Can you take Toernquist's inaccessible away?

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Abstract

We prove that $ZF + DC + $ "There are no mad families" is equiconsistent with ZFC.[1]

Introduction

We study the possibility of the non-existence of mad families in models of $ZF + DC$. Recall that $\mathcal{F} \subseteq [\omega]^\omega$ is mad if $A, B \in \mathcal{F} \to |A \cap B| < \aleph_0$, and $\mathcal{F}$ is maximal with respect to this property. Assuming the axiom of choice, it's easy to construct mad families, thus leading to natural investigations concerned with the definability of mad families. By a classical result of Mathias [Ma], mad families can't be analytic (as opposed to the classical regularity properties, there might be Π_1^1 mad families, which is the case when $V = L$ [Mi]). The possibility of the non-existence of mad families was demonstrated by Mathias who proved the following result:

Theorem [Ma]: Suppose there is a Mahlo cardinal, then there is a model of $ZF + DC + $ "There are no mad families".

For a long time it was not known whether there are mad families in Levy's model (aka Solovay's model). This problem was recently settled by Toernquist:

Theorem [To]: There are no mad families in Levy's model.

Toernquist's proof is based on a new proof of the fact that mad families can't be analytic. It's now natural to wonder whether it's possibe to eliminate the large cardinal assumption from Toernquist's result. Our main result in this paper shows that the answer is positive:

Theorem: $ZF + DC_{\aleph_1} + $ "There are no mad families" is equiconsistent with ZFC.

Two other related families of interest are maximal eventually different families and maximal cofinitary groups. For a long time it was not known whether such families can be analytic, and whether there are models of $ZF + DC$ where no such families exist. We intend to settle those problems in a subsequent paper.

The proof

Hypothesis 1: 1. $\lambda = \lambda^{<\mu}$, $\mu = cf(\mu)$, $\alpha < \mu \to |\alpha|^{\aleph_1} < \mu$, $\aleph_0 < \theta = \theta^{\aleph_1} < \kappa = cf(\kappa) \leq \mu$ and $\alpha < \kappa \to |\alpha|^{\aleph_1} < \kappa$.

For example, assuming GCH, the hypothesis holds for $\mu = \aleph_3 = \kappa$, $\lambda = \aleph_4$ and $\theta = \aleph_2$.

2. For transparency, we may assume CH.

Definition 2: 1. Let $K = \{\mathbb{P} : \mathbb{P}$ is a ccc forcing notion such that $\Vdash_{\mathbb{P}} "MA_{\aleph_1}"\}$.

2. Let $\leq_K$ be the partial order $\lessdot$ on K.

3. We say that $(\mathbb{P}_\alpha : \alpha < \alpha_*)$ is $\leq_K$-increasing continuous if $\mathbb{P}_\alpha \in K$ for every $\alpha < \alpha_*$, $\alpha < \beta \to \mathbb{P}_\alpha \lessdot \mathbb{P}_\beta$ and if $\beta < \alpha_*$ is a limit ordinal then $\underset{\gamma<\beta}{\cup} \mathbb{P}_\gamma \lessdot \mathbb{P}_\beta$.

Claim 3: 1. $(K, \leq_K)$ has the amalgamation property.

2. If $\mathbb{P}_1$ is a ccc forcing notion, then there is $\mathbb{P}_2 \in K$ such that $\mathbb{P}_1 \lessdot \mathbb{P}_2$ and $|\mathbb{P}_2| \leq |\mathbb{P}_1|^{\aleph_1} + 2^{\aleph_1}$.

3. If $(\mathbb{P}_\alpha : \alpha < \delta)$ is $\leq_K$-increasing continuous and δ is a limit ordinal, then $\underset{\alpha<\delta}{\cup} \mathbb{P}_\alpha \models ccc$, hence by (2) there is $\mathbb{P}_\delta \in K$ such that $(\mathbb{P}_\alpha : \alpha < \delta)\char94(\mathbb{P}_\delta)$ is $\leq_K$-increasing continuous.

4. If $\mathbb{P} \in K$ and $X \subseteq \mathbb{P}$ such that $|X| < \mu$, then there exists $\mathbb{Q} \in K$ such that $X \subseteq \mathbb{Q}$, $\mathbb{Q} \leq_K \mathbb{P}$ and $|\mathbb{Q}| \leq 2^{\aleph_1} + |X|^{\aleph_1}$.

Proof: 1. Suppose that $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2 \in K$ and $f_l : \mathbb{P}_0 \to \mathbb{P}_l$ $(l = 1, 2)$ are complete embeddings. Let $\mathbb{P}_1 \times_{f_1,f_2} \mathbb{P}_2$ be the amalgamation of $\mathbb{P}_1$ and $\mathbb{P}_2$ over $\mathbb{P}_0$ (as in [RoSh672]), i.e. $\{(p_1, p_2) \in \mathbb{P}_1 \times \mathbb{P}_2 : (\exists p \in \mathbb{P}_0)(p \Vdash_{\mathbb{P}} "p_1 \in \mathbb{P}_1/f_1(\mathbb{P}_0) \wedge p_2 \in \mathbb{P}_2/f_2(\mathbb{P}_0)")\}$. As $\mathbb{P}_0 \lessdot \mathbb{P}_1 \times_{f_1,f_2} \mathbb{P}_2$, $\Vdash_{\mathbb{P}_0} "MA_{\aleph_1}"$ and $MA_{\aleph_1}$ implies that every ccc forcing notion is Knaster (and recalling that being Knaster is preserved under products), it follows that $\mathbb{P}_1 \times_{f_1,f_2} \mathbb{P}_2 \models ccc$, and by (2) we're done.

2. $\mathbb{P}_2$ is obtained as thee composition of $\mathbb{P}_1$ with the ccc forcing notion of cardinality $|\mathbb{P}_1|^{\aleph_1} + 2^{\aleph_1}$ forcing $MA_{\aleph_1}$.

4. As in the proof of subclaim 1 in caim 6 (see next page). $\square$

Claim 4: There is a ccc forcing notion $\mathbb{P}$ of cardinality λ such that:

1. For every $X \subseteq \mathbb{P}$, $|X| < \mu \to (\exists \mathbb{Q} \in K)(X \subseteq \mathbb{Q} \lessdot \mathbb{P} \wedge |\mathbb{Q}| < \mu)$.

2. If $\mathbb{P}_1, \mathbb{P}_2 \in K$ have cardinality $< \mu$, $\mathbb{P}_1 \lessdot \mathbb{P}_2$ and f_1 is a complete embedding of $RO(\mathbb{P}_1)$ into $RO(\mathbb{P})$, then there is $f_1 \subseteq f_2$ that is a complete embedding of $RO(\mathbb{P}_2)$ into $RO(\mathbb{P})$.

Proof: We choose $\mathbb{P}_\alpha \in K$ by induction on $\alpha < \lambda$, such that the sequence is $\leq_K$ −increasing continuous and each $\mathbb{P}_\alpha$ has cardinality λ, as follows:

1. For limit α we choose $\mathbb{P}_\alpha \in K$ such that $\bigcup_{\beta<\alpha} \mathbb{P}_\beta \lessdot \mathbb{P}_\alpha$. We can do it by claim 3(2) and the induction hypothesis.

2. For $\alpha = \beta + 1$, we let $((\mathbb{P}_1^\gamma, \mathbb{P}_2^\gamma, f_1^\gamma) : \gamma < \lambda)$ be an enumeration of all triples as in 4(2) for $\mathbb{P}_\beta$. We construct a $\leq_K$ −increaasing continuous sequence $(\mathbb{P}_\gamma^* : \gamma \leq \lambda)$ by induction as follows: $\mathbb{P}_0^* = \mathbb{P}_\beta$. $\mathbb{P}_{\gamma+1}^*$ is the result of a K−amalgamation for the γth triple, and for limit γ we define $\mathbb{P}_\gamma^*$ as in (1). Finally, we let $\mathbb{P}_\alpha = \mathbb{P}_\lambda^*$.

Note that by claim 3(4), requirement (1) is satisfied for every forcing notion from K, hence it's enough to guarantee that requirement (2) is satisfied. It's now easy to see that $\mathbb{P} = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha$ is as required. $\square$

Definition/Claim 5: Let $\mathbb{P}$ be the forcing notion from claim 4 and let $G \subseteq \mathbb{P}$ be generic over V. In $V[G]$, let $V_1 = HOD(\mathbb{R}^{<\kappa})$, then $V_1 \models ZF + DC_{<\kappa}$. $\square$

Main claim 6: There are no mad families in V_1.

Proof: Let $\underset{\sim}{\mathcal{F}}$ be a canonical $\mathbb{P}$-name of a mad family (i.e. a canonical $\mathbb{P}$-name of a family of subsets of ω), and let $\bar{\eta}$ be a sequence of length $< \kappa$ of canonical $\mathbb{P}$-names of reals such that $\underset{\sim}{\mathcal{F}}$ is definable over V using $\bar{\eta}$. Let $K_\mathbb{P} = \{\mathbb{Q} \in K : \mathbb{Q} \lessdot \mathbb{P} \wedge |\mathbb{Q}| < \kappa\}$. By claim 4(1), there is $\mathbb{Q}_* \in K_\mathbb{P}$ such that $\bar{\eta}$ is a canonical $\mathbb{Q}_*$-name. Let $K_\mathbb{P}^+$ be the set of $\mathbb{Q} \in K_\mathbb{P}$ such that $\mathbb{Q}_* \lessdot \mathbb{Q}$ and $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}$ is a canonical $\mathbb{Q}$-name of a mad family in $V^\mathbb{Q}$, where $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q} = \{\underset{\sim}{a} : \underset{\sim}{a}$ is a canonical $\mathbb{Q}$-name of a subset of ω such that $\Vdash_\mathbb{P} "\underset{\sim}{a} \in \underset{\sim}{\mathcal{F}}"\}$.

Subclaim 1: $K_\mathbb{P}^+$ is $\lessdot$-dense in $K_\mathbb{P}$.

Proof: Let $\mathbb{Q} \in K_\mathbb{P}$ and let $\sigma = |\mathbb{Q}_* + 2|^{\aleph_1} < \kappa$. We choose Z_i by induction on $i < \omega_2$ such that:

a. $Z_i \subseteq \mathbb{P}$ and $|Z_i| \leq \sigma$.

b. $j < i \rightarrow Z_j \subseteq Z_i$.

c. $Z_0 = \mathbb{Q}_* \cup \mathbb{Q}$.

d. If $i = 3j + 1$, then for every canonical name using members of Z_{3j} of an "$MA_{\aleph_1}$ problem" in Z_i we have a name for a solution.

e. If $i = 3j + 2$, then $Z_i \lessdot \mathbb{P}$.

f. If $i = 3j + 3$, then for every canonical Z_{3j+2}-name $\underset{\sim}{a}$ of an infinite subset of ω, there is a canonical Z_i-name $\underset{\sim}{b}$ such that $\Vdash_\mathbb{P} "|\underset{\sim}{a} \cap \underset{\sim}{b}| = \aleph_0 \wedge \underset{\sim}{b} \in \underset{\sim}{\mathcal{F}}"$.

It's now easy to verify that Z_{ω_2} is as required: By (c) and (e), $\mathbb{Q}_* \lessdot Z_{\omega_2} \lessdot \mathbb{P}$, hence also $Z_{\omega_2} \models ccc$. By (a), $|Z_{\omega_2}| < \kappa$. By (d), $\Vdash_{Z_{\omega_2}} "MA_{\aleph_1}"$ (given names for $\aleph_1$ dense sets, we have canonical names depending on $\aleph_1$ conditions, hence there is some $j < \omega_2$ such that they are Z_{3j+2}-names), hence $Z_{\omega_2} \in K$. By (f), $\underset{\sim}{\mathcal{F}} \restriction Z_{\omega_2}$ is a canonical Z_{ω_2}-name of a mad family in $V^{Z_{\omega_2}}$.

We shall now prove that such Z_i can be constructed for $i \leq \omega_2$: For $i = 0$ it's given by (c) and for limit ordinals we simply take the union. For $i = 3j + 1$ and $i = 3j + 3$ we enumerate the canonical names for either the $MA_{\aleph_1}$ problem or the infinite subsets of ω (depending on the stage of the induction), there are $\leq \sigma$ such names. At stage $3j + 1$ we use the fact that $\mathbb{P}$ forces $MA_{\aleph_1}$ in order to extend Z_{3j} using $\mathbb{P}$-names for the solutions of the $MA_{\aleph_1}$-problems. At stage $3j + 3$, we extend the forcing similarly, using the fact that $\underset{\sim}{\mathcal{F}}$ is a name of a mad family. For $i = 3j + 2$, we let Z_{3j+2} be the closure of Z_{3j+1} under the functions $f_1 : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ and $f_2 : [\mathbb{P}]^{[\leq \aleph_0]} \to \mathbb{P}$ where: $f_1(p, q)$ is a common upper bound of p and q if they're compatible, and $f_2(X)$ is incompatible with all members of X provided that X is countable and not predense.

Subclaim 2: If $\mathbb{Q} \in K_{\mathbb{P}}^+$ and $F : \mathbb{Q} \to \mathbb{P}$ is a complete embedding over $\mathbb{Q}_*$, then F maps $\underset{\sim}{\mathcal{F}} \restriction \mathbb{Q}$ to $\underset{\sim}{\mathcal{F}} \restriction F(\mathbb{Q})$.

Proof: As F is the identity over $\mathbb{Q}_*$ and $\underset{\sim}{\mathcal{F}}$ is definable using a $\mathbb{Q}_*$-name.

We now arrive at the two main subclaims:

Subclaim 3: There is a pair $(\mathbb{Q}, \underset{\sim}{D})$ such that:

a. $\mathbb{Q}_* \lessdot \mathbb{Q} \in K_{\mathbb{P}}^+$.

b. $\underset{\sim}{D}$ is a name of a Ramsey ultrafilter on ω.

c. $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D} \cap (\underset{\sim}{F} \restriction \mathbb{Q}) = \emptyset$".

Subclaim 4: Subclaim 3 implies claim 6.

Proof of subclaim 4: Let $\mathbb{M}_D$ be the $\mathbb{Q}$-name for the Mathias forcing restricted to the ultrafilter $\underset{\sim}{D}$. Let $\mathbb{Q}_1 \in \underset{\sim}{K}$ such that $\mathbb{Q} \star \mathbb{M}_D \lessdot \mathbb{Q}_1$ and $|\mathbb{Q}_1| < \kappa$ (such forcing notion exists by $3(2)$), and let $\underset{\sim}{A_1}$ be the $\mathbb{Q}_1$-name for the $\mathbb{M}_D$-generic real.

Let $F_1 : \mathbb{Q}_1 \to \mathbb{P}$ be a complete embedding such that F_1 is thee identity on $\mathbb{Q}$ (such embedding exists by claim 4(2). There is $\mathbb{Q}_1' \in K_{\mathbb{P}}^+$ such that $F_1(\mathbb{Q}_1) \lessdot \mathbb{Q}_1'$ by subclaim 1. There is a pair $(\mathbb{Q}_1'', F_1')$ such that $\mathbb{Q}_1 \lessdot \mathbb{Q}_1''$ and $F_1' : \mathbb{Q}_1'' \to \mathbb{Q}_1'$ is an isomorphism extending F_1'. WLOG $(\mathbb{Q}_1'', F_1') = (\mathbb{Q}_1, F_1)$, so $F_1(\mathbb{Q}_1) \in K_{\mathbb{P}}^+$.

Let $\underset{\sim}{\mathcal{F}_1} = F_1^{-1}(\underset{\sim}{\mathcal{F}} \restriction F_1(\mathbb{Q}_1))$.

As $\Vdash_{F_1(\mathbb{Q}_1)}$ "$\underset{\sim}{\mathcal{F}} \restriction F_1(\mathbb{Q}_1)$ is mad", it follows that $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{\mathcal{F}_1}$ is mad", hence there is some $\underset{\sim}{a_1}$ such that $\underset{\sim}{a_1}$ is a canonical $\mathbb{Q}_1$-name for a subset of ω, $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{a_1} \in \underset{\sim}{\mathcal{F}_1}$" and $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{a_1} \cap \underset{\sim}{A_1}$ is infinite". Recalling the basic property of the forcing $\mathbb{M}_D$, every infnite subset of $\underset{\sim}{A_1}$ is generic, therefore, by considering $\underset{\sim}{A_1} \cap \underset{\sim}{a_1}$ instead of $\underset{\sim}{A_1}$, WLOG $\Vdash_{\mathbb{Q}_1}$ "$\underset{\sim}{A_1} \subseteq \underset{\sim}{a_1}$".

Now let $(\mathbb{Q}_2, \mathbb{M}_D, \underset{\sim}{a_2}, \underset{\sim}{A_2}, \underset{\sim}{\mathcal{F}_2})$ be an isomorphic copy of $(\mathbb{Q}_1, \mathbb{M}_D, \underset{\sim}{a_1}, \underset{\sim}{A_1}, \underset{\sim}{\mathcal{F}_1})$ such that the isomorphism is over $\underset{\sim}{\mathbb{Q}}$. Consider the amalgamation $\mathbb{Q}_3 = \underset{\sim}{\mathbb{Q}_1} \times_{\mathbb{Q}} \underset{\sim}{\mathbb{Q}_2}$. By the

basic properties of $\mathbb{P}$, there is a complete embedding $F_3 : \mathbb{Q}_3 \to \mathbb{P}$ over $\mathbb{Q}$. By the density of $K_{\mathbb{P}}^+$, there is $\mathbb{Q}_4' \in K_{\mathbb{P}}^+$ such that $F_3(\mathbb{Q}_3) \lessdot \mathbb{Q}_4'$. As before, choose $(\mathbb{Q}_4, F_4)$ such that $\mathbb{Q}_3 \lessdot \mathbb{Q}_4$ and $F_4 : \mathbb{Q}_4 \to \mathbb{Q}_4'$ is an isomorphism extending F_3.

Now observe that $\Vdash_{\mathbb{Q}_4}$ "$A_1 \cap A_2$ is infinite": Let $G \subseteq \mathbb{Q}$ be generic, then in $V[G]$ we have: $\mathbb{Q}_3/G = (\mathbb{Q}_1/G) \times (\mathbb{Q}_2/G) \lessdot \mathbb{Q}_4/G$. As $\mathbb{M}_D[G] \lessdot \mathbb{Q}_l/\mathbb{Q}$ ($l = 1, 2$), we have $\mathbb{M}_D[G] \times \mathbb{M}_D[G] \lessdot \mathbb{Q}_3/G$, so it's enough to show that $\Vdash_{\mathbb{M}_D[G] \times \mathbb{M}_D[G]}$ "$|A_1 \cap A_2| = \aleph_0$": Let $((w_1, B_1), (w_2, B_2)) \in \mathbb{M}_D[G] \times \mathbb{M}_D[G]$ and $n < \omega$, so $B_1 \cap B_2 \in D[G]$ is infinite, therefore, there is $n_1 > n, \sup(w_1 \cup w_2)$ such that $n_1 \in B_1 \cap B_2$. Let $q = ((w_1 \cup \{n_1\}, B_1 \setminus (n_1 + 1)), (w_2 \cup \{n_1\}, B_2 \setminus (n_1 + 1)))$, then $p \leq q$ and $q \Vdash$ "$n_1 \in A_1 \cap A_2$".

Therefore, $\Vdash_{\mathbb{Q}_4}$ "$a_1 \cap a_2$ is infinite" (as the intersection contains $A_1 \cap A_2$).

It now follows that $\Vdash_{\mathbb{Q}_4}$ "$a_1 = a_2$": First note that $\Vdash_{F_4(\mathbb{Q}_4)}$ "$F_4(a_1), F_4(a_2) \in \mathcal{F} \upharpoonright F_4(\mathbb{Q}_4)$". Now $F_4(\mathbb{Q}_4) = \mathbb{Q}_4' \in K_{\mathbb{P}}^+$, so $\mathcal{F} \upharpoonright F_4(\mathbb{Q}_4)$ is a canonical $F_4(\mathbb{Q}_4)$-name of a mad family, therefore $\Vdash_{F_4(\mathbb{Q}_4)}$ "$F_4(a_1) = F_4(a_2)$", hence $\Vdash_{\mathbb{Q}_4}$ "$a_1 = a_2$".

It's now enough to show that $\Vdash_{\mathbb{Q}_4}$ "$a_1 = a_2 \in V^{\mathbb{Q}}$": Work in $V[G]$. First note that $\Vdash_{\mathbb{Q}_l/G}$ "A_l is almost contained in every member of $D[G]$, hence (by subclaim 3) it's almost disjoint to every member of $F \upharpoonright \mathbb{Q}$", and also $\Vdash_{\mathbb{Q}_l/G}$ "$a_l \in V^{\mathbb{Q}}$, hence $a_l \in F \upharpoonright \mathbb{Q}$". Now recall that $\Vdash_{\mathbb{Q}_l}$ "$A_l \subseteq a_l$", together we get a contradiction. Therefore, it remains to show that $\Vdash_{\mathbb{Q}_4}$ "$a_1 = a_2 \in V^{\mathbb{Q}}$": By the claim above, $\Vdash_{\mathbb{Q}_3}$ "$a_1 = a_2$". Work in $V[G]$, so a_l is a $\mathbb{Q}_l/G$-name ($l = 1, 2$). Suppose that the claim doesn't hold, then there are $q_1, r_1 \in \mathbb{Q}_1/G$ and $n < \omega$ such that $q_1 \Vdash$ "$n \in a_1$" and $r_1 \Vdash$ "$n \notin a_1$". Let $q_2, r_2 \in \mathbb{Q}_2/G$ be the "conjugates" of (q_1, r_1) (i.e. their images under the isomorphism that was previously mentioned), then $(q_1, r_2) \in \mathbb{Q}_3/G$ forces that $n \in a_1$ and $n \notin a_2$, contradicting the fact tha $\Vdash_{\mathbb{Q}_3}$ "$a_1 = a_2$". This completes the proof of subclaim 4.

Proof of subclaim 3: Let $\sigma = |\mathbb{Q}_*|^{\aleph_1} < \kappa$. We choose $(\mathbb{Q}_\epsilon, A_\epsilon)$ by induction on $\epsilon < \sigma^+$ such that:

a. $\mathbb{Q}_\epsilon \in K_{\mathbb{P}}^+$ and $|\mathbb{Q}_\epsilon| \leq \sigma$.

b. A_ϵ is a cononical $\mathbb{Q}_\epsilon$-name of a subset of ω.

c. $\Vdash_{\mathbb{Q}_\epsilon}$ "A_ϵ is not almost included in a finite union of elements of $\mathcal{F} \upharpoonright \mathbb{Q}_\epsilon$.

d. $(\mathbb{Q}_0, A_0) = (\mathbb{Q}_*, \omega)$. WLOG $\mathbb{Q}_* \in K_{\mathbb{P}}^+$, as $K_{\mathbb{P}}^+$ is $\lessdot$-dense in $K_{\mathbb{P}}$.

e. $(\mathbb{Q}_\zeta : \zeta < \epsilon)$ is $\lessdot$-increasing.

f. $\Vdash_{\mathbb{Q}_\epsilon}$ "$(A_\zeta : \zeta < \epsilon)$ is $\subseteq^*$-decreasing".

g. If $\epsilon = 2\xi + 1$ and $\Lambda_\epsilon \neq \emptyset$ where $\Lambda_\epsilon = \{(\zeta, \underset{\sim}{a}) : \zeta \leq \xi, \underset{\sim}{a}$ is a canonical $\mathbb{Q}_\zeta$-name of a subset of ω such that $\nVdash_{\mathbb{Q}_{2\xi}}$ "$A_{2\xi} \subseteq^* \underset{\sim}{a}$ or $A_{2\xi} \subseteq^* \omega \setminus \underset{\sim}{a}$"$\}$, then letting $\Gamma_\epsilon = \{\zeta : (\zeta, \underset{\sim}{a}) \in \Lambda_\epsilon\}$ and $\zeta_\epsilon = min(\Gamma)$, for some $\underset{\sim}{a_\epsilon}$, $(\zeta_\epsilon, \underset{\sim}{a_\epsilon}) \in \Lambda_\epsilon$ and $\Vdash_{\mathbb{Q}_\epsilon}$ "$A_\epsilon \subseteq^* \underset{\sim}{a_\epsilon}$ or $\underset{\sim}{A_\epsilon} \subseteq^* (\omega \setminus \underset{\sim}{a_\epsilon})$".

h. If $\epsilon = 2\xi + 2$ and $\mathcal{F}_\epsilon \neq \emptyset$ where $\mathcal{F}_\epsilon = \{(\zeta, \underset{\sim}{f}) : \zeta \leq \xi$ and $\underset{\sim}{f}$ is a canonical $\mathbb{Q}_\zeta$-name of a function from $[\omega]^2$ to $\{0, 1\}$ such that $\Vdash_{\mathbb{Q}_{2\xi+1}}$ "$\neg(\exists n)\underset{\sim}{f} \restriction [A_\xi \setminus n]^2$ is constant", $\underset{n<\omega}{\bigwedge} \underset{l<2}{\bigvee} A_{\epsilon-1} \subseteq^* \{i : \underset{\sim}{f}(i, n) = l\}$ and $\underset{l<2}{\bigvee} A_{\epsilon-1} \subseteq^* \{n : (\forall^\infty i \in \underset{\sim}{A_{\epsilon-1}})f(i, n) = l\}$ $\}$, then letting $\Gamma_\epsilon = \{\zeta : (\zeta, \underset{\sim}{f}) \in \mathcal{F}_\epsilon\}$ and $\zeta_\epsilon = min(\Gamma_\epsilon)$, for some $\underset{\sim}{f_\epsilon}$, $(\zeta_\epsilon, \underset{\sim}{f_\epsilon}) \in \mathcal{F}_\epsilon$ and $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{f_\epsilon} \restriction [\underset{\sim}{A_\epsilon}]^2$ is constant".

Subclaim 3a: The above induction can be carried for every $\epsilon < \sigma^+$.

Subclaim 3b: Subclaim 3 is implied by subclaim 3a.

Proof of Subclaim 3b: First we consider the case where $\sigma^+ < \kappa$. Let $\mathbb{Q} = \underset{\epsilon < \sigma^+}{\bigcup} \mathbb{Q}_\epsilon$, note that as $\aleph_2 \leq cf(\sigma^+)$, $\mathbb{Q} \in K_{\mathbb{P}}^+$. By the choice of $\mathbb{Q}_0$, $\mathbb{Q}_* \lessdot \mathbb{Q}$. Now define a $\mathbb{Q}$-name $\underset{\sim}{D} := \{B : B$ is a canonical $\mathbb{Q}$-name of a subset of ω such that $\Vdash_\mathbb{Q}$ "$(\exists \epsilon < \sigma^+)(\underset{\sim}{A_\epsilon} \subseteq^* B)$"$\}$. By (g), $\Vdash_\mathbb{Q}$ "$\underset{\sim}{D}$ is an ultrafilter": For example, in order to see that $\underset{\sim}{D}$ is forced to be upwards closed, suppose that $p_1 \Vdash$ "$\underset{\sim}{B} \subseteq^* \underset{\sim}{A} \subseteq \omega$ and $\underset{\sim}{B} \in \underset{\sim}{D}$", then there are $p_1 \leq p_2$, $n < \omega$ and $\epsilon < \sigma^+$ such that $p_2 \Vdash$ "$\underset{\sim}{B} \setminus n \subseteq \underset{\sim}{A}$ and $\underset{\sim}{A_\epsilon} \setminus n \subseteq \underset{\sim}{B}$". There is a condition p_3 and a canonical name $\underset{\sim}{A_3}$ such that $p_2 \leq p_3$ and $p_3 \Vdash$ "$\underset{\sim}{A} = \underset{\sim}{A_3}$". Let $\{p_{3,i} : i < \omega\}$ be a maximal antichain in $\mathbb{Q}$ such that $p_3 = p_{3,0}$ and let $\underset{\sim}{A_4}$ be the $\mathbb{Q}$-name defined as:

1. $\underset{\sim}{A_4}[G_\mathbb{Q}] = \underset{\sim}{A_3}[G_\mathbb{Q}]$ if $p_{3,0} \in G_\mathbb{Q}$

2. $\underset{\sim}{A_4}[G_\mathbb{Q}] = \underset{\sim}{B}[G_\mathbb{Q}]$ if $p_{3,0} \notin G_\mathbb{Q}$.

Therefore, $\underset{\sim}{A_4}$ is a canonical name for a subset of ω, $\Vdash$ "$\underset{\sim}{A_4} \in \underset{\sim}{D}$" and $p_3 \Vdash$ "$\underset{\sim}{A_4} = \underset{\sim}{A}$".

In order to see that for every $\mathbb{Q}$-name $\underset{\sim}{a} \subseteq \omega$, it's forced that $\underset{\sim}{a} \in \underset{\sim}{D} \vee \omega \setminus \underset{\sim}{a} \in \underset{\sim}{D}$, we have to show that every such name is being handled by clause (g) at some stage of the induction. Suppose that for some name $\underset{\sim}{a}$ it's not the case. Each such name is a $\mathbb{Q}_\zeta$-name for some $\zeta < \sigma^+$, so pick a minimal ζ for which there is such a $\mathbb{Q}_\zeta$-name. Therefore, for every $\epsilon = 2\xi + 1$ such that $\zeta \leq \xi$, $\zeta_\epsilon \leq \zeta$, so at each such stage we're handling a $\mathbb{Q}_\zeta$-name. As $|\mathbb{Q}_\zeta|^{\aleph_0} \leq \sigma$, the number of $\mathbb{Q}_\zeta$-names is at most σ and the number of induction steps is larger, we get a contradiction. Similarly, it follows by (h) that $\Vdash_\mathbb{Q}$ "$\underset{\sim}{D}$ is a Ramsey ultrafilter": Let $\underset{\sim}{f}$ be a $\mathbb{Q}$−name of a function from $[\omega]^2$ to $\{0, 1\}$ (wlog $\underset{\sim}{f}$ is a canonical name). As $\Vdash_\mathbb{Q}$ "$\underset{\sim}{D}$ is an ultrafilter", for every $n < \omega$, $\{\{i : \underset{\sim}{f}(i, n) = l\} : l < 2\}$ is a $\mathbb{Q}$−name of a partition of ω in $V^\mathbb{Q}$, hence

for some $l_{f,n}$, $V^{\mathbb{Q}} \models$ "$\{i : \underset{\sim}{f}(i,n) = \underset{\sim}{l}_{f,n}\} \in \underset{\sim}{D}$", and therefore, for some $\underset{\sim}{\xi} = \underset{\sim}{\xi}_{f,n}$,

$V^{\mathbb{Q}} \models$ "$A_{\underset{\sim}{\xi}} \subseteq^* \{i : \underset{\sim}{f}(i,n) = \underset{\sim}{l}_{f,n}\}$". Now $\{\{n : \underset{\sim}{l}_{f,n} = k\} : k < 2\}$ is a canonical

$\mathbb{Q}$−name of a partition of ω, so again, there is k_f such that $\{n : \underset{\sim}{l}_{f,n} = k_f\} \in \underset{\sim}{D}$, and

there is $\underset{\sim}{\xi_1}$ such that $A_{\underset{\sim}{\xi_1}} \subseteq^* \{n : \underset{\sim}{l}_{f,n} = k_f\}$. As $\mathbb{Q} \models ccc$, there is $\underset{\sim}{\xi} < \sigma^+$ such that

all of the above names are $\mathbb{Q}_\xi$−names and $\Vdash_{\mathbb{Q}_\xi}$ "$\underset{\sim}{\xi_2}, \underset{\sim}{\xi}_{f,n} \leq \xi$ for every $n < \omega$". As

the sequence of the A_ζ is $\subseteq^*$-decreasing, $\underset{\sim}{f}$ has the form of the functions appearing

in requirement (h) of the induction, hence by (h) there is a large homogeneous set
for $\underset{\sim}{f}$.

By (c), it follows that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D} \cap \underset{\sim}{\mathcal{F}} \upharpoonright \mathbb{Q} = \emptyset$"

We now consider the case where $\sigma^+ = \kappa$. In this case we add a slight modification
to our inductive construction: The induction is now on $\epsilon < \sigma$. We fix a partition
$(S_\xi : \xi < \sigma)$ of σ such that $|S_\xi| = \sigma$ and $S_\xi \cap \xi = \emptyset$ for each $\xi < \sigma$. At stage ξ
of the induction we fix enumertions $(a_i^\xi : i \in S_\xi)$ and $(f_i^\xi : i \in S_\xi)$ of the canonical
$\mathbb{Q}_\xi$-names for the subsets of ω and the 2-colorings of $[\omega]^2$ such that for some $\zeta < \xi$,
A_ζ satisfies the condition from (h) with respect to $\underset{\sim}{f_i^\xi}$.

We now replace the original (g) and (h) by (g)' and (h)' as follows:

(g)' If $\epsilon = 2i + 1$ and $i \in S_\xi$ then $\Vdash_{\mathbb{Q}_\epsilon}$ "$A_{2\xi} \subseteq^* \underset{\sim}{a_i^\xi} \vee A_{2\xi} \subseteq^* \omega \setminus \underset{\sim}{a_i^\xi}$".

(h)' If $\epsilon = 2i + 2$ and $i \in S_\xi$ then $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{f_i^\xi}[\underset{\sim}{A_\epsilon}]$ is constant".

Note that $\xi \leq i$ in the clauses above, as $S_\xi \cap \xi = \emptyset$, therefore, at stage $\epsilon = 2i + l$
$(l = 1, 2)$, the names $\underset{\sim}{a_i^\xi}$ and $\underset{\sim}{f_i^\xi}$ are well-defined when $i \in S_\xi$.

As $\aleph_2 \leq cf(\sigma)$, then as before, letting $\mathbb{Q} = \bigcup_{\epsilon < \sigma} \mathbb{Q}_\epsilon$, $\mathbb{Q}_* \lessdot \mathbb{Q} \in K_{\mathbb{P}}^+$. As before, $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$
is a filter", and by clause (c), $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D} \cap \underset{\sim}{\mathcal{F}} \upharpoonright \mathbb{Q} = \emptyset$", and by (g)', $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is an
ultrafilter". By (h)', $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{D}$ is a Ramsey ultrafilter" (the argument is the same as
in the case of $\sigma^+ < \kappa$), so we're done.

Proof of subclaim 3a:

We give the argument for the case $\sigma^+ < \kappa$. The case $\sigma^+ = \kappa$ is essentially the same.

Case I ($\epsilon = 0$): Trivial.

Case II ($\epsilon = 2\xi + 1$): We let $\mathbb{Q}_\epsilon = \mathbb{Q}_{2\xi}$. Pick some $(\zeta_\epsilon, a_\epsilon) \in \Lambda_\epsilon$, the $\mathbb{Q}_\epsilon$-name $\underset{\sim}{A_\epsilon}$
will be defined as follows: If $A_{2\xi} \cap a_\epsilon$ satisfies clause (c) of the induction, then we let
$\underset{\sim}{A_\epsilon} = \underset{\sim}{A_{2\xi}} \cap \underset{\sim}{a_\epsilon}$. Otherwise, let $\underset{\sim}{A_\epsilon} = \underset{\sim}{A_{2\xi}} \setminus \underset{\sim}{a_\epsilon}$. We need to show that $\underset{\sim}{A_\epsilon}$ satifies clause

(c). Suppose not, then both $A_{2\xi} \cap a_\epsilon$ and $A_{2\xi} \setminus a_\epsilon$ don't satisfy clause (c), but then $A_{2\xi}$ is almost included in a finite union of elements of $\mathcal{F} \restriction \mathbb{Q}_{2\xi}$, a contradiction.

Case III ($\epsilon = 2\xi + 2$): Pick some $(\zeta_\epsilon, f_\epsilon) \in \mathcal{F}_\epsilon$. By the definition of $\mathcal{F}_\epsilon$, in $V^{\mathbb{Q}_{\epsilon-1}}$, for every $n < \omega$ there are $l_n^\epsilon < 2$ and $k_n^\epsilon < \omega$ such that for every $k \in A_{\epsilon-1}$, if $k_n^\epsilon \leq k$ then $f(k, n) = l_n^\epsilon$. In addition, there are k_ϵ, l_ϵ such that $k_\epsilon \leq n \in A_{\epsilon-1} \to l_n^\epsilon = l_\epsilon$.

WLOG $k_n^\epsilon < k_{n+1}^\epsilon$ for every $n < \omega$. By the induction hypothesis, as $\Vdash_{\mathbb{Q}_{\epsilon-1}}$ "$\mathcal{F} \restriction \mathbb{Q}_{\epsilon-1}$ is mad" and as A_ϵ satisfies clause (c), there are pairwise distinct $a_{\epsilon,n} \in \mathcal{F} \restriction \mathbb{Q}_{\epsilon-1}$ such that $b_{\epsilon,n} = a_{\epsilon,n} \cap A_{\epsilon-1}$ is infinite for every $n < \omega$. We now choose n_i by induction on i such that:

a. $n_i \in A_{\epsilon-1} \setminus k_\epsilon$.

b. If $i = j + 1$ then $n_i > n_j$ and $n_i > k_{n_j}^\epsilon$.

c. If $i \in (j^2, (j+1)^2)$ then $n_i \in b_{\epsilon, i-j^2}$.

This should suffice: By (a)+(b), $f \restriction \{n_i : i < \omega\}$ is constantly l_ϵ. By (c), $\{n_i : i < \omega\}$ is not almost included in a finite union of elements of $\mathcal{F} \restriction \mathbb{Q}_{\epsilon-1}$: This follows from the fact that for each $n < \omega$, $\{n_i : i < \omega\}$ contains infinitely many members of $b_{\epsilon,n}$, hence of $a_{\epsilon,n}$. As $\{n_i : i < \omega\}$ has infinite intersection with an infinite number of members of $\mathcal{F} \restriction \mathbb{Q}_{\epsilon-1}$, it can't be covered by a finite number of members of $\mathcal{F} \restriction \mathbb{Q}_{\epsilon-1}$.

Therefore, $\mathbb{Q}_\epsilon := \mathbb{Q}_{\epsilon-1}$ and $A_\epsilon := \{n_i : i < \omega\}$ are as required.

Why is it possible to carry the induction? As each $b_{\epsilon,n}$ is infinite, and requirements (a)+(b) only exclude a finite number of elements, this is obviously possible.

Case IV (ϵ is a limit ordinal): We choose $(\mathbb{Q}_{\epsilon,n}, a_{\epsilon,n}, b_{\epsilon,n})$ by induction on $n < \omega$ such that:

a. $\bigcup_{\xi < \epsilon} \mathbb{Q}_\xi \subseteq \mathbb{Q}_{\epsilon,n} \in K_{\mathbb{P}}^+$.

b. If $n = m + 1$ then $\mathbb{Q}_{\epsilon,m} \lessdot \mathbb{Q}_{\epsilon,n}$.

If $n > 0$ then we also require:

c. $a_{\epsilon,n}$ is a $\mathbb{Q}_{\epsilon,n}$-name of a member of $\mathcal{F} \restriction \mathbb{Q}_{\epsilon,n}$.

d. $b_{\epsilon,n}$ is a $\mathbb{Q}_{\epsilon,n}$-name of an infinite subset of ω.

e. $\Vdash_{\mathbb{Q}_{\epsilon,n}}$ "$b_{\epsilon,n} \subseteq a_{\epsilon,n} \wedge \bigwedge_{\zeta < \epsilon} b_{\epsilon,n} \subseteq^* A_\zeta$".

f. $\Vdash_{\mathbb{Q}_{\epsilon,n}}$ "$a_{\epsilon,l} \neq a_{\epsilon,n}$ for $l < n$".

Why can we carry the induction? By the properties of $\mathbb{P}$, there is $\mathbb{Q}_{\epsilon,0} \in K_{\mathbb{P}}^+$ such that $\bigcup_{\zeta < \epsilon} \mathbb{Q}_\zeta \subseteq \mathbb{Q}_{\epsilon,0}$. Let $D_{\epsilon,0}$ be a $\mathbb{Q}_{\epsilon,0}$-name of an ultrafilter containing $\{A_\epsilon : \epsilon < \zeta\}$, let $\mathbb{M}_{D_{\epsilon,0}}$ be the $\mathbb{Q}_{\epsilon,0}$-name for the corresponding Mathias forcing and let w be the name for the generic set of natural numbers added by it. By the properties of $\mathbb{P}$, there is $\mathbb{Q}_{\epsilon,1} \in K_{\mathbb{P}}^+$ such that $\mathbb{Q}_{\epsilon,0} \lessdot \mathbb{Q}_{\epsilon,1}$ and $\mathbb{Q}_{\epsilon,1}$ adds a pseudo-intersection w to $D_{\epsilon,0}$.

There is a $\mathbb{Q}_{\epsilon,1}$-name $a_{\epsilon,1}$ such that $\Vdash_{\mathbb{Q}_{\epsilon,1}}$ "$a_{\epsilon,1} \in \mathcal{F} \restriction \mathbb{Q}_{\epsilon,1} \wedge |a_{\epsilon,1} \cap w| = \aleph_0$". Let $b_{\epsilon,1} = w \cap a_{\epsilon,1}$, then clearly $(\mathbb{Q}_{\epsilon,1}, a_{\epsilon,1}, b_{\epsilon,1})$ are as required. Suppose now that $(\mathbb{Q}_{\epsilon,l}, a_{\epsilon,l}, b_{\epsilon,l})$ were chosen for $l \leq k$. Note that $\Vdash_{\mathbb{Q}_{\epsilon,k}}$ "$\{\omega \setminus \bigcup_{l \leq k} a_{\epsilon,l}\} \cup \{A_\zeta : \zeta < \epsilon\}$ have the FIP". Suppose not, then there is $\zeta < \epsilon$ such that $\Vdash$ "$A_\zeta \subseteq^* \bigcup_{l \leq k} a_{\epsilon,l}$", as $\Vdash_{\mathbb{P}}$ "$\bigwedge_{l \leq k} a_{\epsilon,l} \in \mathcal{F}$", this is a contradiction: It's enough to show that $\Vdash_{\mathbb{P}}$ "A_ζ is not almost contained in a finite union of members of F". Suppose that $p \Vdash_{\mathbb{P}}$ "$A_\zeta \subseteq^* \bigcup_{l \leq k} b_l$" where b_l are elements of $\mathcal{F}$. Let $G \subseteq \mathbb{P}$ be a generic set containing p, then $V[G] \models$ "$A_\zeta[G] \subseteq \bigcup_{l \leq k} b_l[G]$". $G \cap \mathbb{Q}_\zeta$ is generic, $\{b \in \mathcal{F} \restriction \mathbb{Q}_\zeta[G \cap \mathbb{Q}_\zeta] : |b \cap A_\zeta[G \cap \mathbb{Q}_\zeta]| = \aleph_0\}$ is infinite. Therefore, in $V[G]$ there are $b_i \in \mathcal{F}[G]$ ($i < \omega$) such that $|A_\zeta[G] \cap b_i| = \aleph_0$ for each $i < \omega$, so $A_\zeta[G]$ can't be almost covered by a finite number of members of $\mathcal{F}[G]$, which is a contradiction.

Let $D_{\epsilon,k}$ be a $\mathbb{Q}_{\epsilon,k}$-name for a nonprincipal ultrafilter cotaining $\{\omega \setminus \bigcup_{l \leq k} a_{\epsilon,l}\} \cup \{A_\zeta : \zeta < \epsilon\}$, as before, let $\mathbb{Q}_{\epsilon,k+1} \in K_{\mathbb{P}}^+$ such that $\mathbb{Q}_{\epsilon,k} \lessdot \mathbb{Q}_{\epsilon,k+1}$ and $\mathbb{Q}_{\epsilon,k+1}$ adds a pseudo-intersection w_{k+1} to $D_{\epsilon,k}$. Again, $\Vdash_{\mathbb{Q}_{\epsilon,k+1}}$ "There is $a_{\epsilon,k+1} \in \mathcal{F} \restriction \mathbb{Q}_{\epsilon,k+1}$ such that $|w_{k+1} \cap a_{\epsilon,k+1}| = \aleph_0$", now let $b_{\epsilon,k+1} = a_{\epsilon,k+1} \cap w_{k+1}$. It's easy to see that $(\mathbb{Q}_{\epsilon,k+1}, a_{\epsilon,k+1}, b_{\epsilon,k+1})$ are as required.

We shall now prove that there is a forcing notion $\mathbb{Q}_\epsilon \in K_{\mathbb{P}}^+$ and a $\mathbb{Q}_\epsilon$-name A_ϵ such that $\bigcup_{n < \omega} \mathbb{Q}_{\epsilon,n} \subseteq \mathbb{Q}_\epsilon$ and $\Vdash_{\mathbb{Q}_\epsilon}$ "$\bigwedge_{\zeta < \epsilon} A_\epsilon \subseteq^* A_\zeta \wedge (\bigwedge_{n < \omega} |A_\epsilon \cap b_{\epsilon,n}| = \aleph_0)$":

Let $\mathbb{Q}' = \bigcup_{n < \omega} \mathbb{Q}_{\epsilon,n}$, we shall prove that there is a $\mathbb{Q}'$-name for a ccc forcing $\mathbb{Q}''$ that forces the existence of A_ϵ as above, such that $|\mathbb{Q}' * \mathbb{Q}''| < \kappa$:

Let $\mathbb{Q}''$ be the $\mathbb{Q}'$−name for the Mathias forcing $\mathbb{M}_{D'}$, restricted to the filter D' generated by $\{A_\zeta : \zeta < \epsilon\} \cup \{[n, \omega) : n < \omega\}$, so there is a name A^ϵ such that $\Vdash_{\mathbb{Q}' * \mathbb{Q}''}$ "$A^\epsilon \in [\omega]^\omega$, $\bigwedge_{\zeta < \epsilon} A^\epsilon \subseteq^* A_\zeta$ and $\bigwedge_{n < \omega} |A^\epsilon \cap b_{\epsilon,n}| = \aleph_0$". Letting A^ϵ be the generic set added by $\mathbb{M}_{D'}$, in order to show that the last condition holds, we need to show that (in $V^{\mathbb{Q}'}$) if $p \in \mathbb{M}_{D'}$ and $k < \omega$, then there exists a stronger condition q forcing that $k' \in A^\epsilon \cap b_{\epsilon,n}$ for some $k' > k$. Let $p = (w, S)$, as $S \in D'$, there is $\zeta < \epsilon$ and

$l_* < \omega$ such that $A_\zeta \setminus l_* \subseteq S$. As $\underset{\sim}{b}_{\epsilon,n} \subseteq^* \underset{\sim}{A}_\zeta$, there is $sup(w)+k < k' \in \underset{\sim}{b}_{\epsilon,n} \cap \underset{\sim}{A}_\zeta \setminus l_* \cap S$, so we can obviously extend p to a condition q forcing that $k' \in \underset{\sim}{A}^\epsilon \cap \underset{\sim}{b}_{\epsilon,n}$.

By claim 3, there is $\mathbb{Q}^3 \in K$ such that $\mathbb{Q}' * \mathbb{Q}'' \lessdot \mathbb{Q}^3$ and $|\mathbb{Q}^3| \leq \sigma$. By the properties of $\mathbb{P}$, there is a complete embedding $f^3 : \mathbb{Q}^3 \to \mathbb{P}$ such that f^3 is the identity over $\mathbb{Q}_{\epsilon,0}$ (hence over $\mathbb{Q}_*$). Therefore, $\Vdash_\mathbb{P}$ " $\underset{n<\omega}{\wedge} f^3(\underset{\sim}{a}_{\epsilon,n}) \in \underset{\sim}{\mathcal{F}}$". By the (proof of the) density of $K_\mathbb{P}^+$, there is $\mathbb{Q}^4 \in K_\mathbb{P}^+$ such that $f^3(\mathbb{Q}^3) \lessdot \mathbb{Q}^4$ and $|\mathbb{Q}^4| \leq \sigma$. Let $\mathbb{Q}_\epsilon = \mathbb{Q}^4$, $\underset{\sim}{A}_\epsilon = f^3(\underset{\sim}{A}^\epsilon)$, we shall prove that $(\mathbb{Q}_\epsilon, \underset{\sim}{A}_\epsilon)$ are as required. Obviously, $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{A}_\epsilon \in [\omega]^\omega$", and as f^3 is the identity over each $\mathbb{Q}_\zeta$ ($\zeta < \epsilon$), $\Vdash_{\mathbb{Q}_\epsilon}$ " $\underset{\zeta<\epsilon}{\wedge} \underset{\sim}{A}_\epsilon \subseteq^* \underset{\sim}{A}_\zeta$". The other requirements for $\mathbb{Q}_\epsilon$ and $\underset{\sim}{A}_\epsilon$ are trivial. It remains to show that $\Vdash_{\mathbb{Q}_\epsilon}$ "$\underset{\sim}{A}_\epsilon$ is not almost covered by a finite union of elements of $\underset{\sim}{\mathcal{F}} \upharpoonright \mathbb{Q}_\epsilon$". As $\Vdash_{\mathbb{Q}_\epsilon}$ " $\underset{n<\omega}{\wedge} f^3(\underset{\sim}{a}_{\epsilon,n}) \in \underset{\sim}{\mathcal{F}} \upharpoonright \mathbb{Q}_\epsilon$ and $\underset{n \neq m}{\wedge} f^3(\underset{\sim}{a}_{\epsilon,n}) \neq f^3(\underset{\sim}{a}_{\epsilon,m})$, it's enough to show that $\Vdash_{\mathbb{Q}_\epsilon}$ " $\underset{n<\omega}{\wedge} |\underset{\sim}{A}_\epsilon \cap f^3(\underset{\sim}{a}_{\epsilon,n})| = \aleph_0$", which follows from the fact that $\Vdash_{\mathbb{Q}' * \mathbb{Q}''}$ " $\underset{n<\omega}{\wedge} |\underset{\sim}{A}^\epsilon \cap \underset{\sim}{b}_{\epsilon,n}| = \aleph_0$" and the fact that $\Vdash_{\mathbb{Q}_{\epsilon,n}}$ "$\underset{\sim}{b}_{\epsilon,n} \subseteq \underset{\sim}{a}_{\epsilon,n}$". This completes the proof of the induction.

Remark: By the proof of the density of $K_\mathbb{P}^+$ in $K_\mathbb{P}$, whenever we have $\mathbb{Q} \in K_\mathbb{P}$ of cardinality $\leq \sigma$, we can construct $\mathbb{Q}' \in K_\mathbb{P}^+$ such that $\mathbb{Q} \lessdot \mathbb{Q}'$ and $|\mathbb{Q}'| \leq \sigma$. Therefore, at each of the steps in the limit case, it's possible to guarantee that the cardinality of the forcing is $\leq \sigma$. $\square$

Open problems

We intend to present the solutions to the following problems in a subsequent paper:

1. Assuming ZFC, can we construct a model of $ZF + DC +$ "There are no maximal eventually different families"?

2. Are there analytic maximal eventually different families?

Recall that $\mathcal{F} \subseteq \omega^\omega$ is a maximal eventually different family if $f, g \in \mathcal{F} \to f(n) \neq g(n)$ for every large enough n, and $\mathcal{F}$ is maximal with respect to this property. It's noted in [To] that the answer is not known even in Levy's model.

3. Assuming ZFC, can we construct a model of $ZF + DC +$ "There are no maximal cofinitary groups"?

4. Are there analytic cofinitary groups?

Recall that $G \subseteq S_\infty$ is a maximal cofinitary group if G is a group under the composition of functions, for every $Id \neq f \in G$, $|\{n : f(n) = n\}| < \aleph_0$ and G is maximal with respect to these properties. As in the previous case, according to [To], the answer is not known in Levy's model.

More references and remarks on the above problems can be found in [To]

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