}^\iota)^{V^{\mathbb{P}_\alpha}}$".

e) $\mathbf{n} \in \mathbf{N}$ is special.

f) For every α, $\mathbf{n}$ and $\mathbf{n}_\alpha$ are far (i.e. $\eta_1 \in T_{\mathbf{n}} \wedge \eta_2 \in T_{\mathbf{n}_\alpha} \to \lambda_{\eta_1}^{\mathbf{n}} \ll \mu_{\eta_2}^{\mathbf{n}_\alpha}$ or $\lambda_{\eta_2}^{\mathbf{n}_\alpha} \ll \mu_{\eta_1}^{\mathbf{n}}$).

f)(+) For every $\alpha < \alpha_*$ for every l large enough, for some $m \in \{l, l+1\}$ we have:

If $\rho \in T_{\mathbf{n}}, lg(\rho) = l$, $\nu_1, \nu_2 \in T_{\mathbf{n}_{\alpha(l)}}$ and $lg(\nu_1) \leq m < lg(\nu_2)$) then $\lambda_{\mathbf{n}_{\alpha(l)},\nu_1} \ll \mu_{\mathbf{n},\rho}$ and $\lambda_{\mathbf{n},\rho} \ll \mu_{\mathbf{n}_{\alpha(l)},\nu_2}$.

g) $p_* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\rho} \in lim(T_{\mathbf{n}})$".

Proof: For $\eta \in T_{\mathbf{n}}$ define $W_{\mathbf{n},\eta} := \{w : w \subseteq suc_{T_{\mathbf{n}}}(\eta)$ and $i = 1 \rightarrow lg(\eta) \leq nor^{\mathbf{n}}_{\eta}(w)$ and $i = 2 \rightarrow 2 \leq nor^{\mathbf{n}}_{\eta}(w)\}$. For $n < \omega$ define $\Lambda_n = \{\eta \in T_{\mathbf{n}} : lg(\eta) < n\}$, so $T_{\mathbf{n}} = \underset{n<\omega}{\cup} \Lambda_n$. Define $S_n := \{\bar{w} : \bar{w} = (w_\eta : \eta \in \Lambda_n \wedge w_\eta \in W_{\mathbf{n},\eta})\}$ and $S = \underset{n<\omega}{\cup} S_n$. $(S, \leq)$ is a tree with ω levels such that each level is finite and $lim(S) = \{\bar{w} : \bar{w} = (w_\eta : \eta \in T_{\mathbf{n}})$ and $\bar{w} \restriction \Lambda_n \in S_n$ for every $n\}$. For $\bar{w} \in lim(S)$ let $\mathbf{B}_{\bar{w}} := \{\rho \in lim(T_{\mathbf{n}})$: for every n large enough, $\rho \restriction (n+1) \in w_{\rho \restriction n}\}$, so $\mathbf{B}_{\bar{w}} = \underset{m<\omega}{\cup} \mathbf{B}_{\bar{w},m}$ where $\mathbf{B}_{\bar{w},m} = \{\rho \in lim(T_{\mathbf{n}})$: if $m \leq n$ then $\rho \restriction (n+1) \in w_{\rho \restriction n}\}$. We shall prove that

$(*) \Vdash_{\mathbb{Q}^\iota_{\mathbf{n}}}$ "$\underset{\sim}{\eta}^\iota_{\mathbf{n}} \in \mathbf{B}_{\bar{w}}$" for every $\bar{w} \in lim(S)$.

Let $p \in \mathbb{Q}^\iota_{\mathbf{n}}$, we shall prove that for some $p \leq q$ and $m < \omega$, $q \Vdash \underset{\sim}{\eta}^i_{\mathbf{n}} \in \mathbf{B}_{\bar{w},m}$. Let $\nu \in T_p$ such that $lg(\nu)$ is large enough and let $m = lg(\nu)$. Now q will be defined by taking the subtree obtained from the intersection of $T^{[\leq \nu]}_p$ with $(\underset{\nu \leq \rho}{\cup} w_\rho)$. By the co-bigness property, q is a well defined condition, and obviously $q \Vdash \underset{\sim}{\eta}^i_{\mathbf{n}} \in \mathbf{B}_{\bar{w},m}$.

By $(*)$ it suffices to prove that for some $\bar{w} \in lim(S)$, $p_* \nVdash_{\mathbb{P}}$ "$\underset{\sim}{\rho} \in \mathbf{B}_{\bar{w}}$".

Proof: Assume towards contradiction that $p \Vdash$ "$\underset{\sim}{\rho} \in \mathbf{B}_{\bar{w}}$ for every $\bar{w} \in lim(S)$", so there is a sequence $(p_{\bar{w}} : \bar{w} \in lim(S))$ and a sequence $(m(\bar{w}) : \bar{w} \in lim(S))$ such that:

a) $p_* \leq p_{\bar{w}}$.

b) $p_{\bar{w}} \Vdash \underset{\sim}{\rho} \in \mathbf{B}_{\bar{w},m(\bar{w})}$.

By increasing the conditions $p_{\bar{w}}$ if necessary, we may assume WLOG that:

1. $tr(p_{\bar{w}}(\alpha))$ is an object for every $\bar{w}$ and every $\alpha \in Dom(p_{\bar{w}})$.

2. If $\iota = 1$ and $\alpha \in Dom(p_{\bar{w}})$, then $p_{\bar{w}} \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$\nu \in p_{\bar{w}}(\alpha) \rightarrow nor_\nu(Suc_{p_{\bar{w}}(\alpha)}(\nu)) > 2$".

If $\iota = 2$ and $\alpha \in Dom(p_{\bar{w}})$, then for some $m \ll lg(tr(p_{\bar{w}}(\alpha)))$, $p_{\bar{w}} \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \nu \in p_{\bar{w}}(\alpha) \rightarrow 1 + \frac{1}{m} \leq nor(suc_{p_{\bar{w}}(\alpha)}(\nu))$.

In order to prove (1)+(2), we shall prove by induction on $\beta \leq \alpha(*)$ that for every $p \in \mathbb{P}_\beta$ there is $p \leq q \in \mathbb{P}_\beta$ satisfying (2) and forcing a value to the relevant trunks.

The induction step: assume that $\beta = \gamma + 1$. As $p(\gamma)$ is a $\mathbb{P}_\gamma-$name of a condition in $\mathbb{Q}^2_{\mathbf{n}}$, there are $p \restriction \gamma \leq p' \in \mathbb{P}_\gamma$ and ρ such that $p' \Vdash_{\mathbb{P}_\gamma} tr(p(\gamma)) = \rho$. As $p' \Vdash_{\mathbb{P}_\gamma} p(\gamma) \in \mathbb{Q}^2_{\mathbf{n}}$ and by the definition of $\mathbb{Q}^2_{\mathbf{n}}$, there is $p' \leq p''$ and $m \leq \mu_{lg(\rho)}$ such that $p'' \Vdash_{\mathbb{P}_\gamma} \nu \in p(\gamma) \rightarrow 1 + \frac{1}{m} \leq nor(suc_{p(\gamma)}(\nu))$. Now choose $m \ll m_1$, so $p'' \Vdash_{\mathbb{P}_\gamma}$ "there is $\nu \in p(\gamma)$ such that $lg(\nu) = m_1$". Therefore there are $p'' \leq p^*$ and ν of length m_1 such that $p^* \Vdash_{\mathbb{P}_\gamma}$ "$\nu \in p(\gamma) \wedge (\nu \leq \eta \in p(\gamma) \rightarrow 1 + \frac{1}{m} \leq nor(suc_{p(\gamma)}(\eta)))$". By the induction hypothesis, there is $p^* \leq q' \in \mathbb{P}_\gamma$ satisfying (1)+(2). Now define $q := q' \cup (\gamma, p(\gamma)^{[\nu \leq]})$, obviously q is as required. The proof for $\mathbb{Q}^1_{\mathbf{n}}$ is similar.

Now we shall define a partition of $lim(S)$ to $\aleph_0$ sets as follows:

Let $W_{m,u,\bar{\rho}} = \{\bar{w} \in lim(S) : m(\bar{w}) = m, Dom(p_{\bar{w}}) = u \in [\alpha(*)]^{<\aleph_0}, \bar{\rho} = (tr(p_{\bar{w}}(\alpha)) : \alpha \in u)\}$. Choose $(m_*, u_*, \bar{\rho}_*)$ such that $W = W_{m_*,u_*,\bar{\rho}_*} \subseteq lim(S)$ is not meagre. Let $\bar{u}_* \in S$ such that W is no-where meagre above $\bar{u}_*$. Let $l(*)$ be such that $\bar{u}_* \in S_{l(*)}$ and let $lg(\bar{u}_*) := l(*)$.

Denote $\bar{\rho}^* = (\rho^*_\alpha : \alpha \in u_*)$, let $(\alpha_n : n < n(*))$ list u_* in increasing order and let $\alpha_{n(*)} = \alpha(*)$. Therefore, if $\bar{u}_* \leq \bar{w} \in W$ then $Dom(p_{\bar{w}}) = \{\alpha_0, ..., \alpha_{n(*)-1}\}$ and $tr(p_{\bar{w}}(\alpha_n)) = \rho^*_{\alpha_n}$ for every $n < n(*)$.

By our assumption, $\mathbf{n}$ is far from $\mathbf{n}_\alpha$. As increasing $\bar{u}_*$ is not going to change the argument, we may assume that $l(*)$ is large enough so $\bigwedge_{\alpha \in u_*} lg(\rho^*_\alpha) < l(*)$ and if $l < n(*)$, $\nu \in T_{\mathbf{n}}$, $\rho \in T_{\mathbf{n}_{\alpha_l}}$ and $lg(\bar{u}_*) \leq lg(\nu)$, then $\lambda_{\mathbf{n},\nu} \ll \mu_{\mathbf{n}_{\alpha_l},\rho}$ or $\lambda_{\mathbf{n}_{\alpha_l},\rho} \ll \mu_{\mathbf{n},\nu}$. Note that we don't have to assume that $lg(\bar{u}_*) \leq lg(\rho)$: For every $n < n(*)$, there is m_n as guaranteed by $(f)(+)$, with $(\mathbf{n}_{\alpha_n}, lg(\nu), m_n)$ here standing for $(\mathbf{n}_\alpha, l, m)$ there. If $lg(\rho) \leq m_n$, then by taking an arbitrary ν_2 of length $> m_n$, it follows from $(f)(+)$ that $\lambda_{\mathbf{n}_{\alpha(n)},\rho} \ll \mu_{\mathbf{n},\nu}$. If $m_n < lg(\rho)$, then by taking an arbitrary ν_2 of length $\leq m_n$, we get $\lambda_{\mathbf{n},\nu} \ll \mu_{\mathbf{n}_{\alpha(n)},\rho}$.

Recalling $(f)(+)$ (and by increasing $\bar{u}_*$ if necessary), let $(m_n : n < n(*))$ be a series of natural numbers such that $(\mathbf{n}, \mathbf{n}_{\alpha(n)}, lg(\bar{u}_*), m_n)$ satisfy that assumptions of $(f)(+)$ (with $(\mathbf{n}, \mathbf{n}_{\alpha(n)}, lg(\bar{u}_*), m_n)$ here standing for $(\mathbf{n}, \mathbf{n}_\alpha, l, m)$ there).

Let $\Lambda^0_m = \Lambda_{m+1} \setminus \Lambda_m = \{\rho \in T_{\mathbf{n}} : lg(\rho) = m\}$ and let $S^0_m = \{\bar{w} : \bar{w} = (w_\eta : \eta \in \Lambda^0_m)$, for every $\eta \in \Lambda^0_m$, $w_\eta \in W_{\mathbf{n},\eta}\}$.

Recalling that above $\bar{u}_*$, W is nowhere meagre, for every $\bar{v} \in S^0_{l(*)}$ there is $\bar{w}_{\bar{v}} \in W \subseteq lim(\mathbb{S})$ such that $\bar{u}_* \hat{} \bar{v} \leq \bar{w}_{\bar{v}}$.

Choose p_n, U_n by induction on $n \leq n(*)$ such that:

1. $p_n \in \mathbb{P}_{\alpha_n}$.
2. If $m < n$ then $p_m \leq p_n \restriction \alpha_m$.
3. $U_n \subseteq S^0_{l(*)}$.
4. If $m < n$ then $U_n \subseteq U_m$.
5. If E is an equivalence relation on U_n with $\leq \Pi\{|T_{\mathbf{n}_{\alpha_l},m_l}| : n \leq l < n(*)\}$ equivalence classes, then for some $\bar{v}_* \in U_n$, $\cap\{\bigcup_{\rho \in T_{\mathbf{n},l(*)}} w_{\bar{v}_\rho} : \bar{v} \in \bar{v}_*/E\} = \emptyset$.
6. If $\bar{v} \in U_n$ then $p_{\bar{w}_{\bar{v}}} \restriction \alpha(n) \leq p_n$.

Suppose we've carried the induction, then for every $\bar{v} \in U_{n(*)}$, $p_{\bar{w}_{\bar{v}}} = p_{\bar{w}_{\bar{v}} \restriction \alpha_{n(*)}} \leq p_{n(*)}$, hence by the choice of $p_{\bar{w}_{\bar{v}}}$, $p_{n(*)} \Vdash \underset{\sim}{\rho} \in \cap\{\mathbf{B}_{\bar{w}_{\bar{v}}}, m_{\bar{w}_{\bar{v}}} : \bar{v} \in U_{n(*)}\}$. Therefore it's enough to show that $\cap\{\mathbf{B}_{\bar{w}_{\bar{v}}}, m_{\bar{w}_{\bar{v}}} : \bar{v} \in U_{n(*)}\} = \emptyset$. By its definition, $\mathbf{B}_{\bar{w}_{\bar{v}}, m_{\bar{w}_{\bar{v}}}} = lim(T_{\bar{v}})$ where $T_{\bar{v}} = \{\eta \in T_{\mathbf{n}} :$ if $m_{\bar{w}_{\bar{v}}} < lg(\eta)$ then $\eta(m+1) \in w_{\eta \restriction m}$ for every $m_{\bar{w}_{\bar{v}}} \leq m\}$. Therefore, if we show that $\cap\{T_{\bar{v}} \cap T_{\mathbf{n},l(*)+1} : \bar{v} \in U_{n(*)}\} = \emptyset$, then it will follow that $\cap\{lim(T_{\bar{v}}) : \bar{v} \in U_{n(*)}\} = \emptyset$. This follows from part (5) of the induction hypothesis, as $\cap\{\bigcup_{\rho \in T_{\mathbf{n},l(*)}} w_{\bar{v}_\rho} : \bar{v} \in U_{n(*)}\} = \emptyset$. This contradiction proves the claim.

Carrying the induction: For $n = 0$, choose any $p_0 \in \mathbb{P}_{\alpha_0}$ and let $U_0 = S^0_{l(*)}$. It's enough to show that U_0 satisfies (5). Let E be an equivalence relation on U_0 with $m_{**} \leq \Pi\{|T_{\mathbf{n}_{\alpha(l)},m_l}| : l < n(*)\}$ equivalence classes and denote $\Pi\{|T_{\mathbf{n}_{\alpha(l)},m_l}| : l < n(*)\}$ by m_*. For every $m < m_{**}$, denote by $U_{0,m}$ the mth equivalence class of E. Suppose towards contradiction that for every $m < m_{**}$ there is some η_m in $\cap\{\bigcup_\rho w_\rho : \bar{w} \in U_{0,m}\}$. For every m there is ρ_m such that $\eta_m \in suc_{T_{\mathbf{n}}}(\rho_m)$. Choose $\bar{w} = (w_\rho : \rho \in T_{\mathbf{n},l(*)})$ by letting $w_\rho = suc_{T_{\mathbf{n}}}(\rho) \setminus \{\eta_m : m < m_{**} \wedge \rho_m = \rho\}$. We shall prove that $\bar{w} \in U_0$. It will then follow that $\bar{w} \in U_{0,m}$ for some m, therefore $\eta_m \in \bigcup_\rho w_\rho$, contradicting the definition of w_ρ. This proves that U_0 is as required. In order to provvve that $\bar{w} \in U_0$, note that for every ρ, $|suc_{T_{\mathbf{n}}}(\rho) \setminus w_\rho| \leq |\{m : \rho_m = \rho\}| \leq m_{**} \leq m_* = \Pi\{|T_{\mathbf{n}_{\alpha(l)},m_l}| : l < n(*)\} \ll \mu_{\mathbf{n},\rho}$ (the last inequality follows by $(f)(+)$ and the choice of m_l). Therefore, $\bar{w} \in U_0$.

Suppose now that $n = k + 1 \leq n(*)$. Choose $q_k \in \mathbb{P}_{\alpha_k}$ such that $p_k \leq q_k$ and q_k forces a value $\Lambda^k_{\bar{v}}$ to $\{\rho \in p_{\bar{w}_{\bar{v}}}(\alpha_k) : lg(\rho) = m_k + 1\}$ for every $\bar{v} \in U_k$. For every $\rho \in T_{\mathbf{n}_{\alpha_k},m_k+1}$ let $U_{k,\rho} = \{\bar{v} \in U_k : \rho \in \Lambda^k_{\bar{v}}\}$. If $\bar{v} \in U_k$, then q_k forces the value $\Lambda^k_{\bar{v}}$ to $\{\rho \in p_{\bar{w}_{\bar{v}}}(\alpha_k) : lg(\rho) = m_k + 1\}$, hence $U_k = \cup\{U_{k,\rho} : \rho \in T_{\mathbf{n}_\alpha,m_k+1}\}$. WLOG $U_{k,\rho}$ are pairwise disjoint. Now suppose towards contradiction that none of them satisfies requirement (5) of the induction for $k+1$, then each $U_{k,\rho}$ has a counterexample E_ρ, and the union $\bigcup_\rho E_\rho$ is therefore an equivalence relation which is a counterexample to U_k satisfying (5). Therefore, for some ρ, $U_{k,\rho}$ satisfies (5), so choose $U_n = U_{k,\rho}$.

Define $p_n \in \mathbb{P}_{\alpha_k+1} \subseteq \mathbb{P}_{\alpha_n}$ as follows:

1. $p_n \restriction \alpha_k = q_k$.
2. $p_n(\alpha_k) = \cap\{p_{\bar{w}_{\bar{v}}}(\alpha_k)^{[\rho\leq]} : \bar{v} \in U_n\}$.

Now for every $\bar{v} \in U_k$, $p_{\bar{w}_{\bar{v}}} \restriction \alpha_k \leq p_k \leq q_k$, hence $q_k \Vdash_{\mathbb{P}_{\alpha_k}} \nu \in p_{\bar{w}_{\bar{v}}}(\alpha_k) \rightarrow 1 + \frac{1}{m} \leq nor(suc_{p_{\bar{w}_{\bar{v}}}(\alpha_k)}(\nu))$. We shall prove that $q_k \Vdash_{\mathbb{P}_{\alpha_k}} p_n(\alpha_k) \in \mathbb{Q}^2_{\mathbf{n}_\alpha}$. As, $|U_n| \leq |S^0_{l(*)}| \leq 2^{\Sigma\{\lambda_{\mathbf{n},\rho'} : \rho' \in \Lambda^0_{l(*)}\}} < \mu_{\mathbf{n}_{\alpha_k},\rho}$, the assumptions of claim 10 hold, the conclusion follows by the proof of claim 10. A similar argument (using the first part of claim 10) proves the claim for the case of $\mathbb{Q}^1_{\mathbf{n}}$.

So p_n obviously satisfies requirements 1,2 and 6.

□

7. Main measurability claim

We're now ready to prove the main result. We shall first prove that Cohen forcing (hence $\mathbb{Q}^i_{\mathbf{n}}$) turns the ground model set of reals into a null set with respect to our ideal. We will then prove the main result by using a Solovay-type argument.

Claim 22: For $\iota \in \{0, \frac{1}{2}, 1, 2\}$ we have $\Vdash_{Cohen}$ "there is a Borel set $\mathbf{B} \subseteq lim(T_{\mathbf{n}_*})$ such that $lim(T_{\mathbf{n}_*})^V \subseteq \mathbf{B}$ and $\mathbf{B}$ is $(\mathbb{Q}^\iota_{\mathbf{n}_*}, \underset{\sim}{\eta}^\iota_{\mathbf{n}_*})$-null".

Proof: Let $\mathbb{Q}$ be the set of finite functions with domain $\{\eta \in T_{\mathbf{n}_*} : lg(\eta) < k\}$ for some $k < \omega$ such that $f(\rho) \in suc_{T_{\mathbf{n}_*}}(\rho)$. $(\mathbb{Q}, \subseteq)$ is countable and for every $q \in \mathbb{Q}$ there are $q \leq q_1, q_2 \in \mathbb{Q}$ which are incompatibe, hence is equivalent to Cohen forcing. Let $\underset{\sim}{f} := \underset{g \in \underset{\sim}{G}}{\cup} g$. For $f \in S = \Pi\{suc_{T_{\mathbf{n}_*}}(\rho) : \rho \in T_{\mathbf{n}_*}\}$ define $\mathbf{B}_f := \{\eta \in lim(T_{\mathbf{n}_*}) :$ for infinitely many n we have $\eta \restriction (n+1) = f(\eta_n)\}$. For every $n < \omega$ let $\mathbf{B}_{f,n} = \{\eta \in lim(T_{\mathbf{n}_*}) : \eta \restriction (m+1) \neq f(\rho)$ if $n \leq m$ and $n \leq lg(\rho)\}$. Clearly, $\Vdash$ "$\underset{\sim}{f} \in S$", $\mathbf{B}^c_f = \underset{n<\omega}{\cup} \mathbf{B}_{f,n}$, and obviously each $\mathbf{B}_{f,n}$ is Borel, hence $\mathbf{B}_f$ is Borel. For every $\eta \in T_{\mathbf{n}_*}$ let $w_\eta = suc_{T_{\mathbf{n}_*}}(\eta) \setminus \{f(\eta)\}$. As in claim 21, $\Vdash_{\mathbb{Q}^\iota_{\mathbf{n}_*}}$ "$\underset{\sim}{\eta}^\iota_{\mathbf{n}_*} \in \mathbf{B}_{\bar{w}}$" for $\bar{w}$ and $\mathbf{B}_{\bar{w}}$ as in that proof. Hence $\Vdash_{\mathbb{Q}^\iota_{\mathbf{n}_*}}$ "$\underset{\sim}{\eta}^\iota_{\mathbf{n}_*} \notin \mathbf{B}_f$, so $\mathbf{B}_f$ is $(\mathbb{Q}^\iota_{\mathbf{n}_*}, \underset{\sim}{\eta}^\iota_{\mathbf{n}_*})$-null. Let $G \subseteq \mathbb{Q}$ be generic and let $g = \underset{\sim}{f}[G]$, so $\mathbf{B}_g$ is a $(\mathbb{Q}^\iota_{\mathbf{n}_*}, \underset{\sim}{\eta}^\iota_{\mathbf{n}_*})$-null Borel set in $V[G]$. We shall prove that $V[G] \models lim(T_{\mathbf{n}_*})^V \subseteq \mathbf{B}_g$. Let $\eta \in lim(T_{\mathbf{n}_*})^V$ and $m < \omega$, it's enough to show that in V, $\Vdash_{\mathbb{Q}}$ "for some $m \leq k$ and $\rho \in T_{\mathbf{n}_*}$, $\underset{\sim}{f}(\rho) = \eta \restriction (k+1)$". Let $p \in \mathbb{Q}$, we can extend p to a function $p \leq q$ with domain $\{\eta \in T_{\mathbf{n}_*} : lg(\eta) < k\}$ for some $m \leq k$. Now let $q \leq s$ be an extension of q with domain $\{\eta \in T_{\mathbf{n}_*} : lg(\eta) \leq k\}$ such that $s(\eta \restriction k) = \eta \restriction (k+1)$. Obviously, s forces the required conclusion, so we're done.

$\square$

Main conclusion 23: Let $i \in \{1, 2\}$. Let $V \models CH$ and suppose $\aleph_1 < \kappa = cf(\kappa) \leq \mu$. Let L be a linear order of cardinality μ and cofinality κ, such that for every proper initial segment $J \subseteq L$ and $t, s \in L \setminus J$, there is an automorphism π of L over J such that $\pi(s) = t$. Suppose that $\mathbf{q}$ is as in 13(A) such that $L_{\mathbf{q}} = L$ and $\mathbf{m}_t = \mathbf{m}$ for every $t \in L_{\mathbf{q}}$ is a (constant) definition of the forcing $\mathbb{Q}^i_{\mathbf{n}}$, then:

a) $\mathbb{P}_{\mathbf{q}}$ is a c.c.c. forcing notion of cardinality μ.

b) $\Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$2^{\aleph_0} = \mu$".

c) Let $G \subseteq \mathbb{P}_{\mathbf{q}}$ be generic over V, $\eta_t = \underset{\sim}{\eta}_t[G]$ for $t \in L_{\mathbf{q}}$, $X = \{\eta_t : t \in L_{\mathbf{q}}\}$ and let $V[X]$ be the collection of sets hereditarily definable from finite sequences of members of X, then:

(α) $V[X] \models ZF + \neg AC_{\aleph_0}$ and $lim(T_{\mathbf{n}})^{V[X]} = \cup\{lim(T_{\mathbf{n}})^{V[\{\eta_t : t \in u\}]} : u \subseteq L_{\mathbf{q}}$ is finite$\}$.

(β) $(\mathbb{Q}^i_{\mathbf{n}}, \aleph_1)$-measurability: Every $A \subseteq lim(T_{\mathbf{n}})^{V[X]}$ is $I_{\mathbb{Q}^i_{\mathbf{n}}, \aleph_1}$-measurable.

(γ) $\{\eta_t : t \in L_{\mathbf{q}}\} = lim(T_{\mathbf{n}})$ mod $I_{\mathbb{Q}^i_{\mathbf{n}}, \aleph_1}$.

(δ) If $J \subseteq L_{\mathbf{q}}$ is a proper initial segment then $\{\eta_t : t \in J\} \in I_{\mathbb{Q}^i_{\mathbf{n}}, \aleph_1}$.

(ϵ) The ideal $I_{\mathbb{Q}^i_{\mathbf{n}}, \aleph_1}$ is non-trivial.

(ζ) $\aleph_1$ is not collapsed, there is an ω_1-sequence of different reals, and if $V = L$ then $\aleph_1^L = \aleph_1^{V[X]}$.

Proof: Clause a) By the definition of $\mathbb{P}_{\mathbf{q}}$ and claim 17, so $|\mathbb{P}_{\mathbf{q}}| \leq \Sigma\{|\mathbb{P}_{\mathbf{q},J}| : J \subseteq L$ is finite$\} \leq 2^{\aleph_0} + |L|^{<\aleph_0} = 2^{\aleph_0} + \mu = \mu$.

Clause b) By a) we have $\Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$2^{\aleph_0} \leq \mu$", and as $|L| = \mu$ we have $\Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$\mu = |L| \leq |\{\underset{\sim}{\eta}_t : t \in L\}| \leq 2^{\aleph_0}$". Together we're done.

Clause c) (α) By the definitions of $V[X]$ and $\mathbb{P}_{\mathbf{q}}$. In particular, $\neg AC_{\aleph_0}$, as we can use $(A_n : n < \omega)$ where $A_n := \{\{\eta_{t_l} : l < n\} : t_0 <_L ... <_L t_{n-1}\}$.

Clause c)(β) Let $A \in V[X]$ be a subset of $lim(T_{\mathbf{m}_*})$. A is definable in $V[X]$ by a first order formula $\phi(x, \bar{a}, c)$ such that $c \in V$ and $\bar{a} = (\eta_{t_0}, ..., \eta_{t_{n-1}})$ is a finite sequence from X. Let $J = \{s \in L_{\mathbf{q}} : s \leq t_l$ for some $l\}$. For $s \in L \setminus J$ let $L_s = \{t_l : l < n\} \cup \{s\}$, then $L_s \in \mathbf{L}_{\mathbf{q}}$ hence by 14 we have $\mathbb{P}_{L_s} \lessdot \mathbb{P}_{L_{\mathbf{q}}}$. Let $\underset{\sim}{T}_s = TV(\phi(\underset{\sim}{\eta}_s, \bar{a}, c))$, so $\underset{\sim}{T}_s$ is a $\mathbb{P}_{L_{\mathbf{q}}}$-name and actually a $\mathbb{P}_{L_s}$-name.

Let $(p_{s,i} : i < \omega)$ be a maximal antichian in $\mathbb{P}_{L_s}$ and let $W_s \subseteq \omega$ such that $p_{s,i} \Vdash \underset{\sim}{T}_s = true$ if and only if $i \in W_s$. Define the $\mathbb{P}_{\{t_l : l<n\}}$-name $\underset{\sim}{U} := \{i < \omega : p_{s,i} \restriction \{t_l : l < n\} \in \underset{\sim}{G}_{\mathbb{P}_{\{t_l:l<n\}}}\}$.

If $G_0 \subseteq \mathbb{P}_{\{t_l:l<n\}}$ is generic over V and $U = \underset{\sim}{U}[G_0]$, then in $V[G_0]$, $(lim(p_{s,i}(s)[G_0]) : i \in U)$ are pairwise disjoint: by claim 7, if $p, q \in \mathbb{Q}_{\mathbf{n}}^{\iota}$ are incompatible and $\eta \in lim(p)$, then $\eta \notin lim(q)$ (otherwise, WLOG $lg(tr(p)) \leq lg(tr(q))$, and both $tr(p)$ and $tr(q)$ are initial segments of η, hence $tr(p) \leq tr(q) \in T_p$ which is a contradiction by claim 7). Hence it's enough to show that $((p_{s,i}(s)[G_0]) : i \in U)$ is an antichain in $V[G_0]$. Assume towards contradiction that for some $i \neq j \in U$ there is a common upper bound q for $p_{s,i}(s)[G_0]$ and $p_{s,j}(s)[G_0]$. Therefore there is a $\mathbb{P}_{\{t_l:l<n\}}$-name $\underset{\sim}{q}$ and $r \in G_0$ such that $r \Vdash_{\mathbb{P}_{\{t_l:l<n\}}}$ "$p_{s,i}(s), p_{s,j}(s) \leq \underset{\sim}{q}$". Since $i, j \in U$, we have $p_{s,i} \restriction \{t_l : l < n\}, p_{s,j} \restriction \{t_l : l < n\} \in G_0$, and as G_0 is directed, there is a common upper bound $r_1 \in G_0$ for $p_{s,i} \restriction \{t_l : l < n\}, p_{s,j} \restriction \{t_l : l < n\}$ and r. Now let $r^+ := r_1 \cup \{(s, \underset{\sim}{q})\} \in \mathbb{P}_{L_s}$, then obviousy r^+ is a common upper bound (in $\mathbb{P}_{L_s}$) for $p_{s,i}$ and $p_{s,j}$, which contradicts our assumption.

Moreover, $(p_{s,i}(s)[G_0] : i \in U)$ is a maximal antichain: If $q \in \mathbb{Q}_{\mathbf{n}}^{\iota\, V[G_0]}$ is incompatible with $p_{s,i}(s)[G_0]$ for every $i \in U$, then as before, there are $r \in G_0$ and a $\mathbb{P}_{\{t_l:l<n\}}$-name $\underset{\sim}{q}$ such that r forces that $\underset{\sim}{q}$ is incompatible with $p_{s,i}(s)$ for every $i \in U$. As before we can get a member of $\mathbb{P}_{L_s}$ that is incompatible with $(p_{s,i} : i < \omega)$, contradicting its maximality. Hence $(p_{s,i}(s)[G_0] : i \in U)$ is a maximal antichain in $V[G_0]$.

If $s_1, s_1 \in L_{\mathbf{q}} \setminus J$, by the homogeneity assumption, there is an autommorphism f of $L_{\mathbf{q}}$ over J such that $f(s_1) = s_2$. Therefore the natural map induced by f is mapping $\bar{a}$ to itself and $\underset{\sim}{\eta}_{s_1}$ to $\underset{\sim}{\eta}_{s_2}$. Hence $\underset{\sim}{T}_{s_1}$ is mapped to $\underset{\sim}{T}_{s_2}$. As $(\hat{f}(p_{s_1,i}) : i < \omega)$ and W_{s_1} have the same properties (with respect to $\underset{\sim}{T}_{s_2}$) as $(p_{s_2,i} : i < \omega)$ and W_{s_2}, we may assume WLOG that $W_{s_1} = W_{s_2}$ (denote it by W) and $\hat{f}(p_{s_1,i}) = p_{s_2,i}$.

Therefore, if $G_0 \subseteq \mathbb{P}_{\{t_l:l<n\}}$ is generic and $i \in \underset{\sim}{U}[G_0]$, then there is $p_i \in (\mathbb{Q}^{\iota}_{\mathbf{n}})^{V[G_0]}$ and W such that for every $s \in L \setminus J$, $p_{s,i}(s)[G_0] = p_i$ and $W_s = W$.

Work now in $V[G_0]$: Let $B := \cup\{lim(p_i) : i \in W \cap U\}$, so B is a Borel set and we shall prove that $A = B$ modulo the ideal: by clauses $(c)(\gamma) + (c)(\delta)$ proved below, it's enough to show that if $s \in L_{\mathbf{q}} \setminus J$, then $\eta_s \notin A\Delta B$.

Let $s \in L_{\mathbf{q}} \setminus J$ and $i \in U$, then $p_{s,i} \in \mathbb{P}_{L_s}/G_0$ and by the choice of $p_{s,i}$, $p_{s,i} \Vdash_{\mathbb{P}_{L_s}/G_0}$ "$\phi(\underset{\sim}{\eta_s}, \bar{a}, c)$ iff $\underset{\sim}{T_s} = true$ iff $i \in W$". In other words, in $V[G_0]$ we have: $p_i \Vdash_{\mathbb{Q}^{\iota}_{\mathbf{n}}}$ "$\phi(\underset{\sim}{\eta_s}, \bar{a}, c)$ iff $\underset{\sim}{i} \in W$". Since $(p_i : i \in U)$ is a maximal antichain, every $G \subseteq \mathbb{Q}^2_{\mathbf{n}}$ generic over $V[G_0]$ must contain exactly one of the p_i, hence in $V[G_0] : \Vdash_{\mathbb{Q}^{\iota}_{\mathbf{n}}}$ "$\phi(\underset{\sim}{\eta_s}, \bar{a}, c)$ iff $i \in W$ for the p_i such that $p_i \in \underset{\sim}{G}$". Now $p_{s,i}(s) = p_i \in \underset{\sim}{G}$ iff $\underset{\sim}{\eta_s} \in lim(T_{p_{s,i}(s)}) = lim(T_{p_i})$, hence we got $\Vdash_{\mathbb{Q}^2_{\mathbf{n}}}$ "$\phi(\underset{\sim}{\eta_s}, \bar{a}, c)$ iff $i \in W$ where i is such that $\underset{\sim}{\eta_s} \in lim(T_{p_i})$". Therefore $\Vdash_{\mathbb{Q}^{\iota}_{\mathbf{n}}}$ "$\underset{\sim}{\eta_s} \in A$ iff $\underset{\sim}{\eta_s} \in B$".

Clause c)(γ) If $\rho \in lim(T_{\mathbf{m}_*})^{V[X]} \setminus \{\eta_t : t \in L_{\mathbf{q}}\}$, then $\rho \in lim(T_{\mathbf{m}_*})^{V[\{\eta_t : t \in u\}]}$ for some finite u. By claim 21, ρ is not $(\mathbb{Q}^{\iota}_{\mathbf{m}_*}, \underset{\sim}{\eta_{\mathbf{m}_*}})$-generic over V. Therefore, by the definition of $I^2_{\mathbf{m}_*}$, $\Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$lim(T_{\mathbf{m}_*}) \setminus \{\underset{\sim}{\eta_t} : t \in L_{\mathbf{q}}\} \in I^2_{\mathbf{m}_*}$". Why can we use claim 21? Assume that in claim 21 α_* is finite, assumptions $(a) - (e)$ and (g) hold and (f) is replaced by (h) where:

(h) $p_* \Vdash_{\mathbb{P}} \underset{\sim}{\rho} \notin \{\underset{\sim}{\eta_\alpha} : \alpha < \alpha_*\}$.

There is a condition $p_* \leq p_{**}$ and a natural number k such that $p_{**} \Vdash_{\mathbb{P}} \underset{\sim}{\rho} \restriction k \notin \{\underset{\sim}{\eta_\alpha} \restriction k : \alpha < \alpha_*\}$ and p_{**} forces values to $\underset{\sim}{\rho} \restriction k$ and $\underset{\sim}{\eta_\alpha} \restriction k$ $(\alpha < \alpha_*)$, which will be deonted by ρ_* and η^*_α $(\alpha < \alpha_*)$. WLOG $\mathbf{k}(\mathbf{n}, \mathbf{n}_\alpha) \leq k$ for every α where $\mathbf{k}(\mathbf{n}, \mathbf{n}_\alpha)$ is as in the definition of "far".

For $\mathbf{n} \in \mathbf{N}$ and $\eta \in T_{\mathbf{n}}$, let $\mathbf{n}^{[\eta \leq]}$ be the natural restriction of $\mathbf{n}$ to $T^{[\eta \leq]}_{\mathbf{n}}$. Now let $\mathbf{n}_* = \mathbf{n}^{[\rho_* \leq]}$ and $\mathbf{n}^*_\alpha = \mathbf{n}^{[\eta^*_\alpha \leq]}_\alpha$. By the choice of k, $\mathbf{n}_*$ and $\mathbf{n}^*_{\eta_\alpha}$ are far, moreover, they satisfy assumption f(+) of claim 21, and by iterating $\mathbb{Q}^i_{\mathbf{n}^*_\alpha}$ instead, we get the desired conclusion.

Clause c)(δ) By claim 19, each $\mathbb{Q}_{\mathbf{m}_t}$ adds a Cohen real, hence the set of previous generics is included in a null Borel set by claim 22. More precisely: Supppose that $\underset{\sim}{\nu_t}$ is a $\underset{\sim}{\mathbb{Q}_t}$-name of a Cohen real, we shall prove that $\Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$\underset{\sim}{\nu_t}$ is Cohen over $V[(\underset{\sim}{\eta_s} : s < t)]$". Let $p_0 \in \mathbb{P}_{\mathbf{q}}$, let $\mathbb{P}_{<t}$ be defined as $\mathbb{P}_{\mathbf{q}}$ with $L_{<t}$ instead of $L_{\mathbf{q}}$ and let $\underset{\sim}{T}$ be a $\mathbb{P}_{<t}-$name such that: $p_0 \restriction L_{<t} \Vdash_{\mathbb{P}_{<t}}$ "$\underset{\sim}{T} \subseteq \omega^{<\omega}$ is a nowhere dense tree".

a. As $\Vdash_{\mathbb{P}_{<t}}$ " $\Vdash_{\underset{\sim}{\mathbb{Q}_t}}$ "$\underset{\sim}{\nu_t}$ is Cohen over $V^{\mathbb{P}_{<t}}$"", there is $p \leq p_1$ and $\rho \in \omega^{<\omega}$ such that:

1. $p_1(t) = p_0(t)$.

2. For every ν such that $\rho \leq \nu \in \omega^{<\omega}$ and $p_2 \in \mathbb{P}_{<t}$ such that $p_1 \restriction L_{<t} \leq p_2$, there is $p_3 \in \mathbb{P}_{\leq t}$ such that $p_1, p_2 \leq p_3$ and $p_3 \Vdash$ "$\nu \leq \underset{\sim}{\nu_t}$".

b. As $\underset{\sim}{T}$ is a name of a nowhere dense tree, there are $p_2 \in \mathbb{P}_{<t}$ and $\rho \leq \nu \in \omega^{<\omega}$ such that $p_1 \restriction L_{<t} \leq p_2$ and $p_2 \Vdash$ "$\nu \notin \underset{\sim}{T}$".

c. Combining (a) and (b), there is $p_1 \leq p_3$ such that $p_3 \Vdash$ "$\nu \leq \underset{\sim}{\nu_t}$", hence $p_3 \Vdash$ "$\underset{\sim}{\nu_t} \notin lim(\underset{\sim}{T})$", which proves the claim.

Clause c)(ϵ) Every definable set from $I_{\mathbb{Q}^i_{\mathbf{n}},\aleph_1}$ is contained in a union of $\aleph_1$ Borel sets from $I_{\mathbb{Q}^i_{\mathbf{n}},\aleph_1}$, and since the cofinality of L is $\kappa > \aleph_1$, there is a final segment of $\{\eta_t : t \in L\}$ not covered by them.

Clause c)(ζ) $V \models AC$, therefore there is an ω_1-sequence of distinct reals in V. $\mathbb{P}_{\mathbf{q}} \models ccc$, therefore $\aleph_1$ is not collapsed, and that sequence is as required in $V[X]$ as well. If $V = L$, then $\aleph_1^L = \aleph_1^{V[X]}$ follows from ccc.

$\square$

24. Discussion: As our model doesn't sasitfy $AC_{\aleph_0}$, it's natural to ask whether we can improve the result getting a model of $AC_{\aleph_0}$ or even DC. In [F1424] we prove that assuming the existence of a measurable cardinal, we can get a model of $DC(\aleph_1)$. This leads to the following question:

25. Problem 1: Can we improve the current result and get a model of DC without large cardinals?

As the current result gives measurability with respect to the ideal $I_{\mathbf{n},\aleph_1}$, it's natural to ask:

26. Problem 2: Can we get a similar result for the ideal $I_{\mathbf{n},\aleph_0}$?

We intend to address these problems in [F1495].

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