AUTOMORPHISM GROUPS OF COUNTABLE STABLE STRUCTURES

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ABSTRACT. For every countable structure M we construct an $\aleph_0$-stable countable structure N such that $Aut(M)$ and $Aut(N)$ are topologically isomorphic. This shows that it is impossible to detect any form of stability of a countable structure M from the topological properties of the Polish group $Aut(M)$.

1. Introduction

In [1] Rosendal isolates a property of topological groups which he calls (local) (OB) and proves that if M is the countable, saturated model of an $\aleph_0$-stable theory then $Aut(M)$ has (OB). Again in [1], Rosendal asks if the property local (OB) is satisfied by the group of automorphisms of any countable model of an $\aleph_0$-stable theory. In [2] Zielinski answers this question in the negative by exhibiting a countable model of an $\aleph_0$-stable theory whose group of automorphisms is *not* locally (OB).

In the present study we show that any attempt at a topological characterization of the group of automorphisms of a countable stable structure is doomed to fail:

Theorem 1. *For every countable*[1] *structure M there exists an $\aleph_0$-stable countable structure N such that $Aut(M)$ and $Aut(N)$ are topologically isomorphic with respect to the naturally associated Polish group topologies.*

The theory $Th(N)$ of Theorem 1 is NDOP and NOTOP, but this will not be proved here, since it appears to be outside of the scope of this study.

2. Proofs

The main technical tool in the proof of Theorem 1 will be a new notion of interpretability, which we call $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretability. To make the exposition complete we first introduce the classical notion of first-order interpretability (cf. Definition 2), and then define the notion of $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretability (cf. Definition 4).

Definition 2. *Let M and N be models. We say that N is interpretable in M if for some $n < \omega$ there are:*

(1) a $\emptyset$-definable subset D of M^n;

(2) a $\emptyset$-definable equivalence relation on D;

(3) a bijection $\alpha : N \to D/E$ such that for every $m < \omega$ and $\emptyset$-definable subset R of N^m the subset of M^{nm} given by:

$$\hat{R} = \{(\bar{a}_1, ..., \bar{a}_m) \in (M^n)^m : (\alpha^{-1}(\bar{a}_1/E), ..., \alpha^{-1}(\bar{a}_m/E)) \in R\}$$

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[1]In the present paper we consider only structures in a countable language.

is $\emptyset$-definable in M.

Notation 3. *Let τ be a language.*

(1) *For $R \in \tau$ a predicate, we denote by $k(R) = k(R, \tau)$ the arity of R.*
(2) *Given a τ-structure M and a τ-formula $\varphi(\bar{x}) = \varphi(x_0, ..., x_{n-1})$, we let:*

$$\varphi(M) = \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}.$$

(3) *Given a τ-structure M, we denote by $|M|$ the domain of M (although we will be sloppy in distinguishing between the two), and by $||M||$ the cardinality of M.*
(4) *Given a τ-structure M and $A \subseteq M$, we denote by $Aut(M/A)$ the set of automorphisms of M which are the identity on A.*

Definition 4. *Let:*

(i) *τ_ℓ ($\ell = 1, 2$) be relational languages (for simplicity);*
(ii) *$\Delta_{M_\ell} = \Delta_\ell \subseteq \mathfrak{L}_{\omega_1,\omega}(\tau_\ell)$ ($\ell = 1, 2$) be sets of formulas;*
(iii) *$\Delta_2 = \{\varphi \in \mathfrak{L}_{\omega_1,\omega}(\tau_2) : \varphi$ is an atomic τ_2-formula$\}$;*
(iv) *M_ℓ ($\ell = 1, 2$) be τ_ℓ-structures.*

We say that M_2 is Δ_1-interpretable in M_1 by the scheme $\mathfrak{s}$ and function $\bar{F}$ when:

(A) *$\mathfrak{s} = \{\mathfrak{s}(p) : p \in \mathfrak{S}_{M_2}\} \cup \{\mathfrak{s}(R, \bar{p}) : R \in \tau_2, \bar{p} = (p_\ell : \ell < k) \in \mathfrak{S}^{k(R)}_{M_2}\}$, where:*
  (a) *$p \in \mathfrak{S}_{M_2} = \{tp_{\Delta_2}(a, \emptyset, M_2) : a \in M_2\}$;*
  (b) *$\mathfrak{s}(p) = (r_p(\bar{x}_{m(p)}), E_p(\bar{y}_{m(p)}, \bar{z}_{m(p)})) \in \Delta_1 \times \Delta_1$, $m(p) < \omega$, and $E_p(M_1)$ is a non-empty equivalence relation on $r_p(M_1)$;*
  (c) *$\mathfrak{s}(R, \bar{p})$ is a τ_1-formula from Δ_1 of the form $\varphi_{(R,\bar{p})}(\bar{x}^0_{m(p_0)}, ..., \bar{x}^{k-1}_{m(p_{k-1})})$, with $\bar{x}^i_{m(p_i)} = (x^i_0, ..., x^i_{m(p_i)-1})$, for every $i < k$;*

(B) *$\bar{F} = (F_p : p \in \mathfrak{S}_{M_2})$, where:*
  (a) *F_p is a one-to-one function from $p(M_2) = \{a \in M_2 : p = tp_{\Delta_2}(a, \emptyset, M_2)\}$ onto $r_p(M_1)/E_p(M_1)$;*
  (b) *for every predicate R of τ_2 we have: if $k = k(R)$, $\bar{a} \in M_2^k$, and, for every $\ell < k$, $p_\ell = tp_{\Delta_2}(a_\ell, \emptyset, M_2)$, $\bar{b}_\ell \in r_{p_\ell}(M_1)$ and $F_{p_\ell}(a_\ell) = \bar{b}_\ell/E_{p_\ell}(M_1)$, then:*

$$M_2 \models R(a_0, ..., a_{k-1}) \text{ iff } M_1 \models \varphi_{(R,\bar{p})}(\bar{b}_0, ..., \bar{b}_{k-1}).$$

Finally, we say that M_2 is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretable in M_1 when M_2 is Δ_1-interpretable in M_1 by the scheme $\mathfrak{s}$ and function $\bar{F}$ for some Δ_1, $\mathfrak{s}$ and $\bar{F}$.

Fact 5. *Let M and N be models, and suppose that N is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretable in M. Then every $\pi \in Aut(M)$ induces a $\hat{\pi} \in Aut(N)$, and the mapping $\pi \mapsto \hat{\pi}$ is a continuous homomorphism of $Aut(M)$ into $Aut(N)$.*

Proof. Essentially as in the case of first-order interpretability (cf. Definition 2). ■

Fact 6. *Let G and H be Polish group and $\alpha : G \to H$ a group isomorphism. If α is continuous, then α is a topological isomorphism.*

Proof. This is well-known. ■

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let M be a countable model. We construct a countable model N such that:

(1) N is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretable in M (cf. Definition 4);

(2) for every $\pi \in Aut(N)$ there is a unique $\pi_0 \in Aut(M)$ such that $\pi = \hat{\pi}_0$ (cf. Fact 5);
(3) N is $\aleph_0$-stable.

Using Facts 5 and 6, and items (1)-(2) above it follows that $Aut(M)$ and $Aut(N)$ are topologically isomorphic, and thus by (3) we are done.
We then proceed to the construction of a model N as above. First all notice that without loss of generality[2] we can assume that M is a relational structure in a language $\tau(M) = \{P_{(n,\ell)} : n < n_* \leqslant \omega, \ell < \ell_n \leqslant \omega\}$, where the predicates $P_{(n,\ell)}$ are n-ary predicates, and, for transparency, we assume that if $M \models P_{(n,\ell)}(\bar{a})$, then $\bar{a}$ is without repetitions. We construct a structure N in the following language $\tau(N)$:

(i) $c \in \tau(N)$ is a constant;
(ii) $P \in \tau(N)$ is a unary predicate;
(iii) for $n < n_* \leqslant \omega$ and $\ell < \ell_n \leqslant \omega$, $Q_{(n,\ell)} \in \tau(N)$ is a unary predicate;
(iv) for $n < n_* \leqslant \omega$ and $\ell < \ell_n \leqslant \omega$, $E_{(n,\ell)} \in \tau(N)$ is a binary predicate;
(v) for $n < n_* \leqslant \omega$, $\ell < \ell_n \leqslant \omega$ and $\iota < n$, $F_{(n,\ell,\iota)} \in \tau(N)$ is a unary function;
(vi) for $n < n_* \leqslant \omega$, $\ell < \ell_n \leqslant \omega$ and $j < \omega$, $G_{(n,\ell,j)} \in \tau(N)$ is a unary function.

We define the structure N as follows:

(a) $|N|$ (the domain of N) is the disjoint union of:
$$P^N \cup \{c^N = e\} \cup \{Q^N_{(n,\ell)} : n < n_* \leqslant \omega \text{ and } \ell < \ell_n \leqslant \omega\};$$
(b) $P^N = |M|$ (the domain of M);
(c) $Q^N_{(n,\ell)} = \{(n, \ell, i, a_0, ..., a_{n-1}) : a_t \in M, i \leqslant \omega, (a_0, ..., a_{n-1}) \notin P^M_{(n,\ell)} \Rightarrow i < \omega\}$;
(d) $E^N_{(n,\ell)} =$
$$\{((n, \ell, i_1, \bar{a}), (n, \ell, i_2, \bar{a})) : i_1, i_2 \leqslant \omega, (n, \ell, i_t, \bar{a} = a_0, ..., a_{n-1}) \in Q^N_{(n,\ell)}\};$$
(e) for $\iota < n$, $F_{(n,\ell,\iota)}(x) = \begin{cases} a_\iota & \text{if } x = (n, \ell, i, a_0, ..., a_{n-1}), \\ e & \text{otherwise;} \end{cases}$
(f) for $j < \omega$, $G_{(n,\ell,j)}(x) = \begin{cases} (n, \ell, j, a_0, ..., a_{n-1}) & \text{if } x = (n, \ell, i, a_0, ..., a_{n-1}), \\ e & \text{otherwise.} \end{cases}$

We now prove items (1)-(3) from the list at the beginning of the proof. Item (3) is proved in Claim 7. We prove item (2). Let $\pi \in Aut(N)$ and, for $a, b \in M$, let $\pi_0(a) = b$ iff $\pi(a) = b$. Clearly $\pi_0 \in Sym(M)$. For the sake of contradiction, suppose that $\pi_0 \notin Aut(M)$. Replacing π with π^{-1}, we can assume without loss of generality that there are $n < n_* \leqslant \omega$, $\ell < \ell_n \leqslant \omega$, $\bar{a} = (a_0, ..., a_{n-1}) \in M^n$ and $\bar{b} = (b_0, ..., b_{n-1}) \in M^n$ such that $\pi_0(\bar{a}) = \bar{b}$, $M \models P_{(n,\ell)}(\bar{a})$ and $M \not\models P_{(n,\ell)}(\bar{b})$. Then the element $(n, \ell, \omega, a_0, ..., a_{n-1}) \in N$ realizes the type:
$$p = \{F_{(n,\ell,\iota)}(x) = a_\iota : \iota < n\} \cup \{G_{(n,\ell,j)}(x) \neq x : j < \omega\},$$
while the type:
$$q = \{F_{(n,\ell,\iota)}(x) = b_\iota : \iota < n\} \cup \{G_{(n,\ell,j)}(x) \neq x : j < \omega\},$$
is not realized in N, a contradiction. Hence, $\pi_0 \in Aut(M)$ and, easily, $\pi = \hat{\pi}_0$ (cf. Fact 5) and for every $\pi_1 \in Aut(M)$ such that $\pi = \hat{\pi}_1$ we have that $\pi_0 = \pi_1$.

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[2]Recall that in this paper we only consider structures in a countable language.

Finally, we prove item (1). Let $(k_{(n,\ell,i)} : n < n_* \leqslant \omega, \ell < \ell_n \leqslant \omega, i \leqslant \omega)$ be a sequence of natural numbers such that:

$$(n_1, \ell_1, i_1) \neq (n_2, \ell_2, i_2) \text{ implies } 1 < n_1 + k_{(n_1,\ell_1,i_1)} \neq n_2 + k_{(n_2,\ell_2,i_2)}.$$

Let also:

(i') $n + k_{(n,\ell,i)} = m(n, \ell, i)$;
(ii') $\bar{x}_{m(n,\ell,i)} = (x_0, ..., x_{m(n,\ell,i)-1})$;
(iii') $\bar{y}_{m(n,\ell,i)} = (y_0, ..., y_{m(n,\ell,i)-1})$.

Consider now the following formulas:

(A) $\varphi_0(x_0) : x_0 = x_0$;
(B) $\theta_0(x_0, y_0) : x_0 = y_0$;
(C) for $n < n_* \leqslant \omega$, $\ell < \ell_n \leqslant \omega$ and $i < \omega$ let:

$$\varphi_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}) : \bigwedge_{m<m(n,\ell,i)} x_m = x_m,$$

$$\theta_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}, \bar{y}_{m(n,\ell,i)}) : \bigwedge_{m<n} x_m = y_m;$$

(D) for $n < n_* \leqslant \omega$, $\ell < \ell_n \leqslant \omega$ and $i = \omega$ let:

$$\varphi_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}) : \bigwedge_{m<m(n,\ell,i)} x_m = x_m \wedge P_{(n,\ell)}(x_0, ..., x_{n-1}),$$

$$\theta_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}, \bar{y}_{m(n,\ell,i)}) : \bigwedge_{m<n} x_m = y_m.$$

Notice now, that:

(I) $P^N = \varphi_0(M)/\theta_0(M)$;
(II) $Q^N_{n,\ell}$ is in bijection with $\bigcup\{\varphi_{(n,\ell,i)}(M)/\theta_{(n,\ell,i)}(M) : i \leqslant \omega\}$.

Using this observation it is easy to see how to choose Δ_M, $\mathfrak{s}$, and $\bar{F} = (F_p : p \in \mathfrak{S}_N)$ as in Definition 4 so as to witness that N is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretable in M. ■

Claim 7. *Let N be as in the proof of Theorem 1. Then $Th(N)$ is $\aleph_0$-stable.*

Proof. Let N_1 be a countable model of $Th(N)$. It is enough to show that there are only countably many 1-types over N_1. To this extent, let N_2 be an $\aleph_1$-saturated model of $Th(N)$ such that every countable non-algebraic type is realized by $||N_2||$-many elements, and define the following equivalence relation $E^* = E^*_{(N_1,N_2)}$ on N_2:

$$aE^*b \text{ iff } \exists \pi \in Aut(N_2/N_1) \text{ such that } \pi(a) = b.$$

We will show that the relation E^* has $\aleph_0$ equivalence classes, clearly this suffices. To this extent, notice that:

($\star_1$) if π is a permutation of P^{N_2} which is the identity on P^{N_1}, then there is an automorphism $\check{\pi}$ of N_2 over N_1 extending it (recall that N_2 is $\aleph_1$-saturated);

$(\star_2)_{(n,\ell)}$ if $b_1, b_2 \in E_{(n,\ell)}$, $(F_{n,\ell,\iota}(b_1) : \iota < n)$ and $(F_{n,\ell,\iota}(b_2) : \iota < n)$ realize the same $\{=\}$-type over P^{N_1}, and for $t = 1, 2$ we have $b_t \notin \{G_{(n,\ell,j)}(b_t) : j < \omega\}$, then there exists $\pi \in Aut(N_2/N_1)$ such that $\pi(b_1) = b_2$;

$(\star_3)_{(n,\ell,j)}$ if $b_1, b_2 \in E_{(n,\ell)}$, $(F_{n,\ell,\iota}(b_1) : \iota < n)$ and $(F_{n,\ell,\iota}(b_2) : \iota < n)$ realize the same $\{=\}$-type over P^{N_1}, and for $t = 1, 2$ we have $G_{(n,\ell,j)}(b_t) = b_t$, then there exists $\pi \in Aut(N_2/N_1)$ such that $\pi(b_1) = b_2$.

Now, using $(\star_1)$-$(\star_2)_{(n,\ell)}$-$(\star_3)_{(n,\ell,j)}$ and noticing that n, ℓ and j range over countable sets, it is easy to see that the relation E^* defined above has $\aleph_0$ equivalence classes. ■

References

[1] Christian Rosendal. *Large Scale Geometry of Automorphism Groups*. Available on the ArXiv.

[2] Joseph Zielinski. *An Automorphism Group of an ω-Stable Structure that is not Locally (OB)*. Math. Log. Q. **62** (2016), No. 6, 547-551.

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