On the classification of definable ccc forcing notions

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Abstract

We show that for a Suslin ccc forcing notion $\mathbb{Q}$ adding a Hechler real, $ZF + DC_{\omega_1}$ + "All sets of reals are $I_{\mathbb{Q},\aleph_0}$-measurable" implies the existence of an inner model with a measurable cardinal. We also further investigate the forcing notions from [HwSh:1067], showing that some of them add Hechler reals (so the above result applies to them) while others don't add dominating reals.[1]

Our paper can be seen as part of a line of research motivated by the following general problem:

Problem: Classify the nicely definable forcing notions.

For further discussion of that problem, see [Sh:666]. The following problem arises naturally from the results of [HwSh:1067] and [HwSh:1094] (see the first section of this paper for a brief presentation of the results from [HwSh:1067]):

Problem: Classify the Suslin ccc forcing notions according to the consistency strength of of T + "All sets of reals are $I_{\mathbb{Q},\kappa}$-measurable" where $\kappa \in \{\aleph_0, \aleph_1\}$ and $T \in \{Z_*, ZF, ZF + AC_\omega, ZF + DC, ZF + DC_{\omega_1}\}$ (or even $T = ZFC$), and similarly for $T' = T + WO_{\aleph_1}$, where T is as above and $WO_{\aleph_1}$ = "There exists an ω_1-sequence of distincet reals".

Remark: The ideals $I_{\mathbb{Q},\kappa}$ and the notion of I-measurability will be defined in the first section. The theory Z_* is defined in [HwSh:1094].

Following theorem D below, we would now like to find reults that discern between the forcing notions $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$ from [HwSh:1067]. This goal is achieved by first showing that $\mathbb{Q}^1_{\mathbf{n}}$ adds a Hechler real while $\mathbb{Q}^2_{\mathbf{n}}$ doesn't add dominating reals. We then relate this result to the above classification problem by proving that if $\mathbb{Q}$ is a Suslin ccc forcing notion adding a Hechler real, then $ZF + DC_{\omega_1}$ + "All sets of reals are $I_{\mathbb{Q},\aleph_0}$-measurable" implies the existence of an inner model with a measurable cardinal. In particular, this is true for $\mathbb{Q}^1_{\mathbf{n}}$. We intend to extend our result to other Suslin ccc forcing notions not adding Hechler reals in a subsequent paper [F1561].

[1]Date: October 19, 2016

2000 Mathematics Subject Classification: 03E35, 03E40, 03E15, 03E25, 03E55

Keywords: Suslin forcing, creature forcing, regularity properties, axiom of choice, measurable cardinals

Publication 1097 of the second author

Partially supported by European Research Council grant 338821.

By [Sh:176], DC_{ω_1} implies the existence of a non-Lebesgue measurable set. It follows that $ZF + DC_{\omega_1} +$ "All sets of reals are $I_{\mathbb{Q},\aleph_0}$-measurable" is inconsistent when $\mathbb{Q}$ is random real forcing, and it also follows that DC_{ω_1} doesn't hold in Solovay's model or in models of AD. The problem of finding forcing notions $\mathbb{Q}$ for which $ZF + DC_{\omega_1} + I_{\mathbb{Q},\aleph_0}$-measurability is consistent (maybe relative to large cardinals) remains open. In future work [F1424] we shall prove the following result:

Theorem ([F1424]): Suppose there is a measurable cardinal, then in a suitable generic extension there is an inner model of $ZF + DC_{\omega_1} +$ "All sets of reals are $I_{\mathbb{Q}^2_{\mathbf{n}},\aleph_1}-$measurable".

1. Preliminaries

We summarize the basic definitions and results from [HwSh1067].

Convention: For sequences η and ν, we write $\eta \leq \nu$ when η is an initial segment of ν. We write $\eta < \nu$ when η is a proper initial segment of ν.

Definition 1. a. A norm on a set A is a function assigning to each $X \in \mathcal{P}(A) \setminus \{\emptyset\}$ a non-negative real number such that $X_1 \subseteq X_2 \rightarrow nor(X_1) \leq nor(X_2)$.

b. Let $\mathbf{M}$ be the collection of pairs $(\mathbb{Q}, \underset{\sim}{\eta})$ such that $\mathbb{Q}$ is a Suslin ccc forcing notion and $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real.

In [HwSh1067] we gave an explicit construction of parameters $\mathbf{n}$ having the following properties:

Definition 2: Let $\mathbf{N}$ be the set of tuples $\mathbf{n} = (T, nor, \bar{\lambda}, \bar{\mu}) = (T_{\mathbf{n}}, nor_{\mathbf{n}}, \bar{\lambda}_{\mathbf{n}}, \bar{\mu}_{\mathbf{n}})$ such that:

a. T is a subtree of $\omega^{<\omega}$.

b. $\bar{\mu} = (\mu_\eta : \eta \in T)$ is a sequence of non-negative real numbers.

c. $\bar{\lambda} = (\lambda_\eta : \eta \in T)$ is a sequence of pairwise distinct non-zero natural numbers such that:

1. $\lambda_\eta = \{m : \eta\hat{}m \in T\}$, so $T \cap \omega^n$ is finite and non-empty for every n.

2. If $lg(\eta) = lg(\nu)$ and $\eta <_{lex} \nu$ then $\lambda_\eta \ll \lambda_\nu$.

3. If $lg(\eta) < lg(\nu)$ then $lg(\eta) \ll \lambda_\eta \ll \lambda_\nu$.

4. $lg(\eta) \ll \mu_\eta \ll \lambda_\eta$ for $\eta \in T$.

d. For $\eta \in T$, nor_η is a function with domain $\mathcal{P}^-(suc_T(\eta)) = \mathcal{P}(suc_T(\eta)) \setminus \emptyset$ and range $\subseteq \mathbb{R}^+$ such that:

1. nor_η is a norm on $suc_T(\eta)$ (see definition 1).

2. $(lg(\eta) + 1)^2 \leq \mu_\eta \leq nor_\eta(suc_T(\eta))$.

e. $\lambda_{<\eta} := \Pi\{\lambda_\nu : \lambda_\nu < \lambda_\eta\} \ll \mu_\eta$.

f. (Co-Bigness) If $k \in \mathbb{R}^+$, $a_i \subseteq suc_{T_{\mathbf{n}}}(\eta)$ for $i < i(*) \leq \mu_\eta$ and $k + \frac{1}{\mu_\eta} \leq nor_\eta(a_i)$ for every $i < i(*)$, then $k \leq nor_\eta(\underset{i<i(*)}{\cap} a_i)$.

g. If $1 \leq nor_\eta(a)$ then $\frac{1}{2} < \frac{|a|}{|suc_{T_{\mathbf{n}}}(\eta)|}$.

h. If $k + \mu_\eta \leq nor_\eta(a)$ and $\rho \in a$, then $k \leq nor_\eta(a \setminus \{\rho\})$.

Definition 3: A. For $\mathbf{n} \in \mathbf{N}$ we shall define the forcing notions $\mathbb{Q}^1_{\mathbf{n}} \subseteq \mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}} \subseteq \mathbb{Q}^0_{\mathbf{n}}$ as follows:

1. $p \in \mathbb{Q}^0_{\mathbf{n}}$ iff for some $tr(p) \in T_{\mathbf{n}}$ we have:

a. p or T_p is a subtree of $T_{\mathbf{n}}^{[tr(p)\leq]}$ (so it's closed under initial segments) with no maximal node.

b. For $\eta \in lim(T_p)$, $lim(nor_{\eta\restriction l}(suc_{T_p}(\eta \restriction l)) : lg(tr(p)) \leq l < \omega) = \infty$.

c. $2 - \frac{1}{\mu_{tr(p)}} \leq nor(p)$ (where $nor(p)$ is defined in C(b) below).

2. $p \in \mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}}$ if $p \in \mathbb{Q}^0_{\mathbf{n}}$ and $nor_\eta(Suc_p(\eta)) > 2$ for every $tr(p) \leq \eta \in T_p$.

In [HwSh1067] we proved that $\mathbb{Q}^{\frac{1}{2}}_{\mathbf{n}}$ is dense in $\mathbb{Q}^0_{\mathbf{n}}$.

3. $p \in \mathbb{Q}^1_{\mathbf{n}}$ if $p \in \mathbb{Q}^0_{\mathbf{n}}$ and for every $n < \omega$, there exists $k^p(n) = k(n) > lg(tr(p))$ such that for every $\eta \in T_p$, if $k(n) \leq lg(\eta)$ then $n \leq nor_\eta(Suc_p(\eta))$.

B. $\mathbb{Q}^i_{\mathbf{n}} \models p \leq q$ $(i \in \{0, \frac{1}{2}, 1\})$ iff $T_q \subseteq T_p$.

C. a. For $i \in \{0, \frac{1}{2}, 1\}$, $\underset{\sim}{\eta}^i_{\mathbf{n}}$ is the $\mathbb{Q}^i_{\mathbf{n}}-$name for $\cup\{tr(p) : p \in \underset{\sim}{G}_{\mathbb{Q}^i_{\mathbf{n}}}\}$.

b. For $i \in \{0, \frac{1}{2}, 1\}$ and $p \in \mathbb{Q}$ let $nor(p) := sup\{a \in \mathbb{R}_{>0} : \eta \in T_p^+ \to a \leq nor_\eta(suc_{T_p}(\eta))\} = inf\{nor_\eta(suc_{T_p}(\eta)) : \eta \in T_p\}$.

D. For $i \in \{0, \frac{1}{2}, 1\}$ let $\mathbf{m}^i_{\mathbf{n}} = \mathbf{m}_{i,\mathbf{n}} = (\mathbb{Q}^i_{\mathbf{n}}, \underset{\sim}{\eta}^i_{\mathbf{n}}) \in \mathbf{M}$ where $\mathbf{M}$ denotes the set of pairs of the form $(\mathbb{Q}^i_{\mathbf{n}}, \underset{\sim}{\eta}^i_{\mathbf{n}})$.

Definition 4: For $\mathbf{n} \in \mathbf{N}$ we define $\mathbf{m} = \mathbf{m}^2_{\mathbf{n}} = (\mathbb{Q}^2_{\mathbf{n}}, \underset{\sim}{\eta}^2_{\mathbf{n}})$ by:

a. $p \in \mathbb{Q}^2_{\mathbf{n}}$ iff p consists of a trunk $tr(p) \in T_{\mathbf{n}}$, a perfect subtree $T_p \subseteq T_{\mathbf{n}}^{[tr(p)\leq]}$ and a natural number $n \in [1, lg(tr(p)) + 1]$ such that $1 + \frac{1}{n} \leq nor_\eta(suc_{T_p}(\eta))$ for every $\eta \in T_p^+$.

b. Order: reverse inclusion.

c. $\underset{\sim}{\eta}^2_{\mathbf{n}} = \cup\{tr(p) : p \in \underset{\sim}{G}_{\mathbb{Q}^2_{\mathbf{n}}}\}$.

d. If $p \in \mathbb{Q}^2_{\mathbf{n}}$ we let $nor(p) = min\{n : \eta \in T_p \to 1 + \frac{1}{n} \leq nor_\eta(suc_p(\eta))\}$.

We shall now describe some of the basic properties and results on $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$ proven in [HwSh1067]:

Theorem A: For $\mathbf{n} \in \mathbf{N}$, $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$ are Suslin ccc forcing notions.

Theorem B: Assume that $\{p_n : n < \omega\} \subseteq \mathbb{Q}^2_{\mathbf{n}}$, $\bigwedge\limits_{n<\omega} tr(p_n) = \eta$ and $\bigwedge\limits_{n<\omega} nor(p_n) = k$, then there is $p_* \in \mathbb{Q}^2_{\mathbf{n}}$ such that:

a. $tr(p_*) = \eta$ and $nor(p_*) = k$.

b. $p_* \Vdash_{\mathbb{Q}^2_{\mathbf{n}}}$ "$(\exists^\infty n)(p_n \in G_{\mathbb{Q}^2_{\mathbf{n}}})$".

Theorem C: $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$ add a Cohen real.

Theorem D: Let $i \in \{1, 2\}$ and $\mathbf{n} \in \mathbf{N}$. The following is consistent relative to ZFC:

I. ZF

II. Every set of reals equals a Borel set modulo $I_{\mathbb{Q}^i_{\mathbf{n}},\aleph_1}$.

III. There exists an ω_1-sequence of distinct reals.

We shall now define the ideals derived from a forcing notion $\mathbb{Q}$:

Definition 5: a. Let $\mathbb{Q}$ be a forcing notion such that each $p \in \mathbb{Q}$ is a perfect subtree of $\omega^{<\omega}$, $p \leq_{\mathbb{Q}} q$ iff $q \subseteq p$ and the generic real is given by the union of trunks of conditions that belong to the generic set, that is, $\underset{\sim}{\eta} = \bigcup\limits_{p \in \underset{\sim}{G}} tr(p)$ and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \in \omega^\omega$".

Let $\aleph_0 \leq \kappa$, the ideal $I_{\mathbb{Q},\kappa}$ will be defined as the closue under unions of size $\leq \kappa$ of sets of the form $\{X \subseteq \omega^\omega : (\forall p \in \mathbb{Q})(\exists p \leq q)(lim(q) \cap X = \emptyset\}$.

b. Let I be an ideal on the reals, a set of reals X is called I-measurable if there exists a Borel set B such that $X \Delta B \in I$.

2. Dominating reals

In this section we shall discern between $\mathbb{Q}^1_{\mathbf{n}}$ and $\mathbb{Q}^2_{\mathbf{n}}$ by showing that $\mathbb{Q}^1_{\mathbf{n}}$ adds a Hechler real while $\mathbb{Q}^2_{\mathbf{n}}$ doesn't add dominating reals.

Remark: We proved in [HwSh:1067] that $\mathbb{Q}^2_{\mathbf{n}}$ is nw-nep (see [Sh:711] for the definition), and by [Sh:711], such forcing notions don't add dominating reals. Here we provide a direct proof of this fact using the compactness property of $\mathbb{Q}^2_{\mathbf{n}}$.

Claim 6: $\mathbb{Q}^2_{\mathbf{n}}$ doesn't add a dominating real.

Proof: Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded family of reals such that $\{f_\alpha : \alpha \in W\}$ is unbounded for every unbounded $W \subseteq \mathfrak{b}$ (e.g. f_α is $<^*$-increasing). Suppose towards contradiction that $p_* \Vdash$ "$\underset{\sim}{g}$ dominates f_α for every $\alpha < \mathfrak{b}$". For every $\alpha < \mathfrak{b}$ there are $p_* \leq p_\alpha$ and $n_\alpha < \omega$ such that $p_\alpha \Vdash$ "$f_\alpha(n) \leq g(n)$ for every $n_\alpha \leq n$". Therefore, for some $n_* < \omega$, $k_* < \omega$ and $\nu \in \omega^{<\omega}$, the set of $\alpha < \mathfrak{b}$ for which $n_\alpha = n_*$, $nor(p_\alpha) = k_*$ and $tr(p_\alpha) = \nu$ is unbounded, let W be the set of those αs. We will now show that for some $\alpha_0 < \alpha_1 < ... < \alpha_l < ...$ from W and $n_* \leq m$, $(f_{\alpha_l}(m) : l < \omega)$ is strictly increasing:

We need to show that for some m such that $n_* \leq m$, the set $\{f_\alpha(m) : \alpha \in W\}$ is infinite. If it's not true, then for every m such that $n_* \leq m$, there is $h(m)$ such that $max\{f_\alpha(m) : \alpha \in W\} \leq h(m)$, and therefore h dominates $\{f_\alpha : \alpha \in W\}$, contrdicting the fact that W is unbounded.

By theorem B, there is $q \in \mathbb{Q}^2_{\mathbf{n}}$ such that $q \Vdash (\exists^\infty l)(p_{\alpha_l} \in \underset{\sim}{G})$. By the choice of the p_α-s, n_*, W and m, $q \Vdash$ "$f_{\alpha_l}(m) \leq g(m)$ for infinitely many l-s", contradicting the fact that $(f_{\alpha_l}(m) : l < \omega)$ is unbounded. $\square$

We shall now prove that $\mathbb{Q}^1_{\mathbf{n}}$ adds a Hechler real. In order to do that, we shall first prove that $\mathbb{Q}^1_{\mathbf{n}}$ adds a dominating real.

Claim 7: a. $\mathbb{Q}^1_{\mathbf{n}}$ adds a dominating real.

b. $\mathbb{Q}^1_{\mathbf{n}}$ adds a Hechler real.

Proof (of (a)): Let $\eta \in T_{\mathbf{n}}$, for every $k \leq lg(\eta)$ choose $w = w_{\eta,k} \subseteq Suc_{T_{\mathbf{n}}}(\eta)$ such that $nor_\eta(Suc_{T_{\mathbf{n}}}(\eta) \setminus w) = k+1$, $|w|$ is minimal and $w_{\eta,k+1} \subseteq w_{\eta,k}$. The following two observations will be useful:

a. Let $w = w_{\eta,k}$. If $u \subseteq Suc_{T_{\mathbf{n}}}(\eta)$ and $k+2 \leq nor_\eta(u)$, then $u \subseteq Suc_{T_{\mathbf{n}}} \setminus w$ is impossible as $k+1 = nor_\eta(Suc_{T_{\mathbf{n}}} \setminus w)$. Therefore $u \cap w \neq \emptyset$.

b. If $u \subseteq Suc_{T_{\mathbf{n}}}(\eta)$, $l < k$ and $l+1 \leq nor_\eta(u)$, then letting $v = u \setminus w$, we have:

1. $v \subseteq u$ and $v \cap w = \emptyset$.

2. By the co-bigness property, $l \leq nor_\eta(u \cap (Suc_{T_{\mathbf{n}}} \setminus w)) = nor_\eta(v)$ and $v \neq \emptyset$.

3. If $u \subseteq Suc_{T_{\mathbf{n}}}(\eta)$ and $nor_\eta(u) > 2$ then $min\{k, nor_\eta(u)-1\} \leq nor_\eta(u \setminus w)$.

We shall define by induction on $n < \omega$ a $\mathbb{Q}^1_{\mathbf{n}}$-name $\underset{\sim}{\tau}_n$ of a member of $\omega \cup \{\omega\}$ as follows:

a. $n = 0$: We let $\underset{\sim}{\tau}_0 = 0$.

b. $n = m+1$: If $\underset{\sim}{\tau}_m[G] = \omega$ then $\underset{\sim}{\tau}_n[G] = \omega$. Otherwise, we let $\underset{\sim}{\tau}_n[G] = j$ where j is the minimal natural number such that $n, \underset{\sim}{\tau}_m[G] < j$ and $\underset{\sim}{\eta}[G] \restriction (j+1) \in w_{\underset{\sim}{\eta}[G]\restriction j, n}$, if such j exists. Otherwise, we let $\underset{\sim}{\tau}_n[G] = \omega$.

Claim: $\Vdash_{\mathbb{Q}^1_{\mathbf{n}}} \underset{\sim}{\tau}_n < \omega$.

Proof: By induction on n. For $n = 0$ the claim is obvious, so let $n = m+1$. Let $p \in \mathbb{Q}^1_{\mathbf{n}}$, we shall find q above p forcing that $\underset{\sim}{\tau}_n < \omega$. By increasing p if necessary, we may assume wlog that p forces the values $\underset{\sim}{\tau}_0 = j_0 = 0$, $\underset{\sim}{\tau}_1 = j_1$,...,$\underset{\sim}{\tau}_m = j_m$. By the choice of $\underset{\sim}{\tau}_k$ and the induction hypothesis, $j_0 < j_1 < ... < j_m < \omega$. Without loss of generality, $j_m + m + 1 < lg(tr(p))$, and by the definition of $\mathbb{Q}^1_{\mathbf{n}}$, we may assume wlog that $n + 8 < nor_\nu(Suc_{T_p}(\nu))$ for every $tr(p) \leq \nu \in T_p$. By a previous claim, there is $\rho \in w_{tr(p),n} \cap Suc_{T_p}(tr(p))$, and $p \leq p^{[\rho\leq]}$ forces that $\underset{\sim}{\tau}_n \leq lg(tr(p))$, as required.

Claim: If $h \in \omega^\omega$ and $p \in \mathbb{Q}^1_{\mathbf{n}}$, then there is $p \leq q$ such that $q \Vdash$ "$h(n) \leq \underset{\sim}{\tau}_n$ for every large enough n".

Proof: WLOG $0 < h(0) < h(1) < ... < h(i)... (i < \omega)$. As before, wlog $lg(tr(p)) > 8$ and $tr(p) \leq \nu \in T_p \to 8 < nor_\nu(Suc_{T_p}(\nu))$. There is an m such that p forces values $j_0, ..., j_m$ for $\underset{\sim}{\tau}_0, ..., \underset{\sim}{\tau}_m$, respectively, but doesn't force a value for $\underset{\sim}{\tau}_{m+1}$. Choose a sequence $lg(tr(p)) = n_0 < n_1 < n_2...$ such that $h(n_i) < n_{i+1}$ (so $h(i) < n_i$) for every $i < \omega$. We shall define a condition q as follows: $\eta \in T_q$ iff:

1. $\eta \in T_p$.
2. If $l \in [n_i, n_{i+1})$ is such that $l < lg(\eta)$, then $\eta \restriction (l+1) \notin w_{\eta \restriction l, i}$.

T_q is obviously closed downwards. For every $\eta \in T_q$, if $lg(\eta) \in [n_i, n_{i+1})$ then $Suc_{T_q}(\eta) = Suc_{T_p}(\eta) \setminus w_{\eta,i} = Suc_{T_p}(\eta) \cap (Suc_{T_\mathbf{n}}(\eta) \setminus w_{\eta,i})$. Note that by the co-bigness property, T_q is a perfect tree. For every $j < \omega$, let $J = max\{n_{j+1}, k^p(j+1)\}$ (see definition 3(3)). For every $\eta \in T_q$ such that $J \leq lg(\eta)$, there is l such that $j+1 \leq l$ and $lg(\eta) \in [n_l, n_{l+1})$. It now follows that $j = min\{j, l\} \leq nor_\eta(Suc_{T_\mathbf{n}}(\eta) \setminus w_{\eta,l}) = nor_\eta(Suc_{T_q}(\eta))$. Choose $\rho \in T_q$ such that $2 < nor_\nu(Suc_{T_q}(\nu))$ for every $\rho \leq \nu \in T_q$ and define $q' = q^{[\rho \leq]}$. Obviously, $p \leq q' \in \mathbb{Q}^1_\mathbf{n}$. We shall prove that $q' \Vdash$ "$n > m \to h(n) \leq \underset{\sim}{\tau}_n$". Suppose it's not true and we shall derive a contradiction. By the assumption, there is $r \in \mathbb{Q}^1_\mathbf{n}$ such that $q' \leq r$, for some $n > m$, r forces values $j_m < j_{m+1} < ... < j_n$ for $\underset{\sim}{\tau}_m, ..., \underset{\sim}{\tau}_n$ and $j_n < h(n)$. WLOG $j_n < lg(tr(r))$ and denote $\rho = tr(r) \restriction j_n$, $\nu = tr(r) \restriction (j_n + 1)$. $\nu \in Suc_{T_q}(\rho)$, therefore, for the i that satisfies $n_i \leq j_n < n_{i+1}$ we have $\nu \notin w_{\rho,i}$ by the definition of q. As $r \Vdash$ "$\underset{\sim}{\tau}_n = j_n$" and $j_n < lg(tr(r))$, it follows from the definition of $\underset{\sim}{\tau}_n$ that $\nu = tr(r) \restriction (j_n + 1) \in w_{tr(r) \restriction j_n, n} = w_{\rho,n}$. As $m < n \to w_{\rho,n} \subseteq w_{\rho,m}$, it follows that $n < i$. Recall that by the properties of h, it follows that $h(n) < h(i) < n_i \leq j_n$, and therefore, $r \Vdash \underset{\sim}{\tau}_n = j_n > h(n)$, contradicting the choice of r. This contradiction shows that q' is as required.

Proof (of (b)): Let $(\underset{\sim}{\tau}_n : n < \omega)$ be as in the previous proof, we shall define the following $\mathbb{Q}^1_\mathbf{n}$-names:

1. For every $i < \omega$, let $\underset{\sim}{l}_i = max\{l : \underset{\sim}{\eta} \restriction (\underset{\sim}{\tau}_i + 1) \in w_{\underset{\sim}{\eta} \restriction \underset{\sim}{\tau}_i, i+l}\}$.
2. The name $\underset{\sim}{k}_i$ will be defined by induction on i as follows: $\underset{\sim}{k}_i = min\{k : k > i, \bigwedge_{j<i} k > \underset{\sim}{k}_j, \underset{\sim}{l}_k > 1\}$.
3. $\underset{\sim}{\rho} = (\underset{\sim}{\tau}_n + \underset{\sim}{l}_{\underset{\sim}{k}_n} : n < \omega)$.

In the rest of the proof we shall use the following terminology: Let $\mathbb{D}$ be Hechler's forcing. Given $I \subseteq \mathbb{D}$ and $f \in \omega^\omega$, we say that f satisfies I if there exists $(\eta, g) \in I$ such that $\eta \leq f$ and $g(n) \leq f(n)$ for every $n < \omega$.

Let $I = \{(\eta_n, f_n) : n < \omega\} \subseteq \mathbb{D}$ be a maximal antichain and let $p_1 \in \mathbb{Q}^1_\mathbf{n}$, we shall find $q \in \mathbb{Q}^1_\mathbf{n}$ such that $p_1 \leq q$ and $q \Vdash_{\mathbb{Q}^1_\mathbf{n}}$ "$\underset{\sim}{\rho}$ satisfies I". Let $h' \in \omega^\omega$ a function satisfying $\bigwedge_{n<\omega} f_n \leq^* h'$ and $n < h'(n) < h'(n+1)$ for every $n < \omega$. Let $h \in \omega^\omega$ be the function defined by $h(n) = h'(n) + 1$.

By the previous proof, there are p_2 and n_1^* such that $p_1 \leq p_2$ and $p_2 \Vdash$ " $\underset{n_1^* \leq l}{\wedge} h(l) \leq \underset{\sim}{\tau_l}$", wlog $n_1^* \leq lg(tr(p_2))$.

Let $p \in \mathbb{Q}^1_{\mathbf{n}}$, we shall define by induction an increasing sequence $lg(tr(p)) = n_0 < n_1 < ...$ such that the following condition holds:

(**) If $l_1 \in [n_i, n_{i+1})$ and $\eta_1 \in T_p \cap \omega^{l_1}$, then there are $l_2 \in [n_{i+1}, n_{i+2})$ and $\eta_2 \in T_p \cap \omega^{l_2}$ extending η_1, such that for every $l \in [l_1, l_2)$ we have $\eta_2 \restriction (l+1) \notin w_{\eta_2 \restriction l, 0}$ and $\beth_{i+1}(0) < nor_{\eta_2}(Suc_{T_p}(\eta_2))$.

Why can we construct a sequence $(n_i : i < \omega)$ as above? Suppose that n_{i+1} was chosen. For every $l_1 \in [n_i, n_{i+1})$ there is a finite number of $\eta_1 \in T_p \cap \omega^{l_1}$. Given such η_1, construct by induction on $l_1 \leq l$ a sequence $(\nu_l : l_1 \leq l < \omega)$ such that $\nu_{l_1} = \eta_1$ and $\nu_{l+1} \in Suc_{T_p}(\nu_l) \setminus w_{\nu_l, 0}$. As before, wlog $nor_\nu(Suc_{T_p}(\nu)) > 8$ for every $tr(p) \leq \nu \in T_p$ and $lg(tr(p)) > 8$. By the definition of $w_{\nu_l, 0}$, it follows that $Suc_{T_p}(\nu_l) \setminus w_{\nu_l, 0} \neq \emptyset$, and therefore we can continue the construction. Let $\nu = \underset{l_1 \leq l < \omega}{\cup} \nu_l$, then $\underset{n<\omega}{lim}(nor_{\nu \restriction n}(Suc_{T_p}(\nu \restriction n))) = \infty$, and therefore there exists $n_{i+1} \leq n_{i+2}(\eta_1)$ such that $nor_{\nu \restriction m}(Suc_{T_p}(\nu \restriction n)) > \beth_{i+1}(0)$ for every $n_{i+1}(\eta_1) \leq m$. Choose n_{i+2} greater than $n_{i+2}(\eta_1)$ for every η_1 as above. It's easy to see that n_{i+2} is as required.

Fix a sequence $(n_i : i < \omega)$ as above for p_2.

Now choose j_* and j_{**} such that:

a. p_2 forces the values $m_0, ..., m_{j_*-1}$ for $\underset{\sim}{\tau_0}, ..., \underset{\sim}{\tau_{j_*-1}}$.

b. p_2 doesn't force a value for $\underset{\sim}{\tau_{j_*}}$.

c. p_2 forces the values $k_0, ..., k_{j_{**}-1}$ for $\underset{\sim}{k_0}, ..., \underset{\sim}{k_{j_{**}-1}}$.

d. p_2 doesn't force a value for $\underset{\sim}{k_{j_{**}}}$.

Let $\nu_1 = (m_i + l_{l_i} : i < j_{**})$ such that p_2 forces the sequence of values ν_1 for $(\underset{\sim}{\tau_i} + \underset{\sim}{l_{k_i}} : i < j_{**})$, but doesn't force a value for $\underset{\sim}{k_{j_{**}}}$. Choose $h \leq h_* \in \omega^\omega$ increasing fast enough, for example, $h_*(i) = \beth_{h(i)+n_i+8}(0) + max\{m_j : j < j_*\} + 8$. The condition $(\nu_1, \nu_1 \cup h_* \restriction [lg(\nu_1), \omega))$ is compatible with a member of I, so let (ν_2, h_2) be a common upper bound. We need to find an extension of p_2 forcing that $\underset{\sim}{\rho}$ satisfies (ν_2, h_2).

Choose by infuction on $i \in [lg(\nu_1), lg(\nu_2)]$ a condition $p_{3,i}$ such that:

a. $p_{3,i}$ is obtained from p_2 by extending the trunk using an η_i such that $lg(\eta_i) \in [n_i, n_{i+1}]$.

b. $lg(tr(p_{3,i})) < n_{i+1}$.

c. $p_{3,i}$ forces the values $(m_j^* : j < j_* + (i - lg(\nu_1)))$ for $(\underset{\sim}{\tau_j} : j < j_* + (i - lg(\nu_1)))$ (and therefore also forces thee values $(l_j^* : j < j_* + (i - lg(\nu)))$ for $(\underset{\sim}{l_j} : j < j_* + (i - lg(\nu))))$, such that $j \in [j_*, j_* + (i - lg(\nu_1))) \rightarrow l_j \leq 1$.

d. $p_{3,i}$ doesn't force a value for $\underset{\sim}{\tau}_{j_*+(i-lg(\nu_1))}$.

We choose η_i by induction as follows: Suppose that η_i was chosen. Use (**) to choose an appropriate $\eta' \in [n_{i+1}, n_{i+2}]$ for η_i and let $u := Suc_{T_{\mathbf{n}}}(\eta'_i) \setminus Suc_{T_{p_2}}(\eta'_i)$. Suppose that we need to force a value for $\underset{\sim}{\tau}_j$ and assume towards contradiction that $w_{\eta'_i,j} \setminus w_{\eta'_i,j+1} \subseteq u$, then $|w_{\eta'_i,j+1}| \leq \frac{|w_{\eta'_i,j}|}{2} \leq |w_{\eta'_i,j} \setminus w_{\eta'_i,j+1}| \leq |u|$, therefore $nor_{\eta'_i}(Suc_{T_{p_2}}(\eta'_i)) = nor_{\eta'_i}(SucT_{\mathbf{n}}(\eta'_i) \setminus u) \leq i+2$, contradicting the choice of η'_i. Therefore, there exists $\eta_{i+1} \in Suc_{p_2}(\eta'_i) \cap (w_{\eta'_i} \setminus w_{\eta'_i,j+1})$, and it's easy to see that η_{i+1} is as required.

Note that for every $i \in (lg(\nu_1), lg(\nu_2))$, $m^*_{i-1} \leq lg(tr(p_{3,i})) = lg(\eta_i) \leq n_{i+1} \leq h_2(i-1)$. Now choose p_4 such that $p_{3,lg(\nu_2)} \leq p_4$, $max(Ran(\nu_2)) < lg(tr(p_4))$ and p_4 doesn't force a value for $\underset{\sim}{\tau}_{j_*+(lg(\nu_2)-lg(\nu_1))}$ (this can be done easily, for example, by extending $tr(p_{3,lg(\nu_2)})$ at each stage to a sequence outside of the appropriate $w_{\rho,0}$).

Now choose $p_{5,i}$ by induction on $i \in [lg(\nu_1), lg(\nu_2)]$ such that:

a. $p_{5,0} = p_4$.

b. $p_{5,i}$ forces a value for $\underset{\sim}{\tau}_{j_*+(lg(\nu_2)-lg(\nu_1))+(j-lg(\nu_1))}$ iff $j < i$. This value will be denoted by $m_{j_*+(lg(\nu_2)-lg(\nu_1))+(j-lg(\nu_1))}$.

c. For $lg(\nu_1) \leq j < i$, $p_{5,i}$ forces that $\underset{\sim}{k}_{j_{**}+(j-lg(\nu_1))} = j_*+(lg(\nu_2)-lg(\nu_1))+(j-lg(\nu_1))$.

d. For $lg(\nu_1) \leq j < i$, $\nu_2(j) = m_{j_{**}+(j-lg(\nu_1))}$.

Let $p_5 = p_{5,lg(\nu_2)}$. It's easy to see that $p_5 \Vdash$ "$\nu_2 \leq \underset{\sim}{\rho}$". We need to show that we can choose $p_{5,i}$ as above. At stage i of the induction, by the choice of conditions of the form $p_{3,j}$, $l_j \leq 1$ for every $j < j_* + (lg(\nu_2) - lg(\nu_1))$. By the definition of the names $\underset{\sim}{k}_j$, we want $p_{5,i}$ to force that

$\underset{\sim}{\eta} \restriction (\underset{\sim}{\tau}_{j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))}+1) \in w_{\underset{\sim}{\eta} \restriction (\underset{\sim}{\tau}_{j_*+(lg(\nu_2)-lg(\nu_1))+((i+1)-lg(\nu_1))}), j_*+(lg(\nu_2)=lg(\nu_1))+((i-1)-lg(\nu_1))+2}$.

If we can guarantee that, we should satisfy clause (c). Note that $tr(p_{5,i})$ will assume the role of $\underset{\sim}{\eta} \restriction (\underset{\sim}{\tau}_{j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))}+1)$.

In order to satisfy clause (b), we need to guarantee that $tr(p_{5,i}) \in w_{tr(p_{5,i})', j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))}$ (where $tr(p_{5,i})'$ is obtained from $tr(p_{5,i})$ by removing the last element), while every initial segment ν of $tr(p_{5,i})$ avoids $w_{\nu', j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))}$. Finally, in order to satify clause (d), we need to guarantee that the value $nor_{tr(p_{5,i})'}(Suc_{p_4}(tr(p_{5,i})'))$ is large enough such that the following will hold:

$w_{tr(p_{5,i})', j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))+i^*} \setminus w_{tr(p_{5,i})', j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))+i^*+1} \neq \emptyset$ (where $i^* = \nu_2(i-1) - m_{j_{**}+((i-1)-lg(\nu_1))}$). Note that by the choice of the sequence $(n_i : i < \omega)$ and the conditions $p_{3,i}$, it follows that $0 \leq i^*$. As we saw when we chose the conditions $p_{3,i}$, it's enough to guarantee that $j_* + (lg(\nu_2) - lg(\nu_1)) + ((i-1) - lg(\nu_1)) + i^* + 2 < nor_{tr(p_{5,i})'}(Suc_{p_4}(tr(p_{5,i})'))$. Now, for $\nu := tr(p_{5,i-1})$, repeat the argument that appeared in $(**)$ and extend ν to $\eta'_i \in T_{p_4}$ such that for every $l \in (lg(\nu), lg(\eta'_i))$, $\eta'_i \restriction (l+1) \notin w_{\eta'_i \restriction l,0}$, and such that $j_* + (lg(\nu_2) - lg(\nu_1)) + ((i-1) - lg(\nu_1)) + i^* + 2 < nor_{\eta'_i}(Suc_{T_{p_4}}(\eta'_i))$. Now choose $\eta_i \in Suc_{p_{5,i-1}}(\eta'_i) \cap (w_{tr(p_{5,i})',j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))+i^*}) \setminus w_{tr(p_{5,i})',j_*+(lg(\nu_2)-lg(\nu_1))+((i-1)-lg(\nu_1))+i^*+1}$ and define $p_{5,i} := p_{5,i-1}^{[\eta'_i \leq]}$. It's now easy to see that $p_{5,i}$ satisfies each of the above requirements.

Finally, we need to find p_6 above p_5 that forces $h_2(l) \leq \underset{\sim}{\rho}(l)$ for every $lg(\nu_2) \leq l$. We already know, by the choice of p_2, that the condition holds for large enough l, therefore there is at most a finite segment $[lg(\nu_2), m_*]$ that we need to take care of. By the proof of claim 7(a), there is $p_5 \leq p'_6$ such that $tr(p_5) = tr(p'_6)$ and $p'_6 \Vdash$ "$lg(tr(p_5)) \leq \underset{\sim}{\tau_n} \to h_2(n) \leq \underset{\sim}{\tau_n}$". Therefore, we may assume wlog that p_5 forces values for $\underset{\sim}{\tau_n}$ for every $n \in [lg(\nu_2), m_*]$. We need to show that p_5 doesn't force values for $l_{\underset{\sim}{k_n}}$ where $n \in [lg(\nu_2), m_*]$. If we succeed, we can repeat thee argument that lead us from p_2 to p_5 in order to guarantee that $h_2(n) \leq \underset{\sim}{\rho}(n)$ for every $n \in [lg(\nu_2), m_*]$. It's easy to see that this is indeed the case, as during the construction of p_5, the trunk of p_5 is the first place where $l_{\underset{\sim}{k}_{lg(\nu_2)-1}}$ is decided. $\square$

3. The additivity of the ideals derived from a Suslin ccc forcing notion adding a Hechler real

We shall now prove that under $ZF + DC_{\omega_1}$ (or actually under a weaker assumption), if $\mathbb{Q}$ is a Suslin ccc forcing notion adding a Hechler real, then the additivity of $I_{\mathbb{Q},\aleph_0}$ is $\aleph_1$. This will allow us to prove in the next section that $ZF + DC_{\omega_1}$+measurability for the ideal derived from such forcing notions implies the existence of an inner model with a measurable cardinal. A main concept in the following proof is a variant of the rank function for Hechler forcing originally introduced in [GiSh:412].

Claim 8: Assume $ZF + (\exists A \subseteq \omega_1)(\aleph_1 = \aleph_1^{L[A]})$.

a. Let $\mathbb{D}$ be Hechler forcing and let $\underset{\sim}{\eta_{dom}}$ be the canonical generic real, then there exists a sequence $(B_\alpha : \alpha < \aleph_1)$ of elements of $I_{\mathbb{D},\aleph_0}$ such that $\underset{\alpha<\aleph_1}{\cup} B_\alpha \notin I_{\mathbb{D},\aleph_0}$.

b. The above is true for every Suslin ccc forcing notion $\mathbb{Q}$ adding a Hechler real.

Remark: 1. The assumptions of the above claim follow from $ZF + DC_{\omega_1}$.

2. Although in the following proof we shall define the sets $(Y_\epsilon : \epsilon < \omega_1)$ and choose $(\Lambda_\epsilon, Y_\epsilon) \in Y_\epsilon$ for every $\epsilon < \omega_1$, there is no use of DC_{ω_1} and it's enough to assume that $\aleph_1 = \aleph_1^{L[A]}$ for some $A \subseteq \omega_1$: Given such A, as $L[A] \models ZFC$, it follows from the proof that there exists a sequence $((\Lambda_\epsilon, h_\epsilon) : \epsilon < \aleph_1^{L[A]})$ in $L[A]$ as required. As $\aleph_1 = \aleph_1^{L[A]}$ and the requirements on $(\Lambda_\epsilon, h_\epsilon)$ are absolute, the sequence is as required in V.

Proof of 8(a): For $\epsilon < \omega_1$, let Y_ϵ be the set of pairs $(\Lambda, h) = (\Lambda_\epsilon, h_\epsilon)$ such that:

1. $\Lambda_\epsilon \subseteq \omega^{<\omega}$ is a tree.

2. a. If $\rho \in \Lambda_\epsilon$ then $\bigwedge\limits_{k<\omega} \hat{\rho} < k > \in \Lambda_\epsilon$ or $Suc_\Lambda(\rho) = \emptyset$.

b. If $\nu_1, \nu_2 \in \omega^k$, $\nu_1 \in \Lambda_\epsilon$ and $\nu_1(l) \leq \nu_2(l)$ for every $l < k$, then $\nu_2 \in \Lambda_\epsilon$.

c. There is no infinite branch through Λ_ϵ.

3. $h : \Lambda \to \epsilon + 1$ is a function such that:

a. $h_\epsilon(<>) = \epsilon$.

b. $\rho_1 < \rho_2 \in \Lambda \to h_\epsilon(\rho_1) > h_\epsilon(\rho_2)$.

c. If $h_\epsilon(\rho) = \zeta + 1$ then $\bigwedge\limits_{k<\omega} h_\epsilon(\hat{\rho} < k >) = \zeta$.

d. If $h_\epsilon(\rho) = \zeta$ where ζ is a limit ordinal, then $\zeta \leq \underset{k<\omega}{lim}(h_\epsilon(\hat{\rho} < k >))$.

Subclaim: a. $Y_\epsilon \neq \emptyset$ for every $\epsilon < \omega_1$.

b. For every $\epsilon < \zeta < \omega_1$ and $(\Lambda_1, h_1) \in Y_\epsilon$, there exists $(\Lambda_2, h_2) \in Y_\zeta$ such that $\Lambda_1 \subseteq \Lambda_2$.

Proof of subclaim: For $(\Lambda_1, h_1) \in Y_\epsilon$ and $(\Lambda_2, h_2) \in Y_\zeta$, define $(\Lambda, h) = (\Lambda_1, h_1) + (\Lambda_2, h_2)$ as follows:

1. $\Lambda = \Lambda_1 \cup \Lambda_2$.

2. $h(\eta) = h_i(\eta)$ for $\eta \in \Lambda_i \setminus \Lambda_{3-i}$, $h(\eta) = max\{h_1(\eta), h_2(\eta)\}$ for $\eta \in \Lambda_1 \cap \Lambda_2$.

It's easy to see that $(\Lambda, h) \in Y_{max\{\epsilon,\zeta\}}$. We shall now prove the subclaim by induction on $\epsilon < \omega_1$. We shall prove both parts of the subclaim together.

In order to prove clause (a), at stage $\zeta + 1$, choose $(\Lambda_\zeta, h_\zeta) \in Y_\zeta$, take ω copies of (Λ_ζ, h_ζ), join them at the trunk and define $h_{\zeta+1}$ accordingly. At stage ζ where ζ is a limit ordinal, choose an increasing sequence $(\xi_{\zeta,k} : k < \omega)$ with limit ζ, use the induction hypothesis to choose an increasing sequence (with respect to $\subseteq$) $((\Lambda_{\xi_{\zeta,k}}, h_{\xi_{\zeta,k}}) : k < \omega)$ such that $(\Lambda_{\xi_{\zeta,k}}, h_{\xi_{\zeta,k}}) \in Y_{\xi_{\zeta,k}}$, join the trees $\Lambda_{\xi_{\zeta,k}}$ at the trunk and define h_ζ naturally. It's easy to see that the trees and functions that we obtained are as required.

In order to prove clause (b), we proved that at stage ζ, $Y_\zeta \neq \emptyset$. Now, for $\epsilon < \zeta$ and $(\Lambda_\epsilon, h_\epsilon) \in Y_\epsilon$, choose $(\Lambda_\zeta, h_\zeta) \in Y_\zeta$ and define $(\Lambda, h) := (\Lambda_\epsilon, h_\epsilon) + (\Lambda_\zeta, h_\zeta)$, then $\Lambda_\epsilon \subseteq \Lambda$ is as required.

We now fix a sequence $((\Lambda_\epsilon, h_\epsilon) : \epsilon < \omega_1) \in L[A]$ such that $(\Lambda_\epsilon, h_\epsilon) \in Y_\epsilon$.

The following definition is a variant of a definition that appeared in [GiSh:412]:

Definition: Let $p^* = (t^*, f^*) \in \mathbb{D}$ and let $I = \{r_k : k < \omega\}$ be a maximal antichain above p^*. Let $A = \{tr(r_k) : k < \omega\}$. We shall define $rk_{p^*,A}(\rho) \in Ord \cup \{\infty\}$ for every $t^* \leq \rho \in \omega^{<\omega}$ by defining when $\alpha \leq rk_{p^*,A}(\rho)$:

1. $\alpha = 0$: This is always true.

2. $\alpha = 1 : \alpha \leq rk_{p^*,A}(\rho)$ iff for every $l \in [lg(t^*), lg(\rho))$, $f^*(l) \leq \rho(l)$, and there is no $\nu \in A$ such that $p^* \leq (\nu, f^* \restriction [lg(\nu), \omega))$, $lg(\nu) \leq lg(\rho)$ and $l \in [lg(t^*), lg(\nu)) \rightarrow \rho(l) \leq \nu(l)$.

3. $\alpha > 1 : \alpha \leq rk_{p^*,A}(\rho)$ iff for every $\beta < \alpha$, for infinitely many k, $\beta \leq rk_{p^*,A}(\hat{\rho} < k >)$.

Subclaim: a. If $\omega_1 \leq rk_{p^*,A}(\rho)$ then $rk_{p^*,A} = \infty$.

b. If p^*, A and I are as above, then $rk_{p^*,A}(t^*) < \omega_1$.

Proof of clause (a): We shalll prove by induction on $\omega_1 \leq \epsilon$ that if $\omega_1 \leq rk_{p^*,A}(\rho)$ then $\epsilon \leq rk_{p^*,A}(\rho)$. For $\epsilon = \omega_1$ the claim is obvious. Suppose that $\epsilon > \omega_1$ and let $\zeta_k = rk_{p^*,A}(\hat{\rho} < k >)$.

Case I: There exists $\zeta < \omega_1$ such that $\{k : \zeta < \zeta_k\}$ is finite. In this case, $rk_{p^*,A}(\rho) \leq \zeta + 1 < \omega$, a contradiction.

Case II: Suppose that the assumption of case I doesn't hold, let $\zeta_* = sup\{\zeta_k : k < \omega, \zeta_k < \omega_1\} < \omega_1$. The set $u = \{k : \zeta_k > \zeta_*\}$ is infinite, therefore by the choice of ζ_*, if $k \in u$ then $\omega_1 \leq rk_{p^*,A}(\rho)$, and by the induction hypothesis, $\bigwedge\limits_{\zeta<\epsilon} \zeta \leq rk_{p^*,A}(\rho)$. It now follows from the definition of the rank that $\epsilon \leq rk_{p^*,A}(\rho)$.

Proof of clause (b): Suppose that the claim is false, then we can choose $\rho_n \in \omega^{lg(t^*)+n}$ by induction on $n < \omega$ such that $\rho_0 = t^*$, $\omega_1 \leq rk_{p^*,A}(\rho_n)$ and $m < n \rightarrow \rho_m \leq \rho_n$ (here we use subclaim (a) and the definition of $rk_{p^*,A}$). Let $f' = \bigcup\limits_{n<\omega} \rho_n$, then $p' := (t^*, \bigcup\limits_{n<\omega} \rho_n) \in \mathbb{D}$ is above p^*: For every n, $1 \leq \omega_1 \leq rk_{p^*,A}(\rho_n)$, and therefore by the definition of the rank for $\alpha = 1$, $f^*(l) \leq \rho_n(l)$ for every $l \in [lg(t^*), lg(\rho_n)]$. We shall derive a contradiction by showing that p' contradicts each r_k. Suppose towards contradiction that p' is compatible with r_k. As $lg(t^*) \leq lg(tr(r_k))$, we need to find $l \in [lg(t^*), lg(tr(r_k)))$ such that $tr(r_k)(l) < f'(l)$, that is, for $n > lg(tr(r_k))$ we need to find $l \in [lg(t^*), lg(tr(r_k)))$ such that $tr(r_k)(l) < \rho_n(l)$. As $tr(r_k) \in A$, it follows by the definition of "$1 \leq rk_{p^*,A}(\rho_n)$" that there exists such l.

We shall now proceed with the proof of the main claim.

For $\epsilon < \omega_1$, $k < \omega$ and $\Lambda = \Lambda_\epsilon$ as above, we shall define the following objects:

a. $\Omega_{\Lambda,k} = \{\eta_0\hat{}(2n_0 + 1)\hat{}\eta_1\hat{}(2n_1 + 1)\hat{}...\hat{}\eta_k : n_i < \omega$ and each η_i is a maximal element in $\Lambda\}$.

b. $\Omega_\Lambda = \bigcup\limits_{k<\omega} \Omega_{\Lambda,k}$.

c. $\Omega^+_{\Lambda,k} = \{\hat{\nu} < 2n >: \nu \in \Omega_{\Lambda,k}, n < \omega\}$.

d. $\Omega^+_\Lambda = \bigcup\limits_{k<\omega} \Omega^+_{\Lambda,k}$.

e. $I_\Lambda = \{(\eta, \hat{\eta}(0 : n < \omega)) : \eta \in \Omega^+_\Lambda\}$.

f. $B_\epsilon = \{\nu \in \omega^\omega : \eta \in \Omega^+_\Lambda \rightarrow \neg(\eta \leq \nu)\}$.

B_ϵ is Borel, and in order to show that $B_\epsilon \in I_{\mathbb{D},\aleph_0}$, it's enough to show that I_Λ is a maximal antichain (it will follow from this fact that one of the members of Ω^+_Λ is an initial segment of η_{dom}, and therefore, $\eta_{dom} \notin B_\epsilon$).

First, we shall prove that I_Λ is an antichain: Suppose that $\eta \neq \nu \in \Omega^+_\Lambda$. Note that by the definition of $\Omega_{\Lambda,k}$ and the assumption on the maximality of the η_ls in the definition, if $\eta' \in \Omega_{\Lambda,k}$ then it has a unique decomposition into a sequence of maximal elements of Λ separated by odd natural numbers. Suppose towards contradiction that $\eta < \nu \in \Omega^+_\Lambda$, and denote by η' and ν' the initial segments (respectively) obtained by omitting the last element. ν' has a unique decomposition as above, which coincides with the unique decomposition of η' on the relevant initial segments. Suppose now that η' ends with η_k, then in $\nu' \in \Omega_\Lambda$ there will be an odd number appearing after it. On the other hand, $\eta \leq \nu'$, and therefore there is an even number appearing after η_k in ν', a contradiction. Therefore, I_Λ is an antichain.

We shall now prove that I_Λ is a maximal antichain: Let $(\nu, f) \in \mathbb{D}$. If there exists $\eta \in \Omega^+_\Lambda$ such that $\eta \leq \nu$, then $(\eta, \eta\hat{}(0 : n < \omega)) \leq (\nu, f)$ and we're done. Therefore, we may assume that there is no such η. Let $\Omega' = \{<>\} \cup \{\rho : \rho = \eta_0\hat{}< 2n_0 + 1 > \hat{}\eta_1\hat{}< 2n_1 + 1 >\hat{}...\hat{}\eta_{k-1}\hat{}< 2n_{k-1} + 1 >: \eta_0, ..., \eta_{k-1} \in max(\Lambda) \wedge \rho \leq \nu\}$, then $\Omega' \neq \emptyset$ and since ν has finite length, there is an element ρ of Ω' of maximal length. Choose $\nu^1 \in \Lambda_\epsilon$ such that $\rho\hat{}\nu^1 \leq \nu$ and ν^1 is maximal.

Case I: $\rho\hat{}\nu^1 = \nu$. Let k be maximal such that $\nu^2 := \nu^1\hat{}(f(lg(\rho) + lg(\nu^1) + i) : i < k) \in \Lambda_\epsilon$. Note that by the construction of Λ_ϵ, it follows that there is no infinite branch in the tree, since $\nu^2 = \nu^1\hat{}(f(lg(\rho) + lg(\nu^1) + i) : i < k) \in \Lambda_\epsilon$ holds for $k = 0$, it follows that there is such maximal k. By the definition of the successors at each stage in Λ_ϵ (and the definition of "$(\Lambda_\epsilon, h_\epsilon) \in Y_\epsilon$") and by the choice of k, it follows that ν_2 is a maximal element in Λ_ϵ. Let $\nu^3 := \rho\hat{}\nu^2\hat{}< 2f(lg(\rho) + lg(\nu^1) + k) >$, then $\nu \leq \nu^3 \in \Omega^+_\Lambda$ and $(\nu^3, \nu^3\hat{}(0 : n < \omega))$ is compatible with (ν, f).

Case II: $\rho\hat{}\nu^1 < \nu$. Recall that by the definition of Λ_ϵ, $Suc_{\Lambda_\epsilon}(\rho) = \{\rho\hat{}< k >: k < \omega\}$ or $Suc_{\Lambda_\epsilon}(\rho) = \emptyset$, therefore, $\nu^1 \in max(\Lambda_\epsilon)$. We now have two possibilites: If $f(lg(\rho)+ lg(\nu^1))$ is odd, then $\rho\hat{}\nu^1\hat{}< f(lg(\rho) + lg(\nu^1)) >\in \Omega'$, contradicting the maximality of ρ. Therefore, $f(lg(\rho)+lg(\nu^1))$ is even, and therefore, $\rho\hat{}\nu^1\hat{}< f(lg(\rho)+lg(\nu^1)) >\in \Omega^+$, contradicting the assumption that there is no $\eta \in \Omega^+$ such that $\eta \leq \nu$.

We now turn to the main part of the claim: $\bigcup\limits_{\epsilon<\omega_1} B_\epsilon \notin I_{\mathbb{D},\aleph_0}$.

Proof: Suppose towards contrdiction that $\bigcup\limits_{\epsilon<\omega_1} B_\epsilon \in I_{\mathbb{D},\aleph_0}$, then there is a Borel set B such that $\bigcup\limits_{\epsilon<\omega_1} B_\epsilon \subseteq B$ and $\Vdash_{\mathbb{D}}$ "$\underset{\sim}{\eta_{dom}} \notin B$".

By the definition of the ideal, there is a sequence $\bar{p} = (p_{n,l} : n, l < \omega)$ such that $\bar{p_n} = (p_{n,l} : l < \omega)$ is predense for every $n < \omega$ and $B \subseteq (\bigcap\limits_{n<\omega} \bigcup\limits_{l<\omega} set(p_{n,l}))^c$ (where $set(\eta, f)$ is the set of reals g that extend η such that $f(n) \leq g(n)$ for every n).

Fix a countable elementary submodel N of $L_\chi[\bar{p}, (\Lambda_\epsilon : \epsilon < \omega_1)]$ for χ large enough, such that $\bar{p}, (\Lambda_\epsilon : \epsilon < \omega_1) \in N$. Let $\delta(*) = N \cap \omega_1^{L_\chi[\bar{p},(\Lambda_\epsilon:\epsilon<\omega_1)]}$, then $\Lambda = \Lambda_{\delta(*)}$ is well-defined. Let $(\bar{p^*_m} : m < \omega)$ list the predense subsets of $\mathbb{D}$ in N, and for every $m < \omega$, denote $\bar{p^*_m} = (p^*_{m,l} : l < \omega)$. For every $n < \omega$, there exists $j(n) < \omega$ such that $\bar{p_n} = \bar{p^*_{j(n)}}$. We shall choose conditions q_n by induction on $n < \omega$ such that:

a. $q_n = (\nu_n, f_n) \in \mathbb{D} \cap N$.

b. $n = m + 1 \to q_m \leq q_n$.

c. If $n = m + 1$ then there exists l such that $p^*_{m,l} \leq q_n$.

d. $\nu_0 =<>$, and if $n > 0$ then $\nu_n = \rho_n\hat{\ } < 2m_n >$ for some $\rho_n \in \Omega_\Lambda$ and $m_n < \omega$.

Suppose that we can construct such sequence and we shall derive a contradiction: As $(q_n : n < \omega)$ is increasing, $(\nu_n : n < \omega)$ is increasing too, and $\nu := \bigcup_{n<\omega} \nu_n$ is a well-defined function. By the elementarity of N, for every k there is a predense $I \in N$ such that $k < lg(\eta)$ for every $(\eta, f) \in I$. Let $m = m(k)$ such that $I = \{p^*_{m,l} : l < \omega\}$. As there exists an l such that $p^*_{m,l} \leq q_{m+1}$, it follows that $k < lg(\nu_{m+1})$, hence $\nu \in \omega^\omega$. As $(q_n : n < \omega)$ is increasing, it follows that $\nu \in set(q_n)$ for every $n < \omega$. By clause (c) of the induction, for every $n = m+1$ there is $l(n)$ such that $p^*_{m,l(n)} \leq q_n$, therefore $\nu \in set(p^*_{m,l(m+1)})$ for every $m < \omega$, therefore $\nu \in \bigcap_{m<\omega} \bigcup_{l<\omega} set(p^*_{m,l})$ and $\nu \in \bigcap_{n<\omega} \bigcup_{l<\omega} set(p_{n,l})$ (recall that each $\bar{p_n}$ appears also as $\bar{p^*_{j(n)}}$). For every predense $I \in N$, $I = \{p^*_{m,l} : l < \omega\}$ for an appropriate m and $\nu \in \bigcup_{l<\omega} set(p^*_{m,l})$, therefore ν is $(N, \mathbb{D})$-generic and therefore $\nu \notin B$.

We shall now prove that $\nu \in B_{\delta(*)}$, which is a contradiction (since $B_{\delta(*)} \subseteq B$). We need to show that for every $\eta \in \Omega^+_{\Lambda_{\delta(*)}}$, $\neg(\eta \leq \nu)$. Suppose towards contradiction that there exists $\eta \in \Omega^+_{\Lambda_{\delta(*)}}$ such that $\eta \leq \nu$. Choose ν_n long enough such that $\eta < \nu_n \leq \nu$, then by clause (d) of the induction we get to comparable elements of $\Omega^+_{\Lambda_{\delta(*)}}$, contradicting the fact that $I_{\Lambda_{\delta(*)}}$ is an antichain.

It remains to show that we can construct a sequence $(q_n : n < \omega)$ as above. For $n = 0$ there is no problem, so assume that $n = m+1$ and $q_m = (\nu_m, f_m)$ was chosen such that it satisfies the induction hypothesis. Denote $p^* = (t^*, f^*) = (\nu_m, f_m)$. As $\{p^*_{m,l} : l < \omega\}$ is predense, $\{p : (\exists l)(p^*_{m,l} \leq p)\}$ is open and dense, and therefore there exists a maximal antichain $\bar{r} = (r_l : l < \omega)$ above p^* such that each r_l is above some $p^*_{m,k}$. By elementarity, there is such $\bar{r}$ in N. Let $A = \{tr(r_l) : l < \omega\}$, then $(p^*, \bar{r}, A)$ are as in the definition of the rank, and by a previous claim, $rk_{p^*,A}(t^*) < \omega_1$. Note that $\{p^*, A, t^*\} \in N$, therefore $rk_{p^*,A}(t^*) \in N$ and therefore $rk_{p^*,A}(t^*) < \delta(*)$. Let $h_{\delta(*)} : \Lambda_{\delta(*)} \to \delta(*) + 1$ be as in the definition of $\Lambda_{\delta(*)}$ and let Λ' be the set of sequences $\rho \in \Lambda_{\delta(*)}$ satisfying the following properties:

1. $(t^*, f^*) \leq (t^*\hat{\ }\rho, t^*\hat{\ }\rho\hat{\ }f^* \restriction [lg(t^*\hat{\ }\rho), \omega))$.

2. $rk_{p^*,A}(t^*\hat{\ }\rho) < h_{\delta(*)}(\rho)$.

Note that $\Lambda' \neq \emptyset$: As $\delta(*)$ is a limit ordinal, $<>\in \Lambda'$. Let $\alpha_* = min\{rk_{p^*,A}(t^*\hat{\ }\rho) : \rho \in \Lambda'\}$ and choose $\rho_* \in \Lambda'$ such that $\alpha_* = rk_{p^*,A}(t^*\hat{\ }\rho)$. There are two possible cases:

Case I: $\alpha_* = rk_{p^*,A}(t^*\hat{\ }\rho_*) = 0$. By the way we defined the rank, there is $\nu'_{m+1} \in A$ such that $(t^*, f^*) \leq (\nu'_{m+1}, f^* \restriction [lg(\nu'_{m+1}), \omega))$, $lg(\nu'_{m+1}) \leq lg(t^*\hat{\ }\rho)$ and $t^*\hat{\ }\rho(l) \leq$

$\nu'_{m+1}(l)$ for every $lg(t^*) \leq l < lg(\nu'_{m+1})$. There exists l_* such that $\nu'_{m+1} = tr(r_{l_*})$, denote $r_{l_*} = (\nu'_{m+1}, f_{r_{l_*}})$. Choose k_i by induction on $lg(\nu'_{m+1}) \leq i$ such that $f^*(i) \leq k_i$, $f_{r_{l_*}}(i) \leq k_i$ and $\rho'_{m,i} := \nu'_{m+1} \restriction [lg(\nu_m), lg(\nu'_{m+1}))\hat{\ }(k_j : j < i) \in \Lambda_{\delta(*)}$. By the definition of the pairs of the form (Λ, h), it follows that $\nu'_{m+1} \restriction [lg(\nu_m), lg(\nu'_{m+1})) \in \Lambda_{\delta(*)}$. Additionally, as there is no infinite branch in $\Lambda_{\delta(*)}$, it follows that there exists a maximal i for which we can choose k_i as required. Let $\nu' = \nu'_{m+1}\hat{\ }(k_j : j < i+1)\hat{\ } < 2(f_{r_{l_*}}(lg(\nu'_{m+1}) + i) + f^*(lg(\nu'_{m+1}) + i)) >$ and consider the condition $(\nu', \nu' \cup f^{**} \restriction [lg(\nu'), \omega))$ where $f^{**}(i) = max\{f^*(i), f_{r_{l_*}}(i)\}$ for every $i \in [lg(\nu'), \omega)$. It's easy to see that $r_{l_*}, q_m \leq (\nu', \nu' \cup f^{**} \restriction [lg(\nu'), \omega))$ and that the requirement from clause (d) in the definition of q_{m+1} is satisfied.

Case II: $\alpha_* = rk_{p^*,A}(t^*\hat{\ }\rho_*) > 0$. $\rho_* \in \Lambda'$, therefore $rk_{p^*,A}(t^*\hat{\ }\rho_*) < h_{\delta(*)}(\rho_*)$, therefore by the definition of $h_{\delta(*)}$, for every k large enough, $rk_{p^*,A}(t^*\hat{\ }\rho_*) < h_{\delta(*)}(\rho_*\hat{\ } < k >)$. Denote $\beta_* = rk_{p^*,A}(t^*\hat{\ }\rho_*)$. By the definition of the rank, the following set is finite: $\{k : \beta_* \leq rk_{p^*,A}(t^*\hat{\ }\rho_*\hat{\ } < k >)\} =: u_1$. In addition, the following set is finite: $\{k : k \leq f^*(lg(t^*\hat{\ }\rho_*)) \vee k \leq f_{r_{l_*}}(lg(t^*\hat{\ }\rho_*))\}$. For every k large enough, $k \in \omega \setminus u_1 \setminus u_2$ and $\beta_* < h_{\delta(*)}(\rho_*\hat{\ } < k >)$. For such k, since $k \notin u_1$, $rk_{p^*,A}(t^*\hat{\ }\rho_*\hat{\ } < k >) < \beta_*$. By the definition of the rank, $rk_{p^*,A}(t^*\hat{\ }\rho_*\hat{\ } < k >) < rk_{p^*,A}(t^*\hat{\ }\rho_*)$ for every k large enough. Therefore, for every k large enough, $\rho_*\hat{\ } < k > \in \Lambda'$ and $rk_{p^*,A}(t^*\hat{\ }\rho_*\hat{\ } < k >) < \alpha_*$, contradicting the minimality of α_*.

This completes the proof of claim 8(a).

Proof of 8(b): Let f be a Borel function such that $\Vdash_{\mathbb{Q}}$ "$f(\underset{\sim}{\eta}) = \underset{\sim}{\eta}_{dom}$". Consider the sequence $(f^{-1}(B_\alpha) : \alpha < \omega_1)$ where $(B_\alpha : \alpha < \omega_1)$ is the sequence constructed in the proof of 8(a). $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\eta} \notin f^{-1}(B_\alpha)$" for every $\alpha < \omega_1$, and therefore $f^{-1}(B_\alpha) \in I_{(\mathbb{Q},\underset{\sim}{\eta}),\aleph_0}$. Let N be a countable model of ZFC^* containing the relevant objects, we need to find $g \subseteq \mathbb{Q} \cap N$ such that G is $(N,\mathbb{Q})$-generic and $\underset{\sim}{\eta}[G] \in \bigcup_{\alpha<\omega_1} f^{-1}(B_\alpha)$. Let $H \subseteq \mathbb{D} \cap N$ be $(N,\mathbb{D})$-generic such that $\underset{\sim}{\eta}_{dom}[H] \in \bigcup_{\alpha<\omega_1} B_\alpha$, it suffices to construct $H \subseteq G \subseteq \mathbb{Q} \cap N$ which is $(N,\mathbb{Q})$-generic. Now $N[H]$ is a model of sufficiently many axioms, and over $N[H]$ there is a generic $G' \subseteq \mathbb{Q}/\underset{\sim}{\eta}_{dom}[H]$, so the required conclusion follows. $\square$

4. A measurable cardinal from regularity properties and DC_{ω_1}

In this section we shall prove a general criterion for the existence of an inner model for a measurable cardinal under the assumptions DC_{ω_1} + "all sets of reals have certain regularity properties".

Claim 9: The following conditions imply the existence of an inner model of ZFC with a measurable cardinal:

a. $V \models ZF$.

b. $V_0 \subseteq V$ is an inner model of ZF.

c. In V_0, $\mathbb{Q}$ is a (definition of) Suslin ccc forcing and $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real.

d. I is a σ-ideal on the reals extending $I_{(\mathbb{Q},\underset{\sim}{\eta}),\aleph_0}$.

e. $(B_\alpha : \alpha < \lambda)$ is a sequence of sets from $I_{(\mathbb{Q},\underset{\sim}{\eta}),\aleph_0}$ such that $\cup\{B_\alpha : \alpha < \lambda\} \notin I$.

f. $\mathcal{P}(\omega^\omega)/I \models ccc$ (which means that there is no sequence $(B_\alpha : \alpha < \omega_1)$ of elements of $\mathcal{P}(\omega^\omega)/I$ such that $(\alpha \neq \beta \wedge B'_\alpha \in B_\alpha \wedge B'_\beta \in B_\beta) \to B'_\alpha \cap B'_\beta \in I)$, or just:

f$^-$: There is no sequence $(B'_\alpha : \alpha < \aleph_1)$ of I-positive sets such that $B'_\alpha \cap B'_\beta \in I$ for every $\alpha \neq \beta < \aleph_1$.

Remark: The condition in clause (f) implies the condition in clause (f$^-$), and they're equivalent under AC_{ω_1}.

Remark: There is no essential use of assumption (b) in the proof, but it might be more transparent.

Proof: For every $\alpha < \lambda$, let $B'_\alpha = B_\alpha \setminus \underset{\beta<\alpha}{\cup} B_\beta$. As $\cup\{B_\alpha : \alpha < \lambda\} \notin I$, there is $u \subseteq \lambda$ such that $\{B'_\alpha : \alpha \in u\} \notin I$ and $|u|$ is minimal (there is no problem about the AC as u is a set of ordinals). Fix an enumeration $(\xi_\alpha : \alpha < |u|)$ of u and let $B''_\alpha = B'_{\xi_\alpha}$ for $\alpha < |u|$. $(B''_\alpha : \alpha < |u|)$ is a sequence of pairwise disjoint sets whose union is I-positive. Let I' be the ideal on $|u|$ consisting of sets X such that $\cup\{B''_\alpha : \alpha \in X\} \in I$.

We claim that I' is σ-complete: Suppose that $(X_n : n < \omega) \in V$ such that $X_n \in I'$ for $n < \omega$, but $Y := \underset{n<\omega}{\cup} X_n \notin I'$. Let $A_n = \cup\{B''_\alpha : \alpha \in X_n\}$, then $(A_n : n < \omega) \in V$ and each A_n is in I, but as $Y \notin I'$, it follows that $\underset{n<\omega}{\cup} A_n \notin I$, contradicting its σ-completeness. Therefore, I' is σ-complete.

We now work in $L[I']$. Let $J := I' \cap L[I'] \in L[I']$, then clearly $L[I'] \models$ "J is an $\aleph_1^V$-complete ideal on $|u|$". We shall prove that $L[I'] \models$ "$\mathcal{P}(|u|)/J \models \aleph_1^V - cc$". Suppose not, then in $L[I']$ there is a sequence $(A_\alpha : \alpha < \aleph_1^V)$ of J-positive sets such that $\alpha < \beta \to A_\alpha \cap A_\beta \in J$. As J is $\aleph_1^V$-complete in $L[I']$, we may assume WLOG that $\alpha < \beta \to A_\alpha \cap A_\beta = \emptyset$. In V, let $(C_\epsilon : \epsilon < \aleph_1^V)$ be the sequence defined by $C_\epsilon = \cup\{B''_\alpha : \alpha \in A_\epsilon\}$. Obviously, $\epsilon \neq \zeta \to C_\epsilon \cap C_\zeta = \emptyset$. Suppose that $C_\epsilon = \cup\{B''_\alpha : \alpha \in A_\epsilon\} \in I$, then $A_\epsilon \in I'$ by the definition of I', and as $A_\epsilon \in L[I']$, it follows that $A_\epsilon \in J$, contradicting the choise of A_ϵ. Therefore, each C_ϵ is I-positive, contradicting the fact that $V \models$ "$\mathcal{P}(\omega^\omega)/I \models ccc$" (and contradicting (f$^-$)).

Therefore, $L[I'] \models$ "J is an $\aleph_1^V$-complete ideal on $|u|$ such that $\mathcal{P}(|u|)/J \models \aleph_1^V - cc$", and the existence of an inner model for a measurable cardinal follows. $\square$

Our goal now is to use DC_{ω_1} and regularity properties in order to derive the countable chain condition for $\mathcal{P}(\omega^\omega)/I_{\mathbb{Q},\aleph_0}$ from the fact that $Borel(\omega^\omega)/I_{\mathbb{Q},\aleph_0}$ is ccc.

The following definition is of interest in the absence of choice:

Definition 10: a. We say that a forcing notion $\mathbb{Q}$ satisfies the strong chain condition (scc) if there is no uncountable[2] set $\{X_s : s \in S\} \subseteq \mathcal{P}(\mathbb{Q})$ such that $X_s \neq \emptyset$ for each $s \in S$, and for every $s \neq t \in S$, if $p \in X_s$ and $q \in X_t$ then p and q are incompatible. We define the strong chain condition for boolean algebras similarly.

[2]So it may be non well-orderable in the absence of choice.

b. Given a Boolean algebra B and an ideal I, we say that $(B, I) \models scc^-$ if there is no uncountable collection $\{X_s : s \in S\} \subseteq \mathcal{P}(B)$ of nonempty subsets of B such that each of them is disjoint to I and $(s \neq t \in S \wedge B_s \in X_s \wedge B_t \in X_t) \rightarrow B_s \wedge B_t \in I$.

c. Given a Boolean algebra B and an ideal I, we say that (B, I) satisfies the weak countable chain condition (ccc^-) if it satisfies the property appearing in clause f$^-$ of claim 9, where instead of a set of cardinality $\aleph_1$ we have an uncountable set.

d. Similarly, we can define $\kappa - scc$ and $\kappa - scc^-$ for a cardinal κ.

Observation 11: a. *scc* is equivalent to *ccc* under DC_{ω_1}. In addition, note that in the above definition it doesn't follow that $\aleph_1 \leq |S|$.

b. Similarly, $\kappa - scc$ is equivalent to $\kappa - cc$ under AC_κ.

The following seems like a natural question:

Question 12: Assume ZF. Is ccc equivalent to scc for Suslin forcing notions?

We shall address the above problem in future work.

Claim 13 (ZF)**:** Let $\mathbb{Q}$ be a Suslin scc forcing notion, then $(Borel(\omega^\omega), I_{\mathbb{Q},\aleph_0}) \models ccc^-$. Moreover, $(Borel(\omega^\omega), I_{\mathbb{Q},\aleph_0}) \models scc^-$.

Proof: We shall first prove ccc^-. Let $B = Borel(\omega^\omega)$ and suppose that $\{B_s : s \in S\}$ is a collection of $I_{\mathbb{Q},\aleph_0}-$positive Borel sets such that $s \neq t \in S \rightarrow B_s \cap B_t \in I_{\mathbb{Q},\aleph_0}$. For every $s \in S$, let $X_s = \{p \in \mathbb{Q} : p \Vdash \underset{\sim}{\eta} \in B_s\}$. As each B_s is positive, $X_s \neq \emptyset$. Obviously, if $s \neq t \in S$, $p \in X_s$ and $q \in X_t$, then p and q are incompatible. As $\mathbb{Q} \models scc$, it follows that S is countable, hence $(B, I_{\mathbb{Q},\aleph_0}) \models ccc^-$.

As for scc^-, suppose that $\{X_s : s \in S\} \subseteq \mathcal{P}(B)$ is a collection of non-empty subsets of B, each consisting of positive sets, such that $s \neq t \in S \wedge B_s \in X_s \wedge B_t \in X_t \rightarrow B_s \cap B_t \in I_{\mathbb{Q},\aleph_0}$. For each $s \in S$ let P_s be the set of conditions $p \in \mathbb{Q}$ that force "$\underset{\sim}{\eta} \in B'$" for some $B' \in X_s$. Now the rest of the argument is similar to the previous case. $\square$

Claim 14: Assume $ZF + DC_{\omega_1}$. If $\mathbb{Q}$ is a Suslin ccc forcing notion (that is, there are no $\aleph_1$ pairwise incompatible conditions), then $Borel(\omega^\omega)/I_{\mathbb{Q},\aleph_0} \models ccc$.

Proof: By DC_{ω_1}, $\mathbb{Q}$ is *scc*. By the previous claim, it follows that $(Borel(\omega^\omega), I_{\mathbb{Q},\aleph_0}) \models ccc^-$. By DC_{ω_1}, it follows that $Borel(\omega^\omega)/I_{\mathbb{Q},\aleph_0} \models ccc$: Suppose that $\{X_s : s \in S\} \subseteq Borel(\omega^\omega)/I_{\mathbb{Q},\aleph_0}$ is an uncountable antichain, by DC_{ω_1} it follows that it has a subset of size $\aleph_1$, and from this set we can choose representatives and get a contradiction to ccc^-. $\square$

Corollary 15: There is an inner model with a measurable cardinal when the following conditions hold:

1. $\mathbb{Q}$ is a Suslin ccc forcing notion and $\underset{\sim}{\eta}$ is a $\mathbb{Q}-$name of a real.

2. $ZF + DC_{\omega_1} +$ "All sets of reals are $I_{\mathbb{Q},\aleph_0}-$measurable".

3. There is a sequence $(B_\alpha : \alpha < \lambda)$ of sets from $I_{(\mathbb{Q},\underset{\sim}{\eta}),\aleph_0}$ such that $\cup\{B_\alpha : \alpha < \lambda\} \notin I_{(\mathbb{Q},\underset{\sim}{\eta}),\aleph_0}$.

Proof: Let $I = I_{(\mathbb{Q},\underset{\sim}{\eta}),\aleph_0}$. By claim 9, it's enough to prove that $(\mathcal{P}(\omega^\omega), I) \models ccc^-$. Suppose that $\{X_s : s \in S\}$ is an uncountable collection of I-positive sets such that $s \neq t \in S \to X_s \cap X_t \in I$. For every $s \in S$, let $P_s := \{B \subseteq \omega^\omega : B$ is a Borel set such that $B = X_s \bmod I\}$. By our assumption, each P_s is non-empty. By DC_{ω_1}, there is an injection $f : \aleph_1 \to S$, so there is a sequence $(P_{f(\alpha)} : \alpha < \aleph_1)$, and again by $DC_{\aleph_1}$, there is a sequence $(B_\alpha : \alpha < \aleph_1)$ such that $B_\alpha \in P_{f(\alpha)}$ for each $\alpha < \aleph_1$. Obviously, $(B_\alpha : \alpha < \aleph_1)$ witnesses that $Borel(\omega^\omega)/I$ doesn't satisfy ccc, contradicting claim 14. $\square$

Corollary 16: a. Let $\mathbb{Q}$ be a Suslin ccc forcing notion with generic $\underset{\sim}{\eta}$ adding a Hechler real, then $ZF + DC_{\omega_1} +$ "Every set of reals is $I_{(\mathbb{Q},\underset{\sim}{\eta}),\aleph_0}$-measurable" implies the existence of an inner model with a measurable cardinal.

b. The above claim is true for $(\mathbb{Q}^1_{\mathbf{n}}, \underset{\sim}{\eta}^1_{\mathbf{n}})$ where $\mathbf{n} \in \mathbf{N}$.

Proof: a. By corollary 15 and claim 8(b).

b. By (a) and claim 7(b). $\square$

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