SPECIALISING ARONSZAJN TREES WITH STRONG AXIOM A AND HALVING

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ABSTRACT. We construct creature forcings that specialise a given Aronszajn tree that have strong Axiom A.

We work with tree creature forcing. The creatures live on the Aronszajn tree, are normed and have the halving property. We show that our models fulfil

$$\aleph_1 = \mathfrak{d} < \mathrm{unif}(\mathcal{M}) = \aleph_2 = 2^\omega.$$

1. INTRODUCTION

We establish a notion of forcing with strong Axiom A that specialises an Aronszajn tree and makes the ground model reals a meagre set.

Solovay and Tennenbaum [19] specialised Aonszajn trees by finite approximations. Later Shelah [18, Ch. V] found a way to specialise Aronszajn trees without adding reals. Now we are interested in an intermediate way, an ${}^\omega\omega$-bounding forcing (see Def. 4.1) that adds reals.

We use creature forcing. Creature forcing tries to enlarge and systemise the family of very nice forcings. There is "the book on creature forcing" [15], and the work is extended in [14, 16, 8, 17, 9, 3, 4]. Our exposition is self-contained with respect to the creature technique.

At first glance we cannot replace the countable reservoir of creatures in [15] by an uncountable set. However, this was first done in [11] where we applied the theory of creatures for specialising an Aronszajn tree. Unfortunately [11] contained some inaccuracies, and we hope that we give a more detailed and clearer presentation here. Here we rework the forcings from [11] and develop their use further. The norm of creatures (see Definitions 2.5) we shall use is natural for specialising Aronszajn trees, cf. [18, Ch. V, §6]. In the terminology of [15], the forcing conditions are liminf tree creature forcings. The trees in the forcing conditions are finitely branching and endless, that is, do not have maximal nodes.

Definition 1.1. (1) $(\mathbf{T}, <_\mathbf{T})$ is called an Aronszajn tree if

(a) $|\mathbf{T}| = \aleph_1$,

(b) $(\mathbf{T}, <_\mathbf{T})$ is a partial order such that for any $t \in \mathbf{T}$, $\mathrm{pred}(t) = \{s \in \mathbf{T} : s <_\mathbf{T} t\}$ is well-ordered; our trees may have countably many $<_\mathbf{T}$-minimal elements,

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(c) for $\alpha < \omega_1$, the level $\mathbf{T}_\alpha = \{t \in \mathbf{T} : \text{pred}(t) \cong \alpha\}$ of $(\mathbf{T}, <_\mathbf{T})$ is a subset of $\{\beta \in \omega_1 : \alpha\omega \le \beta < (\alpha+1)\omega\}$,

(d) $(\mathbf{T}, <_\mathbf{T})$ has no uncountable branch,

(e) $(\mathbf{T}, <_\mathbf{T})$ is normal, i.e., for every $\alpha < \beta < \omega_1$ for every $t \in \mathbf{T}_\alpha$ there is $t' \in \mathbf{T}_\beta$ such that $t <_\mathbf{T} t'$.

(2) A function $f\colon \mathbf{T} \to \omega$ is called a specialisation of $(\mathbf{T}, <_\mathbf{T})$ or we say f specialises $(\mathbf{T}, <_\mathbf{T})$ if $\forall s, t \in \mathbf{T}(s <_\mathbf{T} t \to f(s) \ne f(t))$ (see [6, p. 244]). An Aronszajn tree $(\mathbf{T}, <_\mathbf{T})$ is special if it has a specialisation function.

(3) A Souslin tree is an Aronszajn tree in which all antichains are countable.

Nahman Aronszajn was the first to construct a tree with properties (a) to (e). Kurepa [10] coined the name Aronszajn tree and introduced Aronszajn trees in the literature.

If an Aronszajn tree is special then it is the union of countably many antichains, and hence the tree is not a Souslin tree. Shelah showed [18, Ch. IX] that "There is no Souslin tree" does not imply "All Aronszajn trees are special".

In forcing the *larger* condition is the *stronger* one. We recall the definition of Axiom A and of strong Axiom A.

Definition 1.2. A notion of forcing $(\mathbb{Q}, \le)$ is said to have Axiom A if there are quasi orders $\le_n$, $n \in \omega$, over $\mathbb{Q}$ with the following properties:

(1) $\le_{n+1}$ is a subrelation of $\le_n$ for any n, and $\le_0$ is a subrelation of $\le$.

(2) $(\mathbb{Q}, \le, (\le_n)_{n\in\omega})$ has the fusion property, i.e., for any sequence $\langle p_n : n < \omega\rangle$ such that $p_n \le_n p_{n+1}$ there is $q \in \mathbb{Q}$ such that for any n, $q \ge_n p_n$.

(3) For any $n \in \omega$, $p \in \mathbb{Q}$ and maximal antichain A in $\mathbb{Q}$ there is a $q \ge_n p$ such that q is compatible with at most countably many $a \in A$.

Definition 1.3. A notion of forcing $(\mathbb{Q}, \le)$ is said to have strong Axiom A if it has Axiom A and in item (3) the number of compatible elements is finite.

Theorem 1.4. *Given an Aronszajn tree* $\mathbf{T}$*, there is a notion of forcing* $(\mathbb{Q}_\mathbf{T}, \le, (\le_n)_{n\in\omega})$ *with the following properties:*

(a) $\mathbb{Q}_\mathbf{T}$ *specialises* $\mathbf{T}$.

(b) $\mathbb{Q}_\mathbf{T} \subseteq H(\aleph_1)$.

(c) $(\mathbb{Q}_\mathbf{T}, \le, (\le_n)_{n\in\omega})$ *has strong Axiom A.*

(d) Let D *be dense and open in* $\mathbb{Q}_\mathbf{T}$*,* $n \in \omega$*,* $p = ((T^p, <_p), \langle \mathbf{c}_{p,t} : t \in T^p\rangle) \in \mathbb{Q}_\mathbf{T}$*. Then there is* $q \ge_n p$ *and there is* $m \in \omega$ *such that*

$$\begin{array}{rl} \{q^{\langle t\rangle} : t \in (T^q)^{[m]}\} & \text{is predense above } q \\ (\forall t \in (T^q)^{[m]}) & (q^{\langle t\rangle} \in D) \text{ and} \\ q \restriction (T^q)^{[m]} & \text{is a finite structure with finite signature.} \end{array}$$

(e) $\mathbb{Q}_\mathbf{T}$ *adds a real that makes the ground model reals a meagre set.*

Remark 1.5. The list of properties is redundant: Property (d) implies strong Axiom A. We state (d) because it describes the underlying structure. The components of a condition $p \in \mathbb{Q}_\mathbf{T}$ and the notions in Property (d) will be explained in the coming sections.

We give an overview over the paper: In Sections 2 to 5 we prove Theorem 1.4 by a forcing $\mathbb{Q}_{\mathbf{T}}$. In Section 2 we introduce creatures. In Section 3 we define a notion of forcing, an iterand, and show that is specialises a given Aronszajn tree. We show that the smooth conditions are dense. In Section 4 we prove Theorem 1.4 for all but item (e). In Section 5 we prove that the forcing with $\mathbb{Q}_{\mathbf{T}}$ makes the ground model reals a meagre set and thus finish the proof of Theorem 1.4. Section 2 and Section 3 conclude with some results on the halving property, that are not used in Theorem 1.4. There was some hope that strong halving properties would allow to establish a name for an Ostaszewski club sequence (see [13]) in the extension. This stays open.

The background on proper forcing can be found in [1, 18].

2. Tree creatures

In this section we define the tree creatures which will be used in the next section to describe the branching of the countable trees that will serve as forcing conditions. We define three important operations that can be performed on creatures:

– gluing together creatures (Lemmata 2.13 and 2.14),
– extending the domains of the partial specialisations in the set of possibilities of a creature (Lemma 2.15),
– extending the basis of a creature together with thinning out the set of possibilities (Lemma 2.16) and extending the elements in the set of possibilities.

We shall define the forcing conditions only in the next section. They will endless finitely branching tagged trees, such that each node and its immediate successors in the tree are described by a creature (see Def. 2.9). Roughly speaking, in our context, a creature $\mathbf{c}$ will be a tree of height 2 of partial specialisation functions whose root it labelled by a pair $(i(\mathbf{c}), k(\mathbf{c}))$ of natural numbers.

We let χ stand for some regular cardinal larger than $(2^{\aleph_2})^+$ and let $H(\chi)$ denote the set of sets of hereditary cardinality less than χ. We use the symbol $<^*_\chi$ for some some well-order on this set. The symbol $\mathcal{H}(\chi)$ denotes the structure $(H(\chi), \in, <^*_\chi)$.

Throughout this work we make the following assumption that $(\mathbf{T}, <_{\mathbf{T}})$ is an Aronszajn tree as in Def. 1.1(1).

We define the following finite approximations of specialisation maps:

Definition 2.1. For $u \subseteq \mathbf{T}$ and $n < \omega$ we let

$$\mathrm{spec}_n(u) = \{\eta \mid \eta \colon u \to [0, n) \ \wedge \ (\eta(x) = \eta(y) \to \neg(x <_{\mathbf{T}} y))\}.$$

We let $\mathrm{spec}(u) = \bigcup_{n<\omega} \mathrm{spec}_n(u)$, $\mathrm{spec}_n = \mathrm{spec}^{\mathbf{T}}_n = \bigcup\{\mathrm{spec}_n(u) \,:\, u \subset \mathbf{T}, u \text{ finite}\}$, $\mathrm{spec} = \mathrm{spec}^{\mathbf{T}} = \bigcup\{\mathrm{spec}(u) \,:\, u \subset \mathbf{T}, u \text{ finite}\}$.

Definition 2.2. (1) We choose three sequences of natural numbers $\langle n_{k,i} : i < \omega\rangle$, $k = 1, 2, 3$, such that the following growth conditions are fulfilled:

$$\begin{aligned} 2 \le n_{1,i} &\le n_{2,i} \le n_{3,i}, \\ 2^{(n_{3,i})^2} &< n_{1,i+1}. \end{aligned}$$

(2) Jumping ahead to Definition 2.4, these numbers bound the size of a simple i-creature, $i \geq 1$, in the following way:
  (a) the number $n_{1,i-1}$ bounds the size of the domain of the partial specialisation function that is the basis of the creature,
  (b) the basis of the creature is an element of $\mathrm{spec}^{\mathbf{T}}_{n_{2,i-1}}$,
  (c) the number $n_{3,i}$ bounds the number of the possibilities of the creature,
  (d) each function in the possibilities is an element of $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$,
  (e) the number $n_{1,i}$ bounds the size of domain each function in the possibilities of the creature.
(3) We fix the $n_{k,i}$, $k = 1, 2, 3$, $i < \omega$, for the rest of this work. The number $n_{1,i}$ is an upper bound for any kind of norm of an i-creature.

We compare with the book [15] in order to justify the use of the name "creature". We extend the framework developed there in order to allow for the approximation of uncountable domains $\mathbf{T}$.

Definition 2.3. (1) ([15, 1.1.1]) We let $\mathbf{H} = \langle \mathbf{H}(i) : i \in \omega \rangle$, and let $\mathbf{H}(i)$ be sets. A triple $\mathbf{c} = (\mathrm{nor}[\mathbf{c}], \mathrm{val}[\mathbf{c}], \mathrm{dis}[\mathbf{c}])$ is a *weak creature for* $\mathbf{H}$ if the following holds:
  (a) $\mathrm{nor}[\mathbf{c}] \in \mathbb{R}^{\geq 0}$.
  (b) Let $\lhd$ be the strict initial segment relation. $\mathrm{val}[\mathbf{c}]$ is a non-empty subset of
$$\Big\{\langle x, y\rangle \in \bigcup_{m_0 < m_1 < \omega} [\prod_{i < m_0} \mathbf{H}(i) \times \prod_{i < m_1} \mathbf{H}(i)] \,:\, x \lhd y\Big\}.$$
  (c) $\mathrm{dis}[\mathbf{c}] \in H(\chi)$.
(2) nor stands for norm, val stands for value, and dis stands for distinguish.

A creature is a weak creature with additional properties. In the creatures for the forcing $\mathbb{Q}_{\mathbf{T}}$ the component $\mathrm{dis}[\mathbf{c}]$ is a pair $(i(\mathbf{c}), k(\mathbf{c}))$ of natural numbers. More properties are specified in 2.4 to 2.10.

The set val is a non-empty subset of $\{\langle x, y\rangle \in \mathrm{spec}^{\mathbf{T}} \times \mathrm{spec}^{\mathbf{T}} : x <_T y\}$ for some strict partial order $<_T$ as in Definition 3.1 and $\mathbf{H}(i) = \mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$. The members of $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$ are finite partial functions, but the set $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$ is uncountable. Often properness of a tree creature forcing follows from the countability of the sets $\mathbf{H}(i)$, $i \in \omega$, and our analogue to $\mathbf{H}(i)$ is the uncountable set $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$. In Section 4 we shall prove that the notions of forcing we introduce are proper for other reasons.

Creatures with $|\mathrm{dom}(\mathrm{val}[\mathbf{c}]))| = 1$ are called tree creatures. As common in the work with tree creatures we write $\mathrm{pos}(\mathbf{c})$ for $\mathrm{rge}(\mathrm{val}[\mathbf{c}])$ and call $\mathrm{pos}(\mathbf{c})$ the set of possibilities for $\mathbf{c}$.

Definition 2.4. A *simple creature* is a tuple $\mathbf{c} = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}))$ with the following properties:
(a) The first component, $i(\mathbf{c})$, is called the kind of $\mathbf{c}$ and is just a natural number. A (simple) creature $\mathbf{c}$ is called a (simple) i-creature if $i(\mathbf{c}) = i$.

(b) The second component, $\eta(\mathbf{c})$, is called the base of $\mathbf{c}$. We require ($\eta(\mathbf{c}) = \emptyset$ and $i(\mathbf{c}) = 0$) or ($i(\mathbf{c}) = i > 0$ and $0 \neq |\operatorname{dom}(\eta(\mathbf{c}))| \leq n_{1,i-1}$, and $\eta(\mathbf{c}) \in \operatorname{spec}_{n_{2,i-1}}$).

(c) $\operatorname{pos}(\mathbf{c})$ is a non-empty subset of $\{\eta \in \operatorname{spec}_{n_{2,i}} : \eta(\mathbf{c}) \subsetneqq \eta \wedge |\operatorname{dom}(\eta)| \leq n_{1,i}\}$ and $|\operatorname{pos}(\mathbf{c})| \leq n_{3,i}$.

We reserve the name "creature" for a simple creature that is expanded by another coordinate, a natural number. In the wider realm of creatures, simple i-creatures and i-creatures, having a singleton base, can be counted as tree-creating creatures.

For a non-negative real number r we let $m = [r]$ be the largest natural number such that $m \leq r$. We let log denote the logarithm function to the base 2.

The following definition has ideas from [18, Ch. V, §6] and is the most important definition in this work.

Definition 2.5. (1) For a simple i-creature $\mathbf{c}$ we define $\operatorname{nor}^0(\mathbf{c})$ as the maximal natural number $m \leq n_{1,i}$ such that $m = 0$ or

(α) if $a \subseteq n_{2,i}$ and $|a| \leq m$ and $B_0, \dots, B_{m-1}$ are branches of $\mathbf{T}$, then there is $\nu \in \operatorname{pos}(\mathbf{c})$ such that

$$(\forall x \in (\bigcup_{\ell<m} B_\ell \cap \operatorname{dom}(\nu)) \setminus \operatorname{dom}(\eta(\mathbf{c})))(\nu(x) \notin a),$$

(β) $\max\{|\operatorname{dom}(\nu)| : \nu \in \operatorname{pos}(\mathbf{c})\} \leq \frac{n_{1,i}}{m}$,

(γ) $|\operatorname{pos}(\mathbf{c})| \leq \frac{n_{3,i}}{m}$.

(2) If $\operatorname{nor}^0(\mathbf{c}) > 1$ we define $\operatorname{nor}^1(\mathbf{c}) = \log(\operatorname{nor}^0(\mathbf{c}))$, otherwise $\operatorname{nor}^1(\mathbf{c}) = 0$.

Remark 2.6. Note that in (α), only finitely many m-tuples of branches of $\mathbf{T}$ need to be checked, indeed, only the part of $\mathbf{T}$ intersected with $\bigcup\{\operatorname{dom}(\eta) : \eta \in \operatorname{pos}(\mathbf{c})\}$ matters for computing the norm.

Sometimes it is useful not only to know that $\operatorname{nor}^0(\mathbf{c}) \geq m$ but also to pin down a norm exactly.

Lemma 2.7. *Suppose that* $\operatorname{nor}^0(\mathbf{c}) = m$ *and* $m' < m$. *Then there is a subset* $p \subseteq \operatorname{pos}(\mathbf{c})$ *such that the subcreature* $\mathbf{c}' = \mathbf{c} \restriction p = (i(\mathbf{c}), \eta(\mathbf{c}), p)$ *fulfils* $\operatorname{nor}^0(\mathbf{c}') = m'$.

Proof. For a simple i-creature $\mathbf{c}$ we define $\operatorname{nor}^{0,0}(\mathbf{c})$ as the maximal natural number $m \leq n_{1,i}$ such that $m = 0$ or if $a \subseteq n_{2,i}$ and $|a| \leq m$ and $B_0, \dots, B_{m-1}$ are branches of $\mathbf{T}$, then there is $\nu \in \operatorname{pos}(\mathbf{c})$ such that $(\forall x \in (\bigcup_{\ell<m} B_\ell \cap \operatorname{dom}(\nu)) \setminus \operatorname{dom}(\eta(\mathbf{c})))(\nu(x) \notin a)$. By the relationship between nor^0 and $\operatorname{nor}^{0,0}$ and since taking a subcreature does not decrease the nor^0 if in its computation clause β or clause γ is decisive, the lemma follows from the following statement: Suppose that $\operatorname{nor}^{0,0}(\mathbf{c}) = m$. Then there is a subset $p \subseteq \operatorname{pos}(\mathbf{c})$ such that the subcreature $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}), p)$ fulfils $\operatorname{nor}^{0,0}(\mathbf{c}') = m - 1$. For proving the latter statement, we first take $p' \subseteq \operatorname{pos}(\mathbf{c})$ such that it is minimal with $\operatorname{nor}^{0,0}(\mathbf{c} \restriction p') = m$. Then we remove one element, call it ν, from p' and call the outcome p. By minimality $\operatorname{nor}^{0,0}(\mathbf{c} \restriction p) \leq m-1$. We show $\operatorname{nor}^{0,0}(\mathbf{c} \restriction p) \geq m-1$. If $m \geq 1$, then $p \neq \emptyset$. We assume $m \geq 2$. Let $a \subseteq n_{2,i}$ and $|a| = m-1$ and $B_1, \dots, B_{m-1}$ be given. We

take $a \cup \{\nu(x)\}$ for an $x \in \mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta(\mathbf{c}))$, it does not matter which. We take B_m so that $\nu(x) \in B_m$. Then by $\mathrm{nor}(\mathbf{c}) = m$ in $\mathrm{pos}(\mathbf{c})$ there is $\nu' \in \mathrm{pos}(\mathbf{c})$ such that

$$(\forall y \in (B_1 \cup \cdots \cup B_m) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\nu'(y) \notin a \cup \{\nu\}).$$

Thus $\nu' \neq \nu$ and we have $\nu' \in p$. ⊣

Some of the requirements on the norms in the conditions of Lemmata 2.13 to 2.16 are easy to fulfil. Most of the time the requirement Def. 2.5(1)(α) is the hardest one.

Definition 2.8. We let $\ell_i = 2^{\prod_{j \leq i} n_{3,j}}$.

Definition 2.9. An *i-creature* is a tuple $\mathbf{c} = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}), k(\mathbf{c}))$ such that

(1) $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}))$,

(2) $k(\mathbf{c}) \in \omega$.

Definition 2.10. (1) For $\ell \in \omega \setminus \{0\}$, $f_\ell : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+_0$ is defined by cases as follows

$$f_\ell(n, k) := \begin{cases} 0, & \text{if } n = 0 \text{ or } \log(n) \leq k + 1; \\ \frac{\log(\log(n) - k)}{\ell}, & \text{else.} \end{cases}$$

(2) We let $f = \langle f_\ell : \ell < \omega \rangle$. For an i-creature $\mathbf{c} = (\mathbf{c}', k(\mathbf{c}))$ with $\mathrm{nor}^0(\mathbf{c}') > 0$ we define its f-norm

$$\mathrm{nor}_f(\mathbf{c}) = f_{\ell_i}(\mathrm{nor}^0(\mathbf{c}'), k(\mathbf{c})). \tag{2.1}$$

(3) We write $\mathrm{nor}^0(\mathbf{c})$ for $\mathrm{nor}^0(\mathbf{c}')$.

Remark 2.11. We took f similarly to the functions used in [17, Section 3]. We chose ℓ_i so large that it ensures a suitable strong form of halving, see 2.19 and 2.20.

The following estimate is a step towards bigness (Lemma 2.18) and the halving property for creatures (Def. 2.19, Lemma 2.20):

Lemma 2.12. *If* $f_1(n, k) \geq \log(2)$ *then* $f_\ell(\frac{n}{2}, k) \geq f_\ell(n, k) - \frac{1}{\ell}$.

Proof. As $\log(\frac{n}{2}) = \log(n) - 1$ we have $\ell \cdot f_\ell(\frac{n}{2}, k) = \log(\log(n) - 1 - k) \geq \log(\frac{\log(n) - k}{2}) = \log(\log(n) - k) - 1 = \ell \cdot f_\ell(n, k) - 1$. For the inequality between the second and the third term we use $f_1(n, k) = \log(\log(n) - k) \geq \log 2$. Hence $\log(n) - k \geq 2$ and thus $\log(n) - 1 - k \geq \frac{\log(n) - k}{2}$. ⊣

The next lemma shows that we can extend the possibilities of a creature and at the same time decrease the norm of the creature only by a small amount.

Lemma 2.13. *Assume that*

(a) $\eta^* \in \mathrm{spec}$,

(b) $\mathbf{c}$ *is an i-creature with base* η^*, $\mathrm{nor}^0(\mathbf{c}) > 0$,

(c) $k^* > 0$,

(d) for each $\eta \in \mathrm{pos}(\mathbf{c})$ *we have: either* $k_\eta = k^*$ *and for each* $k < k^*$ *we are given* $\eta \subseteq \rho_{\eta,k} \in \mathrm{spec}_{n_{2,i}}$ *with* $|\mathrm{dom}(\rho_{\eta,k})| < n_{1,i}$ *or* $k_\eta = 1$ *and* $\rho_{\eta,0} = \eta$*,*

(e) for each $\eta \in \mathrm{pos}(\mathbf{c})$*, if* $k_\eta = k^* > 1$*, and if* $k_1 < k_2 < k^*$ *and* $x_1 \in \mathrm{dom}(\rho_{\eta,k_1}) \setminus \mathrm{dom}(\eta)$ *and* $x_2 \in \mathrm{dom}(\rho_{\eta,k_2}) \setminus \mathrm{dom}(\eta)$*, then* x_1*,* x_2 *are* $<_{\mathbf{T}}$*-incomparable,*

(f) $\ell^* = \max\{|\mathrm{dom}(\rho_{\eta,k})| : \eta \in \mathrm{pos}(\mathbf{c})) \wedge k < k^*\}$.

Then

(α) *There is an i-creature* $\mathbf{d}$ *given by*

$$\begin{aligned} \mathrm{pos}(\mathbf{d}) &= \{\rho_{\eta,k} : k < k^*_\eta, \eta \in \mathrm{pos}(\mathbf{c})\}, \\ \eta(\mathbf{d}) &= \eta^*, \\ k(\mathbf{d}) &= k(\mathbf{c}). \end{aligned}$$

(β) *We have* $\mathrm{nor}^0(\mathbf{d}) \geq m_0 \overset{\mathrm{def}}{=} \min\left\{\mathrm{nor}^0(\mathbf{c}), [\frac{n_{1,i}}{\ell^*}], \left[\frac{n_{3,i}}{k^* \cdot |\mathrm{pos}(\mathbf{c})|}\right], k^* - 1\right\}$.

Proof. First we check Definition 2.4(1). Clauses (a), (b), and (c) follow immediately from the premises of the lemma.

Now for the norm: We check clause (α) of Definition 2.5(1). Let branches $B_0, \dots, B_{m_0-1}$ of $\mathbf{T}$ and a set $a \subseteq n_{2,i}$ be given, $|a| \leq m_0$. Since $m_0 \leq \mathrm{nor}^0(\mathbf{c})$, there is some $\eta \in \mathrm{pos}(\mathbf{c})$ such that $(\forall x \in \left(\bigcup_{\ell<m_0} B_\ell\right) \cap \mathrm{dom}(\eta) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\eta(x) \notin a)$. We fix such an η. If $k^*_\eta = 1$, we are done. Now for each $\ell < m_0$, we let

$$w_{\eta,\ell} = \{j < k^* : \exists x \in B_\ell \cap \mathrm{dom}(\rho_{\eta,j}) \setminus \mathrm{dom}(\eta)\}.$$

Now we have that $|w_{\eta,\ell}| \leq 1$ because otherwise we would have $k_1 < k_2 < k^*$ in $w_{\eta,\ell}$ and $x_i \in B_\ell \cap \mathrm{dom}(\rho_{\eta,k_i}) \setminus \mathrm{dom}(\eta)$, $i = 1, 2$. Such witnesses x_1 and x_2 would be $<_{\mathbf{T}}$-comparable, in contradiction to requirement (e) of this lemma.

Since $m_0 < k^*$, there is some $j \in k^* \setminus \bigcup_{\ell<m_0} w_{\eta,\ell}$. For such a j, $\rho_{\eta,j}$ is as required.

We check clause (β) of Definition 2.5(1). We take any $\rho_{\eta,k}$. Then we have

$$\frac{|\mathrm{dom}(\rho_{\eta,k})|}{n_{1,i}} \leq \frac{\ell^*}{n_{1,i}} \leq \frac{1}{[\frac{n_{1,i}}{\ell^*}]} \leq \frac{1}{m_0},$$

as $m_0 \leq [\frac{n_{1,i}}{\ell^*}]$. Clause ($\gamma$) of Definition 2.5(1) follows from $m_0 \leq \left[\frac{n_{3,i}}{k^* \cdot |\mathrm{pos}(\mathbf{c})|}\right]$. ⊣

Now we restate the previous lemma for applications to nor_f:

Lemma 2.14. *Assume that*

(a) $\eta^* \in \mathrm{spec}$*,*

(b) $\mathbf{c}$ *is an i-creature with base* η^**,* $\log(\log(n_{1,i})) \geq \mathrm{nor}_f(\mathbf{c}) > 2$*,*

(c) $k^* = [\sqrt{\mathrm{nor}^0(\mathbf{c})}]$ *(and it is really* nor^0 *here),*

(d) for each $\eta \in \mathrm{pos}(\mathbf{c})$ *we have: either* $k_\eta = k^*$ *and for each* $k < k^*$ *we are given* $\eta \subseteq \rho_{\eta,k} \in \mathrm{spec}_{n_{2,i}}$ *with* $|\mathrm{dom}(\rho_{\eta,k})| < \frac{n_{1,i}}{2^{(2^m + k(\mathbf{c}))}}$ *or* $k_\eta = 1$ *and* $\rho_{\eta,0} = \eta$*,*

(e) for each $\eta \in \mathrm{pos}(\mathbf{c})$*, if* $k_\eta = k^* > 1$*, and if* $k_1 < k_2 < k^*$ *and* $x_1 \in \mathrm{dom}(\rho_{\eta,k_1}) \setminus \mathrm{dom}(\eta)$ *and* $x_2 \in \mathrm{dom}(\rho_{\eta,k_2}) \setminus \mathrm{dom}(\eta)$*, then* x_1*,* x_2 *are* $<_{\mathbf{T}}$*-incomparable,*

Then

(α) *There is an* i*-creature* $\mathbf{d}$ *given by*

$$\begin{aligned} \mathrm{pos}(\mathbf{d}) &= \{\rho_{\eta,k} : k < k^*, \eta \in \mathrm{pos}(\mathbf{c})\}, \\ \eta(\mathbf{d}) &= \eta^*, \\ k(\mathbf{d}) &= k(\mathbf{c}). \end{aligned}$$

(β) *We have* $\mathrm{nor}_f(\mathbf{d}) \geq \min(m, \mathrm{nor}_f(\mathbf{c}) - 1)$.

Proof. By definition $[\frac{n_{3,i}}{|\mathrm{pos}(\mathbf{c})|}] \geq \mathrm{nor}^0(\mathbf{c})$. Hence $[\frac{n_{3,i}}{k^* \cdot |\mathrm{pos}(\mathbf{c})|}] \geq \sqrt{\mathrm{nor}^0(\mathbf{c})}$. By the previous lemma we have have $\mathrm{nor}^0(\mathbf{d}) \geq \min(\sqrt{\mathrm{nor}^0(\mathbf{c})} - 1, 2^{(2^m + k(\mathbf{c}))})$ and hence $\mathrm{nor}_f(\mathbf{d}) \geq \min(m, \mathrm{nor}_f(\mathbf{c}) - 1)$. ⊣

The previous lemma will be used only in Section 4 in the proof of properness in Lemma 4.8. Indeed, its premise (e) is like a step in the proof that the specialisation of Aronszajn trees by finite approximations has the c.c.c.. For a proof, see e.g. [7, Lemma 16.18] or [18, Ch. III, Theorem 5.4].

The following two lemmata will be used in the next section in the proof that the smooth conditions are dense. The latter property is used in the proof of properness as well.

Lemma 2.15. *Suppose that* $\mathbf{c}$*,* m*,* m' *are as follows:*

(a) $\mathbf{c}$ *is an* i*-creature,*

(b) $7 \leq \mathrm{nor}^0(\mathbf{c}) = m \leq \sqrt{n_{1,i}}$*,*

(c) $x \in \mathbf{T}$*,*

(d) $m' = [\sqrt{m}]$.

Then there is some i*-creature* $\mathbf{d}$ *such that*

(1) $\eta(\mathbf{d}) = \eta(\mathbf{c})$*,* $k(\mathbf{c}) = k(\mathbf{d})$*,*

(2) $\mathrm{pos}(\mathbf{d}) \subseteq \{\nu \in \mathrm{spec}^{\mathbf{T}} : (\exists \eta \in \mathrm{pos}(\mathbf{c}))(\eta \subseteq \nu \wedge \mathrm{dom}(\nu) = \mathrm{dom}(\eta) \cup \{x\})\}$*,*

(3) $\mathrm{nor}^0(\mathbf{d}) \geq \min(\frac{m}{m'+1}, m')$.

Proof. For each $\eta \in \mathrm{pos}(\mathbf{c})$ we choose $m' + 1$ elements from $n_{2,i} \setminus \mathrm{rge}(\eta)$, and put them into a set E_η. By (b) and (d) this set is not empty: $|\mathrm{rge}(\eta)| \leq |\mathrm{dom}(\eta)| \leq \frac{n_{1,i}}{m} \leq n_{2,i} - \sqrt{m'} - 2$ for $m \geq 7$. For each $a \in [n_{2,i}]^{m'}$ we let $\{z_{\eta,a}\} = E_\eta \setminus a$. Then we set $\nu_{\eta,a} = \eta \cup \{(x, z_{\eta,a})\}$. Since $z_{\eta,a} \notin \mathrm{rge}(\eta)$, $\nu_{\eta,a}$ is a partial specialisation. We set $\eta(\mathbf{d}) = \eta(\mathbf{c})$, $k(\mathbf{d}) = k(\mathbf{c})$ and

$$\mathrm{pos}(\mathbf{d}) = \{\nu_{\eta,a}) : \eta \in \mathrm{pos}(\mathbf{c}), a \in [n_{2,i}]^{m'}\}.$$

We show that $\mathbf{d}$ is as required. Now we check the norm: Let m'' be the smallest integer $\geq \min(\frac{m}{m'+1}, m')$. For clause ($\alpha$) of Definition 2.5(1), let $B_0, \dots, B_{m''-1}$ be branches of $\mathbf{T}$ and let $a \subseteq n_{2,i}$, $|a| \leq m''$. We have to find $\nu \in \mathrm{pos}(\mathbf{d})$ such that $(\forall \ell < m'')(\forall y \in \mathrm{dom}(\nu) \cap B_\ell \setminus \mathrm{dom}(\eta(\mathbf{c})))(\nu(y) \notin a)$. We add a branch

$B_{m''}$ with $x \in B_{m''}$. Since $m'' \le m' = [\sqrt{m}] \le m$, by premise (b), we find $\eta \in \text{pos}(\mathbf{c})$ such that

$$(\forall \ell < m'')(\forall x' \in \text{dom}(\eta) \cap B_\ell \setminus \text{dom}(\eta(\mathbf{c})))(\eta(x') \notin a).$$

We fix this η. By the choice of $z_{\eta,a}$, a and E_η, there is $\nu_{\eta,a} \in \text{pos}(\mathbf{d})$ such that

$$(\forall \ell < m'')(\forall x' \in \text{dom}(\nu_{\eta,a}) \cap B_\ell \setminus \text{dom}(\eta(\mathbf{c})))(\nu_{\eta,a}(x') \notin a).$$

Now for item (β) of Def. 2.5(1): Every element of pos($\mathbf{d}$) is just by one larger than an element of pos($\mathbf{c}$). So we have $\max\{\frac{|\text{dom}(\nu)|}{n_{1,i}} : \nu \in \text{pos}(\mathbf{d})\} \le \max\{\frac{|\text{dom}(\nu)|+1}{n_{1,i}} : \nu \in \text{pos}(\mathbf{c})\} \le \frac{1}{m} + \frac{1}{n_{1,i}} \le \frac{1}{m} + \frac{1}{m^2} \le \frac{1}{m'} \le \frac{1}{m''}$.

Now for item (γ) of Def. 2.5(1): The norm drops from m to $\ge \left[\frac{m}{m'+1}\right]$ by replacing each $\eta \in \text{pos}(\mathbf{c})$ by $\le m'+1$ elements. ⊣

Suppose that we have extended the partial specialisation functions in the set of possibilities of a creature as in one of the previous lemmas. Then we want that these extended functions can serve as bases for suitable creatures as well. This is provided by the next lemma. The number i from Lemma 2.15 will now appear in Lemma 2.16 as $i-1$, since in the latter lemma new creatures $\mathbf{d}$ are constructed from simple creatures $\mathbf{c}$ by extending the base of $\mathbf{c}$.

Lemma 2.16. *Assume that*

(a) $\mathbf{c}$ *is an* i*-creature.*

(b) $\eta^* \supseteq \eta(\mathbf{c})$, $\eta^* \in \text{spec}_{n_{2,i-1}}$ *(note that we do not suppose that* $\eta^* \in \text{pos}(\mathbf{c})$*). Furthermore we assume* $|\text{dom}(\eta^*)| \le n_{1,i-1}$.

(c) For any $\nu \in \text{pos}(\mathbf{c})$, $\text{dom}(\eta^*) \cap \text{dom}(\nu) = \text{dom}(\eta(\mathbf{c}))$.

We set

$$\ell_2^* = |\text{dom}(\eta^*) \setminus \text{dom}(\eta(\mathbf{c}))|,$$

and

$$\begin{aligned} Y =& \{y \,:\, \exists \nu(\nu \in \text{pos}(\mathbf{c}) \wedge y \in \text{dom}(\nu) \wedge \\ & (\exists x)(x \in \text{dom}(\eta^*) \setminus \text{dom}(\eta(\mathbf{c})) \wedge x \le_{\mathbf{T}} y))\}; \\ \ell_1^* =& |Y|, \end{aligned}$$

and in addition to (a), (b), and (c) we assume that $\ell_1^* + \ell_2^* < \text{nor}^0(\mathbf{c}) < \sqrt{n_{1,i}}$. *We define* $\mathbf{d}$ *by* $\eta(\mathbf{d}) = \eta^*$, $k(\mathbf{d}) = k(\mathbf{c})$ *and*

$$\text{pos}(\mathbf{d}) = \{\nu \cup \eta^* \,:\, \nu \in \text{pos}(\mathbf{c}) \ \wedge \ \nu \cup \eta^* \in \text{spec}_{n_{2,i}} \ \wedge |\text{dom}(\nu \cup \eta^*)| < n_{1,i}\}.$$

Then

(α) $\mathbf{d}$ *is an* i*-creature.*

(β) $\text{nor}^0(\mathbf{d}) \ge \min(\text{nor}^0(\mathbf{c}) - \ell_2^* - \ell_1^*, \frac{\text{nor}^0(\mathbf{c})}{2})$.

Proof. Item (α) here follows from the requirements on η^* and from the estimates on the norm. For item (β), we set $k = \text{nor}^0(\mathbf{c}) - \ell_1^* - \ell_2^*$. We first consider Definition 2.5(1) (α): We let $B_0, \dots, B_{k-1}$ be branches of $\mathbf{T}$ and $a \subseteq n_{2,i(\mathbf{c})}$, $|a| \le k$. We set $\ell^* = \ell_1^* + \ell_2^*$. We let $\langle y_\ell \,:\, \ell < \ell_1^*\rangle$ list Y without repetition. Let $B_k, \dots, B_{k+\ell_1^*-1}$ be branches of $\mathbf{T}$ such that $y_\ell \in B_{k+\ell}$ for $\ell < \ell_1^*$. Let

$\langle x_\ell : \ell < \ell_2^* \rangle$ list $\mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c}))$. Take for $\ell < \ell_2^*$, $B_{k+\ell_1^*+\ell}$ such that $x_\ell \in B_{k+\ell_1^*+\ell}$. We set $a' = a \cup \{\eta^*(x_\ell) : \ell < \ell_2^*\}$. Since $\mathrm{nor}^0(\mathbf{c}) \geq k + \ell^*$ there is some $\nu \in \mathrm{pos}(\mathbf{c})$ such that $\forall x \in ((\mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta(\mathbf{c}))) \cap \bigcup_{\ell < k+\ell^*} B_\ell)(\nu(x) \notin a')$. Then, if $x \in \mathrm{dom}(\nu \cup \eta^*) \setminus \mathrm{dom}(\eta^*)$, we have $(\nu \cup \eta^*)(x) \notin a$. We have to show that $\nu \cup \eta^*$ is a partial specialisation: Since η^* and ν are specialisation maps extending $\eta(\mathbf{c})$, we have to consider only the case $x \in \mathrm{dom}(\eta^*) \setminus \mathrm{dom}(\eta(\mathbf{c}))$ and $y \in \mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta^*)$ and $(y <_{\mathbf{T}} x \vee x \leq_{\mathbf{T}} y)))$. If $x \leq_{\mathbf{T}} y$ then $y \in Y$, then we have $\nu(y) \neq \eta^*(x_\ell)$ for all $\ell < \ell_2^*$ by the choice of $B_{k+\ell_1^*+\ell}$ for $\ell < \ell_2^*$. If $y <_{\mathbf{T}} x$, then y is in a branch leading to some $x = x_\ell$ for some $\ell < \ell_2^*$, and hence again $\nu(y) \neq \eta^*(x)$.

Moreover, for item (β) in Definition 2.5, $|\mathrm{dom}(\nu \cup \eta^*)| \leq \frac{n_{1,i}}{\mathrm{nor}^0(\mathbf{c})} + \ell_2^* \leq \frac{n_{1,i}}{\mathrm{nor}^0(\mathbf{c})} + \mathrm{nor}^0(\mathbf{c}) \leq 2\frac{n_{1,i}}{\mathrm{nor}^0(\mathbf{c})}$.

For item (γ) we do not have anything to check, since $|\mathrm{pos}(\mathbf{d})| \leq |\mathrm{pos}(\mathbf{c})|$. $\dashv$

Remark 2.17. (1) Apparently the premises of the previous lemma are hard to fulfil. In the proofs of the density properties we add ℓ_2^* points to the domain of the functions in the set of possibilities of a creature with sufficiently high norm. Moreover, $\ell_1^* \leq |u|$, were u is the set that sticks out of $\mathbf{T}_{<\alpha(p)}$ (see Definition 3.3(A) clause (c)). We will suppose that $|u|$ and ℓ_2^* are small in comparison to $\mathrm{nor}^0(\mathbf{c})$, so that the premises for Lemma 2.16 are fulfilled.

(2) Only $\ell_2^* = 1$ is used (namely, in the proof of Lemma 3.11) since we can fill in the elements of the Aronszajn tree in the domains of partial specialisations in conditions one by one.

The next lemma will help to find large homogeneous subtrees of the trees built from creatures that will later be used as forcing conditions.

Lemma 2.18. *The 2-bigness property* [15, Definition 2.3.2]. *If* $\mathbf{c}$ *is a* i*-creature with* $\mathrm{nor}^1(\mathbf{c}) \geq m+1$, *and* $\mathbf{c}_1$, $\mathbf{c}_2$ *are* i*-creatures such that* $\mathrm{pos}(\mathbf{c}) = \mathrm{pos}(\mathbf{c}_1) \cup \mathrm{pos}(\mathbf{c}_2)$ *and* $\eta(\mathbf{c}) = \eta(\mathbf{c}_1) = \eta(\mathbf{c}_2)$ *and* $k(\mathbf{c}) = k(\mathbf{c}_1) = k(\mathbf{c}_2)$, *then* $\mathrm{nor}^1(\mathbf{c}_1) \geq m$ *or* $\mathrm{nor}^1(\mathbf{c}_2) \geq m$. *Under the same premises we have: If* $m \geq 1$ *and* $\mathrm{nor}_f(\mathbf{c}) \geq m+1$ *then* $\mathrm{nor}_f(\mathbf{c}_1) \geq m$ *or* $\mathrm{nor}_f(\mathbf{c}_2) \geq m$.

Proof. We let $j = 2^m$, We suppose that $\mathrm{nor}^0(\mathbf{c}_1) < j$ and $\mathrm{nor}^0(\mathbf{c}_2) < j$ and derive a contradiction: For $\ell = 1, 2$ let branches $B_0^\ell, \dots, B_{j-1}^\ell$ and sets $a^\ell \subseteq n_{2,i}$, $|a^\ell| \leq j$, exemplify this.

Let $a = a^1 \cup a^2$ and let, by $\mathrm{nor}^0(\mathbf{c}) \geq 2j$, $\eta \in \mathrm{pos}(\mathbf{c})$ be such that for all $x \in (\mathrm{dom}(\eta) \cap \bigcup_{\ell=1,2} \bigcup_{i=0}^{j-1} B_i^\ell) \setminus \mathrm{dom}(\eta(\mathbf{c}))$ we have $\eta(x) \notin a$. But then for that $\ell \in \{1, 2\}$ for which $\eta \in \mathrm{pos}(\mathbf{c}_\ell)$ we get a contradiction to $\mathrm{nor}^0(\mathbf{c}_i) < j$. Hence for $i = 0$ or for $i = 1$, $\mathrm{nor}^1(\mathbf{c}_i) \geq m$.

The inequality also holds for nor_f by the definition of f_ℓ: $f_\ell(\frac{n}{2}, k) \geq f_\ell(n, k) - 1$ for $f_\ell(n, k) \geq \log(2)$. $\dashv$

Now for the first time we make use of the coordinate $k(\mathbf{c})$ of our creatures. The next lemma states that the creatures have the halving property. Originally the halving property was introduced in [15, 2.2.7]. Our version is similar to the strong form of halving in [17, Def 3.1].

Definition 2.19. (A) Let $\mathbf{c}$ be an i-creature as in Definition 2.9. Let $\ell \in \omega\setminus\{0\}$. We say $\mathbf{c}^*$ is an $\frac{1}{\ell}$-*half of* $\mathbf{c}$ if the following hold:

(1) $i(\mathbf{c}^*) = i(\mathbf{c})$, $\eta(\mathbf{c}^*) = \eta(\mathbf{c})$, $\mathrm{nor}_f(\mathbf{c}^*) \geq \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}$, $\mathrm{pos}(\mathbf{c}^*) = \mathrm{pos}(\mathbf{c})$, $k(\mathbf{c}^*) \geq k(\mathbf{c})$.

(2) *If* $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}'))$ satisfies
  (i) there is a a map $\pi\colon \{\eta(\mathbf{c}')\} \cup \mathrm{pos}(\mathbf{c}') \to \{\eta(\mathbf{c}^*)\} \cup \mathrm{pos}(\mathbf{c}^*)$ such that for each $\nu \in \mathrm{pos}(\mathbf{c}')$, $\nu \supseteq \pi(\nu)$, [1] or $\mathbf{c}'$ is just any $i(\mathbf{c})$-creature
  (ii) $k(\mathbf{c}') \geq k(\mathbf{c}^*)$ and
  (iii) $\mathrm{nor}_f(\mathbf{c}') > 0$

  then $\mathbf{c}_0 = (i(\mathbf{c}), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}_0))$ is an $i(\mathbf{c})$-creature with $\mathrm{nor}_f(\mathbf{c}_0) \geq \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}$.

(B) Let $\ell \in \omega \setminus \{0\}$. We say K has the $\frac{1}{\ell}$-halving property if for each creature $\mathbf{c} \in K$ there is an $\frac{1}{\ell}$-half of $\mathbf{c}$.

(C) Let $\mathbf{c}$ be an i-creature as in Definition 2.9. Let $\mathrm{nor}_f(\mathbf{c}) > 1$ and let $\ell = \ell_i$. We say $\mathbf{c}^*$ is the *standard* $\frac{1}{\ell}$-*half of* $\mathbf{c}$ if the following hold:

$$\mathbf{c}^* = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}), \left\lceil \frac{\log(\mathrm{nor}^0(\mathbf{c})) + k(\mathbf{c})}{2} \right\rceil).$$

(D) Let $\mathbf{c}'$ be an i-creature as in Definition 2.9. Let $\mathbf{c}^*$ be its standard $\frac{1}{\ell_i}$-half. Let $\mathrm{nor}_f(\mathbf{c}') > 1$ and let $\ell = \ell_i$ and $k(\mathbf{c}') \geq \left\lceil \frac{\log(\mathrm{nor}^0(\mathbf{c}))+k(\mathbf{c})}{2} \right\rceil = k(\mathbf{c}^*)$. We say $\mathbf{c}_0$ is the *standard de-halving of* $\mathbf{c}'$ *with respect to* $(\mathbf{c}, \mathbf{c}^*)$ if $\mathbf{c}_0 = (i, \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}))$. We write

$$\mathbf{c}_0 = \text{de-halve}(\mathbf{c}', \mathbf{c}, \mathbf{c}^*).$$

Lemma 2.20. *If* $\mathbf{c} \in K$ *is an* i*-creature with* $\mathrm{nor}_f(\mathbf{c}) > 0$, *it has the* $\frac{1}{\ell_i}$*-halving property.*

Proof. Let $\mathbf{c}$ be an i-creature, $\ell = \ell_i$. We let $\mathbf{c}^* = (i(\mathbf{c}), s(\mathbf{c}), \mathrm{pos}(\mathbf{c}), k(\mathbf{c}^*))$ with

$$k(\mathbf{c}^*) = \left\lceil \frac{\log(\mathrm{nor}^0(\mathbf{c})) + k(\mathbf{c})}{2} \right\rceil.$$

Since $\mathrm{nor}_f(\mathbf{c}) > 0$, we have $k(\mathbf{c}^*) \geq k(\mathbf{c})$. Then

$$\begin{aligned}
\mathrm{nor}_f(\mathbf{c}^*) &= \frac{\log\left(\log(\mathrm{nor}^0(\mathbf{c}^*)) - k(\mathbf{c}^*)\right)}{\ell} \\
&\geq \frac{\log(\frac{\log(\mathrm{nor}^0(\mathbf{c}))-k(\mathbf{c})}{2})}{\ell} \\
&= \frac{\log(\log(\mathrm{nor}^0(\mathbf{c})) - k(\mathbf{c})) - 1}{\ell} \\
&= \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}.
\end{aligned}$$

[1] This stronger form of premise (i) and hence weaker form of clause (2) is used in the de-halving lemma 3.16

Now let $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}'))$ be any creature with $k(\mathbf{c}') \geq k(\mathbf{c}^*)$, and $\mathrm{nor}_f(\mathbf{c}') > 0$. Then we take $\mathbf{c}_0 = (i(\mathbf{c}), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}))$ and get

$$\begin{aligned}
\mathrm{nor}_f(\mathbf{c}_0) &= \frac{\log\big(\log(\mathrm{nor}^0(\mathbf{c}_0)) - k(\mathbf{c})\big)}{\ell} \\
&\geq \frac{\log(\log(\mathrm{nor}^0(\mathbf{c}')) - k(\mathbf{c}') + \frac{\log(\mathrm{nor}^0(\mathbf{c}))-k(\mathbf{c})}{2} - 1)}{\ell} \\
&\geq \frac{\log(\log(\mathrm{nor}^0(\mathbf{c}))) - k(\mathbf{c})) - 1}{\ell} \\
&= \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}.
\end{aligned}$$

From line 1 to 2 we used $-k(\mathbf{c}) \geq -k(\mathbf{c}') - \frac{\log(\mathrm{nor}^0(\mathbf{c}))-k(\mathbf{c})}{2} - 1$, since $k(\mathbf{c}') > \frac{\log(\mathrm{nor}^0(\mathbf{c}))+k(\mathbf{c})}{2} - 1$, as $k(\mathbf{c}') \geq k(\mathbf{c}^*)$. From line 2 to 3 we use $\mathrm{nor}_f(\mathbf{c}') > 0$ implies $\log(\mathrm{nor}^2(\mathbf{c}')) - k(\mathbf{c}') \geq 1$. $\dashv$

Why is this strong form of halving useful? Later we define the half of a conditions in Def. 3.15 and the de-halve of conditions in Lemma 3.16. Roughly spoken, decisions that are taken by some $r \geq_0 \mathrm{half}(p)$ with $\mathrm{nor}_f(\mathbf{c}_{r,t}) > 0$ for any $t \in T^r$ are also taken by the "de-half" of r with respect to p (see Lemma 3.16). Then de-halve$(r, p, 0) \geq_0 p$ and $\mathrm{nor}_f(\mathbf{c}_{\text{de-halve}(r,p,0),t})$ is sufficiently large for any t. This idea is carried further at the end of the next section.

3. Tree forcings with creatures

Now we construct a notion of forcing from the creatures introduced in the previous section.

Definition 3.1. Let $t = \langle \eta_0, \dots, \eta_{n-1} \rangle$ denote a strictly increasing sequence of finite length of finite partial specialisations. We let, for $n \geq 1$, $\mathrm{first}(t) = \eta_0$, $\mathrm{last}(t) = \eta_{n-1}$ denote the first and the last entry of t, and $\lg(t) = n$ denote the length of t.

Now we consider endless trees $(T, <_T)$ of elements (nodes) of the form $t = \langle \eta_0, \dots, \eta_n \rangle$, ordered by end extension. A partial specialisation η can appear in two different nodes. Below we define a notion of forcing with labelled trees $\langle \mathbf{c}_t : t \in (T, <_T) \rangle$ as components of conditions. The notation with the angled brackets $\langle \mathbf{c}_t : t \in (T, <_T) \rangle$ denotes a structure $(T, <_T)$ together with function $\mathbf{c} \colon T \to \mathbf{V}$ with $\mathbf{c}(t) = \mathbf{c}_t$. A condition has the form $p = (i(p), (T^p, <_{T^p}), \langle \mathbf{c}_{p,t} : t \in (T^p, <_{T^p}) \rangle)$. To every node t of such the finitely branching endless tree $(T^p, <_p) = (T^p, <_{T^p})$ we attach a creature $\mathbf{c}_{p,t}$ from Definition 2.4. This gives $\mathbb{Q}_{\mathbf{T}}$. We consider only assignments $t \mapsto \mathbf{c}_t$ that fulfil $\eta(\mathbf{c}_t) = \mathrm{last}(t)$.

We recall some notions about trees:

Definition 3.2.

(1) A *tree* $(T, <_T)$ is a non-empty set T with a partial order $<_T$ such that for $t \in T$, $\{s \in T : s <_T t\}$ is a finite linear order.

(2) We define the *set of immediate successors of s in T* by

$$\mathrm{suc}_T(s) = \{t \in T : s <_T t \wedge \neg(\exists r \in T)(s <_T r <_T t)\}.$$

(3) The *restriction of T to nodes that are comparable with s* is
$$T^{\langle s\rangle} = \{t \in T \,:\, s \leq_T t \vee s \leq_T t\}.$$
(4) A tree is called *endless* if $\max(T) = \{s \in T \,:\, \neg(\exists t \in T)(s <_T t)\} = \emptyset$.
(5) Now let $(\mathbf{T}, <_{\mathbf{T}})$ be an Aronszajn tree as in Def. 1.1 and let $i \in \omega \setminus \{0\}$. A tree $(T, <_T)$ is an *$(i, \mathbf{T})$-tree* if
   (a) $(T, <_T)$ is endless.
   (b) $T \subseteq \{\langle \eta_0, \eta_1, \dots, \eta_n\rangle \in {}^{\omega>}(\mathrm{spec}^{\mathbf{T}}) \,:\, (\forall i < j \leq n)(\eta_i \subsetneq \eta_j)\}$. Elements $t = \langle \eta_0, \dots, \eta_n\rangle$ of T are also called nodes of T.
   (c) The tree order $\leq_T$ is just the initial segment relation $\trianglelefteq$: $s \trianglelefteq t$ iff $t \restriction \lg(s) = s$. We write $\triangleleft$ for the corresponding strict relation.
   (d) In T there is a least element, called the *root*, $\mathrm{rt}(T)$, which has the form $\langle \eta_0\rangle$ and $\eta_0 \in \mathrm{spec}^{\mathbf{T}}_{n_{2,i-1}}$. We also write η_0 instead of $\langle \eta_0\rangle$. The root counts as sequence of length 1, and is the unique element of $T^{[1]}$, the level number 1 of T.
   (e) If $\mathrm{rt}(T) \triangleleft \bar\nu \triangleleft \bar\eta$ and $\bar\eta \in T$, then $\bar\nu \in T$.
   (f) $(T, <_T)$ is finitely branching tree of height ω.
(6) The *set of branches through T* is
$$\lim(T) = \{\langle \eta_k \,:\, k < \omega\rangle \,:\, (\forall n)\langle \eta_0, \dots, \eta_n\rangle \in T\}$$
(7) A subset F of T is called a *front of T* if every branch of T passes through this set, and the set consists of $<_T$-incomparable elements.

Definition 3.3. Let $\mathbf{T}$ be an Aronszajn tree. We define a notion of forcing $\mathbb{Q} = \mathbb{Q}_{\mathbf{T}}$ with set of elements $\mathbb{Q}$ and a preorder $\leq_{\mathbb{Q}}$.

(A) $p \in \mathbb{Q}$ if $p = (i(p), (T^p, <_p), \langle \mathbf{c}_{p,t} \,:\, t \in (T^p, <_{T^p})\rangle)$ has the following properties:
   (a) $i(p) \in \omega \setminus \{0\}$, $T^p \subseteq {}^{\omega>}\mathrm{spec}^{\mathbf{T}}$. We write $\mathrm{dom}(p) = T^p$. We require that $(T^p, <_p)$ is a $(i(p), \mathbf{T})$-tree. The elements of T are of the form $\langle \eta_0, \dots, \eta_n\rangle$ such that $\eta_j \subsetneq \eta_{i+1}$ and $\eta_j \in \mathrm{spec}^{\mathbf{T}}_{n_{2,i(p)+j-1}}$ for $j \geq 0$. The tree ordering $\leq_T$ is end extension.
   (b) For $n \in \omega$, the *n-th level of T* is
$$T^{[n]} = \{t \in T \,:\, \lg(t) = n\}.$$
   We also write $p^{[m]}$ instead of $(T^p)^{[m]}$, $\mathrm{suc}_p(s)$ for $\mathrm{suc}_{T^p}(s)$ and call a front of T^p also a front of p. We let $\mathrm{rt}(p) = \mathrm{rt}(T^p)$. So $T^{[1]} = \{\mathrm{rt}(p)\}$. For any $1 \leq \ell < \omega$ and $s \in (T^p)^{[\ell]}$ there is an $i(p) + \ell - 1$-creature

$$(*)\qquad \begin{aligned} &\mathbf{c}_{p,s} \text{ such that}\\ &\eta(\mathbf{c}_{p,s}) = \mathrm{last}(s),\\ &\mathrm{pos}(\mathbf{c}_{p,s}) = \{\mathrm{last}(t) \,:\, t \in (T^p)^{[\ell+1]} \,:\, t \in \mathrm{suc}_p(s)\}.\end{aligned}$$

   (c) There is $\alpha = \alpha(p) \in \omega_1$ such that the following holds:[2] For some $0 \leq h < \omega$ for every $t \in (T^p)^{[h]}$ there is a finite set $u_t \subseteq \mathbf{T} \setminus \mathbf{T}_{<\alpha}$ such

[2]This condition is used in Lemma 3.11. It is crucial for the fact that smooth conditions are dense. Only for smooth conditions we have fusion. So the properness proof in Lemma 4.8

that for every ω-branch $\langle \eta_\ell : \ell < \omega\rangle$ of T^p satisfying $\eta_h = \mathrm{last}(t)$ we have $\bigcup_{\ell\in\omega} \mathrm{dom}(\eta_\ell) = \mathbf{T}_{<\alpha} \cup u_t$. We let $h(p)$ be the least such h.

(d) For every ω-branch $\langle \eta_\ell : \ell \in \omega\rangle$ of T^p with $t_\ell = \langle \eta_0, \dots, \eta_\ell\rangle$ we have $\lim_{\ell\to\omega} \mathrm{nor}^0(\mathbf{c}_{p,t_\ell}) = \omega$.

(B) The order $\le = \le_{\mathbb{Q}}$ is given by letting $p \le q$ (q is stronger than p, we follow the Jerusalem convention) if there is a projection $\mathrm{pr}_{q,p}$ which satisfies

(a) $\mathrm{pr}_{q,p}$ is a function from T^q to T^p such that every $t \in T^q$, $\mathrm{lg}(t) + i(q) = \mathrm{lg}(\mathrm{pr}_{q,p}(t)) + i(p)$. (And hence $i(\mathbf{c}_{q,t}) = i(\mathbf{c}_{p,\mathrm{pr}_{q,p}(t)})$.)

(b) If $t \in T^q$ then $\mathrm{last}(t) \supseteq \mathrm{last}(\mathrm{pr}_{q,p}(t))$. This holds of course not only for the last element of the sequence t but for all elements, since T^q is downward closed.

(c) If t_1, t_2 are both in $\mathrm{dom}(q)$ then $t_1 \le_q t_2$, if $\mathrm{pr}_{q,p}(t_1) \le_p \mathrm{pr}_{q,p}(t_2)$.

(d) For any $\ell \in \omega$: If $s_1 \in (T^q)^{[\ell]}$ and $s_2 \in (T^q)^{[\ell+1]}$ and $s_1 <_q s_2$, $\mathrm{pr}_{q,p}(s_2) = t_2$, $\mathrm{pr}_{q,p}(s_1) = t_1$, then $\mathrm{dom}(\mathrm{last}(t_2)) \cap \mathrm{dom}(\mathrm{last}(s_1)) = \mathrm{dom}(\mathrm{last}(t_1))$.

(e) $k(\mathbf{c}_{q,t}) \ge k(\mathbf{c}_{p,\mathrm{pr}_{q,p}(t)})$.

The projection in general is neither injective nor surjective.

In all our fusion constructions to come we will have $i(p) = i(q)$, so the counting with the lengths of nodes is not too difficult.

We give some informal description of the $\le$-relation in $\mathbb{Q}$: The stronger condition's domain is via $\mathrm{pr}_{q,p}$ mapped homomorphically w.r.t. the tree orders into $T^{p^{\langle \mathrm{pr}_{q,p}(\mathrm{rt}(q))\rangle}}$. The projection is in general neither one-to-one nor onto. The root can grow as well. According to Def. 3.3(B)(a), the projection preserves the i of the respective creatures, i.e. level of the node plus the i-number of the tree. The partial specialisation functions sitting on the nodes of the tree are extended (possibly by more than one extension per function) in q as to compared with the ones attached to the image under $\mathrm{pr}_{q,p}$ according to Def. 3.3(B)(b), but by Def. 3.3(B)(a) the extensions are so small and so few that they preserve the kind i of the creature given by the node and its successors, and according to Def. 3.3(B)(d) the new part of the domain of the extension is disjoint from the domains of the old partial specialisation functions living higher up in the projection of the new tree to the old tree.

Lemma 3.4. $\mathbb{Q}_{\mathbf{T}} \neq \emptyset$.

Proof. We assume w.l.o.g. that the Aronszajn tree $\mathbf{T}$ has ω be $\mathbf{T}_0$. We build T^p, $\mathbf{c}_{p,t}$ by induction on the height of T^p. For each $i \ge 1$, each node $t = \langle \eta_{t,0}, \dots, \eta_{t,i-1}\rangle$ at level i has $\mathrm{nor}^0(\mathbf{c}_{p,t}) \ge i$ for $i \ge 1$ and $\mathrm{dom}(\eta_{t,i-1}) = i$ (independently of t, so that Def. 3.3(A)(d) will be fulfilled). Recall our choice, $n_{2,i} \ge n_{1,i} \ge 2^{i^2}$ and $n_{1,i+1} \ge 2^{(n_{2,i})^2}$.

We start with $T^{[1]} = \{\emptyset\}$ and let $\eta(\mathbf{c}_{p,\emptyset}) = \emptyset$. Given $T^{[i]}$ and $t \in T^{[i]}$, we take $\mathrm{pos}(\mathbf{c}_{p,t}) = \{\eta(\mathbf{c}_{p,t}) \cup \{(\max(\mathrm{dom}(\eta(\mathbf{c}_{p,t}))) + 1, k)\} : k \le i\}$.

---

hinges on this clause. The existence of the finite u_t, that has the same size for all branches in a cone of T^p, is used to show that the number ℓ_1^* in Lemma 2.13 is small relative to the norm of sufficiently many creatures in a forcing condition.

To compute the norm is easy since all branches in the Aronszajn tree intersected with $\mathrm{dom}(\eta)$ for $\eta \in \mathrm{pos}(\mathbf{c})$ have length 1. This defines $T^{[i+1]}$.

A more detailed proof with a less flat part of the Aronszajn tree taken as the union of the domains of the components of the nodes of T^p is given in Lemma 3.11. $\dashv$

Definition 3.5. Let $p \in \mathbb{Q}$.

(1) For $t = \langle \eta_0, \dots, \eta_n \rangle \in T^p$ we let $q = p^{\langle t \rangle}$ be given by
  (a) $T^q := \{s : \langle \eta_0, \dots, \eta_{n-1} \rangle\hat{\ } s \in T^p, \mathrm{first}(t) = \eta_n\}$. In particular, $\mathrm{rt}(q) = \eta_n$.
  (b) $\mathrm{pr}_{q,p}(s) = \langle \eta_0 \dots, \eta_{n-1} \rangle\hat{\ } s$ for $s \in T^q$.
  (c) $\mathbf{c}_{q,s} = \mathbf{c}_{p,\mathrm{pr}_{q,p}(s)}$ for $s \in T^q$.

(2) For $n \in \omega \setminus \{0\}$, we let $p \uparrow n = \langle \langle t, \mathbf{c}_{p,t} \rangle : t \in ((T^p)^{[<n]}, <_p \restriction ((T^p)^{[<n]})^2) \rangle$.

Note that

$$(*) \qquad \begin{aligned} &p \uparrow n \text{ determines } T^p \restriction \{t : \mathrm{ht}_p(t) \le n\}, \\ &\text{since } \mathbf{c}_{p,t} \text{ determines } \mathrm{pos}(\mathbf{c}_{p,t}). \end{aligned}$$

Definition 3.6. (1) $p \in \mathbb{Q}$ is called *normal* if for every ω-branch $\langle \eta_\ell : \ell \in \omega \rangle$ of T^p with $t_\ell = \langle \eta_0, \dots, \eta_\ell \rangle$ the sequence $\langle \mathrm{nor}(\mathbf{c}_{p,t_\ell}) : \ell \in \omega \rangle$ is non-decreasing.

(2) $p \in \mathbb{Q}$ is called *smooth* if in clause (A)(c) of Definition 3.3 the number h is 0 and u_t is empty.

(3) $p \in \mathbb{Q}$ is called *weakly smooth* if in clause (A)(c) of Definition 3.3 the number h is 0.

Fact 3.7. *(1) Def. 3.3(B)(d) does not only hold for ℓ and $\ell + 1$ but for any finite difference of levels.*

(2) If $p \le q$ and p is weakly smooth with witness u then
$(t \in T^q \wedge \lg(t) \ge 1) \to \mathrm{dom}(\mathrm{last}(t)) \cap (\mathbf{T}_{<\alpha(p)} \cup u) = \mathrm{dom}(\mathrm{last}(\mathrm{pr}_{q,p}(t)))$.

Proof. (2): If p is weakly smooth, then all branches of T^p have the same union of domains of their entries, and hence the condition $\mathrm{dom}(\mathrm{last}(s_1)) \supseteq \mathrm{dom}(\mathrm{last}(t_1))$ is fulfilled t_1 and t_2 from Def. 3.3(B)(d) are in the range of $\mathrm{pr}_{q,p}$ or not. $\dashv$

Definition 3.8. (1) For $0 \le n < \omega$ we define the partial order $\le_{f,n} = \le_n$ on $\mathbb{Q}$ by letting $p \le_n q$ if
  (i) $p \le q$.
  (ii) $\mathrm{rt}(p) = \mathrm{rt}(q)$ and $i(p) = i(q)$.
  (iii) $p \uparrow n = q \uparrow n$ for $n \ge 1$.
  (iv) For any projection $\mathrm{pr}_{q,p}$ witnessing $p \le q$:

$$\begin{aligned} &(\forall t \in T^q)\big(\mathbf{c}_{q,t} \neq \mathbf{c}_{p,\mathrm{pr}_{q,p}(t)} \\ &\to (\mathrm{nor}_f(\mathbf{c}_{p,\mathrm{pr}_{q,p}(t)}) \ge n \wedge \mathrm{nor}_f(\mathbf{c}_{q,t}) \ge n)\big). \end{aligned} \tag{3.1}$$

Recall the definition of nor_f: For $t \in p^{[j]}$ we take $f_{\ell_{i(p)+j}}$ as in Def. 2.10(2) and in Choice 2.8.

(2) We define $\leq^0_n$ analogously, with nor^0 in item (iv).

Note that $\mathbf{c}_{p,\mathrm{rt}(p)} = \mathbf{c}_{q,\mathrm{rt}(p)}$ is a requirement on two levels in T^p and in T^q. So property (iii) says that also on the level n the two trees still coincide. We state and prove some basic properties of the notions defined above.

Lemma 3.9. *(1)* $(\mathbb{Q}, \leq_{\mathbb{Q}})$ *is a partial order.*

(2) For every p we have that $\lim_{n\to\omega} \min\{\mathrm{nor}(\mathbf{c}_{p,t}) : t \in (T^p)^{[n]}\} = \infty$.

(3) If $p \in \mathbb{Q}$, $\ell \in \omega$, and $t \in p^{[\ell]}$ then $|\mathrm{dom}(\mathrm{last}(t))| < n_{2,i(p)+\ell-2}$.

If $q \geq p$ and both are smooth then $\alpha(q) \geq \alpha(p)$.

Proof. (1) Given $p \leq q$ and $q \leq r$ we define $\mathrm{pr}_{r,p} = \mathrm{pr}_{q,p} \circ \mathrm{pr}_{r,p}$. It is easily seen that this function is as required.

(2) This follows from König's lemma: Since T^p is finitely branching, there is a branch though every infinite subset.

(3) Follows from Definitions 2.4 and 3.3. ⊣

Lemma 3.10. *Let $\langle n_i : i \in \omega\rangle$ be a strictly increasing sequence of natural numbers. We assume that for every i, $q_i \leq_{n_i} q_{i+1}$ and that each p_i is smooth. Then $q = \bigcup_{i<\omega} q_i \restriction (T^{q_i})^{[n_{i-1},n_i)} \in \mathbb{Q}$ and for all i, $q \geq_{n_i} q_i$ and $\alpha(q) = \sup\{\alpha(p_i) : i < \omega\}$.*

Proof. Clear. We remark that smoothness is necessary. ⊣

Now we need to know that the set of smooth conditions is dense in $\mathbb{Q}_{\mathbf{T}}$. We prove this in the next lemma by a more general fusion construction that works for arbitrary p_0. The finite sets sticking out of $\alpha(p_n)$ in the sense of Def. 3.3(A)(c) are gradually filled up, each finite part to the same $\alpha(q)$. Actually, already after finitely many filling up steps the unions of the domains above t for $t \in p^{[h]}$ are the same.

Lemma 3.11. *If $p \in \mathbb{Q}$, $\alpha \in \omega_1$, $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in T^q\} \subseteq \mathbf{T}_{<\alpha}$ and $n < \omega$ then there is q such that*

(1) $p \leq_n q$,

(2) q is smooth and $\alpha(q) = \alpha$,

(3) for each branch b of T^q, $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in b\} = \mathbf{T}_{<\alpha}$.

Proof. We write the proof for nor^0. Wlog, we assume that $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in T^p\} \neq \mathbf{T}_{<\alpha}$.

Since $\bigcup\{\mathrm{dom}(\eta) : \exists t \in T^p, \mathrm{last}(t) = \eta\} \subsetneq \mathbf{T}_{<\alpha}$ we have $\alpha(p) < \alpha$. By Def. 3.3(A)(c) there is some $h < \omega$ for every $t \in p^{[h]}$ there is $u_t \in [\mathbf{T} \setminus \mathbf{T}_{<\alpha(p)}]^{<\omega}$ such that for every ω-branch $\langle \eta_\ell : \ell < \omega\rangle$ of T^p with $\langle \eta_0, \dots, \eta_h\rangle = t$ we have $\bigcup_{\ell\in\omega} \mathrm{dom}(\eta_\ell) = \mathbf{T}_{<\alpha(p)} \cup u_t$.

We fix such a h and such u_t, $t \in p^{[h]}$. Now for each $t \in p^{[h]}$ separately we perform the following inductive filling up: Fix $t \in p^{[h]}$. Let $\{x^t_\ell : \ell < \omega\}$

enumerate $\mathbf{T}_{<\alpha} \setminus (u_t \cup T_{<\alpha(p)})$. We assume $n \geq (|u_t|+1)^2$ for every $t \in T^{[h]}$. We let $p_{t,0} = p^{\langle t \rangle}$, $n_0 = n$.

By induction on $\ell \in \omega$ we choose $p_{t,\ell}$ and n_ℓ with the following properties:

(a) $p_{t,\ell+1}^{[\leq n_{\ell+1}]} = p_{t,\ell}^{[\leq n_{\ell+1}]}$,

(b) $p_{t,\ell+1} \geq_{n+\ell} p_{t,\ell}$

(c) $\alpha(p_{t,\ell}) = \alpha(p)$,

(d) for every branch b of $p_{t,\ell}$, $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in b \cap p_{t,\ell}^{[n_\ell]}\} \supseteq \{x^t_{\ell'} : \ell' \leq \ell\}$ and $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in b\} = u_t \cup T_{<\alpha(p)} \cup \{x^t_{\ell'} : \ell' \leq \ell\}$.

Step from ℓ to $\ell+1$: We find $n_{\ell+1} < \omega$ such that

$(*)_1$ $n + \ell + 1 \leq n_{\ell+1}$,

$(*)_2$ for every $s \in (T^{p_{t,\ell}})^{[\geq n_{\ell+1}]}$, we have $\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},s}) \geq (n + |u_t| + \ell + 1)^2$,

$(*)_3$ $n_{\ell+1} \geq n_\ell$.

For each $s \in (T^{(p_{t,\ell})})^{[n_{\ell+1}]}$ let

$$w_s^+ = \{r : s <_{p_{t,\ell}} r \in T^{p_{t,\ell}} \wedge \mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},r}) > \ell + n_{\ell+1} + \mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},s})\}.$$

Now we consider the front

$$w_s = \{r \in w_s^+ : (\neg \exists z)(s <_{p_{t,\ell}} z <_{p_{t,\ell}} r \wedge z \in w_s^+)\},$$
$$w_t = \bigcup\{w_s : s \in (T^{p_{t,\ell}})^{[n_{\ell+1}]}\}$$

For each $r \in w_t$ and for each $\mathbf{c}_{p_t,r}$ we perform the operation from Lemma 2.15 and get a creature as $\mathbf{d}$ there with $x^t_\ell \in \mathrm{dom}(\tilde{\varrho})$ for every $\tilde{\varrho} \in \mathrm{pos}(\mathbf{d})$, and $\mathbf{d}$ serves as $\mathbf{c}_{p_{t,\ell+1},r}$.

Then we have for each such $\tilde{\varrho}$

$$(\boxtimes) \qquad \begin{aligned} &|\,\mathrm{dom}(\tilde{\varrho}) \setminus \mathrm{dom}(\varrho)| \leq 1, \text{ and} \\ &|\{y : (\exists \tilde{\eta} \in \mathrm{pos}(\mathbf{c}_{p_{t,\ell},r}))(y \in \mathrm{dom}(\tilde{\eta}) \wedge x^t_\ell <_{\mathbf{T}} y)\}| \leq \\ &|u_t| + \ell \leq \frac{\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},r})}{2}, \end{aligned}$$

since only $y \notin T_{<\alpha(p)}$ can be in the latter set. Hence the inequalities in the premises of Lemma 2.15 are fulfilled and $\mathrm{nor}^0(\mathbf{d}) \geq n + |u_t| + \ell + 1$. Then we can go on with Lemma 2.15 and change $\mathbf{c}_{p_{t,\ell},r'}$ (that corresponds to $\mathbf{c}$ there) into $\mathbf{c}_{p_{t,\ell+1},r'}$ (that corresponds to $\mathbf{d}$ there) with $\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell+1},r'}) \geq \frac{1}{2}\mathrm{nor}(\mathbf{c}_{p_{t,\ell},r'})$ for all immediate successors $r' \geq_{p_{t,\ell}} r$ as there. In order to fulfil the premise $\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},r}) < \sqrt{n_{1,i(p)+\lg(r)-1}}$, if necessary we go to a subcreature according to Lemma 2.7. Note that $n_{1,n_{\ell+1}} \geq 2^{{n_{\ell+1}}^2}$. Now we use Lemma 2.16 one level higher such that the specialisations in $\mathrm{pos}(\mathbf{c}_{p_{t,\ell+1},r'})$ become the bases of thinned out creatures in $p_{t,\ell+1}$ of norm at least the square root of the creatures in $p_{t,\ell}$. After having worked upwards through all of $T^{p_{t,\ell}}$ in this way, we get $p_{t,\ell+1}$. Since in the transition from $p_{t,\ell}$ to $p_{t,\ell+1}$ at each node in $T^{p_{t,\ell+1}}$ the norm drops at most once ans to at least half of its former value, we have $p_{t,\ell+1} \geq_{n+\ell} p_{t,\ell}$.

By Def. 3.3(A)(d) and (B)(b) we have for every branch b of $T^{p_{t,\ell+1}}$,

$$\bigcup \mathrm{dom}(\mathrm{last}(t')) : t' \in b\} = T_{<\alpha(p)} \cup u_t \cup \{x^t_i : i \leq \ell + 1\}.$$

This concludes the step from ℓ to $\ell+1$.

Now we let q be such that

$$q^{[<h]} = p^{[<h]} \text{ and } q^{\langle t\rangle} = \bigcup\{(p_{t,\ell})^{[n_\ell,n_{\ell+1})} \,:\, \ell \in \omega\}.$$

The condition q is smooth, and for every branch b of T^q we have

$$\bigcup \text{dom}(\text{last}(t')) \,:\, t' \in b\} = T_{<\alpha}.$$

$\dashv$

Definition 3.12. $\mathbb{Q}^{\rm s}_{\mathbf{T}}$ is the partial order of smooth conditions in $\mathbb{Q}_{\mathbf{T}}$, with the order as in $\mathbb{Q}_{\mathbf{T}}$.

So in the forcing sense, $\mathbb{Q}_{\mathbf{T}}$ and $\mathbb{Q}^{\rm s}_{\mathbf{T}}$ are equivalent.

Convention 3.13. *From now on we assume that all conditions are smooth.*

Corollary 3.14. *Forcing with $\mathbb{Q}_{\mathbf{T}}$ specialises $\mathbf{T}$.*

Proof. For any $\alpha \in \mathbf{T}$, $\{p \in \mathbb{Q}_{\mathbf{T}} \,:\, \alpha \in \text{dom}(\text{last}(\text{rt}(p)))\}$ is dense in $\mathbb{Q}_{\mathbf{T}}$. Let G be $\mathbb{Q}_{\mathbf{T}}$-generic over $\mathbf{V}$, then

$$f_{\mathbb{Q}_{\mathbf{T}}}[G] = \bigcup\{\langle p, \text{last}(t)\rangle \,:\, t = \text{rt}(p), p \in G\}$$

is a specialisation function for $\mathbf{T}$. $\dashv$

Definition 3.15. (1) Let $p \in \mathbb{Q}_{\mathbf{T}}$ be such that for any $t \in T^p$, $\text{nor}_f(\mathbf{c}_{p,t}) > 0$.
We say $q = \text{half}(p)$ if $i(q) = i(p)$, $T^q = T^q$, and for any $t \in T^q$ we have

$$(\forall \ell \geq \ell_0)(\forall t \in (T^q)^{[h]})\big(\mathbf{c}_{q,t} = \text{half}_{\frac{1}{\ell_{i(p)+h}}}(\mathbf{c}_{p,t})\big)$$

where the operation of taking the $\frac{1}{\ell}$-half of a creature was defined in Definition 2.19 (C). The numbers ℓ_i were defined in 2.8.

(2) Let $p \in \mathbb{Q}_{\mathbf{T}}$ and $n \in \omega$. Let ℓ_0 be minimal such that for any $t \in (T^p)^{[\geq \ell_0]}$, $\text{nor}_f(\mathbf{c}_{p,t}) \geq n+1$. Then we define $\text{half}_{n+1}(p) = q$ as follows:

$$(\forall t \in (T^q)^\ell)\big(\ell_0 \leq \ell = \ell_{i(q)+\text{ht}_q(t)} \;\rightarrow\; \mathbf{c}_{q,t} = \text{half}_{\frac{1}{\ell_{i+\text{ht}_p(t)}}}(\mathbf{c}_{p,t})\big)$$

$$(\forall t \in (T^q)^{[<\ell_0]})(\mathbf{c}_{q,t} = \mathbf{c}_{p,t}).$$

Note that according to the choices made by Definitions 2.8 and 2.2, $\ell_0 \geq 2$.

We recall the definition of de-halving a creature, Def. 2.19. Now we can also "de-halve" a condition:

Lemma 3.16. *The de-halving lemma. We assume*

(a) $a \in \omega$ and $p_1 \in \mathbb{Q}_{\mathbf{T}}$ are such that for any $t \in T^{p_1}$, $\text{nor}_f(\mathbf{c}_{p_1,t}) > a + \frac{1}{2}$,

(b) $q_1 = \text{half}(p_1)$,

(c) $q_2 \geq_0 q_1$ and

(d) for any $t \in T^{q_2}$, $\text{nor}_f(\mathbf{c}_{q_2,t}) > 0$.

Then we define p_2 as follows:

(1) $i(p_2) = i(p_1)$,

(2) $T^{p_2} = T^{q_2}$,

(3) *We let* $i_* = \min\{\ell : (\forall t \in (T^{q_2})^{[\ell]}) \operatorname{nor}_f(\mathbf{c}_{q_2,t}) \geq a + \frac{1}{2}\}$. *If* $\operatorname{pr}_{q_2,q_1}(t) = s$ *and* $\ell < i_*$ *and* $t \in (T^{q_2})^{[\ell]}$, *then* $\mathbf{c}_{p_2,t} = \text{de-halve}(\mathbf{c}_{q_2,t}, \mathbf{c}_{p_1,s})$. *If* $\operatorname{pr}_{q_2,q}(t) = s$ *and* $\ell \geq i_*$ *and* $t \in (T^{q_2})^{[\ell]}$, *then* $\mathbf{c}_{p_2,t} = \mathbf{c}_{q_2,t}$. *We write*

$$p_2 = \text{de-halve}(q_2, p_1, a).$$

Then $p_2 \Vdash q_2 \in G$ *and* $p_2 \geq_0 p_1$ *and* $\forall t \in T^{p_2}$, $\operatorname{nor}_f(\mathbf{c}_{p_2,t}) \geq a + \frac{1}{2} - \frac{1}{\ell_{i(p_1)}}$. *Moreover we have* $p_2^{\langle t \rangle} \geq_a p_1^{\langle t \rangle}$ *for any* $t \in T^{p_2}$.

The proof follows directly from the definitions. As mentioned, the application of halving and de-halving for $\mathbb{Q}_{\mathbf{T}}$ is left for future work.

4. Proof of Theorem 1.4 (b) to (d)

In this section we prove (in Lemma 4.9) that $(\mathbb{Q}, \leq, (\leq_n)_{n\in\omega})$ has properties (b) to (d) from Theorem 1.4. Thus $\mathbb{Q}_{\mathbf{T}}$ is ${}^{\omega}\omega$-bounding. This implication is proved in Section 3.1 of [15]. For the reader's convenience we recall the definition.

Definition 4.1. Let $\mathbb{P}$ be a notion of forcing. $\mathbb{P}$ is called ${}^{\omega}\omega$-bounding if for any sufficiently large regular cardinal χ and any $M \prec (H(\chi), \in)$ with $\mathbb{P} \in M$ for any $p \in \mathbb{P} \cap M$ and $\underset{\sim}{f} \in M$ that is a name for a function from ω to ω there is an $(M, \mathbb{P})$-generic condition $q \geq p$ and there is $g \in \mathbf{V} \cap {}^{\omega}\omega$ such that $q \Vdash \forall n \underset{\sim}{f}(n) \leq g(n)$. A condition q is $(M, \mathbb{P})$-generic if for any $D \in M$ that is a dense subset of $\mathbb{P}$ we have $q \Vdash M \cap \underset{\sim}{G} \cap D \neq \emptyset$.

Lemma 4.2. *(1) If* $p \in \mathbb{Q}$, $n \in \omega$, *and* $\{t_0, \ldots, t_n\}$ *is a front of* p, *then* $\{p^{\langle t_0 \rangle}, \ldots, p^{\langle t_n \rangle}\}$ *is predense above* p.

(2) If $\{t_0, \ldots, t_n\}$ *is a front of* p *and* $p^{\langle t_\ell \rangle} \leq q_\ell \in \mathbb{Q}$ *and there is* $\alpha < \omega_1$ *such that* $\alpha(q_\ell) = \alpha$ *for* $\ell \leq n$, *then there is* $q \geq p$ *with* $\{t_0, \ldots, t_n\} \subseteq T^q$ *such that for all* ℓ *we have that* $q^{\langle t_\ell \rangle} = q_\ell$.

(3) If $n \in \omega$ *and* $\{t_0, \ldots, t_m\}$ *is a front of* p *and* $\lg(t_\ell) \geq n$ *and* $p^{\langle t_\ell \rangle} \leq_0 q_\ell \in \mathbb{Q}$ *and there is* $\alpha < \omega_1$ *such that* $\alpha(q_\ell) = \alpha$ *for* $\ell \leq m$ *and*

– *for all* $\ell \leq m$, $(\forall s \in T^{q_\ell})(s \geq_{q_\ell} t_\ell \to \operatorname{nor}^0(\mathbf{c}_{q_\ell,s}) \geq n)$,

– *for all* $s \in T^p$ *if* $\operatorname{nor}^0(\mathbf{c}_{p,s}) < n$ *then* $(\exists \ell \leq m)(s <_p t_\ell)$,

then *there is* $q \geq_n p$ *with* $\{t_0, \ldots, t_m\} \subseteq T^q$ *such that for all* ℓ *we have that* $q^{\langle t_\ell \rangle} = q_\ell$ *and* $\{t_0, \ldots, t_m\}$ *is a front of* q.

The 2-bigness of our creatures is used to find stronger conditions that are homogeneous with respect to a downwards closed set:

Lemma 4.3. *If* $p \in \mathbb{Q}$, *and* $X \subseteq T^p$ *is* $<_p$*-downwards closed, then there is some* q *such that*

(a) $p \leq_0 q$, *and either* $(\forall \ell)((T^q)^{[\ell]} \subseteq X)$ *or* $(\forall^\infty \ell)((T^q)^{[\ell]} \cap X = \emptyset)$,

(b) $T^q \subseteq T^p$, *for* $t \in T^q$, $\mathbf{c}_{q,t} = \mathbf{c}_{p,t} \restriction T^q$ *(which means that the set of possibilities* $\operatorname{pos}(\mathbf{c}_{q,t})$ *is just those* $\eta \in \operatorname{pos}(\mathbf{c}_{p,t})$ *with* $t\hat{\ }\eta \in T^q$*)*,

(c) for every $t \in T^q$, if $\mathbf{c}_{q,t} \neq \mathbf{c}_{p,t}$, then $\mathrm{nor}^1(\mathbf{c}_{q,t}) \geq \mathrm{nor}^1(\mathbf{c}_{p,t}) - 1$ *and* $\mathrm{nor}_f(\mathbf{c}_{q,t}) \geq \mathrm{nor}_f(\mathbf{c}_{p,t}) - 1$.

Proof. We will choose $T^q \subseteq T^p$. For each $\ell \geq 1$ we first choose by downward induction on $j \leq \ell$ a colouring $f_{\ell,j}$ of $(T^p)^{[j]}$ with two colours, 0 and 1. For $t \in (T^p)^{[\ell]}$ we set $f_{\ell,\ell}(t) = 0$ iff $t \in X$ and $f_{\ell,\ell}(t) = 1$ otherwise.

Suppose that $f_{\ell,j}$ is defined. For $s \in (T^p)^{[j-1]}$ we have

$$\begin{aligned}\mathrm{pos}(\mathbf{c}_{p,s}) =& \{\nu \in \mathrm{pos}(\mathbf{c}_{p,s}) \,:\, f_{\ell,j}(\nu) = 0\} \cup \\ & \{\nu \in \mathrm{pos}(\mathbf{c}_{p,s}) \,:\, f_{\ell,j}(\nu) = 1\}\end{aligned}$$

For $m = 0, 1$, we let $\mathbf{c}_{p,s,m} = (i(\mathbf{c}_{p,s}), \eta, \{\nu \in \mathrm{pos}(\mathbf{c}_{p,s}) \,:\, f_{\ell,j}(s\hat{\ }\nu) = m\})$. By Lemma 2.18 there is $m \in \{0, 1\}$ such that $\mathrm{nor}^1(\mathbf{c}_{p,s,m}) \geq \mathrm{nor}^1(\mathbf{c}_{p,s}) - 1$. Now we colour $s \in (T^p)^{[j-1]}$ as follows: $f_{\ell,j-1}(s) = m$ iff $m \in \{0, 1\}$ is minimal such that $\mathrm{nor}^1(\mathbf{c}_{p,s,m}) \geq \mathrm{nor}^1(\mathbf{c}_{p,s}) - 1$. We work downwards until we come to the root of p and keep $f_{\ell,0}(\mathrm{rt}(p))$ and $\mathbf{c}_{p,s,m}$ in our memory.

We repeat the procedure of the downwards induction on j for larger and larger ℓ. Since X is downwards closed, we have

$$(*) \qquad \forall \ell \forall j \leq \ell \forall s \in (T^p)^{[j]} \ (f_{\ell+1,j}(s) = 0 \to f_{\ell,j}(s) = 0).$$

For each fixed ℓ, these statements are proved by easy downward induction on j.

Case 1: There are infinitely many ℓ such that $f_{\ell,0}(\mathrm{rt}(p)) = 0$. If there are infinitely many ℓ such that $f_{\ell,0}(\mathrm{rt}(p)) = 0$, then by $(*)$ this holds for all ℓ. Since for each fixed m there are only finitely many possible $\langle f_{m,j}(s) \,:\, s \in (T^p)^{[j]}, j \leq m\rangle$, by König's lemma we find an infinite subsequence $\langle \ell_k \,:\, k < \omega\rangle$ such that for each k for all $k' \geq k$ for all $j \leq \ell_k$, for all $s \in (T^p)^{[j]}$, $f_{\ell_{k'},j}(s) = f_{\ell_k,j}(s)$. So we have for every k, $f_{\ell_k,0}(\mathrm{rt}(p)) = 0$. We let

$$T^q = \{s \in T^p \,:\, (\forall j, k)((j \leq \ell_k \wedge s \in T^{[j]}) \to (\forall k' \geq k)(f_{\ell_{k'},j}(s) = f_{\ell_k,j}(s) = 0))\}.$$

Then $T^q \subseteq X$ and $\mathrm{rt}(q) = \mathrm{rt}(p)$. By our choice of $f_{\ell,j}$, by the case assumption $\forall \ell f_{\ell,0}(\mathrm{rt}(p)) = 0$ and the norm drops at most one in the transition from p to q. So q is as required.

Case 2: There is ℓ such that $\forall \ell' \geq \ell$, $f_{\ell',0}(\mathrm{rt}(p)) = 1$. Let ℓ be minimal with this property. Since for each fixed m there are only finitely many possible $\langle f_{m,j}(s) \,:\, s \in (T^p)^{[j]}, j \leq m\rangle$, by König's lemma we find an infinite subsequence $\langle \ell_k \,:\, k < \omega\rangle$ such that for each k for all $k' \geq k$ for all $j \leq \ell_k$, for all $s \in (T^p)^{[j]}$, $f_{\ell_{k'},j}(s) = f_{\ell_k,j}(s)$. So we have for every k, $f_{\ell_k,0}(\mathrm{rt}(p)) = 0$. We let

$$\begin{aligned}T^q = \{s \in T^p \,:& (\forall j, k)((\ell \leq j \leq \ell_k \wedge s \in T^{[j]}) \to \\ & (\forall k' \geq k)(f_{\ell_{k'},j}(1) = f_{\ell_k,j}(s) = 1))\}.\end{aligned}$$

Then $T^q \subseteq X$ and $\mathrm{rt}(q) = \mathrm{rt}(p)$. By our choice of $f_{\ell,j}$, by the case assumption the norm drops at most one in the transition from p to q. So q is as required. ⊣

We now improve the property $p \leq_0 q$ in Lemma 4.3 to $p \leq_n q$, and therefore we have to weaken the homogeneity property in item $(a)(iii)$ with $n+1$ instead of n and p instead of q of Lemma 4.3.

Lemma 4.4. *If $p \in \mathbb{Q}$, $n \in \omega$, and $X \subseteq T^p$ is downward closed, then there is some q such that*

(a) $p \leq_n q$, and there is a front $\{t_0, \dots, t_j\}$ such that

(i) $\{t \in T^p : \mathrm{nor}_f(\mathbf{c}_{p,t}) \leq n\} \subseteq \{t \in T^q : (\exists i \leq j)(t \leq_q t_i)\}$,

(ii) for all $i \leq j$ we have: either $\{s \in T^q : s \geq_p t_i\} \subseteq X$ or $(\forall^\infty \ell)(\{s \in (T^q)^{[\ell]} : s \geq_p t_i\} \cap X = \emptyset)$,

(iii) and for all $i \leq j$, $t \geq_q t_i$, $\mathrm{nor}^0(\mathbf{c}_{q,t}) \geq n$.

(b) $T^q \subseteq T^p$ and $\mathbf{c}_{q,t} = \mathbf{c}_{p,t} \restriction T^q$,

(c) for every $t \in T^q$, if $\mathbf{c}_{q,t} \neq \mathbf{c}_{p,t}$, then $\mathrm{nor}^1(\mathbf{c}_{q,t}) \geq \mathrm{nor}^1(\mathbf{c}_{p,t}) - 1$ and $\mathrm{nor}_f(\mathbf{c}_{q,t}) \geq \mathrm{nor}_f(\mathbf{c}_{p,t}) - 1$.

Proof. We choose a front of p as in (a) and use Lemma 4.3 for each $p^{\langle t_i \rangle}$. ⊣

Definition 4.5. Let $\nu_0, \nu_1 \in \mathrm{spec}$. We say ν_0 *is isomorphic to* ν_1 *over* $\mathbf{T}_{<\alpha}$ if there is some injective partial function $f \colon \mathbf{T} \to \mathbf{T}$ such that $(\forall x, y \in \mathrm{dom}(f))(x <_{\mathbf{T}} y \leftrightarrow f(x) <_{\mathbf{T}} f(y))$ and $\mathrm{dom}(\nu_0) \cup \mathbf{T}_{<\alpha} \subseteq \mathrm{dom}(f)$ and $f \restriction \mathbf{T}_{<\alpha} = \mathrm{id}$ and $f[\mathrm{dom}(\nu_0)] = \mathrm{dom}(\nu_1)$ and $\nu_0(x) = \nu_1(f(x))$ for all $x \in \mathrm{dom}(\nu_0)$.

Fact 4.6. *(1) Being isomorphic over $\mathbf{T}_{<\alpha}$ is an equivalence relation.*

(2) For each $\alpha < \omega_1$, there are only countably many isomorphism types for $\eta \in \mathrm{spec}^{\mathbf{T}}$ over $\mathbf{T}_{<\alpha}$.

Definition 4.7. Let $q \in \mathbb{Q}_{\mathbf{T}}$ (recall that this means smooth), $\ell, n \in \omega$, $t \in T^q$, $\varrho \in \mathrm{spec}^{\mathbf{T}}$. Let $\underset{\sim}{\tau}$ be $\mathbb{Q}$-name for an ordinal.

Let $\boxplus_{q,\ell,t,\varrho,n}(\underset{\sim}{\tau})$ abbreviate the following statement:

(i) $q \in \mathbb{Q}_{\mathbf{T}}$, $t \in (T^q)^{[\ell]}$, $n \in \omega$, and

(ii) $(\forall s \geq_q t)(\mathrm{nor}_f(\mathbf{c}_{q,s}) \geq n+1)$, and

(iii) if there are finite partial specialisations ϱ' and ϱ'' and a condition $q' \geq q$ with $\mathrm{rt}(q') = \mathrm{last}(t) \cup \varrho' \cup \varrho''$, $i(q') = i(q) + \lg(t) - 1$, $|\,\mathrm{rt}(q')| \leq \frac{n_{1,i(q')}}{2^{(2^n + k(\mathbf{c}_{q,t}))}}$ and $\mathrm{nor}_f(\mathbf{c}_{q',t''}) \geq n+1$ for every $t'' \in T^{q'}$ and ϱ' and ϱ are isomorphic over $T_{<\alpha(q)}$, q' forces a value to $\underset{\sim}{\tau}$, *then* $q^{\langle t \rangle}$ forces a value to $\underset{\sim}{\tau}$.

Lemma 4.8. *Suppose that $p \in \mathbb{Q}_{\mathbf{T}}$ and that $n < \omega$. Let $N \prec \mathcal{H}(\chi)$ be countable and let $N \cap \omega_1 = \delta_*$, $p \in N$, $\mathbf{T} \in N$. Let $\underset{\sim}{\tau} \in N$ be a $\mathbb{Q}_{\mathbf{T}}$-name of an ordinal. For every $n \in \omega$ there is a $q \in \mathbb{Q}_{\mathbf{T}}$ such that*

(a) $p \leq_n q$,

(b) $\alpha(q) = \delta_*$,

(c) If $\varrho \in \mathrm{spec}^{\mathbf{T}}$ and $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$ then for any i infinitely many $\ell \in [i, \omega)$ we have $\forall t \in (T^q)^{[\ell]}$ $\boxplus_{q,\ell,t,\varrho,n+i}(\underset{\sim}{\tau})$.

Proof. Let $\langle \varrho_m : m \in \omega \rangle$ list representatives of the possible isomorphism types over $\mathbf{T}_{<\delta_*}$ of an $\varrho \in \mathrm{spec}^{\mathbf{T}}$ such that $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$ such that each type is represented infinitely often.

Let $\langle \alpha_i : i < \omega \rangle$ be an increasing sequence of ordinals that converges to δ_*. We choose (p_i, ℓ'_i) by induction on i with the following properties:

(1) $p_i \in \mathbb{Q}_{\mathbf{T}} \cap N$,
(2) $\ell'{}_i < \ell'{}_{i+1}$,
(3) $p_0 = p$, $p_i \leq_{n+i} p_{i+1}$, $p_i^{[<\ell'_i]} = p_{i+1}^{[<\ell'_i]}$,
(4) $(\forall t \in (T^{p_i})^{[\geq \ell'_i]})(\mathrm{nor}_f(\mathbf{c}_{p_i,t}) \geq n+i+1)$,
(5) For any $s \in (T^{p_i})^{[\ell'_i]}$ we define

$$\Lambda^2_{i,s} = \{\nu \in \mathrm{pos}(\mathbf{c}_{p_i,s}) \,:\, (\alpha)_{i,s,\nu} \text{ holds}\},$$

where

$(\alpha)_{i,s,\nu}$ There is a prolongation $\tilde{\varrho}_i$ of ϱ_i that is disjoint from ν and compatible with ν such that there are unboundedly many $\gamma \in \delta_*$ such that: [3]
There are $\tilde{\varrho} \in N$ and a smooth $r_0 \in N$, $r_0 \geq p_i$
such that $\mathrm{rt}(r_0) = \nu \dot{\cup} \tilde{\varrho}$, $i(r_0) = i(p_i) + \lg(s)$, $|\mathrm{rt}(r_0)| \leq \frac{n_{1,i(r_0)}}{2^{(2^{n+i}+k(\mathbf{c}_{p_i,t}))}}$,
and $\tilde{\varrho}$ and $\tilde{\varrho}_i$ realise the same type over $\mathbf{T}_{<\alpha(p_i)}$, $\mathrm{dom}(\tilde{\varrho}) \cap \gamma = \emptyset$, and r_0 forces a value to $\underset{\sim}{\tau}$ and for all $t \in T^{r_0}$, $\mathrm{nor}_f(\mathbf{c}_{r_0,t}) \geq n+i+1$. By $\dot{\cup}$ we denote the disjoint union. This ends $(\alpha)_{i,s,\nu}$.

We let

$$\Lambda^1_{s,i} = \mathrm{pos}(\mathbf{c}_{p_i,s}) \setminus \Lambda^2_{s,i}.$$

We demand: If $\nu \in \Lambda^2_{s,i}$ then

$$(\forall t \in (T^{p_{i+1}})^{[\ell'_i+1]})\Big((t >_{p_{i+1}} s \wedge \mathrm{pr}_{p_{i+1},p_i}(t) = s\hat{\ }\langle\nu\rangle) \rightarrow p_{i+1}^{\langle t\rangle} \text{ forces a value to } \underset{\sim}{\tau}\Big).$$

This ends item (5).

We show that there is such a sequence $\langle p_i, \ell'_i \,:\, i < \omega\rangle$. Assume we are given p_i. Then we choose ℓ'_i such that $(\forall t \in (T^{p_i})^{[\geq \ell'_i]})(\mathrm{nor}_f(\mathbf{c}_{p_i,t}) \geq (n+i+1)$. For every $s \in (T^{p_i})^{[\ell'{}_i]}$ we divide $\mathrm{pos}(\mathbf{c}_{p_i,s})$ into $\Lambda^1_{s,i}$ and $\Lambda^2_{s,i}$. Assume that $(\alpha)_{i,s,\nu}$ holds.

Explanation: We fix for each of cofinally many $\gamma \in \delta_*$ a condition $r_0 = r_{0,\gamma} \in N$ as in $(\alpha)_{i,s,\nu}$. The core of the construction is the definition of a preliminary part of p_{i+1} that is composed of ν, k parts. We choose $\varrho_{\nu,k} \supseteq \nu$ for $k = 0, \dots, m+i$, $\nu \in \Lambda^2_{i,s}$ and $p_{i+1,s,\nu,k} \geq p_i^{\langle(s\hat{\ }\langle\nu\rangle)\rangle}$ with root $\varrho_{\nu,k}$ that is isomorphic to $\nu \cup \tilde{\varrho}$ and a common $\alpha(p_{i+1,s,\nu,k})$ for all s and (ν,k) and then graft the conditions $(p_{i+1,s,\nu,k})$ at the node $s\hat{\ }\langle\nu\rangle$ into $p_i^{\langle s\hat{\ }\langle\nu\rangle} =: p_{i+1,s,\nu,k}$. This implies that $p_{i+1} = \bigcup\{p_{i+1,s,\nu,k} \,:\, s \in (T^p)^{[\ell'_i]}, \nu \in \Lambda^2_{i,s}, k \leq [\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_s,s})}]\}$ fulfils

$$p_{i+1,s,\nu,k} = p_{i+1}^{\langle(s\hat{\ }\langle\varrho_{\nu,k}\rangle)\rangle}$$

and $p_i \uparrow \ell'_{i+1} - 1 = p_{i+1} \uparrow \ell'_{i+1} - 1$. In the stronger condition p_{i+1}, the place of ν in $<_{p_i}$ will be taken by

$$\varrho_{\nu,k} = \nu \cup \varrho'_{\nu,k}, \qquad k = 0, 1, \dots, [\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}],$$

where the $\varrho'_{\nu,k}$ still have to be defined, see (p1) to (p7) below. The order $<_{p_{i+1}}$ is defined such that for any $k \leq [\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}]$, $\mathrm{pr}_{p_{i+1},p_i}(s\hat{\ }\langle\varrho_{\nu,k}\rangle) = s\hat{\ }\langle\nu\rangle$ and

[3] Everything depends also on s, but we do not introduce an index s.

$\mathrm{pr}_{p_{i+1},p_i} \restriction (p_{i+1})^{\langle (s\hat{}\langle \varrho_{\nu,k}\rangle)\rangle})$ is a projection witnessing $(p_{i+1})^{\langle (s\hat{}\langle \varrho_{\nu,k}\rangle)\rangle} \geq p_i^{\langle s\hat{}\nu\rangle}$. We will show that there are such $\varrho_{\nu,k}$ with the additional property that for each k, $\varrho'_{\nu,k} = \varrho_{\nu,k} \setminus \nu$ is over $\mathbf{T}_{<\alpha(p_i)}$ isomorphic to $\tilde{\varrho}_i$ and such that there is $p_{i+1} \in \mathbb{Q}_{\mathbf{T}}$ such that for any $s \in (T^{p_i})^{[\ell'_i]} = (T^{p_i+1})^{[\ell'_i]}$

$(*)_1$ $\mathrm{pos}(\mathbf{c}_{p_{i+1},s}) = \{\varrho_{\nu,k} : k < n+i+1\}$,

$(*)_2$ if $\nu \in \Lambda^2_{i,s}$, then

$$(\forall s\hat{}\langle \varrho_{\nu,k}\rangle \in p_{i+1}^{[\ell'i+1]})\Big((s\hat{}\langle \varrho_{\nu,k}\rangle >_{p_{i+1}} s\hat{}\langle \nu\rangle \wedge \mathrm{pr}_{p_{i+1},p_i}(s\hat{}\langle \varrho_{\nu,n}\rangle) = s\hat{}\langle \nu\rangle) \rightarrow p_{i+1}^{\langle (s\hat{}\langle \varrho_{\nu,k}\rangle)\rangle} \text{ forces a value to } \underset{\sim}{\tau}\Big),$$

and

$(*)_3$ $p_{i+1}^{\langle s\rangle} \geq_{n+i} p_i^{\langle s\rangle}$.

Mishappenings like $\mathrm{nor}_f(\mathbf{c}_{p_{i+1},s}) < \mathrm{nor}_f(\mathbf{c}_{p_i,s}) - 1$ must be prevented.

Of course we only have $p_{i+1}^{\langle s\hat{}\langle \varrho_{\nu,k}\rangle)\rangle} \geq p_i^{\langle s\hat{}\nu\rangle}$, not even with $\geq_0$ by the choice of $\varrho_{\nu,k}$. A remedy is to lengthen the ν in many disjoint and isomorphic ways to $\varrho_{\nu,k}$, and then to use an old fact about uncountably many disjoint finite subsets of Aronszajn trees and Lemma 2.14. This ends the explanation of the envisaged construction.

We continue the construction: By the case assumption, there are $\varrho'_{\nu,k}$, r_k, $1 \leq k < \omega$, with the following properties

(p1) $\varrho'_{\nu,k}$ and $\tilde{\varrho}_i$ have the same type over $\mathbf{T}_{<\alpha(p_i)}$,

(p2) $\mathrm{dom}(\varrho'_{\nu,k}) \cap \alpha(p_i) = \emptyset$, and $\mathrm{dom}(\varrho'_{\nu,k}) \cap \mathrm{dom}(\varrho'_{\nu,k'}) = \emptyset$ for $k \neq k'$,

(p3) $\varrho'_{\nu,k} \in N$,

(p4) $r_k \in N$ forces a value to $\underset{\sim}{\tau}$, r_k is smooth,

(p5) $\mathrm{rt}(r_k) = \nu \cup \varrho'_{\nu,k} =: \varrho_{\nu,k}$, $i(r_k) = i(p_i) + \lg(s)$,

(p6) for all $\forall t' \in (T^{r_k})^{[\geq \ell'_i]}$, $\mathrm{nor}_f(\mathbf{c}_{r_k,t'}) \geq n+i+1$,

(p7) $|\mathrm{dom}(\varrho_{\nu,k})| \leq \frac{n_{1,i}}{2^{(2^{n+i}+k(\mathbf{c}_{p_i},t))}}$.

By a fact on uncountably many disjoint finite subsets in an Aronszajn tree (see e.g. [18, Ch. III, Thm 5.4] or [7, Lemma 18.10]), applied in N iteratively $n+i$ times, we can have additionally

(p8) and such that for $k \neq k'$, $k, k' \leq \sqrt{n+i}$, any $t \in \mathrm{dom}(\varrho'_{\nu,k})$ and any $t' \in \mathrm{dom}(\varrho'_{\nu,k'})$ are $\leq_{\mathbf{T}}$-incomparable.

Then we pick for each $k < \omega$ some $\varrho'_{\nu,k}$ and r_k and let $\varrho_{\nu,k} := \nu \cup \varrho'_{\nu,k}$, a preliminary condition smooth $p'_{i+1,s,\nu,k} = r_k$. We have $s\hat{}\langle \varrho_{\nu,k}\rangle = \mathrm{rt}(T^{p'_{i+1,s.\nu,k}})$. We glue the preliminary conditions together above s and get $(p'_{i+1})^{\langle (s\hat{}\langle \varrho_{\nu,k}\rangle)\rangle} = p'_{i+1,s,\nu,k}$ and $\mathrm{pr}_{p'_{i+1},p_i}(s\hat{}\langle \varrho_{\nu,k}\rangle) = s\hat{}\langle \nu\rangle$. We use only $k = 0, \dots, n+i$. The outcome p'_{i+1} of the gluing procedure might not be smooth, indeed not be a condition at all, because each $p'_{i+1,s,\nu,k}$ has its own $\alpha(p'_{i+1,s,\nu,k})$ and Def. 3.3(A)(c) might be missing. Clause (p4) guarantees $(*)_2$ for $p_{i+1}^{\prime\langle(s\hat{}\langle\varrho_{\nu,k}\rangle)}$, and by Lemma 2.14 the requirement $(*)_3$ is fulfilled by $p_{i+1}^{\prime\langle s\rangle}$ for all the relevant s, ν, k.

This finishes the particular construction of p'_{i+1} above the projection's preimage of $s\hat{}\langle\nu\rangle$ for $\nu \in \Lambda^2_{s,i}$. For $\nu \in \Lambda^1_{i,s}$ we let p'_{i+1} above $s\hat{}\langle\nu\rangle$ be just p_i above $s\hat{}\langle\nu\rangle$.

Thereafter we take $\alpha'_{i+1} \in [\alpha_{i+1}, \delta_*)$ sufficiently large so that for each $\nu \in \Lambda^2_{s,i}$ and each $k \le [\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}]$ and each $\nu \in \Lambda^1_{s,i}$ (with $\varrho_{\nu,0} = \nu$) a smooth $p_{i+1s,\nu,k} \ge_{n+i} p'_{i+1,s,\nu,k}$ such that for every $s\hat{}\langle\nu\rangle \ge_{p_i} s$, $s\hat{}\langle\nu\rangle \in (T^{p_i})^{[\ell'{}_i+1]}$, $k = 0,\dots,n+i$ for all $\varrho_{\nu,k}$, $\alpha(p_{i+1,s,\nu,k}) = \alpha'_{i+1} \ge \alpha_{i+1}$. Such a condition exists by Lemma 3.11. We perform all the filling up from the proof of the latter lemma strictly above level ℓ'_{i+1}.

Gluing all these smooth conditions with the same α'_{i+1} gives

$$\begin{aligned} p_{i+1} = \bigcup \{ p_{i+1,s,\nu,k} \,:\, s \in p_i^{[\ell'_i]}, \nu \in \Lambda^1_{s,i}\} \cup \\ \{p_{i+1,s,\nu,k} \,:\, k \le [\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}], s \in p_i^{[\ell'_i]}, \nu \in \Lambda^2_{s,i}\} \end{aligned}$$

together in a natural way finally gives p_{i+1} with $\alpha(p_{i+1}) = \alpha'_{i+1}$. Since $(p')^{[\le \ell'_{i+1}]}_{i+1} = p^{[\le \ell'_{i+1}]}_{i+1}$, the properties $(*)_{1,2,3}$ of p'_{i+1} hold also for p_{i+1}. Now by Lemma 2.14,

$$\mathrm{nor}_f(\mathbf{c}_{p_{i+1},s}) \ge n+i \text{ and } p_{i+1} \ge_{n+i} p_i. \tag{4.1}$$

So we finished the inductive choice of $\langle (p_i, \ell'_i) \,:\, i < \omega\rangle$ with properties (1) to (5).

We let q be the fusion of the p_i.

We show that q is as desired as in the lemma: Let $\varrho \in \mathrm{spec}^{\mathbf{T}}$, i be given with $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$. Then at any of the infinitely many steps i of the above construction in which the isomorphism type of ϱ is invoked, for any $t = s\hat{}\langle\varrho_{\nu,k}\rangle \in (T^q)^{[\ell'_i+1]}$, $\boxplus_{q,\ell',t,\varrho,n+i}(\underset{\sim}{\tau})$ is ensured by properties $(*)_{1,2,3}$ and (1) to (5). $\dashv$

In the next lemma we turn conclusion (c) of the previous lemma into a stronger property, by strengthening a condition with the help of the homogeneity property from Lemma 4.4. The property in (b) in the next lemma is a version of "continuous reading of names" that yields a strong version of Axiom A.

Lemma 4.9. *Suppose that $\mathbb{Q} = \mathbb{Q}_{\mathbf{T}}$, $p \in \mathbb{Q}$, $n < \omega$, and $\underset{\sim}{\tau}$ is a $\mathbb{Q}$-name of an ordinal. Then there is a $q \in \mathbb{Q}$ such that*

(a) $p \le_n q$,

(b) for some $\ell \in \omega$ we have that for every $t \in (T^q)^{[\ell]}$ the condition $q^{\langle t\rangle}$ forces a value to $\underset{\sim}{\tau}$.

Proof. Let $N \prec H(\chi)$ be such that $\mathbb{Q}_{\mathbf{T}}, p, \underset{\sim}{\tau} \in N$. We take $q \ge_{n+1} p$ in the role of q from the previous lemma applied to N, $\delta_* = N \cap \omega_1$ and $\underset{\sim}{\tau} \in N$ and p, so (a), (b), (c) of the conclusion of Lemma 4.8 hold for p and q.

Then we define for $k \in \omega$,

$$X_{\underset{\sim}{\tau}}(q,k,n) = \big\{t : t \in \bigcup_{k' \geq k} (T^{p'_0})^{[k']} \wedge (\exists q')$$
$$\big(q^{\langle t \rangle} \leq_0 q' \wedge (q' \text{ forces a value to } \underset{\sim}{\tau}) \wedge$$
$$(\forall t' \in T^{q'})(t' \geq_{q'} t \to \mathrm{nor}_f(\mathbf{c}_{q',t'}) \geq n+1)\big)\big\}.$$

For $\tilde{n} < \omega$, $p_1, p_2 \in \mathbb{Q}$, $t \in T^r$, we denote the following property:

$$(*)^{\tilde{n},t}_{p_1,p_2} \qquad \begin{array}{l} (p_1)^{\langle t \rangle} \leq_0 p_2 \wedge \\ \forall t'(t \leq t' \in T^{(p_2)} \to \mathrm{nor}^0(\mathbf{c}_{p_2,t'}) \geq \tilde{n}+1) \wedge \\ (p_2 \text{ forces a value to } \underset{\sim}{\tau}). \end{array}$$

Note that $(T^{q'})^{[\ell]} \subseteq X(q,k,n)$ implies $\forall t \in (T^{q'})^{[\ell]}(\exists q'')(*)^{n,t}_{q,q''}$.

Choose

(1) k such that $t \in (T^q)^{[\geq k]} \to \mathrm{nor}_f(\mathbf{c}_{q,t}) > n+2$,
(2) $q' \geq_{n+1} q$ is chosen as in Lemma 4.4 applied to q, the front $(T^q)^{[k]}$ and $X = \mathrm{dom}(q) \setminus X(q,k,n+1)$ which is downwards closed.

We show that $\forall^\infty \ell \forall t \in (T^{q'})^{[\ell]} (q')^{\langle t \rangle}$ forces a value to $\underset{\sim}{\tau}$. Note that also q' has with respect to p the properties from the previous lemma.

First case: In Lemma 4.4(a) we get $\forall \ell (T^{q'})^{[\ell]} \subseteq X$. We show that this does not happen. We work with $i = 0$ in conclusion (c) of Lemma 4.8.

Suppose $t \in T^{q'}$ is such that $(\forall t' \in (T^{q'})(t' \geq_q t \to \mathrm{nor}(\mathbf{c}_{q',t'}) \geq n+1)$. Then, by the definition of X, for any $t' \geq_Q t$, $(q')^{\langle t' \rangle}$ does not force a value to τ. However, $\alpha(q) = \alpha(q') = \alpha((q')^{\langle t \rangle}) = \delta_* = N \cap \omega_1$. We take any $q'' \geq (q')^{\langle t \rangle}$ that forces a value to τ. Without loss of generality we can assume that for all $t' \in T^{q''}$, $(t' \geq_{q''} t^- \to \mathrm{nor}_f(\mathbf{c}_{q'',t'}) \geq n+1)$, where $t^- <_{q''} t$ is the direct predecessor of t. Then $\varrho := \mathrm{rt}(r) \setminus \mathrm{last}(t)$ has $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$ by Fact 3.7 (2). Moreover $|\,\mathrm{rt}(r)| < \frac{n_{1,i(r)}}{2^{(2^n + k(\mathbf{c}_{q'',t}))}}$ by the assumption on the norms. Then since q' has the properties of the previous lemma we get there are infinitely many ℓ such that such that $(\forall t' \in (T^q)^{[\ell]} \cap T^{(q'')^{\langle t \rangle}})(\boxplus_{q,\ell,t',\varrho,n})$ and in clause (iii) of the statement $\boxplus_{q,\ell,t',\varrho,n}$ the premise is fulfilled. So we have $(T^{q'})^{[\ell]} \not\subseteq X$.

Second case: In Lemma 3.3(a) we get $(\forall^\infty \ell)((T^{q'})^{[\ell]} \cap X = \emptyset)$. By the definition of $X(q,k,n) = T^q \setminus X$ we are done. ⊣

Conclusion 4.10. *$\mathbb{Q}_{\mathbf{T}}$ is a proper ${}^\omega\omega$-bounding forcing that specialises the Aronszajn tree $\mathbf{T}$.*

The following result was also established by Hirschorn [5] and in [11].

Corollary 4.11. *It is consistent relative to ZFC that all Aronszajn trees are special (hence there are no Souslin trees) and $\mathfrak{d} = \aleph_1$ and $2^\omega = \aleph_2$.*

Proof. We shall show that there is an iterated forcing with iterands of the form $\mathbb{Q}_{\mathbf{T}}$ such that in the extension every Aronszajn in special. Since the letter $\mathbb{Q}$ is reserved for the $\mathbb{Q}_{\mathbf{T}}$, we use that symbol $\mathbb{Q}'$ for the iterands. Every element of a forcing extension has a proper class of names. Canonical names provide for a small set of representatives. We first recall the notion of canonical names. Let $\mathbb{P}$ be a notion of forcing. For $a \in \mathbf{V}$ we let $\check{a} = \{\langle p, \check{b}\rangle : b \in a, p \in \mathbb{P}\}$. For a $\mathbb{P}$-name τ we define its name rank $\mathrm{rk}_n(\tau)$ by induction as follows:

$$\mathrm{rk}_n(\tau) = \sup\{\mathrm{rk}_n(\sigma) + 1 : (\exists p)(\langle \sigma, p\rangle \in \tau\}.$$

In addition we define the revised name rank as

$$\mathrm{rk}_r(\tau) = \begin{cases} 0, & \text{if } \exists a \in \mathbf{V}\check{a} = \tau; \\ \sup\{\mathrm{rk}_r(\sigma) + 1 : \exists p\langle p, \sigma\rangle \in \tau\}, & \text{else.} \end{cases}$$

Finally we define the $\mathbf{V}$-rank $\mathrm{rk}_{\mathbf{V}}$ for $x \in \mathbf{V}[G]$,

$$\mathrm{rk}_{\mathbf{V}}(x) = \begin{cases} 0, & \text{if } x \in \mathbf{V}; \\ \sup\{\mathrm{rk}_{\mathbf{V}}(y) + 1 : y \in x\}, & \text{else.} \end{cases}$$

Let $\mathrm{rk}(x) = \sup\{\mathrm{rk}(y) + 1 : y \in x\}$ be the usual rank function.

A $\mathbb{P}$-name τ with $\mathrm{rk}_n(\tau) = \alpha$ is called *canonical* if

(a) for any β, $\mathbb{P} \Vdash \mathrm{rk}(\tau) \leq \beta$ implies $\beta \geq \alpha$ and $\mathbb{P} \Vdash \mathrm{rk}_{\mathbf{V}}(\tau) \leq \beta$ implies $\mathrm{rk}_r(\tau) \leq 1 + \beta$ and

(b) if $\mathbb{P}$ has the λ -c.c. and $\mathbb{P} \Vdash |\tau| < \lambda$, then $|\tau| < \lambda$ and

(c) any σ such that for some $p \in \mathbb{P}$, $\langle p, \sigma\rangle \in \tau$ is canonical as well (see [18, Ch. I, §5]).

Shelah [18, Ch. I, Theorem 5.13] proves that every $x \in \mathbf{V}[G]$ has a canonical name. The canonical names are in general not unique.

We write $\mathbf{V}^{\mathbb{P}}$ for any $\mathbf{V}[G]$ with a generic G. In the special case of an Aronszajn tree $(\omega_1, <_{\mathbf{T}})$ in the sense of Definition 1.1 in an extension $\mathbf{V}^{\mathbb{P}}$ with $\omega_1^{\mathbf{V}^{\mathbb{P}}} = \omega_1^{\mathbf{V}}$ we have for example a $\mathbb{P}$-canonical name of $<_{\mathbf{T}}$ the form

$$\tau = \{\langle p, (\alpha, \beta, i)\rangle : \alpha, \beta \in \omega_1, i \in \{0, 1\}, p \in A_{\alpha,\beta}\}$$

with maximal antichains $A_{\alpha,\beta} \subseteq \mathbb{P}$ such that for each $p \in A_{\alpha,\beta}$, $p \Vdash_{\mathbb{P}} \alpha <_{\mathbf{T}} \beta$ if $\langle p, (\alpha, \beta, 1)\rangle \in \tau$, and $p \Vdash_{\mathbb{P}} \neg\alpha <_{\mathbf{T}} \beta$ if $\langle p, (\alpha, \beta, 0)\rangle \in \tau$. If $\mathbb{P} \in H(\aleph_2)$ then $\tau \in H(\aleph_2)$. This ends the review of canonical names.

We start with a ground model of $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Let $b \colon \aleph_2 \to H(\aleph_2)$ be a surjective function such that each element of $H(\aleph_2)$ has cofinally many preimages under b. Since $2^{\aleph_1} = \aleph_2$, such a function exists.

We argue that it suffices to specialise any Aronszajn in the sense of Definition 1.1. Any normal Aronszajn tree has an isomorphic copy that meets Definition 1.1. Every Aronszajn $\mathbf{T}$ tree contains a normal subtree

$$\mathbf{T}_{\text{normal}} = \{t \in \mathbf{T} : |\{s \in \mathbf{T} : s \geq_{\mathbf{T}} t\}| = \aleph_1\}.$$

Any specialisation of the normal part can be extended to a specialisation of the whole tree.

Recall, we write $\mathbb{P} \Vdash \varphi$ to denote that any element of $\mathbb{P}$ forces φ. We prove by induction on $\alpha \leq \omega_2$ that there is a countable support iteration

$$\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \mathbb{Q}'_\beta : \beta < \omega_2, \alpha \leq \omega_2\rangle$$

with the following properties:

(1) $\forall \alpha < \omega_2$, $\mathbb{P}_\alpha \in H(\aleph_2)$, $\mathbb{P}_0 = \{1\}$,

(2) $\forall \alpha < \omega_2$,

$$\mathbb{P}_\alpha \Vdash_{\mathbb{P}_\alpha} \text{"if } b(\alpha) \text{ is a } \mathbb{P}_\alpha\text{-name of an Aronszajn tree then } \mathbb{Q}'_\alpha \text{ is a } \mathbb{P}_\alpha\text{-name for } \mathbb{Q}_{b(\alpha)}, \text{ otherwise } \mathbb{Q}'_\alpha = \{1\}\text{"},$$

(3) for $\alpha \le \omega_2$,

$$\mathbb{P}_\alpha = \{f : \mathrm{supp}(f) \subseteq \alpha, \mathrm{supp}(f) \text{ is countable and } \forall \beta \in \mathrm{supp}(f)(f(\beta) \text{ is a canonical } \mathbb{P}_\beta\text{-name and } \mathbb{P}_\beta \Vdash f(\beta) \in \mathbb{Q}'_\beta)\},$$

(4) $\forall \alpha \le \omega_2$, $\mathbb{P}_\alpha$ is ${}^\omega\omega$-bounding (and hence proper, see Def. 4.1).

For carrying the induction we use the following important preservation properties:

Under CH for an Aronszajn tree in the sense of Definition 1.1 the forcing order $\mathbb{Q}_{\mathbf{T}}$ from Definition 3.3 is of size $\le \aleph_1^{\aleph_0} = \aleph_1$ and is a subset of $H(\aleph_1)$, so $\mathbb{Q}_{\mathbf{T}} \in H(\aleph_2)$. Here the cardinals $\aleph_1$ and $\aleph_2$ are taken in the ground model.

The countable support limit of proper forcings in proper by [18, Ch. III, Theorem 3.1]. Hence for any $\gamma \le \omega_2$, $\aleph_1^{\mathbf{V}[G_\gamma]} = \aleph_1$.

We cite [18, Ch. III, Theorem 4.1]: Assume that κ is regular and uncountable and that $(\forall \alpha < \kappa)(\alpha^{\aleph_0} < \kappa)$. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \kappa, \alpha \le \kappa \rangle$ be a countable support iteration such that for any $\alpha < \kappa$, $\mathbb{P}_\alpha \Vdash |\mathbb{Q}_\alpha| < \kappa$. Then $\mathbb{P}_\kappa$ has the κ-c.c. and for any $\alpha < \kappa$, $\mathbb{P}_\alpha$ has a dense subset of size $< \kappa$ and $\mathbb{P}_\alpha \Vdash 2^{\aleph_0} < \kappa$. We apply this theorem under CH with $\kappa = \aleph_2$ (from the ground model) and thus get $\aleph_2^{\mathbf{V}[G]} = \aleph_2$ and that at any stage $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash$ CH.

The preservation of $\aleph_1$ and of $\aleph_2$ yields: At any stage $\alpha < \omega_2$, ω_1 in the sense of the stage is the ω_1 from the ground model. If $\mathbb{P}_\alpha \in H(\aleph_2)$, then also any canonical $\mathbb{P}_\alpha$-name of a a condition in $\mathbb{Q}_{\mathbf{T}}$ for an Aronszajn tree $\mathbf{T}$ is an element of $H(\aleph_2)$. Hence according to clause (3) of the recursive definition, $\mathbb{P}_{\alpha+1} \subseteq H(\aleph_2)$. Since the iterand $\mathbb{Q}'_\alpha$ is forced to be an element of $H(\aleph_2)$, by the CH there are few canonical names for elements of the iterand $\mathbb{Q}'_\alpha$. Thus by (3) we get $|\mathbb{P}_{\alpha+1}| < \aleph_2$ and $\mathbb{P}_{\alpha+1} \in H(\aleph_2)$. For limit steps $\alpha < \omega_2$, if $\mathbb{P}_\beta \in H(\aleph_2)$ for $\beta < \alpha$, then by the CH again there are few functions with countable support and hence by (3), $\mathbb{P}_\alpha \in H(\aleph_2)$.

Let $\mathbb{P}$ be a countable support iteration an for for any $i < \kappa$, $\mathbb{P}_i \Vdash$ "$\mathbb{Q}_i$ is ${}^\omega\omega$-bounding". Then by [18, Ch. VI, Section 4] also the limit $\mathbb{P}_\kappa$ is ${}^\omega\omega$-bounding.

Hence we know that an iterated forcing $\mathbb{P}_{\omega_2}$ with properties (1) to (4) exists. Fix any such $\mathbb{P}_{\omega_2}$. Let G be a $\mathbb{P}_{\omega_2}$-generic filter over $\mathbf{V}$. We let $G_\alpha = G \cap \mathbb{P}_\alpha$.

We prove that $\mathbb{P}_{\omega_2}$ forces that any Aronszajn is special. Let $\mathbf{T} = \sigma[G] \in \mathbf{V}[G]$ be an Aronszajn tree in the extension in the sense of Def. 1.1 and let $p \in \mathbb{P}_{\alpha_0} \cap G$ force this. Since $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{V}[G]}$ and since $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c., by [18, Ch. III, Theorem 4.1] there is an $\alpha_1 \in [\alpha_0, \omega_2)$ such that $\mathbf{T}$ has a canonical $\mathbb{P}_{\alpha_1}$-name τ and $p \Vdash_{\mathbb{P}_{\alpha_1}} \sigma = \tau$. Now for any $\gamma \ge \alpha_1$, $p \Vdash_{\mathbb{P}_\gamma}$ "τ is an Aronszajn tree", since this is forced by p in the forcing $\mathbb{P}_{\omega_2}$ and being an Aronszajn tree is downwards absolute if $\aleph_1$ is the same and $\mathbb{P}_\gamma$ is a complete suborder of $\mathbb{P}_{\omega_2}$. Moreover $\tau[G_{\alpha_1}] = \tau[G_\gamma]$ for any $\gamma \in [\alpha_1, \omega_2]$. Since for any $\gamma \in \omega_2$, $b \restriction [\gamma, \omega_2)$ is surjective onto $H(\aleph_2)$ and since canonical $\mathbb{P}_\alpha$-names for subsets of $\omega_1 \times \omega_1$

are elements of $H(\aleph_2)$, by property (2) of $\mathbb{P}_{\omega_2}$ there is some $\alpha \geq \alpha_1$ such that $b(\alpha) = \tau$ (so really the same names) and

$$p \Vdash_{\mathbb{P}_\alpha} b(\alpha) = \tau \text{ is an Aronszajn tree and } \mathbb{Q}'_\alpha = \mathbb{Q}_\tau.$$

Now we use again Shelah's result on canonical names: Each $f(\alpha)$ that is forced to be an element of $\mathbb{Q}_{b(\alpha)}$ has a canonical $\mathbb{P}_\alpha$-name. This ensures that p forces that $\mathbb{P}_{\alpha+1}$ that is defined according to (3) is equivalent to forcing with $\mathbb{Q}_{b(\alpha)}$ over $\mathbf{V}^{\mathbb{P}_\alpha}$. By the choice of the iterand $\mathbb{Q}'_\alpha = \mathbb{Q}_{b(\alpha)}$ under the condition p, by Conclusion 4.10 the same condition $p \in \mathbb{P}_{\alpha+1} \cap G$ forces in $\mathbb{P}_{\alpha+1}$ that τ is special via the generic specialisation function $f_{\mathbb{Q}'_\alpha}$ from Corollary 3.14. The property "$f_{\mathbb{Q}'_\alpha}[G_{\alpha+1}]$ specialises $\tau[G_{\alpha_1}]$" is upwards absolute from $\mathbf{V}[G_{\alpha+1}]$ to $\mathbf{V}[G]$. Hence it holds also in $\mathbf{V}[G]$. This concludes the proof that $\mathbb{P}_{\omega_2}$ forces that any Aronszajn tree is special.

In any extension by an ${}^\omega\omega$-bounding (and hence proper) forcing the set ${}^\omega\omega \cap \mathbf{V}$ stays a dominating family and therefore $\mathfrak{d} \leq |({}^\omega\omega) \cap \mathbf{V}|^{\mathbf{V}[G]} = \omega_1$. Of course $\mathfrak{d}$ is uncountable, hence $\mathfrak{d} = \aleph_1$ in $\mathbf{V}[G]$.

In Theorem 5.5 we prove that for each $\mathbf{q} \in K_1$, $\mathbb{P}_\mathbf{q} \Vdash \operatorname{unif}(\mathcal{M}) = \aleph_2$. The forcing $\mathbb{P}_{\omega_2}$ given by the above definition is in K_1. Hence in $\mathbf{V}[G]$, $2^\omega \geq \aleph_2$. By the already mentioned theorem [18, Ch. III, Theorem 4.1], $\mathbb{P}_{\omega_2} \Vdash 2^\omega \leq \aleph_2$. $\dashv$

5. $\mathbb{Q}_\mathbf{T}$ makes the ground model reals meagre

Let the set of reals $\mathbb{R}$ carry the usual order topology. A subset $A \subseteq \mathbb{R}$ is called meagre if it is the union of countably many nowhere dense sets. The uniformity of the ideal of meagre sets is defined as

$$\operatorname{unif}(\mathcal{M}) = \min\{|A| \,:\, A \subseteq \mathbb{R}, A \text{ is not meagre}\}.$$

Moore, Hrušák and Džamonja [12] showed that $\diamondsuit(\mathbb{R}, \mathcal{M}, \not\in)$ — a strengthening of $\operatorname{unif}(\mathcal{M}) = \aleph_1$, which says

$$(\forall \text{ Borel } F\colon {}^{\omega_1>}2 \to \text{ meagre } F_\sigma)(\exists \langle g_\delta \,:\, \delta \in \omega_1, \delta \text{ limit}\rangle)$$
$$\forall x \in {}^{\omega_1}2 \{\alpha \in \omega_1 \,:\, g_\alpha \not\in F(x \restriction \alpha)\} \text{ is stationary.}$$

— implies that there is a Souslin tree. A function $F\colon {}^{<\omega_1} \to$ meagre F_σ is called Borel if for each infinite countable α the layer $F \restriction 2^\alpha$ is Borel in the natural topologies on 2^α and the set of F_σ sets.

We assume $2^{\aleph_1} = \aleph_2$ and we let $\mathbb{P}_{\omega_2}$ be a countable support iteration of $\mathbb{Q}_{\mathbf{T}_\alpha}$, with a suitable bookkeeping so that for each $\beta < \omega_2$, each $\mathbb{P}_\beta$ name of an Aronszajn tree is named after stage β. Our forcing $\mathbb{P}_{\omega_2}$ is fairly definable, hence the proofs sketched in [12] support the conjecture: If $\mathbb{P}_{\omega_2}$ forced $\operatorname{unif}(\mathcal{M}) = \aleph_1$ then it would also force $\diamondsuit(\mathbb{R}, \mathcal{M}, \not\in)$. Here we show that $\mathbb{P}$ indeed forces $\operatorname{unif}(\mathcal{M}) = \aleph_2$.

We assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$.

Definition 5.1. Let $\mathbf{T}$ be a standard Aronszajn tree $\mathbf{T}$ and $p \in \mathbb{Q}_\mathbf{T}$.

(1) We say p is *diverse* if for any $s \in T^p$ for any $t_1 \neq t_2 \in \operatorname{suc}_{T^p}(s)$, the partial specialisations $\nu_i = \operatorname{last}(t_i)$ are contradictory, that means $(\exists \gamma_1 \in$

$\mathrm{dom}(\nu_1))(\exists\gamma_2 \in \mathrm{dom}(\nu_2))((\gamma_1 <_{\mathbf{T}} \gamma_2 \wedge \nu_1(\gamma_1) = \nu_2(\gamma_2)) \vee (\gamma_2 <_{\mathbf{T}} \gamma_1 \wedge \nu_1(\gamma_1) = \nu_2(\gamma_2) \vee (\gamma_1 = \gamma_2 \wedge \nu_1(\gamma_1) \neq \nu_2(\gamma_2))$.

(2) A condition $p \in \mathbb{Q}_{\mathbf{T}}$ is called *weakly diverse* if for any $s \in T^p$ there is some $h \in \omega$ such that for any $t_1 \neq t_2 \in \mathrm{suc}_{T^p}(s)$ for any extensions t_1^* of t_1 and t_2^* of t_2 to level $\lg(s)+1+h$ we have $\mathrm{last}(t_1^*)$ and $\mathrm{last}(t_2^*)$ are contradictory.

Lemma 5.2. *For a standard Aronszajn tree* $\mathbf{T}$ *the following hold:*

(1) There is a diverse $p \in \mathbb{Q}_{\mathbf{T}}$.

(2) We assume

(a) $p \in \mathbb{Q}_{\mathbf{T}}$ *is diverse.*

(b) $p \Vdash$ *“ there is a unique branch* $\langle t_i : i \in [i(p),\omega)\rangle$ *of* T^p *such that* $\bigcup\{\mathrm{last}(t_i) : i \in [i(p),\omega)\} \subseteq \bigcup\{\mathrm{last}(\mathrm{rt}(r)) : r \in G_{\mathbb{Q}_{\mathbf{T}}}\}$.

(c) For $s \in T^p$ *the sequence* $\langle t_{s,\ell} : \ell \in \mathrm{pos}(\mathbf{c}_{p,s})\rangle$ *lists* $\mathrm{suc}_{T^p}(s)$.

(d) The $\mathbb{Q}_{\mathbf{T}}$*-name* $\underset{\sim}{\varrho}$ *is a name for an element of* ${}^\omega\omega$ *such that* $i \geq i(p) \to t_{i+1} = t_{i,\varrho(i)}$, *and* $\varrho(i) = 0$ *for* $i < i(p)$.

Under the assumptions (a) to (d) we have $p \Vdash_{\mathbb{Q}_{\mathbf{T}}} \underset{\sim}{\varrho} \in {}^\omega\omega$ *is eventually different from any* $\eta \in ({}^\omega\omega)^{\mathbf{V}}$.

(3) The set of weakly diverse $p \in \mathbb{Q}_{\mathbf{T}}$ *is dense (not used).*

(4) Similarly to (2) for any weakly diverse p *there is a list of infinitely many levels and a name for a eventually different real.*

Proof. (1) The condition given in Lemma 3.4 is diverse. (2) Let $\eta \in {}^\omega\omega \cap \mathbf{V}$. We let for $n \geq i(p)$, $D_{\eta,n} = \{q \geq p : (\forall i \geq n)(\forall s \in q^{[i]})(t_{s,\eta(i)} \notin \mathrm{suc}_{T^q}(s)\}$. It is easy to see that $\bigcup_{n\geq i} D_n$ is dense above p. And any $q \in D_n$ forces that $\underset{\sim}{\varrho}(i) \neq \eta(i)$ for $i \geq n$. ⊣

Lemma 5.3. *Let* $p \in \mathbb{Q}_{\mathbf{T}}$ *be diverse. Then* $p \Vdash ({}^\omega 2)^{\mathbf{V}}$ *is meagre.*

Proof. The conditions p forces that the generic real $\underset{\sim}{\varrho}$ that is constructed from p and an enumeration as in (2)(c) of the previous lemma is eventually different from any real in the ground model. By [2, Theorem 2.4.7] a forcing makes the ground model reals meagre iff it adds an eventually different real. ⊣

Definition 5.4. K_1 is the class of countable support iterations $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \omega_2, \alpha \leq \omega_2\rangle$ and $\mathbb{Q}_\beta = \mathbb{Q}_{\underset{\sim}{\mathbf{T}}_\beta}$ and $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\omega_2}$ where $\underset{\sim}{\mathbf{T}}_\beta$ is a $\mathbb{P}_\beta$-name of a standard Aronszajn tree (as in Def.1.1) and for every $\alpha < \omega_2$ and $\mathbb{P}_\alpha$-name $\underset{\sim}{\mathbf{T}}$ of a standard Aronszajn tree there is some $\beta \in [\alpha,\omega_2)$ such that $\mathbb{P}_\beta \Vdash$ “$\mathbf{T}_\beta = \underset{\sim}{\mathbf{T}}$ if $\mathbf{T}$ is an Aronszajn tree”. (Note that $\mathbb{P}_\beta/\mathbb{P}_\alpha$ may add an ω_1-branch to $\underset{\sim}{\mathbf{T}}$.)

Theorem 5.5. *If* $\mathbf{q} \in K_1$ *then* $\mathbb{P}_{\mathbf{q}} \Vdash \mathrm{unif}(\mathcal{M}) = \aleph_2$.

Proof. It is enough to prove for $\alpha < \omega_2$ that $\mathbb{P}_q \Vdash$ “$({}^\omega 2)^{\mathbf{V}[\mathbb{P}_\alpha]}$ is meagre”. Let $p \in \mathbb{P}_{\mathbf{q}}$. Then there is $\beta < \omega_2$, $\beta = \alpha + i \notin \mathrm{dom}(p)$ for some $i < \omega_1$. We let $q = p \cup \{(\beta, p_*)\}$ and $q \restriction \beta \Vdash p_* = p(\beta)$ is a diverse condition. So by [2], $q \Vdash_{\mathbb{P}_{\mathbf{q}}}$ “$({}^\omega 2)^{\mathbf{V}[\mathbb{P}_\beta]}$ is meagre”. Then also $q \Vdash_{\mathbb{P}_{\mathbf{q}}}$ “$({}^\omega 2)^{\mathbf{V}[\mathbb{P}_\alpha]}$ is meagre”. As $\alpha < \omega_2$

and $p \in \mathbb{P}_{\mathbf{q}}$ were arbitrary we are done. $\dashv$

This concludes the proof of Theorem 1.4. $\dashv_{1.4}$

Remark 5.6. Except for the work on the halving property, all other technical steps can be performed with simple creatures, because there we never changed the value $k(\mathbf{c})$ of a creature in a condition when strengthening a condition according to the demands. So Theorem 1.4 can be proved with a slightly simpler relative of $\mathbb{Q}_{\mathbf{T}}$ in which the nodes in the conditions p are described by simple creatures.

References

[1] Uri Abraham. Proper forcing. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.
[2] Tomek Bartoszyński and Haim Judah. *Set Theory, On the Structure of the Real Line*. A K Peters, 1995.
[3] Arthur Fischer, Martin Goldstern, Jakob Kellner, and Saharon Shelah. Creature forcing and five cardinal characteristics of the continuum. *Archive for Mathematical Logic*, accepted; Baumgartner issue. arxiv:1402.0367.
[4] Martin Goldstern, Diego Mejia, and Saharon Shelah. The left side of Cichoń's Diagram. *Proceedings of the AMS*, accepted. arxiv:math.LO/1504.04192.
[5] James Hirschorn. Random trees under CH. *Israel J. Math.*, 157:123–154, 2007.
[6] Thomas Jech. *Set Theory*. Addison Wesley, 1978.
[7] Thomas Jech. *Set Theory. The Third Millenium Edition, revised and expanded*. Springer, 2003.
[8] Jakob Kellner and Saharon Shelah. Decisive creatures and large continuum. *J. Symbolic Logic*, 74:73–104, 2009.
[9] Jakob Kellner and Saharon Shelah. Creature forcing and large continuum: the joy of halving. *Arch. Math. Logic*, 51(1-2):49–70, 2012.
[10] Duro Kurepa. Ensembles ordonnés et ramifiés. *Publ. Math. Univ. Belgrade*, 4:1–138, 1935.
[11] Heike Mildenberger and Saharon Shelah. Specializing Aronszajn trees by countable approximations. *Archive for Math. Logic*, 42:627 –648, 2003.
[12] Justin Tatch Moore, Michael Hrušák, and Mirna Džamonja. Parametrized $\diamondsuit$ principles. *Trans. Amer. Math. Soc.*, 356:2281–2306, 2004.
[13] Adam J. Ostaszewski. A perfectly normal countably compact scattered space which is not strongly zero-dimensional. *J. London Math. Soc. (2)*, 14(1):167–177, 1976.
[14] Andrzej Rosłanowski and Saharon Shelah. Norms on possibilities II, more ccc ideals on 2^ω. *J. Appl. Anal.*, 3, 1997.
[15] Andrzej Rosłanowski and Saharon Shelah. *Norms on Possibilities I: Forcing with Trees and Creatures*, volume 141 (no. 671) of *Memoirs of the American Mathematical Society*. AMS, 1999.
[16] Andrzej Rosłanowski and Saharon Shelah. Measured creatures. *Israel J. Math.*, 151:61–110, 2006.
[17] Andrzej Rosłanowski, Saharon Shelah, and Otmar Spinas. Nonproper products. *Bull. Lond. Math. Soc.*, 44(2):299–310, 2012.
[18] Saharon Shelah. *Proper and Improper Forcing, 2nd Edition*. Springer, 1998.
[19] Robert Solovay and Stanley Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. Math.*, 94:201–245, 1971.

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