Maximal independent sets in Borel graphs and large cardinals

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Abstract

We construct a Borel graph G such that $ZF + DC +$ "There are no maximal independent sets in G" is equiconsistent with $ZFC +$ "There exists an inaccessible cardinal".[1]

Introduction

The main result of this note is motivated by our recent study of maximal almost disjoint families and their relatives. Recall that $\mathcal{F} \subseteq [\omega]^\omega$ is a MAD family if $A \neq B \in \mathcal{F} \to |A \cap B| < \aleph_0$, and $\mathcal{F}$ is maximal with respect to this property. Maximal eventually different (MED) families are the analog of MAD families where elements of $[\omega]^\omega$ are replaced by graphs of functions from ω to ω, namely, $f, g \in \omega^\omega$ are eventually different if $f(n) \neq g(n)$ for large enough n, and $\mathcal{F} \subseteq \omega^\omega$ is a MED family if the elements of $\mathcal{F}$ are pairwise eventually different and $\mathcal{F}$ is maximal with respect to this property.

Questions on the (non-)existence and definability of such families have attracted considerable interest for decades. The first results were obtained by Mathias who proved the following theorem:

Theorem [Ma]: There are no analytic MAD families.

As for the possibility of the non-existence of MAD families, the following result was recently proved by the authors (earlier such results were proven by Mathias in [Ma] and by Toernquist in [To] using Mahlo and inaccessible cardinals, respectively):

Theorem [HwSh:1090]: $ZF+DC+$"There are no MAD families" is equiconsistent with ZFC.

Quite surprisingly, the situation for MED families turns out to be different:

Theorem [HwSh:1089]: Assuming ZF, there exists a Borel MED family.

A possible approach to explaining the above difference is via Borel combinatorics. The study of Borel and analytic graphs was initiated by Kechris, Solecki and Todorcevic in [KST], and has been a source of fruitful research ever since (see [KM] for a survey of recent results). The above questions on MAD families are connected

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to Borel combinatorics due to the following observation: There exist Borel graphs G_{MAD} and H_{MED} such that there exists a MAD (MED) family iff there exists a maximal independent set in G_{MAD} (H_{MED}). Therefore, we might try to explain the above difference of MAD and MED families by pursuing the general problem of classifying Borel graphs according to the consistency strength of $ZF+DC+$"There are no maximal independent sets in G".

The main goal of this note is to show that for some Borel graphs G, $ZF+DC+$"There are no maximal independent sets in G" has large cardinal strength.

The main result

Definition 1: We shall define a Borel graph $G=(V,E)$ as follows:

a. V is the set of reals r that code the following objects:

1. A linear order I_r of the element of ω or some $n<\omega$.

2. A sequence $(s_{r,\alpha} : \alpha \in I_r)$ of pairwise distinct reals.

3. A sequence of functions $(f_{r,a} : a \in I_r)$ such that each $f_{r,a}$ is an injective function from $I_{r,<a} := \{b \in I_r : b <_{I_r} a\}$ onto some initial segment of ω.

b. Given $r_1 \neq r_2 \in V$ and $b \in I_{r_2}$, let $X_{r_1,r_2,b}$ be the set of pairs $(a_1,a_2) \in I_{r_1} \times I_{r_2,<b}$ such that $s_{r_1,a_1} = s_{r_2,a_2}$.

c. Given $r_1 \neq r_2 \in V$, $\neg(r_1 E r_2)$ holds iff one of the following holds:

1. There exists $b \in I_{r_2}$ such that $X_{r_1,r_2,b}$ is an isomorphism from I_{r_1} to $I_{r_2,<b}$ which also commutes with $f_{-,-}$.

2. There exists $b \in I_{r_1}$ such that $X_{r_2,r_1,b}$ is an isomorphism from I_{r_2} to $I_{r_1,<b}$ which also commutes with $f_{-,-}$.

Definition 2: Given $r_1 \neq r_2 \in V$, we say that r_2 extends r_1 and denote it by $r_1 <_G r_2$ when $\neg(r_1 E r_2)$ and clause (1) holds in definition 1(c).

Claim 3 ($ZF+DC$)**:** Let $X \subseteq V$ be an independent set.

a. X is linearly ordered by $<_G$.

b. If X is countable then X is not a maximal independent set.

Proof: a. Obvious.

b. By clause (a), there is a linear order I such that $X=\{r_i : i \in I\}$ and $i <_I j$ iff $r_i <_G r_j$. For every $i<j \in I$, let $F_{i,j}$ be the isomorphism from I_{r_i} to a proper initial segment of I_{r_j} witnessing $r_i <_G r_j$. Let I_r be the direct limit of the system $(I_{r_i}, F_{j,k} : i,j,k \in I, j<k)$. For $a \in I_r$, let $s_{r,a}$ be $s_{r_i,a'}$ where $a' \in I_{r_i}$ is some representative of a, and define $f_{r,a}$ similarly. Let $r \in V$ be a real coding I_r, $(s_{r,a} : a \in I_r)$ and $(f_{r,a} : a \in I_r)$, then $\neg(rEr_i)$ for every $r_i \in X$. $\square$

Theorem 4: $ZF+DC+$"There is no maximal independent set in G" is equiconsistent with $ZFC+$"There exists an inaccessible cardinal".

Theorem 4 will follow from the following claims:

Claim 5 $(ZF+DC)$**:** If there exists $a \in \omega^\omega$ such that $\aleph_1 = \aleph_1^{L[a]}$, then there exists a maximal independent set in G.

Claim 6: There is no maximal independent set in G in Levy's model (aka Solovay's model).

Remark: While the set of vertices of G is denoted by V, the set-theoretic universe will be denoted by $\mathbf{V}$.

Proof of claim 5: Let $(s_\alpha : \alpha < \omega_1^{L[a]}) \in L[a]$ be a sequence of pairwise distinct reals, and let $\bar{f}^* = (f_\alpha^* : \alpha < \omega_1^{L[a]}) \in L[a]$ be a sequence of functions such that each f_α^* is an injective function from α onto $|\alpha| \leq \omega$. For each $\alpha < \omega_1^{L[a]}$, let $r_\alpha \in (\omega^\omega)^{L[a]}$ be the $<_{L[a]}$ $-$first real that codes $(\alpha, (s_\beta : \beta < \alpha), \bar{f}^* \restriction \alpha)$. The sequences $(s_\alpha : \alpha < \omega_1^{L[a]})$, $(f_\alpha^* : \alpha < \omega_1^{L[a]})$ and $(r_\alpha : \alpha < \omega_1^{L[a]})$ belong to $\mathbf{V}$, and as $\omega_1 = \omega_1^{L[a]}$, their length is ω_1.

It's easy to see that $\{r_\alpha : \alpha < \omega_1^{L[a]}\}$ is a well-defined set and is an independent subset of V, we shall prove that it's a maximal independent set. Let $r \in V \setminus \{r_\alpha : \alpha < \omega_1^{L[a]}\}$ and suppose towards contradiction that $\neg(rEr_\alpha)$ for every $\alpha < \omega_1^{L[a]}$. There are two possible cases:

Case I: $r_\alpha <_G r$ for every $\alpha < \omega_1^{L[a]}$. In this case, I_r is a linear order, and each $\alpha < \omega_1^{L[a]}$ embeds into I_r as an initial segment. Therefore, $\omega_1 = \omega_1^{L[a]}$ embeds into I_r as an initial segment, a contradiction.

Case II: $r <_G r_\alpha$ for some $\alpha < \omega_1^{L[a]}$. Let α be the minimal ordinal with this property, then α necessarily has the form $\beta+1$. If $r = r_\beta$, then we get a contradiction to the choice of r. If $r \neq r_\beta$, then it's easy to see that rEr_β, contradicting our assumption. $\square$

Proof of claim 6: Let κ be an inaccessible cardinal and let $\mathbb{P} = Coll(\aleph_0, <\kappa)$, we shall prove that $\Vdash_{\mathbb{P}}$ "There is no maximal independent set in G from $HOD(\mathbb{R})$". Suppose towards contradiction that $p \in \mathbb{P}$ forces that $\underset{\sim}{X}$ is such a set. Let $\mathbb{Q}$ be a forcing notion such that $\mathbb{Q} \lessdot \mathbb{P}$, $|\mathbb{Q}| < \kappa$, $p \in \mathbb{Q}$ and $\underset{\sim}{X}$ is definable using a parameter from $\mathbb{R}^{\mathbf{V}^{\mathbb{Q}}}$. By the properties of the Levy collapse, we may assume wlog that $\mathbb{Q} = \{0\}$ and $p = 0$. If $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{X} \subseteq (\omega^\omega)^{\mathbf{V}}$", then $\Vdash_{\mathbb{P}}$ "$|\underset{\sim}{X}| = \aleph_0$", and by claim 3, $\underset{\sim}{X}$ is not a maximal independent set in $\mathbf{V}^{\mathbb{P}}$, a contradiction. Therefore, there exist $p_1 \in \mathbb{P}$ and $\underset{\sim}{r_1}$ such that $p_1 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{r_1} \in \underset{\sim}{X} \wedge \underset{\sim}{r_1} \notin \mathbf{V}$". Let $\mathbb{Q}_1 \lessdot \mathbb{P}$ be a forcing of cardinality $< \kappa$ such that $p_1 \in \mathbb{Q}_1$ and $\underset{\sim}{r_1}$ is a $\mathbb{Q}_1$-name. For $l = 2, 3$ let $(\mathbb{Q}_l, p_l, \underset{\sim}{r_l})$ be isomorphic copies of $(\mathbb{Q}_1, p_1, \underset{\sim}{r_1})$ such that $\prod\limits_{n=1,2,3} \mathbb{Q}_n \lessdot \mathbb{P}$ (identifying $\mathbb{Q}_1$ with its canonical image in the product). Choose $(p_1, p_2) \leq (q_1, q_2)$ such that $(q_1, q_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2}$ "$\underset{\sim}{r_1} \neq \underset{\sim}{r_2}$". As $(q_1, q_2) \Vdash_{\mathbb{Q}_1 \times \mathbb{Q}_2}$ "$\underset{\sim}{r_1}, \underset{\sim}{r_2} \in \underset{\sim}{X}$", then wlog (q_1, q_2) forces that $\underset{\sim}{r_1} <_G \underset{\sim}{r_2}$ as witnessed by an isomorphism from $I_{\underset{\sim}{r_1}}$ to $I_{\underset{\sim}{r_2}, <s}$ for some $s \in I_{\underset{\sim}{r_2}}$. Let $q_3 \in \mathbb{Q}_3$ be the conjugate of

q_1, then (q_2, q_3) forces (in $\mathbb{Q}_2 \times \mathbb{Q}_3$) that $\underset{\sim}{r_2}, \underset{\sim}{r_3} \in \underset{\sim}{X}$ and $\underset{\sim}{r_3} <_G \underset{\sim}{r_2}$ as witnessed by an isomorphism from $I_{\underset{\sim}{r_3}}$ to $I_{\underset{\sim}{r_2},<s}$. Now pick $(q_1, q_2, q_3) \leq (q_1', q_2', q_3')$ that forces in addition that $\underset{\sim}{r_1} \neq \underset{\sim}{r_3}$, then necessarily it forces that $\underset{\sim}{r_1} E \underset{\sim}{r_3}$, a contradiction. $\square$

Open problems

Notation: Given a Borel graph G, let $\psi(G)$ be the statement "There are no maximal independent sets in G".

Problem 1: Classify the Borel graphs according to the consistency strength of $ZF + DC + \psi(G)$.

As the above problem seems to be quite difficult at the moment, it might be reasonable to consider the following subproblems first:

Problem 2: What are the possibilites (in terms of large cardinal strength) for the consistency strength of $ZF + DC + \psi(G)$?

Problem 3: Find combinatorial/descriptive set theoretic/model theoretic properties ϕ_1 and ϕ_2 such that:

a. $\phi_1(G_{MAD})$.

b. $\phi_2(H_{MED})$.

c. $\phi_1(G) \rightarrow ZF + DC + \psi(G)$ is equiconsistent with ZFC.

d. $\phi_2(G) \rightarrow ZF + DC \vdash \neg\psi(G)$.

e. ϕ_1 and ϕ_2 are satisfied by a large collection of Borel graphs.

A solution to problem (3) would explain the difference between MAD and MED families that was discussed in the introduction.

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