COLLOQUIUM MATHEMATICUM

VOL. 79 1999 NO. 1

ON FULL SUSLIN TREES

BY

SAHARON SHELAH
(JERUSALEM, NEW BRUNSWICK, N.J., AND MADISON, WISC.)

0. Introduction. In the present paper we answer a combinatorial question of Kunen listed in Arnie Miller's Problem List. We force, e.g. for the first strongly inaccessible Mahlo cardinal λ, a full (see 1.1(2)) λ-Suslin tree and we remark that the existence of such trees follows from $\mathbf{V} = \mathbf{L}$ (if λ is Mahlo strongly inaccessible). This answers [Mi91, Problem 15.13].

Our notation is rather standard and compatible with those of classical textbooks on Set Theory. However, in forcing considerations, we keep the older tradition that

a stronger condition is the larger one.

We will keep the following conventions concerning use of symbols.

NOTATION 0.1. (1) λ, μ will denote cardinal numbers and $\alpha, \beta, \gamma, \delta, \xi, \zeta$ will be used to denote ordinals.

(2) Sequences (not necessarily finite) of ordinals are denoted by ν, η, ϱ (with possible indices).

(3) The length of a sequence η is $\lg(\eta)$.

(4) For a sequence η and an ordinal $\alpha \le \lg(\eta)$, $\eta{\restriction}\alpha$ is the restriction of the sequence η to α (so $\lg(\eta{\restriction}\alpha) = \alpha$). If a sequence ν is a proper initial segment of a sequence η then we write $\nu \vartriangleleft \eta$ (and $\nu \trianglelefteq \eta$ has the obvious meaning).

(5) A tilde indicates that we are dealing with a name for an object in forcing extension (like $\underset{\sim}{x}$).

1. Full λ-Suslin trees. A subset T of ${}^{\alpha>}2$ is an *α-tree* whenever (α is a limit ordinal and) the following three conditions are satisfied:

- $\langle\rangle \in T$, if $\nu \vartriangleleft \eta \in T$ then $\nu \in T$,
- $\eta \in T$ implies $\eta^\frown\langle 0\rangle, \eta^\frown\langle 1\rangle \in T$, and

1991 *Mathematics Subject Classification*: 03E05, 03E35.

We thank the NSF for partially supporting this research under grant #144-EF67. Publication No 624.

• for every $\eta \in T$ and $\beta < \alpha$ such that $\lg(\eta) \leq \beta$ there is $\nu \in T$ such that $\eta \trianglelefteq \nu$ and $\lg(\eta) = \beta$.

A λ-*Suslin tree* is a λ-tree $T \subseteq {}^{\lambda>}2$ in which every antichain is of size less than λ.

Definition 1.1. (1) For a tree $T \subseteq {}^{\alpha>}2$ and an ordinal $\beta \leq \alpha$ we let

$$T_{[\beta]} := T \cap {}^{\beta}2 \quad \text{and} \quad T_{[<\beta]} := T \cap {}^{\beta>}2.$$

If $\delta \leq \alpha$ is limit then we define

$$\lim_\delta T_{[<\delta]} := \{\eta \in {}^{\delta}2 : (\forall \beta < \delta)(\eta{\restriction}\beta \in T)\}.$$

(2) An α-tree T is *full* if for every limit ordinal $\delta < \alpha$ the set $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ has at most one element.

(3) An α-tree $T \subseteq {}^{\alpha>}2$ has *true height* α if for every $\eta \in T$ there is $\nu \in {}^{\alpha}2$ such that

$$\eta \triangleleft \nu \quad \text{and} \quad (\forall \beta < \alpha)(\nu{\restriction}\beta \in T).$$

We will show that the existence of full λ-Suslin trees is consistent assuming the cardinal λ satisfies the following hypothesis.

Hypothesis 1.2. (a) λ is a strongly inaccessible (Mahlo) cardinal,

(b) $S \subseteq \{\mu < \lambda : \mu$ is a strongly inaccessible cardinal$\}$ is a stationary set,

(c) $S_0 \subseteq \lambda$ is a set of limit ordinals,

(d) for every cardinal $\mu \in S$, $\diamondsuit_{S_0 \cap \mu}$ holds true.

Further in this section we will assume that λ, S_0 and S are as above and we may forget to repeat these assumptions.

Let us recall that the *diamond principle* $\diamondsuit_{S_0 \cap \mu}$ postulates the existence of a sequence $\bar{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$ (called *a* $\diamondsuit_{S_0\cap\mu}$*-sequence*) such that $\nu_\delta \in {}^{\delta}2$ (for $\delta \in S_0 \cap \mu$) and

$$(\forall \nu \in {}^{\mu}2)[\text{the set } \{\delta \in S_0 \cap \mu : \nu{\restriction}\delta = \nu_\delta\} \text{ is stationary in } \mu].$$

Now we introduce a forcing notion $\mathbb{Q}$ and its relative $\mathbb{Q}^*$ which will be used in our proof.

Definition 1.3. (1) A condition in $\mathbb{Q}$ is a tree $T \subseteq {}^{\alpha>}2$ of a true height $\alpha = \alpha(T) < \lambda$ (see 1.1(3); so α is a limit ordinal) such that $\|\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}\| \leq 1$ for every limit ordinal $\delta < \alpha$; the order on $\mathbb{Q}$ is defined by $T_1 \leq T_2$ if and only if $T_1 = T_2 \cap {}^{\alpha(T_1)>}2$ (so it is the end-extension order).

(2) For a condition $T \in \mathbb{Q}$ and a limit ordinal $\delta < \alpha(T)$, let $\eta_\delta(T)$ be the unique member of $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ if there is one, otherwise $\eta_\delta(T)$ is not defined.

(3) Let $T \in \mathbb{Q}$. A function $f : T \to \lim_{\alpha(T)}(T)$ is called a *witness* for T if $(\forall \eta \in T)(\eta \triangleleft f(\eta))$.

(4) A condition in $\mathbb{Q}^*$ is a pair (T,f) such that $T\in\mathbb{Q}$ and $f:T\rightarrow \lim_{\alpha(T)}(T)$ is a witness for T; the order on $\mathbb{Q}^*$ is defined by $(T_1,f_1)\leq(T_2,f_2)$ if and only if $T_1\leq_{\mathbb{Q}} T_2$ and $(\forall\eta\in T_1)(f_1(\eta)\trianglelefteq f_2(\eta))$.

PROPOSITION 1.4. (1) *If* $(T_1,f_1)\in\mathbb{Q}^*$, $T_1\leq_{\mathbb{Q}} T_2$ *and*

$(*)$ *either* $\eta_{\alpha(T_1)}(T_2)$ *is not defined or it does not belong to* $\mathrm{rang}(f_1)$

then there is $f_2:T_2\rightarrow\lim_{\alpha(T_2)}(T_2)$ *such that* $(T_1,f_1)\leq(T_2,f_2)\in\mathbb{Q}^*$.

(2) *For every* $T\in\mathbb{Q}$ *there is a witness* f *for* T.

Proof. Should be clear. ■

PROPOSITION 1.5. (1) *The forcing notion* $\mathbb{Q}^*$ *is* $(<\lambda)$*-complete, in fact any increasing chain of length* $<\lambda$ *has the least upper bound in* $\mathbb{Q}^*$.

(2) *The forcing notion* $\mathbb{Q}$ *is strategically* γ*-complete for each* $\gamma<\lambda$.

(3) *Forcing with* $\mathbb{Q}$ *adds no new sequences of length* $<\lambda$. *Since* $\|\mathbb{Q}\|=\lambda$, *forcing with* $\mathbb{Q}$ *preserves cardinal numbers, cofinalities and cardinal arithmetic.*

Proof. (1) It is straightforward: suppose that $\langle(T_\zeta,f_\zeta):\zeta<\xi\rangle$ is an increasing sequence of elements of $\mathbb{Q}^*$. Clearly we may assume that $\xi<\lambda$ is a limit ordinal and $\zeta_1<\zeta_2<\xi\Rightarrow\alpha(T_{\zeta_1})<\alpha(T_{\zeta_2})$. Let $T_\xi=\bigcup_{\zeta<\xi}T_\zeta$ and $\alpha=\sup_{\zeta<\xi}\alpha(T_\zeta)$. Clearly, the union is increasing and T_ξ is a full α-tree. For $\eta\in T_\xi$ let $\zeta_0(\eta)$ be the first $\zeta<\xi$ such that $\eta\in T_\zeta$ and let $f_\xi(\eta)=\bigcup\{f_\zeta(\eta):\zeta_0(\eta)\leq\zeta<\xi\}$. By the definition of the order on $\mathbb{Q}^*$ we see that the sequence $\langle f_\zeta(\eta):\zeta_0(\eta)\leq\zeta<\xi\rangle$ is $\triangleleft$-increasing and hence $f_\xi(\eta)\in\lim_\alpha(T_\xi)$. Plainly, the function f_ξ witnesses that T_ξ has true height α, and thus $(T_\xi,f_\xi)\in\mathbb{Q}^*$. It should be clear that (T_ξ,f_ξ) is the least upper bound of the sequence $\langle(T_\zeta,f_\zeta):\zeta<\xi\rangle$.

(2) For our purpose it is enough to show that for each ordinal $\gamma<\lambda$ and a condition $T\in\mathbb{Q}$ the second player has a winning strategy in the following game $\mathcal{G}_\gamma(T,\mathbb{Q})$. (Also we can let Player I choose T_ξ for ξ odd.)

The game lasts γ moves and during a play the players, called I and II, choose successively open dense subsets $\mathcal{D}_\xi$ of $\mathbb{Q}$ and conditions $T_\xi\in\mathbb{Q}$. At stage $\xi<\gamma$ of the game, Player I chooses an open dense subset $\mathcal{D}_\xi$ of $\mathbb{Q}$ and Player II answers playing a condition $T_\xi\in\mathbb{Q}$ such that

$$T\leq_{\mathbb{Q}} T_\xi,\quad(\forall\zeta<\xi)(T_\zeta\leq_{\mathbb{Q}} T_\xi),\quad\text{and}\quad T_\xi\in\mathcal{D}_\xi.$$

The second player wins if he always has legal moves during the play.

Let us describe the winning strategy for Player II. At each stage $\xi<\gamma$ of the game he plays a condition T_ξ and writes down a function f_ξ such that $(T_\xi,f_\xi)\in\mathbb{Q}^*$. Moreover, he keeps an extra obligation that $(T_\zeta,f_\zeta)\leq_{\mathbb{Q}^*}(T_\xi,f_\xi)$ for each $\zeta<\xi<\gamma$.

So arriving at a non-limit stage of the game he takes the condition (T_ζ,f_ζ) he constructed before (or just (T,f), where f is a witness for T, if this is the first move; by 1.4(2) we can always find a witness). Then he chooses $T^*_\zeta \geq_{\mathbb{Q}} T_\zeta$ such that $\alpha(T^*_\zeta) = \alpha(T_\zeta) + \omega$ and $(T^*_\zeta)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta)$. Thus $\eta_{\alpha(T_\zeta)}(T^*_\zeta)$ is not defined. Now Player II takes $T_{\zeta+1} \geq_{\mathbb{Q}} T^*_\zeta$ from the open dense set $\mathcal{D}_{\zeta+1}$ played by his opponent at this stage. Clearly $\eta_{\alpha(T_\zeta)}(T_{\zeta+1})$ is not defined, so Player II may use 1.4(1) to choose $f_{\zeta+1}$ such that $(T_\zeta, f_\zeta) \leq_{\mathbb{Q}^*} (T_{\zeta+1}, f_{\zeta+1}) \in \mathbb{Q}^*$.

At a limit stage ξ of the game, the second player may take the least upper bound $(T'_\xi, f'_\xi) \in \mathbb{Q}^*$ of the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ (exists by (1)) and then apply the procedure described above.

(3) Follows from (2) above. ■

Definition 1.6. Let $\underset{\sim}{\mathbf{T}}$ be the canonical $\mathbb{Q}$-name for a generic tree added by forcing with $\mathbb{Q}$:

$$\Vdash_{\mathbb{Q}} \underset{\sim}{\mathbf{T}} = \bigcup \{T : T \in \underset{\sim}{G}_{\mathbb{Q}}\}.$$

It should be clear that $\underset{\sim}{\mathbf{T}}$ is (forced to be) a full λ-tree. The main point is to show that it is λ-Suslin and this is done in the following theorem.

Theorem 1.7. $\Vdash_{\mathbb{Q}}$ "*$\underset{\sim}{\mathbf{T}}$ is a λ-Suslin tree*".

Proof. Suppose that $\underset{\sim}{A}$ is a $\mathbb{Q}$-name such that

$$\Vdash_{\mathbb{Q}} \text{"}\underset{\sim}{A} \subseteq \underset{\sim}{\mathbf{T}} \text{ is an antichain"},$$

and let T_0 be a condition in $\mathbb{Q}$. We will show that there are $\mu < \lambda$ and a condition $T^* \in \mathbb{Q}$ stronger than T_0 such that $T^* \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A} \subseteq \underset{\sim}{\mathbf{T}}_{[<\mu]}$" (and thus it forces that the size of $\underset{\sim}{A}$ is less than λ).

Let $\underset{\sim}{\mathbf{A}}$ be a $\mathbb{Q}$-name such that

$$\Vdash_{\mathbb{Q}} \text{"}\underset{\sim}{\mathbf{A}} = \{\eta \in \underset{\sim}{\mathbf{T}} : (\exists \nu \in \underset{\sim}{A})(\nu \trianglelefteq \eta) \text{ or } \neg(\exists \nu \in \underset{\sim}{A})(\eta \trianglelefteq \nu)\}\text{"}.$$

Clearly, $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbf{A}} \subseteq \underset{\sim}{\mathbf{T}}$ is dense open".

Let χ be a sufficiently large regular cardinal ($\beth_7(\lambda^+)^+$ is enough).

Claim 1.7.1. *There are $\mu \in S$ and $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that*:

(a) $\underset{\sim}{A}, \underset{\sim}{\mathbf{A}}, S, S_0, \mathbb{Q}, \mathbb{Q}^*, T_0 \in \mathfrak{B}$,
(b) $\|\mathfrak{B}\| = \mu$ *and* ${}^{\mu>}\mathfrak{B} \subseteq \mathfrak{B}$,
(c) $\mathfrak{B} \cap \lambda = \mu$.

Proof. First construct inductively an increasing continuous sequence $\langle \mathfrak{B}_\xi : \xi < \lambda \rangle$ of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ such that $\underset{\sim}{A}, \underset{\sim}{\mathbf{A}}, S, S_0, \mathbb{Q}, \mathbb{Q}^*, T_0 \in \mathfrak{B}_0$ and for every $\xi < \lambda$,

$$\|\mathfrak{B}_\xi\| = \mu_\xi < \lambda, \quad \mathfrak{B}_\xi \cap \lambda \in \lambda, \quad \text{and} \quad {}^{\mu_\xi \geq}\mathfrak{B}_\xi \subseteq \mathfrak{B}_{\xi+1}.$$

Note that for a club E of λ, for every $\mu \in S \cap E$ we have

$$\|\mathfrak{B}_\mu\| = \mu, \quad {}^{\mu>}\mathfrak{B}_\mu \subseteq \mathfrak{B}_\mu, \quad \text{and} \quad \mathfrak{B}_\mu \cap \lambda = \mu.$$

Choose $\mu \in S \cap E$ and let $\mathfrak{B} = \mathfrak{B}_\mu$.

Let $\mu \in S$ and $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be given by 1.7.1. We know that $\diamondsuit_{S_0 \cap \mu}$ holds, so fix a $\diamondsuit_{S_0\cap\mu}$-sequence $\overline{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$.

Let

$$\underset{\sim}{\mathcal{I}} := \{T \in \mathbb{Q} : T \text{ is incompatible (in } \mathbb{Q}\text{) with } T_0 \text{ or:}$$
$$T \geq T_0 \text{ and } T \text{ decides the value of } \underset{\sim}{\mathbf{A}} \cap {}^{\alpha(T)>}2 \text{ and}$$
$$(\forall \eta \in T)(\exists \varrho \in T)(\eta \trianglelefteq \varrho \ \&\ T \Vdash_{\mathbb{Q}} \varrho \in \underset{\sim}{\mathbf{A}})\}.$$

Claim 1.7.2. *$\underset{\sim}{\mathcal{I}}$ is a dense subset of $\mathbb{Q}$.*

Proof. Should be clear (remember 1.5(2)).

Now we choose by induction on $\xi < \mu$ a continuous increasing sequence $\langle (T_\xi, f_\xi) : \xi < \mu \rangle \subseteq \mathbb{Q}^* \cap \mathfrak{B}$.

Step: $i = 0$. T_0 is already chosen and it belongs to $\mathbb{Q} \cap \mathfrak{B}$. We take any f_0 such that $(T_0, f_0) \in \mathbb{Q}^* \cap \mathfrak{B}$ (exists by 1.4(2)).

Step: limit ξ. Since ${}^{\mu>}\mathfrak{B} \subseteq \mathfrak{B}$, the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is in $\mathfrak{B}$. By 1.5(1) it has the least upper bound (T_ξ, f_ξ) (which belongs to $\mathfrak{B}$).

Step: $\xi = \zeta + 1$. First we take the (unique) tree T^*_ξ of true height $\alpha(T^*_\xi) = \alpha(T_\zeta) + \omega$ such that $T^*_\xi \cap {}^{\alpha(T_\zeta)>}2 = T_\zeta$ and: if $\alpha(T_\zeta) \in S_0$ and $\nu_{\alpha(T_\zeta)} \notin \mathrm{rang}(f_\zeta)$ then $(T^*_\xi)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta) \setminus \{\nu_{\alpha(T_\zeta)}\}$, otherwise $(T^*_\xi)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta)$.

Let $T_\xi \in \mathbb{Q} \cap \underset{\sim}{\mathcal{I}}$ be strictly above T^*_ξ (exists by 1.7.2). Clearly we may choose such T_ξ in $\mathfrak{B}$. Now we have to define f_ξ. We do it by 1.4, but additionally we require that

$$\text{if} \quad \eta \in T_\xi \quad \text{then} \quad (\exists \varrho \in T_\xi)(\varrho \triangleleft f_\xi(\eta) \ \&\ T \Vdash_{\mathbb{Q}} \text{“}\varrho \in \underset{\sim}{\mathbf{A}}\text{”}).$$

Plainly the additional requirement causes no problems (remember the definition of $\underset{\sim}{\mathcal{I}}$ and the choice of T_ξ) and the choice can be done in $\mathfrak{B}$.

There are no difficulties in carrying out the induction. Finally we let

$$T_\mu := \bigcup_{\xi<\mu} T_\xi \quad \text{and} \quad f_\mu = \bigcup_{\xi<\mu} f_\xi.$$

By the choice of $\mathfrak{B}$ and μ we are sure that T_μ is a μ-tree. It follows from 1.5(1) that $(T_\mu, f_\mu) \in \mathbb{Q}^*$, so in particular the tree T_μ has enough μ branches (and belongs to $\mathbb{Q}$).

Claim 1.7.3. *For every $\varrho \in \lim_\mu(T_\mu)$ there is $\xi < \mu$ such that*

$$(\exists \beta < \alpha(T_{\xi+1}))(T_{\xi+1} \Vdash_{\mathbb{Q}} \text{“}\varrho{\restriction}\beta \in \underset{\sim}{\mathbf{A}}\text{”}).$$

Proof. Fix $\varrho \in \lim_\mu(T_\mu)$ and let

$$S^*_\nu := \{\delta \in S_0 \cap \mu : \alpha(T_\delta) = \delta \text{ and } \nu_\delta = \varrho{\restriction}\delta\}.$$

Plainly, the set S^*_ν is stationary in μ (remember the choice of $\overline{\nu}$). By the definition of the T_ξ's (and by $\varrho \in \lim_\mu(T_\mu)$) we conclude that for every $\delta \in S^*_\nu$,

$$\text{if} \quad \eta_\delta(T_{\delta+1}) \text{ is defined} \quad \text{then} \quad \varrho{\restriction}\delta \neq \eta_\delta(T_\mu) = \eta_\delta(T_{\delta+1}).$$

But $\varrho{\restriction}\delta = \nu_\delta$ (as $\delta \in S^*_\nu$). So look at the inductive definition: necessarily for some $\varrho^*_\delta \in T_\delta$ we have $\nu_\delta = f_\delta(\varrho^*_\delta)$, i.e. $\varrho{\restriction}\delta = f_\delta(\varrho^*_\delta)$. Now, $\varrho^*_\delta \in T_\delta = \bigcup_{\xi<\delta} T_\xi$ and hence for some $\xi(\delta) < \delta$, we have $\varrho^*_\delta \in T_{\xi(\delta)}$. By Fodor's lemma we find $\xi^* < \mu$ such that the set

$$S'_\nu := \{\delta \in S^*_\nu : \xi(\delta) = \xi^*\}$$

is stationary in μ. Consequently, we find ϱ^* such that the set

$$S^+_\nu := \{\delta \in S'_\nu : \varrho^* = \varrho^*_\delta\}$$

is stationary (in μ). But the sequence $\langle f_\xi(\varrho^*) : \xi^* \le \xi < \mu\rangle$ is $\trianglelefteq$-increasing, and hence the sequence ϱ is its limit. Now we easily obtain the claim using the inductive definition of the (T_ξ, f_ξ)'s.

It follows from the definition of $\underset{\sim}{\mathbf{A}}$ and 1.7.3 that

$$T_\mu \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{A} \subseteq T_\mu\text{”}$$

(remember that $\underset{\sim}{A}$ is a name for an antichain of $\underset{\sim}{\mathbf{T}}$), and hence

$$T_\mu \Vdash_{\mathbb{Q}} \text{“}\|\underset{\sim}{A}\| < \lambda\text{”},$$

finishing the proof of the theorem. ■

Definition 1.8. A λ-tree T is S_0-*full*, where $S_0 \subseteq \lambda$, if for every limit $\delta < \lambda$,

- if $\delta \in \lambda \setminus S_0$ then $T_{[\delta]} = \lim_\delta(T)$,
- if $\delta \in S_0$ then $\|T_{[\delta]} \setminus \lim_\delta(T)\| \le 1$.

Corollary 1.9. *Assuming Hypothesis* 1.2:

(1) *The forcing notion* $\mathbb{Q}$ *preserves cardinal numbers*, *cofinalities and cardinal arithmetic.*

(2) $\Vdash_{\mathbb{Q}}$ “$\underset{\sim}{\mathbf{T}} \subseteq {}^{\lambda>}2$ *is a* λ-*Suslin tree which is full and even* S_0-*full*”. [*So, in* $\mathbf{V}^{\mathbb{Q}}$, *in particular we have*: *for every* $\alpha < \beta < \mu$, *for all* $\eta \in T \cap {}^\alpha 2$ *there is* $\nu \in T \cap {}^\beta 2$ *such that* $\eta \vartriangleleft \nu$, *and for a limit ordinal* $\delta < \lambda$, $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ *is either empty or has a unique element* (*and then* $\delta \in S_0$).]

Proof. By 1.5 and 1.7. ■

Of course, we do not need to force.

Definition 1.10. Let $S_0, S \subseteq \lambda$. A sequence $\langle (C_\alpha, \nu_\alpha) : \alpha < \lambda \text{ limit}\rangle$ is called a *squared diamond sequence* for (S, S_0) if for each limit ordinal $\alpha < \lambda$,

(i) C_α is a club of α disjoint from S,
(ii) $\nu_\alpha \in {}^\alpha 2$,

(iii) if $\beta \in \mathrm{acc}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$ and $\nu_\beta \lhd \nu_\alpha$,
(iv) if $\mu \in S$ then $\langle \nu_\alpha : \alpha \in C_\mu \cap S_0 \rangle$ is a diamond sequence.

PROPOSITION 1.11. *Assume* (*in addition to* 1.2)

(e) *there exists a squared diamond sequence for* (S, S_0).

Then there is a λ*-Suslin tree* $T \subseteq {}^{\lambda>}2$ *which is* S_0*-full.*

Proof. Look carefully at the proof of 1.7. ■

COROLLARY 1.12. *Assume that* $\mathbf{V} = \mathbf{L}$ *and* λ *is Mahlo strongly inaccessible. Then there is a full* λ*-Suslin tree.*

Proof. Let $S \subseteq \{\mu < \lambda : \mu \text{ is strongly inaccessible}\}$ be a stationary non-reflecting set. By Beller and Litman [BeLi80], there is a square $\langle C_\delta : \delta < \lambda \text{ limit}\rangle$ such that $C_\delta \cap S = \emptyset$ for each limit $\delta < \lambda$. As in Abraham, Shelah and Solovay [AShS 221, §1] we can also have the squared diamond sequence. ■

REFERENCES

[AShS 221] U. Abraham, S. Shelah and R. M. Solovay, *Squares with diamonds and Souslin trees with special squares*, Fund. Math. 127 (1987), 133–162.

[BeLi80] A. Beller and A. Litman, *A strengthening of Jensen's □ principles*, J. Symbolic Logic 45 (1980), 251–264.

[Mi91] A. Miller, *Arnie Miller's problem list*, in: H. Judah (ed.), Set Theory of the Reals, Israel Math. Conf. Proc., Bar-Ilan University, 1993, 645–654.

Institute of Mathematics
The Hebrew University of Jerusalem
91904 Jerusalem, Israel

Department of Mathematics
Rutgers University
New Brunswick, New Jersey 08854
U.S.A.

Mathematics Department
University of Wisconsin–Madison
Madison, Wisconsin 53706
U.S.A.

Received 2 October 1996