RECONSTRUCTING STRUCTURES WITH THE STRONG SMALL INDEX PROPERTY UP TO BI-DEFINABILITY

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Dedicated to the memory of Matti Rubin

Abstract. Let $\mathbf{K}$ be the class of countable structures M with the strong small index property and locally finite algebraicity, and $\mathbf{K}_*$ the class of $M \in \mathbf{K}$ such that $acl_M(\{a\}) = \{a\}$ for every $a \in M$. For homogeneous $M \in \mathbf{K}$, we introduce what we call the expanded group of automorphisms of M, and show that it is second-order definable in $Aut(M)$. We use this to prove that for $M, N \in \mathbf{K}_*$, $Aut(M)$ and $Aut(N)$ are isomorphic as abstract groups if and only if $(Aut(M), M)$ and $(Aut(N), N)$ are isomorphic as permutation groups. In particular, we deduce that for $\aleph_0$-categorical structures the combination of strong small index property and no algebraicity implies reconstruction up to bi-definability, in analogy with Rubin's well-known $\forall\exists$-interpretation technique of [7]. Finally, we show that every finite group can be realized as the outer automorphism group of $Aut(M)$ for some countable $\aleph_0$-categorical homogeneous structure M with the strong small index property and no algebraicity.

1. Introduction

Reconstruction theory deals with the problem of reconstruction of countable structures from their automorphism groups. The first degree of reconstruction that it is usually dealt with is the so-called *reconstruction up to bi-interpretability.* The second and stronger degree of reconstruction is known as *reconstruction up to bi-definability.* In group theoretic terms, the first degree of reconstruction corresponds to reconstruction of *topological group isomorphisms* from isomorphisms of abstract group, while the second degree of reconstruction corresponds to reconstruction of *permutation group isomorphisms* from isomorphisms of abstract group. Two independent techniques lead the scene in this field: the (strong) small index property (see e.g. [4]) and Rubin's $\forall\exists$-interpretation [7].

On the reconstruction up to up to bi-interpretability side the cornerstones of the theory are the following two results:

Theorem (Rubin [7]). *Let M and N be countable $\aleph_0$-categorical structures and suppose that M has a $\forall\exists$-interpretation. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-interpretable.*

Theorem (Lascar [5]). *Let M and N be countable $\aleph_0$-categorical structures and suppose that M has the small index property. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-interpretable.*

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On the reconstruction up to up to bi-definability side, all the known results are based on the following theorem of Rubin:

Theorem (Rubin [7]). *Let M and N be countable $\aleph_0$-categorical structures with no algebraicity and suppose that M has a $\forall\exists$-interpretation. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-definable.*

In particular, on the small index property side there is no result that pairs with the last cited result of Rubin. In this paper we fill this gap proving the following:

Theorem 1. *Let $\mathbf{K}_*$ be the class of countable structures M satisfying:*
(1) M has the strong small index property;
(2) for every finite $A \subseteq M$, $acl_M(A)$ is finite;
(3) for every $a \in M$, $acl_M(\{a\}) = \{a\}$;
Then for $M, N \in \mathbf{K}_*$, $Aut(M)$ and $Aut(N)$ are isomorphic as abstract groups if and only if $(Aut(M), M)$ and $(Aut(N), N)$ are isomorphic as permutation groups.

Thus deducing an analog of Rubin's result on reconstruction up to bi-definability:

Corollary 2. *Let M and N be countable $\aleph_0$-categorical structures with the strong small index property and no algebraicity. Then $\pi : Aut(M) \cong Aut(N)$ if and only if M and N are bi-definable. Furthermore, letting $f : M \to N$ witness bi-definability, the isomorphism $\pi : Aut(M) \cong Aut(N)$ is induced by f.*

For a structure M satisfying the conclusion of Theorem 1 it is easy to determine the outer automorphism group of $Aut(M)$, in fact any $f \in Aut(Aut(M))$ is induced by a permutation of M. For example, as already noted by Rubin in [7], using this fact it is easy to see that for R_n the n-coloured random graph ($n \geqslant 2$) we have that $Out(Aut(R_n)) \cong Sym(n)$. Similarly, but in a different direction, one easily sees that for M_n the K_n-free random graph ($n \geqslant 3$) we have that $Aut(M_n)$ is complete. We show here that in this setting any finite group can occur:

Theorem 3. *Let K be a finite group. Then there exists a countable $\aleph_0$-categorical homogeneous structure M with the strong small index property and no algebraicity such that $K \cong Out(Aut(M))$.*

Our main technical tool is what we call the *expanded group of automorphism* of an homogeneous structure M with the strong small index property and locally finite algebraicity. This powerful object encodes the combinatorics of $Aut(M)$-stabilizers of such a structure M, and it is a crucial ingredient of our proof of Theorem 1. In Theorem 12 we show that the expanded group of automorphism is second-order definable in $Aut(M)$.

2. The Expanded Group of Automorphisms

In this section we introduce the expanded group of automorphisms of M (for certain M), and show that it is second-order definable in $Aut(M)$.

Given a structure M and $A \subseteq M$, and considering $Aut(M) = G$ in its natural action on M, we denote the pointwise (resp. setwise) stabilizer of A under this action by $G_{(A)}$ (resp. $G_{\{A\}}$). Also, we denote the subgroup relation by $\leqslant$.

Definition 4. *Let M be a structure and $G = Aut(M)$.*
(1) We say that a is algebraic (resp. definable) over $A \subseteq M$ in M if the orbit of a under $G_{(A)}$ is finite (resp. trivial).

(2) The algebraic closure *of $A \subseteq M$ in M, denoted as $acl_M(A)$, is the set of elements of M which are algebraic over A.*
(3) The definable closure *of $A \subseteq M$ in M, denoted as $dcl_M(A)$, is the set of elements of M which are definable over A.*

Definition 5. *Let M be a countable structure and $G = Aut(M)$.*
(1) We say that M (or G) has the small index property *(SIP) if every subgroup of $Aut(M)$ of index less than 2^ω contains the pointwise stabilizer of a finite set $A \subseteq M$.*
(2) We say that M (or G) has the strong small index property *(SSIP) if every subgroup of $Aut(M)$ of index less than 2^ω lies between the pointwise and the setwise stabilizer of a finite set $A \subseteq M$.*

Hypothesis 6. *Throughout this section, let M be a countable homogeneous structure with the strong small index property and locally finite algebraicity, i.e. for every finite $A \subseteq M$ we have $|acl_M(A)| < \omega$.*

We denote by $\mathbf{A}(M) = \{acl_M(B) : B \subseteq_{fin} M\}$, and by $\mathbf{EA}(M) = \{(K, L) : K \in \mathbf{A}(M) \text{ and } L \leqslant Aut(K)\}$.
Let $(K, L) \in \mathbf{EA}(M)$, we define:

$$G_{(K,L)} = \{f \in Aut(M) : f \restriction K \in L\}.$$

Notice that if $L = \{id_K\}$, then $G_{(K,L)} = G_{(K)}$, i.e. it equals the pointwise stabilizer of K, and that if $L = Aut(K)$, then $G_{(K,L)} = G_{\{K\}}$, i.e. it equals the setwise stabilizer of K. We then let:

$$\mathcal{PS}(M) = \{G_{(K)} : K \in \mathbf{A}(M)\} \text{ and } \mathcal{SS}(M) = \{G_{(K,L)} : (K, L) \in \mathbf{EA}(M)\}.$$

The crucial point is the following:

Lemma 7. *Let $\mathcal{G} = \{H \leqslant G : [G : H] < 2^\omega\}$. Then $\mathcal{G} = \mathcal{SS}(M)$.*

Proof. The containment from right to left is trivial. Let then $H \leqslant G$ with $[G : H] < 2^\omega$. By the strong small index property, there is finite $K \subseteq M$ such that $G_{(K)} \leqslant H \leqslant G_{\{K\}}$. It follows that $G_{(acl_M(K))} \leqslant H \leqslant G_{\{acl_M(K)\}}$, and so without loss of generality we can assume that $K \in \mathbf{A}(M)$. First of all we claim that $G_{(K)} \trianglelefteqslant G_{\{K\}}$. In fact, for $g \in G_{\{K\}}$, $h \in G_{(K)}$ and $a \in K$, we have $ghg^{-1}(a) = gg^{-1}(a) = a$, since $g^{-1}(a) \in K$ and $h \in G_{(K)}$. Furthermore, for $g, h \in G_{\{K\}}$, we have $g^{-1}h \in G_{(K)}$ iff $g \restriction K = h \restriction K$. Hence, the map $f : gG_{(K)} \mapsto g \restriction K$, for $g \in G_{\{K\}}$, is such that:

$$f : G_{\{K\}}/G_{(K)} \cong Aut(K),$$

since every $f \in Aut(K)$ extends to an automorphism of M. Thus, by the fourth isomorphism theorem we have $H = G_{(K,L)}$ for $L = \{f \restriction K : f \in H\}$. ■

Proposition 8. *Let $H_1, H_2 \in \mathcal{SS}(M)$. The following conditions are equivalent:*
(1) $H_1 \trianglelefteqslant H_2$ and $[H_2, H_1] < \omega$;
(2) there is $K \in \mathbf{A}(M)$ and $L_1 \trianglelefteqslant L_2 \leqslant Aut(K)$ such that $H_i = G_{(K,L_i)}$ for $i = 1, 2$.

Proof. The proof of (2) implies (1) is immediate, since by the normality of L_1 in L_2 we have that, for $g \in G_{(K,L_2)}$ and $h \in G_{(K,L_1)}$, $ghg^{-1} \restriction K \in L_1$, while the fact that $[H_2, H_1] < \omega$ follows from the proof of Lemma 7. We show that (1) implies (2). By assumption, $H_i = G_{(K_i,L_i)}$ for $(K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$.
$(*)_1$ $K_2 \subseteq K_1$.

Suppose not, and let $a \in K_2 - K_1$ witness this. Then we can find $f \in G$ such that $f \restriction K_1 = id_{K_1}$ and $f(a) \notin K_2$. It follows that $f \in H_1 - H_2$, a contradiction.

$(*)_2$ $K_1 \subseteq K_2$.

Suppose not, and let $f_n \in G$, for $n < \omega$, such that $f_n \restriction K_2 = id_{K_2}$, and in addition $\{f_n(K_1 - K_2) : n < \omega\}$ are pairwise disjoint. Then clearly, for every $n < \omega$, $f_n \in H_2$ and $\{f_n H_1 : n < \omega\}$ are distinct, contradicting the assumption $[H_2, H_1] < \omega$.

$(*)_3$ $L_1 \leqslant L_2$.

Suppose not, and let $h \in L_1 - L_2$. Then h extends to an automorphism f of M. Clearly $f \in H_1 - H_2$, a contradiction.

$(*)_4$ $L_1 \trianglelefteq L_2$.

Suppose not, and let $g_i \in L_i$ $(i = 1, 2)$ be such that $g_2 g_1 g_2^{-1} \notin L_1$. Then g_i extends to an automorphism f_i of M $(i = 1, 2)$. Clearly $f_i \in H_i$ $(i = 1, 2)$, and $f_2 f_1 f_2^{-1} \notin H_1$, a contradiction. ■

Proposition 9. *Let $\mathcal{G} = \{H \in \mathcal{SS}(M) :$ there is no $H' \in \mathcal{SS}(M)$, with $H' \subsetneq H, H' \trianglelefteq H$ and $[H, H'] < \omega\}$. Then $\mathcal{PS}(M) = \mathcal{G}$.*

Proof. First we show the containment from left to right. Let $H_2 \in \mathcal{PS}(M)$ and assume that there exists $H_1 \in \mathcal{SS}(M)$ such that $H_1 \subsetneq H_2, H_1 \trianglelefteq H_2$ and $[H_2, H_1] < \omega$. By Proposition 8, $H_i = G_{(K_i, L_i)}$ for $(K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$ and $K_1 = K = K_2$. Now, as $H_2 \in \mathcal{PS}(M)$, $L_2 = \{id_K\}$. Hence, $L_1 = L_2$, and so $H_1 = H_2$, a contradiction. We now show the containment from right to left. Let $H \in \mathcal{G}$, then $H = G_{(K,L)}$ for $(K, L) \in \mathbf{EA}(M)$. If $L \neq \{id_K\}$ then letting $H' = G_{(K, \{id_K\})}$ we have $H' \subsetneq H$, $H' \trianglelefteq H$ and $[H, H'] < \omega$, a contradiction. ■

Let $\mathbf{L}(M)$ be a set of finite groups such that for every $K \in \mathbf{A}(M)$ there is a unique $L \in \mathbf{L}(M)$ such that $L \cong Aut(K)$.

Proposition 10. *Let $L \in \mathbf{L}(M)$ and $H \in \mathcal{SS}(M)$. The following conditions are equivalent:*

(1) $H = G_{(K)} \in \mathcal{PS}(M)$ and $Aut(K) \cong L$;
(2) there is $H' \in \mathcal{SS}(M)$ such that $H \trianglelefteq H'$, $[H', H] < \omega$, H' is maximal under these conditions and $H'/H \cong L$.

Proof. This follows from the proof of Lemma 7 and Proposition 8. ■

Definition 11. *We define the structure $ExAut(M)$, the* expanded group of automorphisms of M*, as follows:*

(1) $ExAut(M)$ is a two-sorted structure;
(2) the first sort has set of elements $Aut(M) = G$;
(3) the second sort has set of elements $\mathbf{EA}(M)$;
(4) we identify $\{(K, id_K) : K \in \mathbf{A}(M)\}$ with $\mathbf{A}(M)$;
(5) the relations are:
  (a) $P_{\mathbf{A}(M)} = \{K \in \mathbf{A}(M)\}$ (recalling the above identification);
  (b) for $L \in \mathbf{L}(M)$, $P_{L(M)} = \{K \in \mathbf{A}(M) : Aut(K) \cong L\}$;
  (c) $P_{\mathbf{L}(M)} = \bigcup_{L \in \mathbf{L}(M)} P_{L(M)}$;
  (d) $\leqslant_{\mathbf{EA}(M)} = \{((K_1, L_1), (K_2, L_2)) : (K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$, $K_1 \leqslant K_2$ and $L_2 \restriction K_1 \leqslant L_1\}$;
  (e) $\leqslant_{\mathbf{A}(M)} = \{(K_1, K_2) : K_i \in \mathbf{A}(M)$ $(i = 1, 2)$ and $K_1 \leqslant K_2\}$;
  (f) $P^{min}_{\mathbf{A}(M)} = \{K \in \mathbf{A}(M) : acl(\emptyset) \neq K \in \mathbf{A}(M)$ is minimal in $(\mathbf{A}(M), \subseteq)\}$;

(6) the operations are:
(f) composition on $Aut(M)$;
(g) for $f \in Aut(M)$ and $K \in \mathbf{A}(M)$, $Op(f, K) = f(K)$;
(h) for $f \in Aut(M)$ and $(K_1, L_1) \in \mathbf{EA}(M)$, $Op(f, (K_1, L_1)) = (K_2, L_2)$ iff $f(K_1) = K_2$ and $L_2 = \{f \restriction K_1 \pi f^{-1} \restriction K_2 : \pi \in L_1\}$.

We say that a set of subsets of a structure N is second-order definable if it is preserved by automorphisms of N. We say that a structure M is second-order definable in a structure N if there is a injective map $\mathbf{j}$ mapping $\emptyset$-definable subsets of M to second-order definable set of subsets N.

Theorem 12. *(1) The map $\mathbf{j}_M = \mathbf{j} : (f, (K, L)) \mapsto (f, G_{(K,L)})$ witnesses second-order definability of $ExAut(M)$ in $Aut(M)$.*
(2) Every $F \in Aut(G)$ has an extension $\hat{F} \in Aut(ExAut(M))$.

Proof. We prove (1).

$(*)_1$ The map $(f, (K, L)) \mapsto (f, G_{(K,L)})$ is one-to-one.

Suppose that $(K_1, L_1) \neq (K_2, L_2) \in \mathbf{EA}(M)$, we want to show that $G_{(K_1,L_1)} \neq G_{(K_2,L_2)}$. Suppose that $K_1 \neq K_2$. By symmetry, we can assume that $K_1 \not\subseteq K_2$. Then there is $f \in G$ such that $f \restriction K_2 = id_{K_2}$ and $f(K_1) \cap K_1 = K_1 \cap K_2$. Thus, $f \in G_{(K_2,L_2)} - G_{(K_1,L_1)}$. Suppose now that $K_1 = K = K_2$ and $L_1 \neq L_2$. By symmetry, we can assume that $L_1 \not\subseteq L_2$. Let $g \in L_1 - L_2$, then g extends to an automorphism f of M. Thus, $f \in G_{(K,L_1)} - G_{(K,L_2)}$.

$(*)_2$ The range $\mathbf{j}(\mathbf{EA}(M)) = \mathcal{SS}(M)$ is mapped onto itself by any $F \in Aut(G)$.

By Lemma 7.

$(*)_3$ The range $\mathbf{j}(P_{\mathbf{A}(M)})) = \mathcal{PS}(M)$ is mapped onto itself by any $F \in Aut(G)$.

By Proposition 9.

$(*)_4$ For $L \in \mathbf{L}(M)$, the range $\mathbf{j}(P_{L(M)}) = \{G_{(K)} : Aut(K) \cong L\}$ is mapped onto itself by any $F \in Aut(G)$.

By Proposition 10.

$(*)_5$ The range $\mathbf{j}(P_{\mathbf{L}}(M)) = \bigcup_{L \in \mathbf{L}(M)} \{G_{(K)} : Aut(K) \cong L\}$ is mapped onto itself by any $F \in Aut(G)$.

Follows from $(*)_4$.

$(*)_6$ The range $\mathbf{j}(\leqslant_{\mathbf{EA}(M)}) = \{G_{(K_1,L_1)} \supseteq G_{(K_2,L_2)} : (K_i, L_i) \in \mathbf{EA}(M)\ (i = 1, 2),\ K_1 \leqslant K_2 \text{ and } L_2 \restriction K_1 \leqslant L_1\}$ is preserved by any $F \in Aut(G)$.

For $(K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$ and $F \in Aut(G)$, obviously we have $\mathbf{j}(K_1, L_1) \supseteq \mathbf{j}(K_2, L_2)$ if and only if $F(\mathbf{j}(K_1, L_1)) \supseteq F(\mathbf{j}(K_2, L_2))$, since F induces an automorphism of $(\mathcal{P}(Aut(G)), \subseteq)$.

$(*)_7$ The range $\mathbf{j}(\leqslant_{\mathbf{A}(M)}) = \{G_{(K_1)} \supseteq G_{(K_2)} : K_1, K_2 \in \mathbf{A}(M), K_1 \leqslant K_2\}$ is preserved by any $F \in Aut(G)$.

As in $(*)_6$, i.e. any $F \in Aut(G)$ induces an automorphism of $(\mathcal{P}(Aut(G)), \subseteq)$.

$(*)_8$ The range $\mathbf{j}(P^{min}_{\mathbf{A}(M)}) = \{H \in \mathcal{PS}(M) : G \neq H \text{ is maximal in } (\mathcal{PS}(M), \subseteq)\}$ is preserved by any $F \in Aut(G)$.

As in $(*)_6$, i.e. any $F \in Aut(G)$ induces an automorphism of $(\mathcal{P}(Aut(G)), \subseteq)$.

$(*)_9$ For any $F \in Aut(G)$, $F(gh) = F(g)F(h)$.

Obvious.

$(*)_{10}$ $\mathbf{j}(Op(f,K)) = fG_{(K)}f^{-1}$ and $F(\mathbf{j}((Op(f,K)))) = Op(F(f), F(\mathbf{j}(K)))$, for any $F \in Aut(G)$.

Observe that:

$$\begin{aligned} F(\mathbf{j}((Op(f,K)))) &= F(fG_{(K)}f^{-1}) \\ &= F(f)F(G_{(K)})(F(f))^{-1} \\ &= F(f)(F(\mathbf{j}(K)) \\ &= Op(F(f), F(\mathbf{j}(K))), \end{aligned}$$

since by $(*)_3$ $\mathcal{PS}(M)$ is mapped onto itself by any $F \in Aut(G)$.

$(*)_{11}$ $\mathbf{j}(Op(f,(K_1,L_1))) = (fG_{(K_1)}f^{-1}, fG_{(K_1,L_1)}f^{-1})$ and $F(\mathbf{j}((Op(f,(K_1,L_1))))) = Op(F(f), F(\mathbf{j}((K_1,L_1))))$, for any $F \in Aut(G)$.

Similar to $(*)_{10}$.

This concludes the proof of (1). Finally, (2) follows directly from (1), in fact for $F \in Aut(Aut(M))$, letting $\hat{F} = \mathbf{j}^{-1}F\mathbf{j}$ we have $\hat{F} \in Aut(ExAut(M))$.

■

3. Reconstruction and Outer Automorphisms

In this section we prove the theorems stated in the introduction.

Let $\mathbf{K}_*$ be the class of countable structures M satisfying:

(1) M has the strong small index property;
(2) for every finite $A \subseteq M$, $acl_M(A)$ is finite;
(3) for every $a \in M$, $acl_M(\{a\}) = \{a\}$;

As in the previous section, we let $G = Aut(M)$. We denote $G_{(\{a\})}$ simply as $G_{(a)}$. The crucial point in asking these additional conditions is the following:

Proposition 13. *Let $M \in \mathbf{K}_*$ be homogeneous, and define:*
(1) $\mathcal{M} = \{G_{(a)} : a \in M\}$;
(2) $\mathcal{G} = \{G_{(K)} \in \mathcal{PS}(M) : acl(\emptyset) \neq K \in \mathbf{A}(M)$ is minimal in $(\mathbf{A}(M), \subseteq)\}$.
Then $\mathcal{M} = \mathcal{G}$.

Proof. This is clear from the work done in the previous section. ■

We will use the suggestive notation $\mathcal{M} = \{G_{(a)} : a \in M\}$ also below.

Definition 14. *Let M and N be structures and consider $Aut(M)$ (resp. $Aut(N)$) as acting naturally on M (resp. N). We say that $(Aut(M), M)$ and $(Aut(N), N)$ are isomorphic as permutation groups if there exists a bijection $f : M \to N$ such that the map $h \mapsto fhf^{-1}$ is an isomorphism from $Aut(M)$ onto $Aut(N)$.*

Proof of Theorem 1. Let $M, N \in \mathbf{K}_*$, and suppose that $F : Aut(M) \cong Aut(N)$. Passing to canonical relational structures (cf. [2, pg. 26]), we can assume without loss of generality that M and N are homogeneous. Now, the isomorphism F induces an isomorphism $\hat{F} : ExAut(M) \cong ExAut(N)$. In particular, $\hat{F}$ maps $P^{min}_{\mathbf{A}(M)}$ onto $P^{min}_{\mathbf{A}(N)}$. Thus, by Proposition 13, we have:

$$\mathbf{j}_M(P^{min}_{\mathbf{A}(M)}) = \mathcal{M} \text{ and } \mathbf{j}_N(P^{min}_{\mathbf{A}(N)}) = \mathcal{N}.$$

Hence, $\hat{F}$ induces the bijection $f : M \to N$:

$$f(a) = \hat{F}(Aut(M)_{(a)}) = Aut(N)_{f(a)} \in \mathcal{N}.$$

Let $G : h \mapsto fhf^{-1}$, for $h \in Aut(M)$. We claim that $G = F$. Let in fact $h \in Aut(M)$, $a, b \in M$ and suppose that $F(h)(f(a)) = f(b)$. Then:

$$\begin{aligned} F(h)(f(a)) = f(b) &\Leftrightarrow F(h)Aut(N)_{(f(a))}(F(h))^{-1} = Aut(N)_{(f(b))} \\ &\Leftrightarrow hAut(M)_{(a)}h^{-1} = Aut(M)_{(b)} \\ &\Leftrightarrow h(a) = b. \end{aligned}$$

So, $fhf^{-1}(f(a)) = fh(a) = f(b)$, as wanted. Hence, $f : M \to N$ witnesses that $(Aut(M), M)$ and $(Aut(N), N)$ are isomorphic as permutation groups. ■

Definition 15. *We say that two structures M and N are bi-definable if there is a bijection $f : M \to N$ such that for every $A \subseteq M^n$, A is $\emptyset$-definable in M if and only if $f(A)$ is $\emptyset$-definable in N.*

Fact 16 ([7], Proposition 1.3)**.** *Let M and N be countable $\aleph_0$-categorical structures. Then the following are equivalent:*

(1) $(Aut(M), M) \cong (Aut(N), N)$;
(2) M and N are bi-definable.

Proof of Corollary 2. Let $M, N \in \mathbf{K}_*$, and suppose that $Aut(M) \cong Aut(N)$. As before, passing to canonical relational structures, we can assume without loss of generality that M and N are homogeneous. Furthermore, since M and N are $\aleph_0$-categorical, this passage preserves definability. Thus, by Theorem 1 and Fact 16 we are done. ■

We now pass to the proof of Theorem 3.

Fact 17 (Frucht's Theorem [3])**.** *Every finite group is the group of automorphisms of a finite graph.*

Proof of Theorem 3. Let Γ be a finite graph on vertex set $\{0, ..., n-1\}$ and

$$L_\Gamma = \{P_\ell : \ell < n\} \cup \{R_{\ell,k} : \ell < k < n \text{ and } \{\ell, k\} \in E_\Gamma\}$$

be such that the P_ℓ are unary predicates and the $R_{\ell,k}$ are binary relations. Let $\mathbf{K}_\Gamma$ be the class of finite L_Γ-models M such that:

(1) $(P^M_\ell : \ell < n)$ is a partition of M;
(2) $R^M_{\ell,k}$ is a symmetric irreflexive relation on $P_\ell \times P_k$.

Notice that $\mathbf{K}_\Gamma$ is a free amalgamation class (cf. [6, Definition 4]). Let M_Γ be the corresponding countable homogeneous structure. By [6, Corollary 2], M_Γ has the strong small index property, and, obviously, M_Γ is $\aleph_0$-categorical and has no algebraicity. Using Corollary 2 it is now easy to see that:

$$Aut(\Gamma) \cong Aut(Aut(M_\Gamma))/Inn(Aut(M_\Gamma)) = Out(Aut(M_\Gamma)).$$

Thus, by Fact 17 we are done. ■

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