Notre Dame Journal of Formal Logic
Volume 60, Number 4, 2019

Specializing Aronszajn Trees with Strong Axiom A and Halving

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Abstract We construct creature forcings with strong Axiom A that specialize a given Aronszajn tree. We work with tree creature forcing. The creatures that live on the Aronszajn tree are normed and have the halving property. We show that our models fulfill

$$\aleph_1 = \mathfrak{d} < \mathrm{unif}(\mathcal{M}) = \aleph_2 = 2^\omega.$$

1 Introduction

We establish a notion of forcing with strong Axiom A that specializes an Aronszajn tree and makes the ground model reals a meager set. Solovay and Tennenbaum in [19] specialized Aronszajn trees by finite approximations. Later, Shelah in [18, Chapter V] found a way to specialize Aronszajn trees without adding reals. Now we are interested in an intermediate way, an ${}^\omega\omega$-bounding forcing (see Definition 4.1) that adds reals.

We use creature forcing. Creature forcing tries to enlarge and systemize the family of very nice forcings. There is "the book on creature forcing" by Rosłanowski and Shelah [15], and the work is extended in Fischer, Goldstern, Kellner, Shelah, Mejía, Rosłanowski, and Spinas in [14], [16], [8], [4], [9], [3], and [17]. Our exposition is self-contained with respect to the creature technique.

At first glance we cannot replace the countable reservoir of creatures in [15] by an uncountable set. However, this was first done by Mildenberger and Shelah in [11], where we applied the theory of creatures for specializing an Aronszajn tree. Unfortunately, [11] contained some inaccuracies, and we hope that we give a more detailed and clearer presentation here. Here we rework the forcings from [11] and

Received November 28, 2016; accepted June 22, 2018
First published online September 25, 2019
2010 Mathematics Subject Classification: Primary 03E15, 03E17; Secondary 03E35, 03D65
Keywords: proper forcing, bounding forcing, Aronszajn tree
 10.1215/00294527-2019-0021

develop their use further. The norm of creatures (see Definition 2.5) we shall use is natural for specializing Aronszajn trees (see [18, Chapter V, Section 6]). In the terminology of [15], the forcing conditions are liminf tree creature forcings. The trees in the forcing conditions are finitely branching and endless, that is, do not have maximal nodes.

Definition 1.1

(1) $(\mathbf{T}, <_{\mathbf{T}})$ is called an *Aronszajn tree* if
  (a) $|\mathbf{T}| = \aleph_1$;
  (b) $(\mathbf{T}, <_{\mathbf{T}})$ is a partial order such that, for any $t \in \mathbf{T}$, $\text{pred}(t) = \{s \in \mathbf{T} : s <_{\mathbf{T}} t\}$ is well ordered; our trees may have countably many $<_{\mathbf{T}}$-minimal elements;
  (c) for $\alpha < \omega_1$, the level $\mathbf{T}_\alpha = \{t \in \mathbf{T} : \text{pred}(t) \cong \alpha\}$ of $(\mathbf{T}, <_{\mathbf{T}})$ is a subset of $\{\beta \in \omega_1 : \alpha\omega \leq \beta < (\alpha + 1)\omega\}$;
  (d) $(\mathbf{T}, <_{\mathbf{T}})$ has no uncountable branch;
  (e) $(\mathbf{T}, <_{\mathbf{T}})$ is normal, that is, for every $\alpha < \beta < \omega_1$ and for every $t \in \mathbf{T}_\alpha$ there is $t' \in \mathbf{T}_\beta$ such that $t <_{\mathbf{T}} t'$.

(2) A function $f : \mathbf{T} \to \omega$ is called a *specialization* of $(\mathbf{T}, <_{\mathbf{T}})$ or we say that f *specializes* $(\mathbf{T}, <_{\mathbf{T}})$ if $\forall s, t \in \mathbf{T}(s <_{\mathbf{T}} t \to f(s) \neq f(t))$ (see Jech [6, p. 244]). An Aronszajn tree $(\mathbf{T}, <_{\mathbf{T}})$ is special if it has a specialization function.

(3) A Souslin tree is an Aronszajn tree in which all antichains are countable.

Nahman Aronszajn was the first to construct a tree with properties (a) through (e). Kurepa in [10] coined the name Aronszajn tree and introduced Aronszajn trees in the literature.

If an Aronszajn tree is special, then it is the union of countably many antichains, and hence, the tree is not a Souslin tree. Shelah showed in [18, Chapter IX] that "There is no Souslin tree" does not imply that "All Aronszajn trees are special."

In forcing, the *larger* condition is the *stronger* one. We recall the definition of Axiom A and of strong Axiom A.

Definition 1.2 A notion of forcing $(\mathbb{Q}, \leq)$ is said to have Axiom A if there are quasiorders $\leq_n$, $n \in \omega$, over $\mathbb{Q}$ with the following properties:

(1) $\leq_{n+1}$ is a subrelation of $\leq_n$ for any n, and $\leq_0$ is a subrelation of $\leq$.
(2) $(\mathbb{Q}, \leq, (\leq_n)_{n\in\omega})$ has the fusion property; that is, for any sequence $\langle p_n : n < \omega\rangle$ such that $p_n \leq_n p_{n+1}$, there is $q \in \mathbb{Q}$ such that, for any n, $q \geq_n p_n$.
(3) For any $n \in \omega$, $p \in \mathbb{Q}$, and maximal antichain A in $\mathbb{Q}$, there is a $q \geq_n p$ such that q is compatible with at most countably many $a \in A$.

Definition 1.3 A notion of forcing $(\mathbb{Q}, \leq)$ is said to have strong Axiom A if it has Axiom A and in item (3) the number of compatible elements is finite.

Theorem 1.4 *Given an Aronszajn tree* $\mathbf{T}$*, there is a notion of forcing* $(\mathbb{Q}_{\mathbf{T}}, \leq, (\leq_n)_{n\in\omega})$ *with the following properties:*

(a) $\mathbb{Q}_{\mathbf{T}}$ *specializes* $\mathbf{T}$.
(b) $\mathbb{Q}_{\mathbf{T}} \subseteq H(\aleph_1)$.
(c) $(\mathbb{Q}_{\mathbf{T}}, \leq, (\leq_n)_{n\in\omega})$ *has strong Axiom A.*
(d) *Let* D *be dense and open in* $\mathbb{Q}_{\mathbf{T}}$*, let* $n \in \omega$*, and let* $p = ((T^p, <_p), \langle \mathbf{c}_{p,t} : t \in T^p\rangle) \in \mathbb{Q}_{\mathbf{T}}$*. Then there is* $q \geq_n p$ *and there is* $m \in \omega$ *such that*

$$\begin{aligned} \{q^{\langle t\rangle} : t \in (T^q)^{[m]}\} &\quad \textit{is predense above } q, \\ \big(\forall t \in (T^q)^{[m]}\big) &\quad (q^{\langle t\rangle} \in D), \textit{ and} \\ q \restriction (T^q)^{[m]} &\quad \textit{is a finite structure with finite signature.} \end{aligned}$$

(e) $\mathbb{Q}_{\mathbf{T}}$ *adds a real that makes the ground model reals a meager set.*

Remark 1.5 The list of properties is redundant: property (d) implies strong Axiom A. We state (d) because it describes the underlying structure. The components of a condition $p \in \mathbb{Q}_{\mathbf{T}}$ and the notions in property (d) will be explained in the coming sections.

We give an overview of the paper. In Sections 2–5, we prove Theorem 1.4 by a forcing $\mathbb{Q}_{\mathbf{T}}$. In Section 2 we introduce creatures. In Section 3, we define a notion of forcing, an iterand, and show that it specializes a given Aronszajn tree. We show that the smooth conditions are dense. In Section 4, we prove Theorem 1.4 for all but item (e). In Section 5, we prove that the forcing with $\mathbb{Q}_{\mathbf{T}}$ makes the ground model reals a meager set and, thus, finish the proof of Theorem 1.4. Sections 2 and 3 conclude with some results on the halving property that are not used in Theorem 1.4. There was some hope that strong halving properties would allow one to establish a name for an Ostaszewski club sequence (see Ostaszewski [13]) in the extension. This stays open. The background on proper forcing can be found in Abraham [1] and Shelah [18].

2 Tree Creatures

In this section we define the tree creatures which will be used in the next section to describe the branching of the countable trees that will serve as forcing conditions. We define three important operations that can be performed on creatures:

- gluing together creatures (Lemmas 2.13 and 2.14),
- extending the domains of the partial specializations in the set of possibilities of a creature (Lemma 2.15),
- extending the basis of a creature together with thinning out the set of possibilities (Lemma 2.16) and extending the elements in the set of possibilities.

We shall define the forcing conditions only in the next section. They will be endless finitely branching tagged trees, in which each node and its immediate successors are described by a creature (see Definition 2.9). Roughly speaking, in our context, a creature $\mathbf{c}$ will be a tree of height 2 of partial specialization functions whose root is labeled by a pair $(i(\mathbf{c}), k(\mathbf{c}))$ of natural numbers.

We let χ stand for some regular cardinal larger than $(2^{\aleph_2})^+$ and let $H(\chi)$ denote the set of sets of hereditary cardinality less than χ. We use the symbol $<^*_\chi$ for some well-order on this set. The symbol $\mathscr{H}(\chi)$ denotes the structure $(H(\chi), \in, <^*_\chi)$.

Throughout this work we make the assumption that $(\mathbf{T}, <_{\mathbf{T}})$ is an Aronszajn tree as in Definition 1.1(1). We define the following finite approximations of specialization maps.

Definition 2.1 For $u \subseteq \mathbf{T}$ and $n < \omega$ we let

$$\mathrm{spec}_n(u) = \big\{\eta \mid \eta\colon u \to [0,n) \ \wedge\ \big(\eta(x) = \eta(y) \to \neg(x <_{\mathbf{T}} y)\big)\big\}.$$

We let $\mathrm{spec}(u) = \bigcup_{n<\omega} \mathrm{spec}_n(u)$, $\mathrm{spec}_n = \mathrm{spec}_n^{\mathbf{T}} = \bigcup\{\mathrm{spec}_n(u) : u \subset \mathbf{T}, u \text{ finite}\}$, and $\mathrm{spec} = \mathrm{spec}^{\mathbf{T}} = \bigcup\{\mathrm{spec}(u) : u \subset \mathbf{T}, u \text{ finite}\}$.

Definition 2.2

(1) We choose three sequences of natural numbers $\langle n_{k,i} : i < \omega\rangle$, $k = 1, 2, 3$, such that the following growth conditions are fulfilled:

$$2 \leq n_{1,i} \leq n_{2,i} \leq n_{3,i},$$
$$2^{(n_{3,i})^2} < n_{1,i+1}.$$

(2) We jump ahead to Definition 2.4 and note that these numbers bound the size of a simple i-creature, $i \geq 1$, in the following way:
   (a) the number $n_{1,i-1}$ bounds the size of the domain of the partial specialization function that is the basis of the creature,
   (b) the basis of the creature is an element of $\mathrm{spec}^{\mathbf{T}}_{n_{2,i-1}}$,
   (c) the number $n_{3,i}$ bounds the number of the possibilities of the creature,
   (d) each function in the possibilities is an element of $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$,
   (e) the number $n_{1,i}$ bounds the size of the domain of each function in the possibilities of the creature.

(3) We fix the $n_{k,i}$, $k = 1, 2, 3$, $i < \omega$, for the rest of this work. The number $n_{1,i}$ is an upper bound for any kind of norm of an i-creature.

We compare with the book [15] in order to justify the use of the name "creature." We extend the framework developed there in order to allow for the approximation of uncountable domains $\mathbf{T}$.

Definition 2.3 ([15, Definition 1.1.1])

(1) We let $\mathbf{H} = \langle \mathbf{H}(i) : i \in \omega\rangle$ and let $\mathbf{H}(i)$ be sets. A triple $\mathbf{c} = (\mathrm{nor}[\mathbf{c}], \mathrm{val}[\mathbf{c}], \mathrm{dis}[\mathbf{c}])$ is a *weak creature for* $\mathbf{H}$ if the following hold.
   (a) $\mathrm{nor}[\mathbf{c}] \in \mathbb{R}^{\geq \mathbf{0}}$.
   (b) Let $\lhd$ be the strict initial segment relation. $\mathrm{val}[\mathbf{c}]$ is a nonempty subset of
   $$\Big\{\langle x, y\rangle \in \bigcup_{m_0 < m_1 < \omega} \Big[\prod_{i<m_0} \mathbf{H}(i) \times \prod_{i<m_1} \mathbf{H}(i)\Big] : x \lhd y\Big\}.$$
   (c) $\mathrm{dis}[\mathbf{c}] \in H(\chi)$.

(2) nor stands for *norm*, val stands for *value*, and dis stands for *distinguish*.

A creature is a weak creature with additional properties. In the creatures for the forcing $\mathbb{Q}_{\mathbf{T}}$ the component $\mathrm{dis}[\mathbf{c}]$ is a pair $(i(\mathbf{c}), k(\mathbf{c}))$ of natural numbers. More properties are specified in Definitions 2.4–2.10.

The set val is a nonempty subset of $\{\langle x, y\rangle \in \mathrm{spec}^{\mathbf{T}} \times \mathrm{spec}^{\mathbf{T}} : x <_T y\}$ for some strict partial order $<_T$ as in Definition 3.1 and $\mathbf{H}(i) = \mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$. The members of $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$ are finite partial functions, but the set $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$ is uncountable. Often the properness of a tree creature forcing follows from the countability of the sets $\mathbf{H}(i)$, $i \in \omega$, and our analogue to $\mathbf{H}(i)$ is the uncountable set $\mathrm{spec}^{\mathbf{T}}_{n_{2,i}}$. In Section 4 we shall prove that the notions of forcing we introduce are proper for other reasons.

Creatures with $|\mathrm{dom}(\mathrm{val}[\mathbf{c}])| = 1$ are called *tree creatures*. As is common in the work with tree creatures, we write $\mathrm{pos}(\mathbf{c})$ for $\mathrm{rge}(\mathrm{val}[\mathbf{c}])$ and call $\mathrm{pos}(\mathbf{c})$ the *set of possibilities* for $\mathbf{c}$.

Definition 2.4 A *simple creature* is a tuple $\mathbf{c} = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}))$ with the following properties:

(a) The first component, $i(\mathbf{c})$, is called the *kind* of $\mathbf{c}$ and is just a natural number. A (simple) creature $\mathbf{c}$ is called a (simple) i*-creature* if $i(\mathbf{c}) = i$.
(b) The second component, $\eta(\mathbf{c})$, is called the *base* of $\mathbf{c}$. We require ($\eta(\mathbf{c}) = \emptyset$ and $i(\mathbf{c}) = 0$) or ($i(\mathbf{c}) = i > 0$, $0 \neq |\mathrm{dom}(\eta(\mathbf{c}))| \leq n_{1,i-1}$, and $\eta(\mathbf{c}) \in \mathrm{spec}_{n_{2,i-1}}$).
(c) $\mathrm{pos}(\mathbf{c})$ is a nonempty subset of $\{\eta \in \mathrm{spec}_{n_{2,i}} : \eta(\mathbf{c}) \subsetneqq \eta \wedge |\mathrm{dom}(\eta)| \leq n_{1,i}\}$ and $|\mathrm{pos}(\mathbf{c})| \leq n_{3,i}$.

We reserve the name "creature" for a simple creature that is expanded by another coordinate, a natural number. In the wider realm of creatures, simple i-creatures and i-creatures, having a singleton base, can be counted as tree-creating creatures.

For a nonnegative real number r we let $m = [r]$ be the largest natural number such that $m \leq r$. We let log denote the logarithm function with base 2.

The following definition has ideas from [18, Chapter V, Section 6] and is the most important definition in this work.

Definition 2.5

(1) For a simple i-creature $\mathbf{c}$ we define $\mathrm{nor}^0(\mathbf{c})$ as the maximal natural number $m \leq n_{1,i}$ such that $m = 0$ or
   (α) if $a \subseteq n_{2,i}$, $|a| \leq m$, and $B_0, \dots, B_{m-1}$ are branches of $\mathbf{T}$, then there is $\nu \in \mathrm{pos}(\mathbf{c})$ such that
$$\Big(\forall x \in \Big(\bigcup_{\ell<m} B_\ell \cap \mathrm{dom}(\nu)\Big) \setminus \mathrm{dom}\big(\eta(\mathbf{c})\big)\Big)\big(\nu(x) \notin a\big);$$
   (β) $\max\{|\mathrm{dom}(\nu)| : \nu \in \mathrm{pos}(\mathbf{c})\} \leq \frac{n_{1,i}}{m}$;
   (γ) $|\mathrm{pos}(\mathbf{c})| \leq \frac{n_{3,i}}{m}$.
(2) If $\mathrm{nor}^0(\mathbf{c}) > 1$, we define $\mathrm{nor}^1(\mathbf{c}) = \log(\mathrm{nor}^0(\mathbf{c}))$; otherwise, $\mathrm{nor}^1(\mathbf{c}) = 0$.

Remark 2.6 Note that, in (α), only finitely many m-tuples of branches of $\mathbf{T}$ need to be checked. Indeed, only the part of $\mathbf{T}$ intersecting with $\bigcup\{\mathrm{dom}(\eta) : \eta \in \mathrm{pos}(\mathbf{c})\}$ matters for computing the norm.

Sometimes it is useful not only to know that $\mathrm{nor}^0(\mathbf{c}) \geq m$ but also to pin down a norm exactly.

Lemma 2.7 *Suppose that* $\mathrm{nor}^0(\mathbf{c}) = m$ *and* $m' < m$. *Then there is a subset* $p \subseteq \mathrm{pos}(\mathbf{c})$ *such that the subcreature* $\mathbf{c}' = \mathbf{c} \restriction p = (i(\mathbf{c}), \eta(\mathbf{c}), p)$ *fulfills* $\mathrm{nor}^0(\mathbf{c}') = m'$.

Proof For a simple i-creature $\mathbf{c}$ we define $\mathrm{nor}^{0,0}(\mathbf{c})$ as the maximal natural number $m \leq n_{1,i}$ such that $m = 0$ or if $a \subseteq n_{2,i}$ and $|a| \leq m$ and $B_0, \dots, B_{m-1}$ are branches of $\mathbf{T}$, then there is $\nu \in \mathrm{pos}(\mathbf{c})$ such that $(\forall x \in (\bigcup_{\ell<m} B_\ell \cap \mathrm{dom}(\nu)) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\nu(x) \notin a)$. By the relationship between nor^0 and $\mathrm{nor}^{0,0}$ and since taking a subcreature does not decrease the nor^0 if in its computation clause β or clause γ is decisive, the lemma follows from the following statement. Suppose that $\mathrm{nor}^{0,0}(\mathbf{c}) = m$. Then there is a subset $p \subseteq \mathrm{pos}(\mathbf{c})$ such that the subcreature $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}), p)$ fulfills $\mathrm{nor}^{0,0}(\mathbf{c}') = m - 1$. For proving the latter statement, we first take $p' \subseteq \mathrm{pos}(\mathbf{c})$ such that it is minimal with $\mathrm{nor}^{0,0}(\mathbf{c} \restriction p') = m$. Then we remove one element, call it ν, from p' and call the outcome p. By minimality $\mathrm{nor}^{0,0}(\mathbf{c} \restriction p) \leq m - 1$. We show $\mathrm{nor}^{0,0}(\mathbf{c} \restriction p) \geq m - 1$. If $m \geq 1$, then $p \neq \emptyset$. We assume that $m \geq 2$. Let $a \subseteq n_{2,i}$, let $|a| = m - 1$, and let $B_1, \dots, B_{m-1}$ be given. We take $a \cup \{\nu(x)\}$ for an $x \in \mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta(\mathbf{c}))$; it does not matter which.

We take B_m so that $\nu(x) \in B_m$. Then by $\operatorname{nor}(\mathbf{c}) = m$ in $\operatorname{pos}(\mathbf{c})$ there is $\nu' \in \operatorname{pos}(\mathbf{c})$ such that

$$\big(\forall y \in (B_1 \cup \cdots \cup B_m) \setminus \operatorname{dom}(\eta(\mathbf{c}))\big)\big(\nu'(y) \notin a \cup \{\nu\}\big).$$

Thus, $\nu' \neq \nu$ and we have $\nu' \in p$. □

Some of the requirements on the norms in the conditions of Lemmas 2.13–2.16 are easy to fulfill. Most of the time the requirement of Definition 2.5(1.α) is the hardest one.

Definition 2.8 We let $\ell_i = 2^{\prod_{j \le i} n_{3,j}}$.

Definition 2.9 An *i-creature* is a tuple $\mathbf{c} = (i(\mathbf{c}), \eta(\mathbf{c}), \operatorname{pos}(\mathbf{c}), k(\mathbf{c}))$ such that

(1) $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}), \operatorname{pos}(\mathbf{c}))$,
(2) $k(\mathbf{c}) \in \omega$.

Definition 2.10

(1) For $\ell \in \omega \setminus \{0\}$, $f_\ell : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_0^+$ is defined by cases as

$$f_\ell(n,k) := \begin{cases} 0 & \text{if } n = 0 \text{ or } \log(n) \le k+1, \\ \frac{\log(\log(n)-k)}{\ell} & \text{otherwise.} \end{cases}$$

(2) We let $f = \langle f_\ell : \ell < \omega \rangle$. For an i-creature $\mathbf{c} = (\mathbf{c}', k(\mathbf{c}))$ with $\operatorname{nor}^0(\mathbf{c}') > 0$ we define its f-norm as

$$\operatorname{nor}_f(\mathbf{c}) = f_{\ell_i}\big(\operatorname{nor}^0(\mathbf{c}'), k(\mathbf{c})\big). \tag{2.1}$$

(3) We write $\operatorname{nor}^0(\mathbf{c})$ for $\operatorname{nor}^0(\mathbf{c}')$.

Remark 2.11 We took f similarly to the functions used in [17, Section 3]. We chose ℓ_i so large that it ensures a suitable strong form of halving (see Definition 2.19, Lemma 2.20).

The following estimate is a step toward bigness (see Lemma 2.18) and the halving property for creatures (see Definition 2.19, Lemma 2.20).

Lemma 2.12 *If $f_1(n,k) \ge \log(2)$, then $f_\ell(\frac{n}{2}, k) \ge f_\ell(n,k) - \frac{1}{\ell}$.*

Proof As $\log(\frac{n}{2}) = \log(n) - 1$ we have

$$\begin{aligned} \ell \cdot f_\ell\Big(\frac{n}{2}, k\Big) &= \log\big(\log(n) - 1 - k\big) \\ &\ge \log\Big(\frac{\log(n) - k}{2}\Big) = \log\big(\log(n) - k\big) - 1 = \ell \cdot f_\ell(n,k) - 1. \end{aligned}$$

For the inequality between the second and the third term we use $f_1(n,k) = \log(\log(n) - k) \ge \log 2$. Hence, $\log(n) - k \ge 2$, and thus, $\log(n) - 1 - k \ge \frac{\log(n)-k}{2}$. □

The next lemma shows that we can extend the possibilities of a creature and at the same time decrease the norm of the creature only by a small amount.

Lemma 2.13 *Assume that*

(a) $\eta^* \in \operatorname{spec}$*;*
(b) $\mathbf{c}$ *is an i-creature with base η^*,* $\operatorname{nor}^0(\mathbf{c}) > 0$*;*
(c) $k^* > 0$*;*

(d) *for each* $\eta \in \mathrm{pos}(\mathbf{c})$ *we have: either* $k_\eta = k^*$ *and for each* $k < k^*$ *we are given* $\eta \subseteq \rho_{\eta,k} \in \mathrm{spec}_{n_{2,i}}$ *with* $|\mathrm{dom}(\rho_{\eta,k})| < n_{1,i}$ *or* $k_\eta = 1$ *and* $\rho_{\eta,0} = \eta$*;*
(e) *for each* $\eta \in \mathrm{pos}(\mathbf{c})$*, if* $k_\eta = k^* > 1$*,* $k_1 < k_2 < k^*$*,* $x_1 \in \mathrm{dom}(\rho_{\eta,k_1}) \setminus \mathrm{dom}(\eta)$*, and* $x_2 \in \mathrm{dom}(\rho_{\eta,k_2}) \setminus \mathrm{dom}(\eta)$*, then* x_1, x_2 *are* $<_{\mathbf{T}}$*-incomparable;*
(f) $\ell^* = \max\{|\mathrm{dom}(\rho_{\eta,k})| : \eta \in \mathrm{pos}(\mathbf{c}) \wedge k < k^*\}$.

Then we have the following.

(α) *There is an i-creature* $\mathbf{d}$ *given by*

$$\begin{aligned} \mathrm{pos}(\mathbf{d}) &= \{\rho_{\eta,k} : k < k^*_\eta, \eta \in \mathrm{pos}(\mathbf{c})\}, \\ \eta(\mathbf{d}) &= \eta^*, \\ k(\mathbf{d}) &= k(\mathbf{c}). \end{aligned}$$

(β) *We have* $\mathrm{nor}^0(\mathbf{d}) \geq m_0 \stackrel{\mathrm{def}}{=} \min\{\mathrm{nor}^0(\mathbf{c}), [\frac{n_{1,i}}{\ell^*}], [\frac{n_{3,i}}{k^* \cdot |\mathrm{pos}(\mathbf{c})|}], k^* - 1\}$.

Proof First we check Definition 2.4(1). Clauses (a), (b), and (c) follow immediately from the premises of the lemma.

Now for the norm, we check clause (α) of Definition 2.5(1). Let branches $B_0, \ldots, B_{m_0-1}$ of $\mathbf{T}$ and a set $a \subseteq n_{2,i}$ be given, $|a| \leq m_0$. Since $m_0 \leq \mathrm{nor}^0(\mathbf{c})$, there is some $\eta \in \mathrm{pos}(\mathbf{c})$ such that $(\forall x \in (\bigcup_{\ell < m_0} B_\ell) \cap \mathrm{dom}(\eta) \setminus \mathrm{dom}(\eta(\mathbf{c})))(\eta(x) \notin a)$. We fix such an η. If $k^*_\eta = 1$, then we are done. Now for each $\ell < m_0$, we let

$$w_{\eta,\ell} = \{j < k^* : \exists x \in B_\ell \cap \mathrm{dom}(\rho_{\eta,j}) \setminus \mathrm{dom}(\eta)\}.$$

Now we have that $|w_{\eta,\ell}| \leq 1$, because otherwise we would have $k_1 < k_2 < k^*$ in $w_{\eta,\ell}$ and $x_i \in B_\ell \cap \mathrm{dom}(\rho_{\eta,k_i}) \setminus \mathrm{dom}(\eta)$, $i = 1, 2$. Such witnesses x_1 and x_2 would be $<_{\mathbf{T}}$-comparable, in contradiction to requirement (e) of this lemma.

Since $m_0 < k^*$, there is some $j \in k^* \setminus \bigcup_{\ell < m_0} w_{\eta,\ell}$. For such a j, $\rho_{\eta,j}$ is as required.

We check clause (β) of Definition 2.5(1). We take any $\rho_{\eta,k}$. Then we have

$$\frac{|\mathrm{dom}(\rho_{\eta,k})|}{n_{1,i}} \leq \frac{\ell^*}{n_{1,i}} \leq \frac{1}{[\frac{n_{1,i}}{\ell^*}]} \leq \frac{1}{m_0},$$

as $m_0 \leq [\frac{n_{1,i}}{\ell^*}]$. Clause ($\gamma$) of Definition 2.5(1) follows from $m_0 \leq [\frac{n_{3,i}}{k^* \cdot |\mathrm{pos}(\mathbf{c})|}]$. □

Now we restate the previous lemma for applications to nor_f.

Lemma 2.14 *Assume that*

(a) $\eta^* \in \mathrm{spec}$*;*
(b) $\mathbf{c}$ *is an i-creature with base* η^**,* $\log(\log(n_{1,i})) \geq \mathrm{nor}_f(\mathbf{c}) > 2$*;*
(c) $k^* = [\sqrt{\mathrm{nor}^0(\mathbf{c})}]$ *(and it is really* nor^0 *here);*
(d) *for each* $\eta \in \mathrm{pos}(\mathbf{c})$ *we have: either* $k_\eta = k^*$ *and for each* $k < k^*$ *we are given* $\eta \subseteq \rho_{\eta,k} \in \mathrm{spec}_{n_{2,i}}$ *with* $|\mathrm{dom}(\rho_{\eta,k})| < \frac{n_{1,i}}{2^{(2^m + k(\mathbf{c}))}}$ *or* $k_\eta = 1$ *and* $\rho_{\eta,0} = \eta$*;*
(e) *for each* $\eta \in \mathrm{pos}(\mathbf{c})$*, if* $k_\eta = k^* > 1$*,* $k_1 < k_2 < k^*$*,* $x_1 \in \mathrm{dom}(\rho_{\eta,k_1}) \setminus \mathrm{dom}(\eta)$*, and* $x_2 \in \mathrm{dom}(\rho_{\eta,k_2}) \setminus \mathrm{dom}(\eta)$*, then* x_1, x_2 *are* $<_{\mathbf{T}}$*-incomparable.*

Then we have the following.

(α) *There is an i-creature* $\mathbf{d}$ *given by*

$$\begin{aligned}\operatorname{pos}(\mathbf{d}) &= \{\rho_{\eta,k} \,:\, k < k^*, \eta \in \operatorname{pos}(\mathbf{c})\},\\ \eta(\mathbf{d}) &= \eta^*,\\ k(\mathbf{d}) &= k(\mathbf{c}).\end{aligned}$$

(β) *We have* $\operatorname{nor}_f(\mathbf{d}) \geq \min(m, \operatorname{nor}_f(\mathbf{c}) - 1)$.

Proof By definition $[\frac{n_{3,i}}{|\operatorname{pos}(\mathbf{c})|}] \geq \operatorname{nor}^0(\mathbf{c})$. Hence, $[\frac{n_{3,i}}{k^* \cdot |\operatorname{pos}(\mathbf{c})|}] \geq \sqrt{\operatorname{nor}^0(\mathbf{c})}$. By the previous lemma we have $\operatorname{nor}^0(\mathbf{d}) \geq \min(\sqrt{\operatorname{nor}^0(\mathbf{c})} - 1, 2^{(2^m + k(\mathbf{c}))})$ and hence $\operatorname{nor}_f(\mathbf{d}) \geq \min(m, \operatorname{nor}_f(\mathbf{c}) - 1)$. □

The previous lemma will be used only in Section 4 in the proof of properness in Lemma 4.8. Indeed, its premise (e) is like a step in the proof that the specialization of Aronszajn trees by finite approximations has the countable antichain condition. For a proof, see, for example, Jech [7, Lemma 16.18] or [18, Chapter III, Theorem 5.4].

The following two lemmas will be used in the next section in the proof that the smooth conditions are dense. The latter property is used in the proof of properness as well.

Lemma 2.15 *Suppose that* $\mathbf{c}$, m, m' *are as follows:*

(a) $\mathbf{c}$ *is an* i*-creature,*
(b) $7 \leq \operatorname{nor}^0(\mathbf{c}) = m \leq \sqrt{n_{1,i}}$,
(c) $x \in \mathbf{T}$,
(d) $m' = [\sqrt{m}]$.

Then there is some i*-creature* $\mathbf{d}$ *such that*

(1) $\eta(\mathbf{d}) = \eta(\mathbf{c})$, $k(\mathbf{c}) = k(\mathbf{d})$,
(2) $\operatorname{pos}(\mathbf{d}) \subseteq \{\nu \in \operatorname{spec}^{\mathbf{T}} \,:\, (\exists \eta \in \operatorname{pos}(\mathbf{c}))(\eta \subseteq \nu \wedge \operatorname{dom}(\nu) = \operatorname{dom}(\eta) \cup \{x\})\}$,
(3) $\operatorname{nor}^0(\mathbf{d}) \geq \min(\frac{m}{m'+1}, m')$.

Proof For each $\eta \in \operatorname{pos}(\mathbf{c})$ we choose $m' + 1$ elements from $n_{2,i} \smallsetminus \operatorname{rge}(\eta)$ and put them into a set E_η. By (b) and (d) this set is not empty:

$$|\operatorname{rge}(\eta)| \leq |\operatorname{dom}(\eta)| \leq \frac{n_{1,i}}{m} \leq n_{2,i} - \sqrt{m'} - 2,$$

for $m \geq 7$. For each $a \in [n_{2,i}]^{m'}$ we let $\{z_{\eta,a}\} = E_\eta \setminus a$. Then we set $\nu_{\eta,a} = \eta \cup \{(x, z_{\eta,a})\}$. Since $z_{\eta,a} \notin \operatorname{rge}(\eta)$, $\nu_{\eta,a}$ is a partial specialization. We set $\eta(\mathbf{d}) = \eta(\mathbf{c})$, $k(\mathbf{d}) = k(\mathbf{c})$, and

$$\operatorname{pos}(\mathbf{d}) = \{\nu_{\eta,a}) \,:\, \eta \in \operatorname{pos}(\mathbf{c}), a \in [n_{2,i}]^{m'}\}.$$

We show that $\mathbf{d}$ is as required. Now we check the norm. Let m'' be the smallest integer greater than or equal to $\min(\frac{m}{m'+1}, m')$. For clause (α) of Definition 2.5(1), let $B_0, \dots, B_{m''-1}$ be branches of $\mathbf{T}$, and let $a \subseteq n_{2,i}$, $|a| \leq m''$. We have to find $\nu \in \operatorname{pos}(\mathbf{d})$ such that $(\forall \ell < m'')(\forall y \in \operatorname{dom}(\nu) \cap B_\ell \smallsetminus \operatorname{dom}(\eta(\mathbf{c})))(\nu(y) \notin a)$. We add a branch $B_{m''}$ with $x \in B_{m''}$. Since $m'' \leq m' = [\sqrt{m}] \leq m$, by premise (b), we find $\eta \in \operatorname{pos}(\mathbf{c})$ such that

$$(\forall \ell < m'')\big(\forall x' \in \operatorname{dom}(\eta) \cap B_\ell \smallsetminus \operatorname{dom}\big(\eta(\mathbf{c})\big)\big)\big(\eta(x') \notin a\big).$$

We fix this η. By the choice of $z_{\eta,a}$, a, and E_η, there is $\nu_{\eta,a} \in \operatorname{pos}(\mathbf{d})$ such that

$$(\forall \ell < m'')\big(\forall x' \in \operatorname{dom}(\nu_{\eta,a}) \cap B_\ell \smallsetminus \operatorname{dom}\big(\eta(\mathbf{c})\big)\big)\big(\nu_{\eta,a}(x') \notin a\big).$$

Now for item (β) of Definition 2.5(1), every element of $\mathrm{pos}(\mathbf{d})$ is just larger by 1 than an element of $\mathrm{pos}(\mathbf{c})$. So we have

$$\max\left\{\frac{|\operatorname{dom}(\nu)|}{n_{1,i}} : \nu \in \mathrm{pos}(\mathbf{d})\right\} \leq \max\left\{\frac{|\operatorname{dom}(\nu)|+1}{n_{1,i}} : \nu \in \mathrm{pos}(\mathbf{c})\right\}$$
$$\leq \frac{1}{m} + \frac{1}{n_{1,i}} \leq \frac{1}{m} + \frac{1}{m^2} \leq \frac{1}{m'} \leq \frac{1}{m''}.$$

Now for item (γ) of Definition 2.5(1), the norm drops from m to at least $[\frac{m}{m'+1}]$ by replacing each $\eta \in \mathrm{pos}(\mathbf{c})$ by at most $m'+1$ elements. □

Suppose that we have extended the partial specialization functions in the set of possibilities of a creature as in one of the previous lemmas. Then we want these extended functions to be able to serve as bases for suitable creatures as well. This is provided by the next lemma. The number i from Lemma 2.15 will now appear in Lemma 2.16 as $i-1$, since in the latter lemma new creatures $\mathbf{d}$ are constructed from simple creatures $\mathbf{c}$ by extending the base of $\mathbf{c}$.

Lemma 2.16 *Assume that we have the following.*

(a) $\mathbf{c}$ *is an* i*-creature.*

(b) $\eta^* \supseteq \eta(\mathbf{c})$, $\eta^* \in \mathrm{spec}_{n_{2,i-1}}$. *(Note that we do not suppose that* $\eta^* \in \mathrm{pos}(\mathbf{c})$.*) Furthermore, we assume that* $|\operatorname{dom}(\eta^*)| \leq n_{1,i-1}$.

(c) *For any* $\nu \in \mathrm{pos}(\mathbf{c})$, $\operatorname{dom}(\eta^*) \cap \operatorname{dom}(\nu) = \operatorname{dom}(\eta(\mathbf{c}))$.

We set

$$\ell_2^* = \big|\operatorname{dom}(\eta^*) \setminus \operatorname{dom}\big(\eta(\mathbf{c})\big)\big|,$$

and

$$Y = \big\{y : \exists \nu\big(\nu \in \mathrm{pos}(\mathbf{c}) \wedge y \in \operatorname{dom}(\nu)$$
$$\wedge (\exists x)\big(x \in \operatorname{dom}(\eta^*) \setminus \operatorname{dom}\big(\eta(\mathbf{c})\big) \wedge x \leq_{\mathbf{T}} y\big)\big)\big\};$$
$$\ell_1^* = |Y|,$$

and in addition to (a), (b), and (c) we assume that $\ell_1^* + \ell_2^* < \mathrm{nor}^0(\mathbf{c}) < \sqrt{n_{1,i}}$. *We define* $\mathbf{d}$ *by* $\eta(\mathbf{d}) = \eta^*$, $k(\mathbf{d}) = k(\mathbf{c})$, *and*

$$\mathrm{pos}(\mathbf{d}) = \big\{\nu \cup \eta^* : \nu \in \mathrm{pos}(\mathbf{c}) \ \wedge\ \nu \cup \eta^* \in \mathrm{spec}_{n_{2,i}} \ \wedge \big|\operatorname{dom}(\nu \cup \eta^*)\big| < n_{1,i}\big\}.$$

Then

(α) $\mathbf{d}$ *is an* i*-creature,*

(β) $\mathrm{nor}^0(\mathbf{d}) \geq \min(\mathrm{nor}^0(\mathbf{c}) - \ell_2^* - \ell_1^*, \frac{\mathrm{nor}^0(\mathbf{c})}{2})$.

Proof Item (α) here follows from the requirements on η^* and from the estimates on the norm. For item (β), we set $k = \mathrm{nor}^0(\mathbf{c}) - \ell_1^* - \ell_2^*$. We first consider item (α) of Definition 2.5(1). We let $B_0, \ldots, B_{k-1}$ be branches of $\mathbf{T}$ and let $a \subseteq n_{2,i(\mathbf{c})}$, $|a| \leq k$. We set $\ell^* = \ell_1^* + \ell_2^*$. We let $\langle y_\ell : \ell < \ell_1^*\rangle$ list Y without repetition. Let $B_k, \ldots, B_{k+\ell_1^*-1}$ be branches of $\mathbf{T}$ such that $y_\ell \in B_{k+\ell}$ for $\ell < \ell_1^*$. Let $\langle x_\ell : \ell < \ell_2^*\rangle$ list $\operatorname{dom}(\eta^*) \setminus \operatorname{dom}(\eta(\mathbf{c}))$. Take, for $\ell < \ell_2^*$, $B_{k+\ell_1^*+\ell}$ such that $x_\ell \in B_{k+\ell_1^*+\ell}$. We set $a' = a \cup \{\eta^*(x_\ell) : \ell < \ell_2^*\}$. Since $\mathrm{nor}^0(\mathbf{c}) \geq k + \ell^*$ there is some $\nu \in \mathrm{pos}(\mathbf{c})$ such that $\forall x \in ((\operatorname{dom}(\nu) \setminus \operatorname{dom}(\eta(\mathbf{c}))) \cap \bigcup_{\ell<k+\ell^*} B_\ell)(\nu(x) \notin a')$. Then, if $x \in \operatorname{dom}(\nu \cup \eta^*) \setminus \operatorname{dom}(\eta^*)$, we have $(\nu \cup \eta^*)(x) \notin a$. We have to show that $\nu \cup \eta^*$ is a partial specialization. Since η^* and ν are specialization maps extending $\eta(\mathbf{c})$, we have to consider only the case in which $x \in \operatorname{dom}(\eta^*) \setminus \operatorname{dom}(\eta(\mathbf{c}))$,

$y \in \mathrm{dom}(\nu) \setminus \mathrm{dom}(\eta^*)$, and $(y <_{\mathbf{T}} x \vee x \leq_{\mathbf{T}} y)$. If $x \leq_{\mathbf{T}} y$, then $y \in Y$, and we have $\nu(y) \neq \eta^*(x_\ell)$ for all $\ell < \ell_2^*$ by the choice of $B_{k+\ell_1^*+\ell}$ for $\ell < \ell_2^*$. If $y <_{\mathbf{T}} x$, then y is in a branch leading to some $x = x_\ell$ for some $\ell < \ell_2^*$, and hence again $\nu(y) \neq \eta^*(x)$.

Moreover, for item (β) in Definition 2.5(1),

$$\left|\mathrm{dom}(\nu \cup \eta^*)\right| \leq \frac{n_{1,i}}{\mathrm{nor}^0(\mathbf{c})} + \ell_2^* \leq \frac{n_{1,i}}{\mathrm{nor}^0(\mathbf{c})} + \mathrm{nor}^0(\mathbf{c}) \leq 2\frac{n_{1,i}}{\mathrm{nor}^0(\mathbf{c})}.$$

For item (γ) we do not have anything to check, since $|\mathrm{pos}(\mathbf{d})| \leq |\mathrm{pos}(\mathbf{c})|$. □

Remark 2.17

(1) Apparently the premises of the previous lemma are hard to fulfill. In the proofs of the density properties we add ℓ_2^* points to the domain of the functions in the set of possibilities of a creature with sufficiently high norm. Moreover, $\ell_1^* \leq |u|$, where u is the set that sticks out of $\mathbf{T}_{<\alpha(p)}$ (see Definition 3.3(A.c)). We will suppose that $|u|$ and ℓ_2^* are small in comparison to $\mathrm{nor}^0(\mathbf{c})$, so that the premises for Lemma 2.16 are fulfilled.

(2) Only $\ell_2^* = 1$ is used (namely, in the proof of Lemma 3.11) since we can fill in the elements of the Aronszajn tree in the domains of partial specializations in conditions one by one.

The next lemma will help to find large homogeneous subtrees of the trees built from creatures that will later be used as forcing conditions.

Lemma 2.18 (The 2-bigness property; see [15, Definition 2.3.2]) *If* $\mathbf{c}$ *is an* i*-creature with* $\mathrm{nor}^1(\mathbf{c}) \geq m+1$, *and* $\mathbf{c}_1$, $\mathbf{c}_2$ *are* i*-creatures such that* $\mathrm{pos}(\mathbf{c}) = \mathrm{pos}(\mathbf{c}_1) \cup \mathrm{pos}(\mathbf{c}_2)$ *and* $\eta(\mathbf{c}) = \eta(\mathbf{c}_1) = \eta(\mathbf{c}_2)$ *and* $k(\mathbf{c}) = k(\mathbf{c}_1) = k(\mathbf{c}_2)$, *then* $\mathrm{nor}^1(\mathbf{c}_1) \geq m$ *or* $\mathrm{nor}^1(\mathbf{c}_2) \geq m$. *Under the same premises we have that if* $m \geq 1$ *and* $\mathrm{nor}_f(\mathbf{c}) \geq m+1$, *then* $\mathrm{nor}_f(\mathbf{c}_1) \geq m$ *or* $\mathrm{nor}_f(\mathbf{c}_2) \geq m$.

Proof We let $j = 2^m$. We suppose that $\mathrm{nor}^0(\mathbf{c}_1) < j$ and $\mathrm{nor}^0(\mathbf{c}_2) < j$ and derive a contradiction. For $\ell = 1, 2$ let branches $B_0^\ell, \dots, B_{j-1}^\ell$ and sets $a^\ell \subseteq n_{2,i}$, $|a^\ell| \leq j$, exemplify this.

Let $a = a^1 \cup a^2$, and let, by $\mathrm{nor}^0(\mathbf{c}) \geq 2j$, $\eta \in \mathrm{pos}(\mathbf{c})$ be such that for all $x \in (\mathrm{dom}(\eta) \cap \bigcup_{\ell=1,2} \bigcup_{i=0}^{j-1} B_i^\ell) \setminus \mathrm{dom}(\eta(\mathbf{c}))$ we have $\eta(x) \notin a$. But then for that $\ell \in \{1, 2\}$ for which $\eta \in \mathrm{pos}(\mathbf{c}_\ell)$ we get a contradiction to $\mathrm{nor}^0(\mathbf{c}_i) < j$. Hence, for $i = 0$ or for $i = 1$, $\mathrm{nor}^1(\mathbf{c}_i) \geq m$. The inequality also holds for nor_f by the definition of f_ℓ: $f_\ell(\frac{n}{2}, k) \geq f_\ell(n, k) - 1$ for $f_\ell(n, k) \geq \log(2)$. □

Now for the first time we make use of the coordinate $k(\mathbf{c})$ of our creatures. The next lemma states that the creatures have the halving property. Originally the halving property was introduced in [15, Definition 2.2.7]. Our version is similar to the strong form of halving in [17, Definition 3.1].

Definition 2.19

(A) Let $\mathbf{c}$ be an i-creature as in Definition 2.9. Let $\ell \in \omega \setminus \{0\}$. We say that $\mathbf{c}^*$ is an $\frac{1}{\ell}$*-half of* $\mathbf{c}$ if the following hold.

(1) $i(\mathbf{c}^*) = i(\mathbf{c})$, $\eta(\mathbf{c}^*) = \eta(\mathbf{c})$, $\mathrm{nor}_f(\mathbf{c}^*) \geq \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}$, $\mathrm{pos}(\mathbf{c}^*) = \mathrm{pos}(\mathbf{c})$, $k(\mathbf{c}^*) \geq k(\mathbf{c})$.

(2) *If* $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}'))$ satisfies

(i) there is a map $\pi\colon \{\eta(\mathbf{c}')\} \cup \mathrm{pos}(\mathbf{c}') \to \{\eta(\mathbf{c}^*)\} \cup \mathrm{pos}(\mathbf{c}^*)$ such that, for each $\nu \in \mathrm{pos}(\mathbf{c}')$, $\nu \supseteq \pi(\nu)$,[1] or $\mathbf{c}'$ is just any $i(\mathbf{c})$-creature,

(ii) $k(\mathbf{c}') \geq k(\mathbf{c}^*)$, and

(iii) $\mathrm{nor}_f(\mathbf{c}') > 0$,

then $\mathbf{c}_0 = (i(\mathbf{c}), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}_0))$ is an $i(\mathbf{c})$-creature with $\mathrm{nor}_f(\mathbf{c}_0) \geq \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}$.

(B) Let $\ell \in \omega \setminus \{0\}$. We say that K has the $\frac{1}{\ell}$*-halving property* if for each creature $\mathbf{c} \in K$ there is an $\frac{1}{\ell}$-half of $\mathbf{c}$.

(C) Let $\mathbf{c}$ be an i-creature as in Definition 2.9. Let $\mathrm{nor}_f(\mathbf{c}) > 1$, and let $\ell = \ell_i$. We say that $\mathbf{c}^*$ is the *standard* $\frac{1}{\ell}$*-half* of $\mathbf{c}$ if the following hold:

$$\mathbf{c}^* = \Big(i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}), \Big\lceil \frac{\log(\mathrm{nor}^0(\mathbf{c})) + k(\mathbf{c})}{2} \Big\rceil\Big).$$

(D) Let $\mathbf{c}'$ be an i-creature as in Definition 2.9. Let $\mathbf{c}^*$ be its standard $\frac{1}{\ell_i}$-half. Let $\mathrm{nor}_f(\mathbf{c}') > 1$, let $\ell = \ell_i$, and let $k(\mathbf{c}') \geq \lceil \frac{\log(\mathrm{nor}^0(\mathbf{c}))+k(\mathbf{c})}{2} \rceil = k(\mathbf{c}^*)$. We say that $\mathbf{c}_0$ is the *standard de-halving of* $\mathbf{c}'$ *with respect to* $(\mathbf{c}, \mathbf{c}^*)$ if $\mathbf{c}_0 = (i, \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}))$. We write

$$\mathbf{c}_0 = \text{de-halve}(\mathbf{c}', \mathbf{c}, \mathbf{c}^*).$$

Lemma 2.20 *If* $\mathbf{c} \in K$ *is an* i*-creature with* $\mathrm{nor}_f(\mathbf{c}) > 0$*, then it has the* $\frac{1}{\ell_i}$*-halving property.*

Proof Let $\mathbf{c}$ be an i-creature, and let $\ell = \ell_i$. We let $\mathbf{c}^* = (i(\mathbf{c}), s(\mathbf{c}), \mathrm{pos}(\mathbf{c}), k(\mathbf{c}^*))$ with

$$k(\mathbf{c}^*) = \Big\lceil \frac{\log(\mathrm{nor}^0(\mathbf{c})) + k(\mathbf{c})}{2} \Big\rceil.$$

Since $\mathrm{nor}_f(\mathbf{c}) > 0$, we have $k(\mathbf{c}^*) \geq k(\mathbf{c})$. Then

$$\begin{aligned} \mathrm{nor}_f(\mathbf{c}^*) &= \frac{\log(\log(\mathrm{nor}^0(\mathbf{c}^*)) - k(\mathbf{c}^*))}{\ell} \\ &\geq \frac{\log(\frac{\log(\mathrm{nor}^0(\mathbf{c}))-k(\mathbf{c})}{2})}{\ell} \\ &= \frac{\log(\log(\mathrm{nor}^0(\mathbf{c})) - k(\mathbf{c}) - 1)}{\ell} \\ &= \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}. \end{aligned}$$

Now let $\mathbf{c}' = (i(\mathbf{c}), \eta(\mathbf{c}), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}'))$ be any creature with $k(\mathbf{c}') \geq k(\mathbf{c}^*)$ and $\mathrm{nor}_f(\mathbf{c}') > 0$. Then we take $\mathbf{c}_0 = (i(\mathbf{c}), \eta(\mathbf{c}'), \mathrm{pos}(\mathbf{c}'), k(\mathbf{c}))$ and get

$$\begin{aligned} \mathrm{nor}_f(\mathbf{c}_0) &= \frac{\log(\log(\mathrm{nor}^0(\mathbf{c}_0)) - k(\mathbf{c}))}{\ell} \\ &\geq \frac{\log(\log(\mathrm{nor}^0(\mathbf{c}')) - k(\mathbf{c}') + \frac{\log(\mathrm{nor}^0(\mathbf{c}))-k(\mathbf{c})}{2} - 1)}{\ell} \\ &\geq \frac{\log(\log(\mathrm{nor}^0(\mathbf{c})) - k(\mathbf{c})) - 1}{\ell} \\ &= \mathrm{nor}_f(\mathbf{c}) - \frac{1}{\ell}. \end{aligned}$$

From line 1 to 2 we use

$$-k(\mathbf{c}) \geq -k(\mathbf{c}') - \frac{\log(\mathrm{nor}^0(\mathbf{c})) - k(\mathbf{c})}{2} - 1,$$

since $k(\mathbf{c}') > \frac{\log(\mathrm{nor}^0(\mathbf{c}))+k(\mathbf{c})}{2} - 1$, as $k(\mathbf{c}') \geq k(\mathbf{c}^*)$. From line 2 to 3 we use that $\mathrm{nor}_f(\mathbf{c}') > 0$ implies that $\log(\mathrm{nor}^2(\mathbf{c}')) - k(\mathbf{c}') \geq 1$. □

Why is this strong form of halving useful? In the next section, we define the half of a conditions in Definition 3.15 and the de-halve of conditions in Lemma 3.16. Roughly speaking, decisions that are taken by some $r \geq_0 \mathrm{half}(p)$ with $\mathrm{nor}_f(\mathbf{c}_{r,t}) > 0$ for any $t \in T^r$ are also taken by the "de-half" of r with respect to p (see Lemma 3.16). Then $\text{de-halve}(r, p, 0) \geq_0 p$ and $\mathrm{nor}_f(\mathbf{c}_{\text{de-halve}(r,p,0),t})$ is sufficiently large for any t. This idea is carried further at the end of the next section.

3 Tree Forcings with Creatures

Now we construct a notion of forcing from the creatures introduced in the previous section.

Definition 3.1 Let $t = \langle \eta_0, \dots, \eta_{n-1} \rangle$ denote a strictly increasing sequence of finite length of finite partial specializations. We let, for $n \geq 1$, $\mathrm{first}(t) = \eta_0$ and $\mathrm{last}(t) = \eta_{n-1}$ denote the first and the last entries of t, respectively, and we let $\mathrm{lg}(t) = n$ denote the length of t.

Now we consider endless trees $(T, <_T)$ of elements (nodes) of the form $t = \langle \eta_0, \dots, \eta_n \rangle$, ordered by end extension. A partial specialization η can appear in two different nodes. Below we define a notion of forcing with labeled trees $\langle \mathbf{c}_t : t \in (T, <_T) \rangle$ as components of conditions. The notation with the angled brackets $\langle \mathbf{c}_t : t \in (T, <_T) \rangle$ denotes a structure $(T, <_T)$ together with function $\mathbf{c}\colon T \to \mathbf{V}$ with $\mathbf{c}(t) = \mathbf{c}_t$. A condition has the form $p = (i(p), (T^p, <_{T^p}), \langle \mathbf{c}_{p,t} : t \in (T^p, <_{T^p}) \rangle)$. To every node t of such a finitely branching endless tree $(T^p, <_p) = (T^p, <_{T^p})$ we attach a creature $\mathbf{c}_{p,t}$ from Definition 2.4. This gives $\mathbb{Q}_{\mathbf{T}}$. We consider only assignments $t \mapsto \mathbf{c}_t$ that fulfill $\eta(\mathbf{c}_t) = \mathrm{last}(t)$.

We recall some notions about trees.

Definition 3.2

(1) A *tree* $(T, <_T)$ is a nonempty set T with a partial order $<_T$ such that, for $t \in T$, $\{s \in T : s <_T t\}$ is a finite linear order.

(2) We define the *set of immediate successors of s in T* by

$$\mathrm{suc}_T(s) = \big\{t \in T : s <_T t \wedge \neg(\exists r \in T)(s <_T r <_T t)\big\}.$$

(3) The *restriction of T to nodes that are comparable with s* is

$$T^{\langle s \rangle} = \{t \in T : s \leq_T t \vee s \leq_T t\}.$$

(4) A tree is called *endless* if $\max(T) = \{s \in T : \neg(\exists t \in T)(s <_T t)\} = \emptyset$.

(5) Now let $(\mathbf{T}, <_{\mathbf{T}})$ be an Aronszajn tree as in Definition 1.1, and let $i \in \omega \setminus \{0\}$. A tree $(T, <_T)$ is an *$(i, \mathbf{T})$-tree* if the following hold.

(a) $(T, <_T)$ is endless.

(b) $T \subseteq \{\langle \eta_0, \eta_1, \dots, \eta_n \rangle \in {}^{\omega>}(\mathrm{spec}^{\mathbf{T}}) : (\forall i < j \leq n)(\eta_i \subsetneq \eta_j)\}$. Elements $t = \langle \eta_0, \dots, \eta_n \rangle$ of T are also called nodes of T.

(c) The tree order $\leq_T$ is just the initial segment relation $\trianglelefteq$: $s \trianglelefteq t$ if and only if $t \restriction \lg(s) = s$. We write $\triangleleft$ for the corresponding strict relation.

(d) In T there is a least element, called the *root*, $\mathrm{rt}(T)$, which has the form $\langle \eta_0 \rangle$ and $\eta_0 \in \mathrm{spec}^{\mathbf{T}}_{n_{2,i-1}}$. We also write η_0 instead of $\langle \eta_0 \rangle$. The root counts as a sequence of length 1 and is the unique element of $T^{[1]}$, the level number 1 of T.

(e) If $\mathrm{rt}(T) \triangleleft \bar{\nu} \triangleleft \bar{\eta}$ and $\bar{\eta} \in T$, then $\bar{\nu} \in T$.

(f) $(T, <_T)$ is a finitely branching tree of height ω.

(6) The *set of branches through* T is

$$\lim(T) = \big\{ \langle \eta_k \, : \, k < \omega \rangle \, : \, (\forall n) \langle \eta_0, \ldots, \eta_n \rangle \in T \big\}.$$

(7) A subset F of T is called a *front of* T if every branch of T passes through this set and the set consists of $<_T$-incomparable elements.

Definition 3.3 Let $\mathbf{T}$ be an Aronszajn tree. We define a notion of forcing $\mathbb{Q} = \mathbb{Q}_{\mathbf{T}}$ with set of elements $\mathbb{Q}$ and a preorder $\leq_{\mathbb{Q}}$.

(A) $p \in \mathbb{Q}$ if $p = (i(p), (T^p, <_p), \langle \mathbf{c}_{p,t} \, : \, t \in (T^p, <_{T^p}) \rangle)$ has the following properties:

(a) $i(p) \in \omega \setminus \{0\}$, $T^p \subseteq {}^{\omega>}\mathrm{spec}^{\mathbf{T}}$. We write $\mathrm{dom}(p) = T^p$. We require that $(T^p, <_p)$ is a $(i(p), \mathbf{T})$-tree. The elements of T are of the form $\langle \eta_0, \ldots, \eta_n \rangle$ such that $\eta_j \subsetneq \eta_{i+1}$ and $\eta_j \in \mathrm{spec}^{\mathbf{T}}_{n_{2,i(p)+j-1}}$ for $j \geq 0$. The tree ordering $\leq_T$ is end extension.

(b) For $n \in \omega$, the *nth level of* T is

$$T^{[n]} = \big\{ t \in T \, : \, \lg(t) = n \big\}.$$

We also write $p^{[m]}$ instead of $(T^p)^{[m]}$, write $\mathrm{suc}_p(s)$ for $\mathrm{suc}_{T^p}(s)$, and call a front of T^p also a front of p. We let $\mathrm{rt}(p) = \mathrm{rt}(T^p)$. So $T^{[1]} = \{\mathrm{rt}(p)\}$.

For any $1 \leq \ell < \omega$ and $s \in (T^p)^{[\ell]}$ there is an $i(p) + \ell - 1$-creature $\mathbf{c}_{p,s}$ such that

$$\eta(\mathbf{c}_{p,s}) = \mathrm{last}(s), \qquad (*)$$

$$\mathrm{pos}(\mathbf{c}_{p,s}) = \big\{ \mathrm{last}(t) \, : \, t \in (T^p)^{[\ell+1]} \, : \, t \in \mathrm{suc}_p(s) \big\}.$$

(c) There is $\alpha = \alpha(p) \in \omega_1$ such that the following holds.[2] For some $0 \leq h < \omega$ for every $t \in (T^p)^{[h]}$ there is a finite set $u_t \subseteq \mathbf{T} \setminus \mathbf{T}_{<\alpha}$ such that for every ω-branch $\langle \eta_\ell \, : \, \ell < \omega \rangle$ of T^p satisfying $\eta_h = \mathrm{last}(t)$ we have $\bigcup_{\ell \in \omega} \mathrm{dom}(\eta_\ell) = \mathbf{T}_{<\alpha} \cup u_t$. We let $h(p)$ be the least such h.

(d) For every ω-branch $\langle \eta_\ell \, : \, \ell \in \omega \rangle$ of T^p with $t_\ell = \langle \eta_0, \ldots, \eta_\ell \rangle$ we have $\lim_{\ell \to \omega} \mathrm{nor}^0(\mathbf{c}_{p,t_\ell}) = \omega$.

(B) The order $\leq = \leq_{\mathbb{Q}}$ is given by letting $p \leq q$ (q is stronger than p—we follow the Jerusalem convention) if there is a projection $\mathrm{pr}_{q,p}$ which satisfies

(a) $\mathrm{pr}_{q,p}$ is a function from T^q to T^p such that, for every $t \in T^q$,

$$\lg(t) + i(q) = \lg\big(\mathrm{pr}_{q,p}(t)\big) + i(p)).$$

(Hence, $i(\mathbf{c}_{q,t}) = i(\mathbf{c}_{p,\mathrm{pr}_{q,p}(t)})$.)

(b) If $t \in T^q$, then $\mathrm{last}(t) \supseteq \mathrm{last}(\mathrm{pr}_{q,p}(t))$. This holds of course not only for the last element of the sequence t but for all elements, since T^q is downward closed.

(c) If t_1, t_2 are both in $\mathrm{dom}(q)$, then $t_1 \leq_q t_2$ if $\mathrm{pr}_{q,p}(t_1) \leq_p \mathrm{pr}_{q,p}(t_2)$.
(d) For any $\ell \in \omega$, if $s_1 \in (T^q)^{[\ell]}, s_2 \in (T^q)^{[\ell+1]}, s_1 <_q s_2, \mathrm{pr}_{q,p}(s_2) = t_2$, and $\mathrm{pr}_{q,p}(s_1) = t_1$, then $\mathrm{dom}(\mathrm{last}(t_2)) \cap \mathrm{dom}(\mathrm{last}(s_1)) = \mathrm{dom}(\mathrm{last}(t_1))$.
(e) $k(\mathbf{c}_{q,t}) \geq k(\mathbf{c}_{p,\mathrm{pr}_{q,p}(t)})$.

The projection in general is neither injective nor surjective. In all our fusion constructions to come we will have $i(p) = i(q)$, so the counting with the lengths of nodes is not too difficult.

We give some informal description of the $\leq$-relation in $\mathbb{Q}$. The stronger condition's domain is via $\mathrm{pr}_{q,p}$ mapped homomorphically with respect to the tree orders into $T^{p^{\langle \mathrm{pr}_{q,p}(\mathrm{rt}(q))\rangle}}$. The projection is in general neither one-to-one nor onto. The root can grow as well. According to Definition 3.3(B.a), the projection preserves the i of the respective creatures, that is, level of the node plus the i-number of the tree. The partial specialization functions sitting on the nodes of the tree are extended (possibly by more than one extension per function) in q so as to compare them with the ones attached to the image under $\mathrm{pr}_{q,p}$ according to Definition 3.3(B.b), but by Definition 3.3(B.a) the extensions are so small and so few that they preserve the kind i of the creature given by the node and its successors, and according to Definition 3.3(B.d) the new part of the domain of the extension is disjoint from the domains of the old partial specialization functions living higher up in the projection of the new tree to the old tree.

Lemma 3.4 *We have that* $\mathbb{Q}_{\mathbf{T}} \neq \emptyset$.

Proof We assume without loss of generality that the Aronszajn tree $\mathbf{T}$ has ω as $\mathbf{T}_0$. We build T^p, $\mathbf{c}_{p,t}$ by induction on the height of T^p. For each $i \geq 1$, each node $t = \langle \eta_{t,0}, \ldots, \eta_{t,i-1}\rangle$ at level i has $\mathrm{nor}^0(\mathbf{c}_{p,t}) \geq i$ for $i \geq 1$ and $\mathrm{dom}(\eta_{t,i-1}) = i$ (independently of t, so that Definition 3.3(A.d) will be fulfilled). Recall our choice, $n_{2,i} \geq n_{1,i} \geq 2^{i^2}$ and $n_{1,i+1} \geq 2^{(n_{2,i})^2}$.

We start with $T^{[1]} = \{\emptyset\}$ and let $\eta(\mathbf{c}_{p,\emptyset}) = \emptyset$. Given $T^{[i]}$ and $t \in T^{[i]}$, we take $\mathrm{pos}(\mathbf{c}_{p,t}) = \{\eta(\mathbf{c}_{p,t}) \cup \{(\max(\mathrm{dom}(\eta(\mathbf{c}_{p,t}))) + 1, k)\} : k \leq i\}$.

To compute the norm is easy since all branches in the Aronszajn tree intersecting with $\mathrm{dom}(\eta)$ for $\eta \in \mathrm{pos}(\mathbf{c})$ have length 1. This defines $T^{[i+1]}$.

A more detailed proof with a less flat part of the Aronszajn tree taken as the union of the domains of the components of the nodes of T^p is given in Lemma 3.11. □

Definition 3.5 Let $p \in \mathbb{Q}$.

(1) For $t = \langle \eta_0, \ldots, \eta_n\rangle \in T^p$ we let $q = p^{\langle t\rangle}$ be given by
  (a) $T^q := \{s : \langle \eta_0, \ldots, \eta_{n-1}\rangle^\frown s \in T^p, \mathrm{first}(t) = \eta_n\}$. In particular, $\mathrm{rt}(q) = \eta_n$,
  (b) $\mathrm{pr}_{q,p}(s) = \langle \eta_0, \ldots, \eta_{n-1}\rangle^\frown s$ for $s \in T^q$,
  (c) $\mathbf{c}_{q,s} = \mathbf{c}_{p,\mathrm{pr}_{q,p}(s)}$ for $s \in T^q$.
(2) For $n \in \omega \setminus \{0\}$, we let $p \uparrow n = \langle\langle t, \mathbf{c}_{p,t}\rangle : t \in ((T^p)^{[<n]}, <_p \restriction ((T^p)^{[<n]})^2)\rangle$.

Note that

$$p \uparrow n \text{ determines } T^p \restriction \{t : \mathrm{ht}_p(t) \leq n\},$$
$$\text{since } \mathbf{c}_{p,t} \text{ determines } \mathrm{pos}(\mathbf{c}_{p,t}).$$

Definition 3.6

(1) $p \in \mathbb{Q}$ is called *normal* if for every ω-branch $\langle \eta_\ell : \ell \in \omega \rangle$ of T^p with $t_\ell = \langle \eta_0, \dots, \eta_\ell \rangle$ the sequence $\langle \text{nor}(\mathbf{c}_{p,t_\ell}) : \ell \in \omega \rangle$ is nondecreasing.
(2) $p \in \mathbb{Q}$ is called *smooth* if in Definition 3.3(A.c) the number h is 0 and u_t is empty.
(3) $p \in \mathbb{Q}$ is called *weakly smooth* if in Definition 3.3(A.c) the number h is 0.

Fact 3.7

(1) *Definition 3.3(B.d) does not only hold for ℓ and $\ell + 1$ but does for any finite difference of levels.*
(2) *If $p \le q$ and p is weakly smooth with witness u, then* $(t \in T^q \wedge \lg(t) \ge 1) \to \text{dom}(\text{last}(t)) \cap (\mathbf{T}_{<\alpha(p)} \cup u) = \text{dom}(\text{last}(\text{pr}_{q,p}(t)))$.

Proof Item (1) is obvious. To prove item (2): If p is weakly smooth, then all branches of T^p have the same union of domains of their entries, and hence, the condition $\text{dom}(\text{last}(s_1)) \supseteq \text{dom}(\text{last}(t_1))$ is fulfilled t_1 and t_2 from Definition 3.3(B.d) are in the range of $\text{pr}_{q,p}$ or not. □

Definition 3.8

(1) For $0 \le n < \omega$ we define the partial order $\le_{f,n} = \le_n$ on $\mathbb{Q}$ by letting $p \le_n q$ if
 (i) $p \le q$,
 (ii) $\text{rt}(p) = \text{rt}(q)$ and $i(p) = i(q)$,
 (iii) $p \upharpoonright n = q \upharpoonright n$ for $n \ge 1$,
 (iv) for any projection $\text{pr}_{q,p}$ witnessing $p \le q$,

$$(\forall t \in T^q)\big(\mathbf{c}_{q,t} \neq \mathbf{c}_{p,\text{pr}_{q,p}(t)} \to \big(\text{nor}_f(\mathbf{c}_{p,\text{pr}_{q,p}(t)}) \ge n \wedge \text{nor}_f(\mathbf{c}_{q,t}) \ge n\big)\big).$$

Recall the definition of nor_f. For $t \in p^{[j]}$ we take $f_{\ell_{i(p)+j}}$ as in Definitions 2.10(2) and 2.8.

(2) We define $\le_n^0$ analogously, with nor^0 in item (iv).

Note that $\mathbf{c}_{p,\text{rt}(p)} = \mathbf{c}_{q,\text{rt}(p)}$ is a requirement on two levels in T^p and in T^q. So property (iii) says that also on the level n the two trees still coincide. We state and prove some basic properties of the notions defined above.

Lemma 3.9

(1) *$(\mathbb{Q}, \le_{\mathbb{Q}})$ is a partial order.*
(2) *For every p we have that* $\lim_{n\to\omega} \min\{\text{nor}(\mathbf{c}_{p,t}) : t \in (T^p)^{[n]}\} = \infty$.
(3) *If $p \in \mathbb{Q}$, $\ell \in \omega$, and $t \in p^{[\ell]}$, then* $|\text{dom}(\text{last}(t))| < n_{2,i(p)+\ell-2}$.
(4) *If $q \ge p$ and both are smooth, then $\alpha(q) \ge \alpha(p)$.*

Proof

(1) Given $p \le q$ and $q \le r$ we define $\text{pr}_{r,p} = \text{pr}_{q,p} \circ \text{pr}_{r,p}$. It is easily seen that this function is as required.

(2) This follows from König's lemma. Since T^p is finitely branching, there is a branch through every infinite subset.

(3) This follows from Definitions 2.4 and 3.3. □

Lemma 3.10 *Let $\langle n_i : i \in \omega \rangle$ be a strictly increasing sequence of natural numbers. We assume that, for every i, $q_i \le_{n_i} q_{i+1}$ and that each p_i is smooth. Then*

$q = \bigcup_{i<\omega} q_i \upharpoonright (T^{q_i})^{[n_{i-1},n_i)} \in \mathbb{Q}$, *and for all* i, $q \geq_{n_i} q_i$ *and* $\alpha(q) = \sup\{\alpha(p_i) : i < \omega\}$.

Proof This is clear. We remark that smoothness is necessary. □

Now we need to know that the set of smooth conditions is dense in $\mathbb{Q}_{\mathbf{T}}$. We prove this in the next lemma by a more general fusion construction that works for arbitrary p_0. The finite sets sticking out of $\alpha(p_n)$ in the sense of Definition 3.3(A.c) are gradually filled up, each finite part to the same $\alpha(q)$. Actually, already after finitely many filling-up steps the unions of the domains above t for $t \in p^{[h]}$ are the same.

Lemma 3.11 *If* $p \in \mathbb{Q}$, $\alpha \in \omega_1$, $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in T^q\} \subseteq \mathbf{T}_{<\alpha}$, *and* $n < \omega$, *then there is* q *such that*

(1) $p \leq_n q$,
(2) q *is smooth and* $\alpha(q) = \alpha$,
(3) *for each branch* b *of* T^q, $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in b\} = \mathbf{T}_{<\alpha}$.

Proof We write the proof for nor^0. Without loss of generality, we assume that $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in T^p\} \neq \mathbf{T}_{<\alpha}$.

Since $\bigcup\{\mathrm{dom}(\eta) : \exists t \in T^p, \mathrm{last}(t) = \eta\} \subsetneq \mathbf{T}_{<\alpha}$ we have $\alpha(p) < \alpha$. By Definition 3.3(A.c), there is some $h < \omega$ such that for every $t \in p^{[h]}$ there is some $u_t \in [\mathbf{T} \setminus \mathbf{T}_{<\alpha(p)}]^{<\omega}$ such that for every ω-branch $\langle \eta_\ell : \ell < \omega\rangle$ of T^p with $\langle \eta_0, \dots, \eta_h\rangle = t$ we have $\bigcup_{\ell\in\omega} \mathrm{dom}(\eta_\ell) = \mathbf{T}_{<\alpha(p)} \cup u_t$.

We fix such an h and such a u_t, $t \in p^{[h]}$. Now for each $t \in p^{[h]}$ separately we perform the following inductive filling up. Fix $t \in p^{[h]}$. Let $\{x^t_\ell : \ell < \omega\}$ enumerate $\mathbf{T}_{<\alpha} \setminus (u_t \cup T_{<\alpha(p)})$. We assume that $n \geq (|u_t| + 1)^2$ for every $t \in T^{[h]}$. We let $p_{t,0} = p^{\langle t\rangle}$, $n_0 = n$.

By induction on $\ell \in \omega$ we choose $p_{t,\ell}$ and n_ℓ with the following properties:

(a) $p^{[\leq n_{\ell+1}]}_{t,\ell+1} = p^{[\leq n_{\ell+1}]}_{t,\ell}$,
(b) $p_{t,\ell+1} \geq_{n+\ell} p_{t,\ell}$,
(c) $\alpha(p_{t,\ell}) = \alpha(p)$,
(d) for every branch b of $p_{t,\ell}$, $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in b \cap p^{[n_\ell]}_{t,\ell}\} \supseteq \{x^t_{\ell'} : \ell' \leq \ell\}$ and $\bigcup\{\mathrm{dom}(\mathrm{last}(t)) : t \in b\} = u_t \cup T_{<\alpha(p)} \cup \{x^t_{\ell'} : \ell' \leq \ell\}$.

Step from ℓ *to* $\ell + 1$. We find $n_{\ell+1} < \omega$ such that

$(*)_1$ $n + \ell + 1 \leq n_{\ell+1}$,
$(*)_2$ for every $s \in (T^{p_{t,\ell}})^{[\geq n_{\ell+1}]}$, we have $\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},s}) \geq (n + |u_t| + \ell + 1)^2$,
$(*)_3$ $n_{\ell+1} \geq n_\ell$.

For each $s \in (T^{(p_{t,\ell})})^{[n_{\ell+1}]}$ let

$$w^+_s = \big\{r : s <_{p_{t,\ell}} r \in T^{p_{t,\ell}} \wedge \mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},r}) > \ell + n_{\ell+1} + \mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},s})\big\}.$$

Now we consider the front

$$w_s = \big\{r \in w^+_s : (\neg\exists z)(s <_{p_{t,\ell}} z <_{p_{t,\ell}} r \wedge z \in w^+_s)\big\},$$
$$w_t = \bigcup\big\{w_s : s \in (T^{p_{t,\ell}})^{[n_{\ell+1}]}\big\}.$$

For each $r \in w_t$ and for each $\mathbf{c}_{p_t,r}$ we perform the operation from Lemma 2.15 and get a creature as $\mathbf{d}$ there with $x^t_\ell \in \mathrm{dom}(\tilde{\varrho})$ for every $\tilde{\varrho} \in \mathrm{pos}(\mathbf{d})$, and $\mathbf{d}$ serves as $\mathbf{c}_{p_{t,\ell+1},r}$.

Then we have for each such $\tilde{\varrho}$

$$\left|\mathrm{dom}(\tilde{\varrho}) \setminus \mathrm{dom}(\varrho)\right| \leq 1, \qquad \text{and}$$
$$\left|\left\{y \,:\, \big(\exists \tilde{\eta} \in \mathrm{pos}(\mathbf{c}_{p_{t,\ell},r})\big)\big(y \in \mathrm{dom}(\tilde{\eta}) \wedge x^t_\ell <_{\mathbf{T}} y\big)\right\}\right|$$
$$\leq |u_t| + \ell \leq \frac{\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},r})}{2}, \tag{$\boxtimes$}$$

since only $y \notin T_{<\alpha(p)}$ can be in the latter set. Hence, the inequalities in the premises of Lemma 2.15 are fulfilled and $\mathrm{nor}^0(\mathbf{d}) \geq n + |u_t| + \ell + 1$. Then we can go on with Lemma 2.15 and change $\mathbf{c}_{p_{t,\ell},r'}$ (which corresponds to $\mathbf{c}$ there) into $\mathbf{c}_{p_{t,\ell+1},r'}$ (which corresponds to $\mathbf{d}$ there) with $\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell+1},r'}) \geq \frac{1}{2}\,\mathrm{nor}(\mathbf{c}_{p_{t,\ell},r'})$ for all immediate successors $r' \geq_{p_{t,\ell}} r$ as there. In order to fulfill the premise $\mathrm{nor}^0(\mathbf{c}_{p_{t,\ell},r}) < \sqrt{n_{1,i(p)+\lg(r)-1}}$, if necessary we go to a subcreature according to Lemma 2.7. Note that $n_{1,n_{\ell+1}} \geq 2^{n_{\ell+1}^2}$. Now we use Lemma 2.16 one level higher such that the specializations in $\mathrm{pos}(\mathbf{c}_{p_{t,\ell+1},r'})$ become the bases of thinned out creatures in $p_{t,\ell+1}$ of norm at least the square root of the creatures in $p_{t,\ell}$. After having worked upward through all of $T^{p_{t,\ell}}$ in this way, we get $p_{t,\ell+1}$. Since in the transition from $p_{t,\ell}$ to $p_{t,\ell+1}$ at each node in $T^{p_{t,\ell+1}}$ the norm drops at most once and to at least half of its former value, we have $p_{t,\ell+1} \geq_{n+\ell} p_{t,\ell}$.

By Definitions 3.3(A.d) and 3.3(B.b) we have, for every branch b of $T^{p_{t,\ell+1}}$,

$$\bigcup\left\{\mathrm{dom}\big(\mathrm{last}(t')\big) \,:\, t' \in b\right\} = T_{<\alpha(p)} \cup u_t \cup \{x^t_i \,:\, i \leq \ell + 1\}.$$

This concludes the step from ℓ to $\ell + 1$.

Now we let q be such that

$$q^{[<h]} = p^{[<h]} \qquad \text{and} \qquad q^{\langle t\rangle} = \bigcup\left\{(p_{t,\ell})^{[n_\ell, n_{\ell+1})} \,:\, \ell \in \omega\right\}.$$

The condition q is smooth, and for every branch b of T^q we have

$$\bigcup\left\{\mathrm{dom}\big(\mathrm{last}(t')\big) \,:\, t' \in b\right\} = T_{<\alpha}. \qquad \square$$

Definition 3.12 We have that $\mathbb{Q}^{\mathrm{s}}_{\mathbf{T}}$ is the partial order of smooth conditions in $\mathbb{Q}_{\mathbf{T}}$, with the order as in $\mathbb{Q}_{\mathbf{T}}$.

So in the forcing sense, $\mathbb{Q}_{\mathbf{T}}$ and $\mathbb{Q}^{\mathrm{s}}_{\mathbf{T}}$ are equivalent.

Convention 3.13 *From now on we assume that all conditions are smooth.*

Corollary 3.14 *Forcing with $\mathbb{Q}_{\mathbf{T}}$ specializes* $\mathbf{T}$.

Proof For any $\alpha \in \mathbf{T}$, $\{p \in \mathbb{Q}_{\mathbf{T}} \,:\, \alpha \in \mathrm{dom}(\mathrm{last}(\mathrm{rt}(p)))\}$ is dense in $\mathbb{Q}_{\mathbf{T}}$. Let G be $\mathbb{Q}_{\mathbf{T}}$-generic over $\mathbf{V}$. Then

$$f_{\mathbb{Q}_{\mathbf{T}}}[G] = \bigcup\left\{\big\langle p, \mathrm{last}(t)\big\rangle \,:\, t = \mathrm{rt}(p), p \in G\right\}$$

is a specialization function for $\mathbf{T}$. $\square$

Definition 3.15

(1) Let $p \in \mathbb{Q}_{\mathbf{T}}$ be such that, for any $t \in T^p$, $\mathrm{nor}_f(\mathbf{c}_{p,t}) > 0$. We say that $q = \mathrm{half}(p)$ if $i(q) = i(p)$, $T^q = T^q$, and for any $t \in T^q$ we have

$$(\forall \ell \geq \ell_0)\big(\forall t \in (T^q)^{[h]}\big)\Big(\mathbf{c}_{q,t} = \mathrm{half}_{\frac{1}{\ell_{i(p)+h}}}(\mathbf{c}_{p,t})\Big),$$

where the operation of taking the $\frac{1}{\ell}$-half of a creature was defined in Definition 2.19(C). The numbers ℓ_i were defined in Definition 2.8.

(2) Let $p \in \mathbb{Q}_{\mathbf{T}}$, and let $n \in \omega$. Let ℓ_0 be minimal such that, for any $t \in (T^p)^{[\geq \ell_0]}$, $\mathrm{nor}_f(\mathbf{c}_{p,t}) \geq n+1$. Then we define $\mathrm{half}_{n+1}(p) = q$ as

$$\big(\forall t \in (T^q)^\ell\big)\big(\ell_0 \leq \ell = \ell_{i(q)+\mathrm{ht}_q(t)} \to \mathbf{c}_{q,t} = \mathrm{half}_{\frac{1}{\ell_{i+\mathrm{ht}_p(t)}}}(\mathbf{c}_{p,t})\big),$$

$$\big(\forall t \in (T^q)^{[<\ell_0]}\big)(\mathbf{c}_{q,t} = \mathbf{c}_{p,t}).$$

Note that, according to the choices made by Definitions 2.8 and 2.2, $\ell_0 \geq 2$.

We recall the definition of de-halving a creature (Definition 2.19). Now we can also "de-halve" a condition.

Lemma 3.16 (The de-halving lemma) *We assume that*

(a) $a \in \omega$ *and* $p_1 \in \mathbb{Q}_{\mathbf{T}}$ *are such that, for any* $t \in T^{p_1}$, $\mathrm{nor}_f(\mathbf{c}_{p_1,t}) > a + \frac{1}{2}$;
(b) $q_1 = \mathrm{half}(p_1)$;
(c) $q_2 \geq_0 q_1$; *and*
(d) *for any* $t \in T^{q_2}$, $\mathrm{nor}_f(\mathbf{c}_{q_2,t}) > 0$.

Then we define p_2 *as follows.*

(1) $i(p_2) = i(p_1)$.
(2) $T^{p_2} = T^{q_2}$.
(3) *We let* $i_* = \min\{\ell : (\forall t \in (T^{q_2})^{[\ell]})\, \mathrm{nor}_f(\mathbf{c}_{q_2,t}) \geq a + \frac{1}{2}\}$. *If* $\mathrm{pr}_{q_2,q_1}(t) = s$, $\ell < i_*$, *and* $t \in (T^{q_2})^{[\ell]}$, *then* $\mathbf{c}_{p_2,t} = \text{de-halve}(\mathbf{c}_{q_2,t}, \mathbf{c}_{p_1,s})$. *If* $\mathrm{pr}_{q_2,q}(t) = s$, $\ell \geq i_*$, *and* $t \in (T^{q_2})^{[\ell]}$, *then* $\mathbf{c}_{p_2,t} = \mathbf{c}_{q_2,t}$. *We write*

$$p_2 = \text{de-halve}(q_2, p_1, a).$$

Then $p_2 \Vdash q_2 \in \underset{\sim}{G}$ *and* $p_2 \geq_0 p_1$, *and* $\forall t \in T^{p_2}$, $\mathrm{nor}_f(\mathbf{c}_{p_2,t}) \geq a + \frac{1}{2} - \frac{1}{\ell_{i(p_1)}}$. *Moreover, we have* $p_2^{\langle t \rangle} \geq_a p_1^{\langle t \rangle}$ *for any* $t \in T^{p_2}$.

The proof follows directly from the definitions. As mentioned, the application of halving and de-halving for $\mathbb{Q}_{\mathbf{T}}$ is left for future work.

4 Proof of Theorems 1.4(b)–1.4(d)

In this section we prove (in Lemma 4.9) that $(\mathbb{Q}, \leq, (\leq_n)_{n \in \omega})$ has properties (b)–(d) from Theorem 1.4. Thus, $\mathbb{Q}_{\mathbf{T}}$ is ${}^\omega\omega$-bounding. This implication is proved in Section 3.1 of [15]. For the reader's convenience we recall the definition.

Definition 4.1 Let $\mathbb{P}$ be a notion of forcing. $\mathbb{P}$ is called *${}^\omega\omega$-bounding* if, for any sufficiently large regular cardinal χ and any $M \prec (H(\chi), \in)$ with $\mathbb{P} \in M$ for any $p \in \mathbb{P} \cap M$ and $\underset{\sim}{f} \in M$ that is a name for a function from ω to ω, there is an $(M, \mathbb{P})$-generic condition $q \geq p$ and there is $g \in \mathbf{V} \cap {}^\omega\omega$ such that $q \Vdash \forall n\, \underset{\sim}{f}(n) \leq g(n)$. A condition q is $(M, \mathbb{P})$-generic if for any $D \in M$ that is a dense subset of $\mathbb{P}$ we have $q \Vdash M \cap \underset{\sim}{G} \cap D \neq \emptyset$.

Lemma 4.2

(1) *If* $p \in \mathbb{Q}$, $n \in \omega$, *and* $\{t_0, \dots, t_n\}$ *is a front of* p, *then* $\{p^{\langle t_0 \rangle}, \dots, p^{\langle t_n \rangle}\}$ *is predense above* p.
(2) *If* $\{t_0, \dots, t_n\}$ *is a front of* p, $p^{\langle t_\ell \rangle} \leq q_\ell \in \mathbb{Q}$, *and there is* $\alpha < \omega_1$ *such that* $\alpha(q_\ell) = \alpha$ *for* $\ell \leq n$, *then there is* $q \geq p$ *with* $\{t_0, \dots, t_n\} \subseteq T^q$ *such that for all* ℓ *we have that* $q^{\langle t_\ell \rangle} = q_\ell$.
(3) *If* $n \in \omega$, $\{t_0, \dots, t_m\}$ *is a front of* p, $\mathrm{lg}(t_\ell) \geq n$, $p^{\langle t_\ell \rangle} \leq_0 q_\ell \in \mathbb{Q}$, *there is* $\alpha < \omega_1$ *such that* $\alpha(q_\ell) = \alpha$ *for* $\ell \leq m$ *and*

– *for all* $\ell \le m$, $(\forall s \in T^{q_\ell})(s \ge_{q_\ell} t_\ell \to \mathrm{nor}^0(\mathbf{c}_{q_\ell,s}) \ge n)$,
– *for all* $s \in T^p$ *if* $\mathrm{nor}^0(\mathbf{c}_{p,s}) < n$, *then* $(\exists \ell \le m)(s <_p t_\ell)$,

then *there is* $q \ge_n p$ *with* $\{t_0, \dots, t_m\} \subseteq T^q$ *such that for all* ℓ *we have that* $q^{\langle t_\ell \rangle} = q_\ell$ *and* $\{t_0, \dots, t_m\}$ *is a front of* q.

The 2-bigness of our creatures is used to find stronger conditions that are homogeneous with respect to a downward closed set.

Lemma 4.3 *If* $p \in \mathbb{Q}$ *and* $X \subseteq T^p$ *is* $<_p$*-downward closed, then there is some* q *such that*

(a) $p \le_0 q$, *and either* $(\forall \ell)((T^q)^{[\ell]} \subseteq X)$ *or* $(\forall^\infty \ell)((T^q)^{[\ell]} \cap X = \emptyset)$,
(b) $T^q \subseteq T^p$, *for* $t \in T^q$, $\mathbf{c}_{q,t} = \mathbf{c}_{p,t} \restriction T^q$ *(which means that the set of possibilities* $\mathrm{pos}(\mathbf{c}_{q,t})$ *is just those* $\eta \in \mathrm{pos}(\mathbf{c}_{p,t})$ *with* $t\hat{\ }\eta \in T^q$*)*,
(c) *for every* $t \in T^q$, *if* $\mathbf{c}_{q,t} \ne \mathbf{c}_{p,t}$, *then* $\mathrm{nor}^1(\mathbf{c}_{q,t}) \ge \mathrm{nor}^1(\mathbf{c}_{p,t}) - 1$ *and* $\mathrm{nor}_f(\mathbf{c}_{q,t}) \ge \mathrm{nor}_f(\mathbf{c}_{p,t}) - 1$.

Proof We will choose $T^q \subseteq T^p$. For each $\ell \ge 1$ we first choose by downward induction on $j \le \ell$ a coloring $f_{\ell,j}$ of $(T^p)^{[j]}$ with two colors, 0 and 1. For $t \in (T^p)^{[\ell]}$ we set $f_{\ell,\ell}(t) = 0$ if and only if $t \in X$ and $f_{\ell,\ell}(t) = 1$ otherwise.

Suppose that $f_{\ell,j}$ is defined. For $s \in (T^p)^{[j-1]}$ we have

$$\mathrm{pos}(\mathbf{c}_{p,s}) = \big\{\nu \in \mathrm{pos}(\mathbf{c}_{p,s}) \,:\, f_{\ell,j}(\nu) = 0\big\} \cup \big\{\nu \in \mathrm{pos}(\mathbf{c}_{p,s}) \,:\, f_{\ell,j}(\nu) = 1\big\}.$$

For $m = 0, 1$, we let $\mathbf{c}_{p,s,m} = (i(\mathbf{c}_{p,s}), \eta, \{\nu \in \mathrm{pos}(\mathbf{c}_{p,s}) \,:\, f_{\ell,j}(s\hat{\ }\nu) = m\})$. By Lemma 2.18 there is $m \in \{0,1\}$ such that $\mathrm{nor}^1(\mathbf{c}_{p,s,m}) \ge \mathrm{nor}^1(\mathbf{c}_{p,s}) - 1$. Now we color $s \in (T^p)^{[j-1]}$ as follows: $f_{\ell,j-1}(s) = m$ if and only if $m \in \{0,1\}$ is minimal such that $\mathrm{nor}^1(\mathbf{c}_{p,s,m}) \ge \mathrm{nor}^1(\mathbf{c}_{p,s}) - 1$. We work downward until we come to the root of p and keep $f_{\ell,0}(\mathrm{rt}(p))$ and $\mathbf{c}_{p,s,m}$ in our memory.

We repeat the procedure of the downward induction on j for larger and larger ℓ. Since X is downward closed, we have

$$\forall \ell \forall j \le \ell \forall s \in (T^p)^{[j]} \big(f_{\ell+1,j}(s) = 0 \to f_{\ell,j}(s) = 0\big).$$

For each fixed ℓ, these statements are proved by easy downward induction on j.

Case 1. There are infinitely many ℓ such that $f_{\ell,0}(\mathrm{rt}(p)) = 0$. If there are infinitely many ℓ such that $f_{\ell,0}(\mathrm{rt}(p)) = 0$, then by $(*)$ this holds for all ℓ. Since for each fixed m there are only finitely many possible $\langle f_{m,j}(s) \,:\, s \in (T^p)^{[j]}, j \le m\rangle$, by König's lemma we find an infinite subsequence $\langle \ell_k \,:\, k < \omega\rangle$ such that, for each k, for all $k' \ge k$, for all $j \le \ell_k$, and for all $s \in (T^p)^{[j]}$, $f_{\ell_{k'},j}(s) = f_{\ell_k,j}(s)$. So we have for every k, $f_{\ell_k,0}(\mathrm{rt}(p)) = 0$. We let

$$T^q = \big\{s \in T^p \,:\, (\forall j,k)\big((j \le \ell_k \wedge s \in T^{[j]}) \to (\forall k' \ge k)\big(f_{\ell_{k'},j}(s) = f_{\ell_k,j}(s) = 0\big)\big)\big\}.$$

Then $T^q \subseteq X$ and $\mathrm{rt}(q) = \mathrm{rt}(p)$. By our choice of $f_{\ell,j}$ and by the case assumption, $\forall \ell\ f_{\ell,0}(\mathrm{rt}(p)) = 0$ and the norm drops at most 1 in the transition from p to q. So q is as required.

Case 2. There is ℓ such that, $\forall \ell' \ge \ell$, $f_{\ell',0}(\mathrm{rt}(p)) = 1$. Let ℓ be minimal with this property. Since for each fixed m there are only finitely many possible $\langle f_{m,j}(s) \,:\, s \in (T^p)^{[j]}, j \le m\rangle$, by König's lemma we find an infinite subsequence $\langle \ell_k \,:\, k < \omega\rangle$

such that, for each k, for all $k' \geq k$, for all $j \leq \ell_k$, and for all $s \in (T^p)^{[j]}$, $f_{\ell_{k'},j}(s) = f_{\ell_k,j}(s)$. So we have, for every k, $f_{\ell_k,0}(\mathrm{rt}(p)) = 0$. We let

$$T^q = \big\{s \in T^p : (\forall j,k)\big((\ell \leq j \leq \ell_k \wedge s \in T^{[j]}) \\ \rightarrow (\forall k' \geq k)\big(f_{\ell_{k'},j}(1) = f_{\ell_k,j}(s) = 1)\big)\big\}.$$

Then $T^q \subseteq X$ and $\mathrm{rt}(q) = \mathrm{rt}(p)$. By our choice of $f_{\ell,j}$ and by the case assumption, the norm drops at most 1 in the transition from p to q. So q is as required. □

We now improve the property $p \leq_0 q$ in Lemma 4.3 to $p \leq_n q$, and therefore we have to weaken the homogeneity property in item Lemma 4.5(a.iii) with $n+1$ instead of n and p instead of q of Lemma 4.3.

Lemma 4.4 *If $p \in \mathbb{Q}$, $n \in \omega$, and $X \subseteq T^p$ is downward closed, then there is some q such that*

(a) *$p \leq_n q$, and there is a front $\{t_0, \ldots, t_j\}$ such that*
  (i) $\{t \in T^p : \mathrm{nor}_f(\mathbf{c}_{p,t}) \leq n\} \subseteq \{t \in T^q : (\exists i \leq j)(t \leq_q t_i)\}$,
  (ii) *for all $i \leq j$ we have: either $\{s \in T^q : s \geq_p t_i\} \subseteq X$ or $(\forall^\infty \ell)(\{s \in (T^q)^{[\ell]} : s \geq_p t_i\} \cap X = \emptyset)$, and*
  (iii) *for all $i \leq j$, $t \geq_q t_i$, $\mathrm{nor}^0(\mathbf{c}_{q,t}) \geq n$;*

(b) *$T^q \subseteq T^p$ and $\mathbf{c}_{q,t} = \mathbf{c}_{p,t} \restriction T^q$;*

(c) *for every $t \in T^q$, if $\mathbf{c}_{q,t} \neq \mathbf{c}_{p,t}$, then $\mathrm{nor}^1(\mathbf{c}_{q,t}) \geq \mathrm{nor}^1(\mathbf{c}_{p,t}) - 1$ and $\mathrm{nor}_f(\mathbf{c}_{q,t}) \geq \mathrm{nor}_f(\mathbf{c}_{p,t}) - 1$.*

Proof We choose a front of p as in (a) and use Lemma 4.3 for each $p^{\langle t_i \rangle}$. □

Definition 4.5 Let $\nu_0, \nu_1 \in \mathrm{spec}$. We say that ν_0 *is isomorphic to* ν_1 *over* $\mathbf{T}_{<\alpha}$ if there is some injective partial function $f : \mathbf{T} \to \mathbf{T}$ such that

$$(\forall x, y \in \mathrm{dom}(f))(x <_{\mathbf{T}} y \leftrightarrow f(x) <_{\mathbf{T}} f(y)),$$

$\mathrm{dom}(\nu_0) \cup \mathbf{T}_{<\alpha} \subseteq \mathrm{dom}(f)$, $f \restriction \mathbf{T}_{<\alpha} = \mathrm{id}$, $f[\mathrm{dom}(\nu_0)] = \mathrm{dom}(\nu_1)$, and $\nu_0(x) = \nu_1(f(x))$ for all $x \in \mathrm{dom}(\nu_0)$.

Fact 4.6

(1) *Being isomorphic over $\mathbf{T}_{<\alpha}$ is an equivalence relation.*

(2) *For each $\alpha < \omega_1$, there are only countably many isomorphism types for $\eta \in \mathrm{spec}^{\mathbf{T}}$ over $\mathbf{T}_{<\alpha}$.*

Definition 4.7 Let $q \in \mathbb{Q}_{\mathbf{T}}$ (recall that this means smooth), let $\ell, n \in \omega$, let $t \in T^q$, and let $\varrho \in \mathrm{spec}^{\mathbf{T}}$. Let $\underset{\sim}{\tau}$ be a $\mathbb{Q}$-name for an ordinal.

Let $\boxplus_{q,\ell,t,\varrho,n}(\underset{\sim}{\tau})$ abbreviate the following statement:

(i) $q \in \mathbb{Q}_{\mathbf{T}}$, $t \in (T^q)^{[\ell]}$, and $n \in \omega$,

(ii) $(\forall s \geq_q t)(\mathrm{nor}_f(\mathbf{c}_{q,s}) \geq n+1)$, and

(iii) if there are finite partial specializations ϱ' and ϱ'' and a condition $q' \geq q$ with $\mathrm{rt}(q') = \mathrm{last}(t) \cup \varrho' \cup \varrho''$, $i(q') = i(q) + \lg(t) - 1$, $|\mathrm{rt}(q')| \leq \frac{n_{1,i(q')}}{2^{(2^n + k(\mathbf{c}_{q,t}))}}$, $\mathrm{nor}_f(\mathbf{c}_{q',t''}) \geq n+1$ for every $t'' \in T^{q'}$, ϱ' and ϱ are isomorphic over $T_{<\alpha(q)}$, and q' forces a value to $\underset{\sim}{\tau}$, *then* $q^{\langle t \rangle}$ forces a value to $\underset{\sim}{\tau}$.

Lemma 4.8 *Suppose that $p \in \mathbb{Q}_{\mathbf{T}}$, and suppose that $n < \omega$. Let $N \prec \mathcal{H}(\chi)$ be countable, let $N \cap \omega_1 = \delta_*$, let $p \in N$, and let $\mathbf{T} \in N$. Let $\underset{\sim}{\tau} \in N$ be a $\mathbb{Q}_{\mathbf{T}}$-name of an ordinal. For every $n \in \omega$ there is a $q \in \mathbb{Q}_{\mathbf{T}}$ such that*

(a) $p \leq_n q$,
(b) $\alpha(q) = \delta_*$,
(c) *if* $\varrho \in \mathrm{spec}^{\mathbf{T}}$ *and* $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$, *then for any* i *and infinitely many* $\ell \in [i, \omega)$ *we have* $\forall t \in (T^q)^{[\ell]}\ \boxplus_{q,\ell,t,\varrho,n+i}(\underset{\sim}{\tau})$.

Proof Let $\langle \varrho_m : m \in \omega\rangle$ list representatives of the possible isomorphism types over $\mathbf{T}_{<\delta_*}$ of a $\varrho \in \mathrm{spec}^{\mathbf{T}}$ such that $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$ and such that each type is represented infinitely often. Let $\langle \alpha_i : i < \omega\rangle$ be an increasing sequence of ordinals that converges to δ_*. We choose (p_i, ℓ'_i) by induction on i with the following properties:

(1) $p_i \in \mathbb{Q}_{\mathbf{T}} \cap N$;
(2) $\ell'_i < \ell'_{i+1}$;
(3) $p_0 = p$, $p_i \leq_{n+i} p_{i+1}$, and $p_i^{[<\ell'_i]} = p_{i+1}^{[<\ell'_i]}$;
(4) $(\forall t \in (T^{p_i})^{[\geq \ell'_i]})(\mathrm{nor}_f(\mathbf{c}_{p_i,t}) \geq n + i + 1)$;
(5) for any $s \in (T^{p_i})^{[\ell'_i]}$ we define

$$\Lambda^2_{i,s} = \big\{\nu \in \mathrm{pos}(\mathbf{c}_{p_i,s}) : (\alpha)_{i,s,\nu} \text{ holds}\big\},$$

where

$(\alpha)_{i,s,\nu}$ There is a prolongation $\tilde{\varrho}_i$ of ϱ_i that is disjoint from ν and compatible with ν such that there are unboundedly many $\gamma \in \delta_*$ such that[3] there are $\tilde{\varrho} \in N$ and a smooth $r_0 \in N$, $r_0 \geq p_i$, such that $\mathrm{rt}(r_0) = \nu \dot{\cup} \tilde{\varrho}$, $i(r_0) = i(p_i) + \lg(s)$, $|\mathrm{rt}(r_0)| \leq \frac{n_{1,i(r_0)}}{2^{(2^{n+i}+k(\mathbf{c}_{p_i,t}))}}$, $\tilde{\varrho}$ and $\tilde{\varrho}_i$ realize the same type over $\mathbf{T}_{<\alpha(p_i)}$, $\mathrm{dom}(\tilde{\varrho}) \cap \gamma = \emptyset$, r_0 forces a value to $\underset{\sim}{\tau}$, and for all $t \in T^{r_0}$, $\mathrm{nor}_f(\mathbf{c}_{r_0,t}) \geq n + i + 1$. By $\dot{\cup}$ we denote the disjoint union. This ends $(\alpha)_{i,s,\nu}$.

We let

$$\Lambda^1_{s,i} = \mathrm{pos}(\mathbf{c}_{p_i,s}) \setminus \Lambda^2_{s,i}.$$

We demand that if $\nu \in \Lambda^2_{s,i}$, then

$$\big(\forall t \in (T^{p_{i+1}})^{[\ell'_i+1]}\big)\Big(\big(t >_{p_{i+1}} s \wedge \mathrm{pr}_{p_{i+1},p_i}(t) = s\hat{\ }\langle\nu\rangle\big) \rightarrow p_{i+1}^{\langle t\rangle} \text{ forces a value to } \underset{\sim}{\tau}\Big).$$

This ends item (5).

We show that there is such a sequence $\langle p_i, \ell'_i : i < \omega\rangle$. Assume that we are given p_i. Then we choose ℓ'_i such that $(\forall t \in (T^{p_i})^{[\geq \ell'_i]})(\mathrm{nor}_f(\mathbf{c}_{p_i,t}) \geq (n + i + 1)$. For every $s \in (T^{p_i})^{[\ell'_i]}$ we divide $\mathrm{pos}(\mathbf{c}_{p_i,s})$ into $\Lambda^1_{s,i}$ and $\Lambda^2_{s,i}$. Assume that $(\alpha)_{i,s,\nu}$ holds.

Explanation. We fix for each of cofinally many $\gamma \in \delta_*$ a condition $r_0 = r_{0,\gamma} \in N$ as in $(\alpha)_{i,s,\nu}$. The core of the construction is the definition of a preliminary part of p_{i+1} that is composed of ν, k parts. We choose $\varrho_{\nu,k} \supseteq \nu$ for $k = 0, \dots, m + i$, $\nu \in \Lambda^2_{i,s}$, $p_{i+1,s,\nu,k} \geq p_i^{\langle (s\hat{\ }\langle\nu\rangle)\rangle}$ with root $\varrho_{\nu,k}$ that is isomorphic to $\nu \cup \tilde{\varrho}$, and a common $\alpha(p_{i+1,s,\nu,k})$ for all s and (ν, k), and then we graft the conditions $(p_{i+1,s,\nu,k})$ at the node $s\hat{\ }\langle\nu\rangle$ into $p_i^{\langle s\hat{\ }\langle\nu\rangle\rangle} =: p_{i+1,s,\nu,k}$. This implies that

$$p_{i+1} = \bigcup\Big\{p_{i+1,s,\nu,k} : s \in (T^p)^{[\ell'_i]}, \nu \in \Lambda^2_{i,s}, k \leq \Big[\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_s,s})}\Big]\Big\}$$

fulfills

$$p_{i+1,s,\nu,k} = p_{i+1}^{\langle(s\hat{}\langle\varrho_{\nu,k}\rangle)\rangle}$$

and $p_i \upharpoonright \ell'_{i+1} - 1 = p_{i+1} \upharpoonright \ell'_{i+1} - 1$. In the stronger condition p_{i+1}, the place of ν in $<_{p_i}$ will be taken by

$$\varrho_{\nu,k} = \nu \cup \varrho'_{\nu,k}, \quad k = 0, 1, \dots, \left[\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}\right],$$

where the $\varrho'_{\nu,k}$ still have to be defined (see (p1)–(p7) below). The order $<_{p_{i+1}}$ is defined such that, for any $k \leq [\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}]$, $\mathrm{pr}_{p_{i+1},p_i}(s\hat{}\langle\varrho_{\nu,k}\rangle) = s\hat{}\langle\nu\rangle$ and $\mathrm{pr}_{p_{i+1},p_i} \restriction (p_{i+1})^{\langle(s\hat{}\langle\varrho_{\nu,k}\rangle)\rangle})$ is a projection witnessing $(p_{i+1})^{\langle(s\hat{}\langle\varrho_{\nu,k}\rangle)\rangle} \geq p_i^{\langle s\hat{}\nu\rangle}$. We will show that there are such $\varrho_{\nu,k}$ with the additional property that, for each k, $\varrho'_{\nu,k} = \varrho_{\nu,k} \setminus \nu$ is over $\mathbf{T}_{<\alpha(p_i)}$ isomorphic to $\tilde{\varrho}_i$ and such that there is $p_{i+1} \in \mathbb{Q}_{\mathbf{T}}$ such that for any $s \in (T^{p_i})^{[\ell'_i]} = (T^{p_i+1})^{[\ell'_i]}$

$(*)_1$ $\mathrm{pos}(\mathbf{c}_{p_{i+1},s}) = \{\varrho_{\nu,k} : k < n + i + 1\}$;

$(*)_2$ if $\nu \in \Lambda^2_{i,s}$, then

$$\left(\forall s\hat{}\langle\varrho_{\nu,k}\rangle \in p_{i+1}^{[\ell'_i+1]}\right)\left(\left(s\hat{}\langle\varrho_{\nu,k}\rangle >_{p_{i+1}} s\hat{}\langle\nu\rangle \wedge \mathrm{pr}_{p_{i+1},p_i}\left(s\hat{}\langle\varrho_{\nu,n}\rangle\right) = s\hat{}\langle\nu\rangle\right)\right.$$
$$\left.\rightarrow p_{i+1}^{\langle(s\hat{}\langle\varrho_{\nu,k}\rangle)\rangle} \text{ forces a value to } \underset{\sim}{\tau}\right);$$

and

$(*)_3$ $p_{i+1}^{\langle s\rangle} \geq_{n+i} p_i^{\langle s\rangle}$.

A drop in norm like $\mathrm{nor}_f(\mathbf{c}_{p_{i+1},s}) < \mathrm{nor}_f(\mathbf{c}_{p_i,s}) - 1$ must be prevented.

Of course we only have $p_{i+1}^{\langle s\hat{}\langle\varrho_{\nu,k}\rangle)\rangle} \geq p_i^{\langle s\hat{}\nu\rangle}$, not even with $\geq_0$ by the choice of $\varrho_{\nu,k}$. A remedy is to lengthen the ν in many disjoint and isomorphic ways to $\varrho_{\nu,k}$ and then to use an old fact about uncountably many disjoint finite subsets of Aronszajn trees and Lemma 2.14. This ends the explanation of the envisaged construction.

We continue the construction. By the case assumption, there are $\varrho'_{\nu,k}$, r_k, $1 \leq k < \omega$, with the following properties:

(p1) $\varrho'_{\nu,k}$ and $\tilde{\varrho}_i$ have the same type over $\mathbf{T}_{<\alpha(p_i)}$,

(p2) $\mathrm{dom}(\varrho'_{\nu,k}) \cap \alpha(p_i) = \emptyset$, and $\mathrm{dom}(\varrho'_{\nu,k}) \cap \mathrm{dom}(\varrho'_{\nu,k'}) = \emptyset$ for $k \neq k'$,

(p3) $\varrho'_{\nu,k} \in N$,

(p4) $r_k \in N$ forces a value to $\underset{\sim}{\tau}$, and r_k is smooth,

(p5) $\mathrm{rt}(r_k) = \nu \cup \varrho'_{\nu,k} =: \varrho_{\nu,k}$, and $i(r_k) = i(p_i) + \lg(s)$,

(p6) for all $\forall t' \in (T^{r_k})^{[\geq \ell'_i]}$, $\mathrm{nor}_f(\mathbf{c}_{r_k,t'}) \geq n + i + 1$,

(p7) $|\mathrm{dom}(\varrho_{\nu,k})| \leq \frac{n_{1,i}}{2^{(2^{n+i}+k(\mathbf{c}_{p_i,t}))}}$.

By a fact about uncountably many disjoint finite subsets in an Aronszajn tree (see, e.g., [18, Chapter III, Theorem 5.4] or [7, Lemma 18.10]), applied in N iteratively $n + i$ times, we can have additionally:

(p8) and such that for $k \neq k'$, $k, k' \leq \sqrt{n+i}$, $t \in \mathrm{dom}(\varrho'_{\nu,k})$, $t' \in \mathrm{dom}(\varrho'_{\nu,k'})$, the two nodes t and t' are $\leq_{\mathbf{T}}$-incomparable.

Then we pick for each $k < \omega$ some $\varrho'_{\nu,k}$ and r_k with properties (p1)-(p8). We set $\varrho_{\nu,k} := \nu \cup \varrho'_{\nu,k}$ and $p'_{i+1,s,\nu,k} = r_k$. We have $s\hat{}\langle\varrho_{\nu,k}\rangle = \mathrm{rt}(T^{p'_{i+1,s,\nu,k}})$. We glue the preliminary conditions together above s and choose p'_{i+1} and $\mathrm{pr}_{p'_{i+1},p_i}$ such that $(p'_{i+1})^{\langle(s\hat{}\langle\varrho_{\nu,k}\rangle)\rangle} = p'_{i+1,s,\nu,k}$ and $\mathrm{pr}_{p'_{i+1},p_i}(s\hat{}\langle\varrho_{\nu,k}\rangle) = s\hat{}\langle\nu\rangle$. We use only $k = 0, \dots, n + i$. The outcome p'_{i+1} of the gluing procedure might not be smooth and, indeed, might not be a condition at all, because each $p'_{i+1,s,\nu,k}$ has its own $\alpha(p'_{i+1,s,\nu,k})$ and Definition 3.3(A.c) might be missing. Clause (p4) guarantees $(*)_2$ for $p'^{\langle(s\hat{}\langle\varrho_{\nu,k}\rangle)}_{i+1}$, and by Lemma 2.14 the requirement $(*)_3$ is fulfilled by $p'^{\langle s\rangle}_{i+1}$ for all the relevant s, ν, k.

This finishes the particular construction of p'_{i+1} above the projection's preimage of $s\hat{}\langle\nu\rangle$ for $\nu \in \Lambda^2_{s,i}$. For $\nu \in \Lambda^1_{i,s}$ we let p'_{i+1} above $s\hat{}\langle\nu\rangle$ be just p_i above $s\hat{}\langle\nu\rangle$.

Thereafter, we take $\alpha'_{i+1} \in [\alpha_{i+1}, \delta_*)$ sufficiently large so that, for each $\nu \in \Lambda^2_{s,i}$, each $k \leq [\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}]$, each $\nu \in \Lambda^1_{s,i}$ (with $\varrho_{\nu,0} = \nu$), a smooth $p_{i+1s,\nu,k} \geq_{n+i} p'_{i+1,s,\nu,k}$ such that, for every $s\hat{}\langle\nu\rangle \geq_{p_i} s$, $s\hat{}\langle\nu\rangle \in (T^{p_i})^{[\ell' i+1]}$, $k = 0, \ldots, n+i$, and for every $\varrho_{\nu,k}$, $\alpha(p_{i+1,s,\nu,k}) = \alpha'_{i+1} \geq \alpha_{i+1}$. Such a condition exists by Lemma 3.11. We perform all the filling up from the proof of the latter lemma strictly above level ℓ'_{i+1}.

Now we define p_{i+1} as follows:

$$p_{i+1} = \bigcup\{p_{i+1,s,\nu,k} \;:\; s \in p_i^{[\ell'_i]}, \nu \in \Lambda^1_{s,i}\}$$
$$\cup \left\{p_{i+1,s,\nu,k} \;:\; k \leq \left[\sqrt{\mathrm{nor}^0(\mathbf{c}_{p_i,s})}\right], s \in p_i^{[\ell'_i]}, \nu \in \Lambda^2_{s,i}\right\}$$

By construction we have $\alpha(p_{i+1}) = \alpha'_{i+1}$. Since $(p')^{[\leq \ell'_{i+1}]}_{i+1} = p^{[\leq \ell'_{i+1}]}_{i+1}$, the properties $(*)_{1,2,3}$ of p'_{i+1} hold also for p_{i+1}. Now by Lemma 2.14,

$$\mathrm{nor}_f(\mathbf{c}_{p_{i+1},s}) \geq n+i \qquad \text{and} \qquad p_{i+1} \geq_{n+i} p_i. \tag{4.1}$$

So we finished the inductive choice of $\langle(p_i, \ell'_i) \;:\; i < \omega\rangle$ with properties (1)–(5).

We let q be the fusion of the p_i's. We show that q is as desired in the lemma. Let $\varrho \in \mathrm{spec}^{\mathbf{T}}$ and i be given with $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$. Then at any of the infinitely many steps i of the above construction in which the isomorphism type of ϱ is invoked, for any $t = s\hat{}\langle\varrho_{\nu,k}\rangle \in (T^q)^{[\ell'_i+1]}$, $\boxplus_{q,\ell',t,\varrho,n+i}(\underset{\sim}{\tau})$ is ensured by properties $(*)_{1,2,3}$ and (1)–(5). □

In the next lemma we turn conclusion (c) of the previous lemma into a stronger property, by strengthening a condition with the help of the homogeneity property from Lemma 4.4. The property in (b) in the next lemma is a version of "continuous reading of names" that yields a strong version of Axiom A.

Lemma 4.9 *Suppose that $\mathbb{Q} = \mathbb{Q}_{\mathbf{T}}$, $p \in \mathbb{Q}$, $n < \omega$, and $\underset{\sim}{\tau}$ is a $\mathbb{Q}$-name of an ordinal. Then there is a $q \in \mathbb{Q}$ such that*

(a) $p \leq_n q$,
(b) *for some $\ell \in \omega$ we have that for every $t \in (T^q)^{[\ell]}$ the condition $q^{\langle t\rangle}$ forces a value to $\underset{\sim}{\tau}$.*

Proof Let $N \prec H(\chi)$ be such that $\mathbb{Q}_{\mathbf{T}}, p, \underset{\sim}{\tau} \in N$. We take $q \geq_{n+1} p$ in the role of q from the previous lemma applied to N, $\delta_* = N \cap \omega_1$, $\underset{\sim}{\tau} \in N$, p, so (a), (b), and (c) from the conclusion of Lemma 4.8 hold for p and q.

Then we define, for $k \in \omega$,

$$X_\tau(q,k,n) = \Big\{t : t \in \bigcup_{k'\geq k} (T^{p'_0})^{[k']}$$
$$\wedge\ (\exists q')\big(q^{\langle t\rangle} \leq_0 q' \ \wedge\ (q' \text{ forces a value to } \underset{\sim}{\tau})$$
$$\wedge\ (\forall t' \in T^{q'})\big(t' \geq_{q'} t \to \mathrm{nor}_f(\mathbf{c}_{q',t'}) \geq n+1\big)\big)\Big\}.$$

For $\tilde n < \omega$, $p_1, p_2 \in \mathbb{Q}$, and $t \in T^r$, we denote the following property:

$$(p_1)^{\langle t\rangle} \leq_0 p_2 \wedge \forall t'\big(t \leq t' \in T^{(p_2)} \to \mathrm{nor}^0(\mathbf{c}_{p_2,t'}) \geq \tilde n + 1\big)$$
$$\wedge\ (p_2 \text{ forces a value to } \underset{\sim}{\tau}). \qquad (*)^{\tilde n,t}_{p_1,p_2}$$

Note that $(T^{q'})^{[\ell]} \subseteq X(q,k,n)$ implies that $\forall t \in (T^{q'})^{[\ell]}(\exists q'')(*)^{n,t}_{q,q''}$.
Choose

(1) k such that $t \in (T^q)^{[\geq k]} \to \mathrm{nor}_f(\mathbf{c}_{q,t}) > n+2$,
(2) $q' \geq_{n+1} q$ as in Lemma 4.4 applied to q, the front $(T^q)^{[k]}$, and $X = \mathrm{dom}(q) \setminus X(q,k,n+1)$, which is downward closed.

We show that $\forall^\infty \ell \forall t \in (T^{q'})^{[\ell]} (q')^{\langle t\rangle}$ forces a value to $\underset{\sim}{\tau}$. Note that also q' has with respect to p the properties from the previous lemma.

First case. In Lemma 4.4(a) we get $\forall \ell (T^{q'})^{[\ell]} \subseteq X$. We show that this does not happen. We work with $i = 0$ in conclusion (c) of Lemma 4.8.

Suppose that $t \in T^{q'}$ is such that $(\forall t' \in T^{q'})(t' \geq_q t \to \mathrm{nor}(\mathbf{c}_{q',t'}) \geq n+1)$. Then, by the definition of X, for any $t' \geq_Q t$, $(q')^{\langle t'\rangle}$ does not force a value to τ. However, $\alpha(q) = \alpha(q') = \alpha((q')^{\langle t\rangle}) = \delta_* = N \cap \omega_1$. We take any $q'' \geq (q')^{\langle t\rangle}$ that forces a value to τ. Without loss of generality we can assume that, for all $t' \in T^{q''}$, $(t' \geq_{q''} t^- \to \mathrm{nor}_f(\mathbf{c}_{q'',t'}) \geq n+1)$, where $t^- <_{q''} t$ is the direct predecessor of t. Then $\varrho := \mathrm{rt}(r) \setminus \mathrm{last}(t)$ has $\mathrm{dom}(\varrho) \cap \delta_* = \emptyset$ by Fact 3.7(2). Moreover, $|\mathrm{rt}(r)| < \frac{n_{1,i(r)}}{2^{(2^n + k(\mathbf{c}_{q'',t}))}}$ by the assumption on the norms. Then since q' has the properties of the previous lemma we get that there are infinitely many ℓ such that $(\forall t' \in (T^q)^{[\ell]} \cap T^{(q'')^{\langle t\rangle}})(\boxplus_{q,\ell,t',\varrho,n})$ and in clause (iii) of the statement $\boxplus_{q,\ell,t',\varrho,n}$ the premise is fulfilled. So we have $(T^{q'})^{[\ell]} \not\subseteq X$.

Second case. In Lemma 3.3(a) we get $(\forall^\infty \ell)((T^{q'})^{[\ell]} \cap X = \emptyset)$. By the definition of $X(q,k,n) = T^q \setminus X$ we are done. □

Conclusion 4.10 *We have that $\mathbb{Q}_{\mathbf{T}}$ is a proper ${}^\omega\omega$-bounding forcing that specializes the Aronszajn tree* $\mathbf{T}$.

The following result was also established by Hirschorn in [5] and in [11].

Corollary 4.11 *It is consistent relative to ZFC that all Aronszajn trees are special (hence, there are no Souslin trees), and $\mathfrak{d} = \aleph_1$ and $2^\omega = \aleph_2$.*

Proof We shall show that there is an iterated forcing with iterands of the form $\mathbb{Q}_{\mathbf{T}}$ such that in the extension every Aronszajn tree is special. Since the letter $\mathbb{Q}$ is reserved for the $\mathbb{Q}_{\mathbf{T}}$'s, we use the symbol $\mathbb{Q}'$ for the iterands. Every element of a forcing extension has a proper class of names. Canonical names provide for a small set of representatives. We first recall the notion of canonical names. Let $\mathbb{P}$ be a notion of forcing. For $a \in \mathbf{V}$ we let $\check a = \{\langle p, \check b\rangle : b \in a, p \in \mathbb{P}\}$. For a $\mathbb{P}$-name τ we define its name rank $\mathrm{rk}_n(\tau)$ by induction as

$$\mathrm{rk}_n(\tau) = \sup\big\{\mathrm{rk}_n(\sigma) + 1 : (\exists p)(\langle \sigma, p\rangle \in \tau\big\}.$$

In addition, we define the revised name rank as

$$\mathrm{rk}_r(\tau) = \begin{cases} 0 & \text{if } \exists a \in \mathbf{V} \check{a} = \tau, \\ \sup\{\mathrm{rk}_r(\sigma) + 1 \,:\, \exists p \langle p, \sigma\rangle \in \tau\} & \text{otherwise.} \end{cases}$$

Finally we define the $\mathbf{V}$-rank $\mathrm{rk}_{\mathbf{V}}$ for $x \in \mathbf{V}[G]$,

$$\mathrm{rk}_{\mathbf{V}}(x) = \begin{cases} 0 & \text{if } x \in \mathbf{V}, \\ \sup\{\mathrm{rk}_{\mathbf{V}}(y) + 1 \,:\, y \in x\} & \text{otherwise.} \end{cases}$$

Let $\mathrm{rk}(x) = \sup\{\mathrm{rk}(y) + 1 \,:\, y \in x\}$ be the usual rank function.

A $\mathbb{P}$-name τ with $\mathrm{rk}_n(\tau) = \alpha$ is called *canonical* if

(a) for any β, $\mathbb{P} \Vdash \mathrm{rk}(\tau) \leq \beta$ implies that $\beta \geq \alpha$ and $\mathbb{P} \Vdash \mathrm{rk}_{\mathbf{V}}(\tau) \leq \beta$ implies that $\mathrm{rk}_r(\tau) \leq 1 + \beta$; and
(b) if $\mathbb{P}$ has the λ-antichain condition $\mathbb{P} \Vdash |\tau| < \lambda$, then $|\tau| < \lambda$; and
(c) for any $\mathbb{P}$-name σ and any $p \in P$, if $\langle \sigma, p\rangle \in \mathbb{P}$, then σ is canonical as well (see [18, Chapter I, Section 5]).

Shelah in [18, Chapter I, Theorem 5.13] proves that every $x \in \mathbf{V}[G]$ has a canonical name. The canonical names are in general not unique.

We write $\mathbf{V}^{\mathbb{P}}$ for any $\mathbf{V}[G]$ with a generic G. In the special case of an Aronszajn tree $(\omega_1, <_{\mathbf{T}})$ in the sense of Definition 1.1 in an extension $\mathbf{V}^{\mathbb{P}}$ with $\omega_1^{\mathbf{V}^{\mathbb{P}}} = \omega_1^{\mathbf{V}}$, we have for example a $\mathbb{P}$-canonical name of $<_{\mathbf{T}}$ of the form

$$\tau = \big\{\big\langle p, (\alpha, \beta, i)\big\rangle \,:\, \alpha, \beta \in \omega_1, i \in \{0, 1\}, p \in A_{\alpha,\beta}\big\}$$

with maximal antichains $A_{\alpha,\beta} \subseteq \mathbb{P}$ such that, for each $p \in A_{\alpha,\beta}$, $p \Vdash_{\mathbb{P}} \alpha <_{\mathbf{T}} \beta$ if $\langle p, (\alpha, \beta, 1)\rangle \in \tau$, and $p \Vdash_{\mathbb{P}} \neg\alpha <_{\mathbf{T}} \beta$ if $\langle p, (\alpha, \beta, 0)\rangle \in \tau$. If $\mathbb{P} \in H(\aleph_2)$, then $\tau \in H(\aleph_2)$. This ends the review of canonical names.

We start with a ground model of $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Let $b \colon \aleph_2 \to H(\aleph_2)$ be a surjective function such that each element of $H(\aleph_2)$ has cofinally many preimages under b. Since $2^{\aleph_1} = \aleph_2$, such a function exists.

We argue that it suffices to specialize any Aronszajn tree in the sense of Definition 1.1. Any normal Aronszajn tree has an isomorphic copy that meets Definition 1.1. Every Aronszajn tree $\mathbf{T}$ contains a normal subtree

$$\mathbf{T}_{\text{normal}} = \big\{t \in \mathbf{T} \,:\, \big|\{s \in \mathbf{T} \,:\, s \geq_{\mathbf{T}} t\}\big| = \aleph_1\big\}.$$

Any specialization of the normal part can be extended to a specialization of the whole tree.

Recall that we write $\mathbb{P} \Vdash \varphi$ to denote that any element of $\mathbb{P}$ forces φ. We prove by induction on $\alpha \leq \omega_2$ that there is a countable support iteration

$$\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \mathbb{Q}'_\beta \,:\, \beta < \omega_2, \alpha \leq \omega_2\rangle$$

with the following properties:

(1) $\forall \alpha < \omega_2, \mathbb{P}_\alpha \in H(\aleph_2), \mathbb{P}_0 = \{1\}$;
(2) $\forall \alpha < \omega_2$,

$$\mathbb{P}_\alpha \Vdash_{\mathbb{P}_\alpha} \text{“if } b(\alpha) \text{ is a } \mathbb{P}_\alpha\text{-name of an Aronszajn tree,}$$
$$\text{then } \mathbb{Q}'_\alpha \text{ is a } \mathbb{P}_\alpha\text{-name for } \mathbb{Q}_{b(\alpha)}\text{; otherwise } \mathbb{Q}'_\alpha = \{1\}\text{”;}$$

(3) for $\alpha \leq \omega_2$,

$$\mathbb{P}_\alpha = \big\{f \,:\, \mathrm{supp}(f) \subseteq \alpha, \mathrm{supp}(f) \text{ is countable, and}$$
$$\forall \beta \in \mathrm{supp}(f)\big(f(\beta) \text{ is a canonical } \mathbb{P}_\beta\text{-name and } \mathbb{P}_\beta \Vdash f(\beta) \in \mathbb{Q}'_\beta\big)\big\};$$

(4) $\forall \alpha \leq \omega_2$, $\mathbb{P}_\alpha$ is ${}^\omega\omega$-bounding (and hence proper; see Definition 4.1).

For carrying the induction we use the following important preservation properties: Under the continuum hypothesis, the forcing order $\mathbb{Q}_{\mathbf{T}}$ from Definition 3.3 is of size at most $\aleph_1^{\aleph_0} = \aleph_1$ and is a subset of $H(\aleph_1)$, so $\mathbb{Q}_{\mathbf{T}} \in H(\aleph_2)$. Here the cardinals $\aleph_1$ and $\aleph_2$ are taken in the ground model.

The countable support limit of proper forcings is proper by [18, Chapter III, Theorem 3.1]. Hence, for any $\gamma \leq \omega_2$, $\aleph_1^{\mathbf{V}[G_\gamma]} = \aleph_1$.

We cite [18, Chapter III, Theorem 4.1]. Assume that κ is regular and uncountable, and assume that $(\forall \alpha < \kappa)(\alpha^{\aleph_0} < \kappa)$. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta \,:\, \beta < \kappa, \alpha \leq \kappa\rangle$ be a countable support iteration such that, for any $\alpha < \kappa$, $\mathbb{P}_\alpha \Vdash |\mathbb{Q}_\alpha| < \kappa$. Then $\mathbb{P}_\kappa$ has the κ-antichain condition, and for any $\alpha < \kappa$, $\mathbb{P}_\alpha$ has a dense subset of size less than κ and $\mathbb{P}_\alpha \Vdash 2^{\aleph_0} < \kappa$. We apply this theorem under the continuum hypothesis with $\kappa = \aleph_2$ (from the ground model) and thus get $\aleph_2^{\mathbf{V}[G]} = \aleph_2$ and that, at any stage $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash \mathrm{CH}$.

The preservation of $\aleph_1$ and of $\aleph_2$ yields that, at any stage $\alpha < \omega_2$, ω_1 in the sense of the stage is the ω_1 from the ground model. If $\mathbb{P}_\alpha \in H(\aleph_2)$, then also any canonical $\mathbb{P}_\alpha$-name of a condition in $\mathbb{Q}_{\mathbf{T}}$ for an Aronszajn tree $\mathbf{T}$ is an element of $H(\aleph_2)$. Hence, according to clause (3) of the recursive definition, $\mathbb{P}_{\alpha+1} \subseteq H(\aleph_2)$. Since the iterand $\mathbb{Q}'_\alpha$ is forced to be an element of $H(\aleph_2)$, by the continuum hypothesis there are few canonical names for elements of the iterand $\mathbb{Q}'_\alpha$. Thus, by (3) we get $|\mathbb{P}_{\alpha+1}| < \aleph_2$ and $\mathbb{P}_{\alpha+1} \in H(\aleph_2)$. For limit steps $\alpha < \omega_2$, if $\mathbb{P}_\beta \in H(\aleph_2)$ for $\beta < \alpha$, then by the continuum hypothesis again there are few functions with countable support, and hence by (3), $\mathbb{P}_\alpha \in H(\aleph_2)$.

Let $\mathbb{P}$ be a countable support iteration, and for any $i < \kappa$, $\mathbb{P}_i \Vdash$ "$\mathbb{Q}_i$ is ${}^\omega\omega$-bounding." Then by [18, Chapter VI, Section 4] also the limit $\mathbb{P}_\kappa$ is ${}^\omega\omega$-bounding.

Hence, we know that an iterated forcing $\mathbb{P}_{\omega_2}$ with properties (1)–(4) exists. Fix any such $\mathbb{P}_{\omega_2}$. Let G be a $\mathbb{P}_{\omega_2}$-generic filter over $\mathbf{V}$. We let $G_\alpha = G \cap \mathbb{P}_\alpha$.

We prove that $\mathbb{P}_{\omega_2}$ forces that any Aronszajn is special. Let $\mathbf{T} = \sigma[G] \in \mathbf{V}[G]$ be an Aronszajn tree in the extension in the sense of Definition 1.1, and let $p \in \mathbb{P}_{\alpha_0} \cap G$ force this. Since $\aleph_1^{\mathbf{V}} = \aleph_1^{\mathbf{V}[G]}$ and since $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-antichain condition, by [18, Chapter III, Theorem 4.1] there is an $\alpha_1 \in [\alpha_0, \omega_2)$ such that $\mathbf{T}$ has a canonical $\mathbb{P}_{\alpha_1}$-name τ and $p \Vdash_{\mathbb{P}_{\alpha_1}} \sigma = \tau$. Now for any $\gamma \geq \alpha_1$, $p \Vdash_{\mathbb{P}_\gamma}$ "τ is an Aronszajn tree," since this is forced by p in the forcing $\mathbb{P}_{\omega_2}$, and being an Aronszajn tree, it is downward absolute if $\aleph_1$ is the same and $\mathbb{P}_\gamma$ is a complete suborder of $\mathbb{P}_{\omega_2}$. Moreover, $\tau[G_{\alpha_1}] = \tau[G_\gamma]$ for any $\gamma \in [\alpha_1, \omega_2]$. Since for any $\gamma \in \omega_2$, $b \restriction [\gamma, \omega_2)$ is surjective onto $H(\aleph_2)$ and since canonical $\mathbb{P}_\alpha$-names for subsets of $\omega_1 \times \omega_1$ are elements of $H(\aleph_2)$, by property (2) of $\mathbb{P}_{\omega_2}$ there is some $\alpha \geq \alpha_1$ such that $b(\alpha) = \tau$ (so really the same names) and

$$p \Vdash_{\mathbb{P}_\alpha} b(\alpha) = \tau \text{ is an Aronszajn tree and } \mathbb{Q}'_\alpha = \mathbb{Q}_\tau.$$

Now we use again Shelah's result on canonical names. Each $f(\alpha)$ that is forced to be an element of $\mathbb{Q}_{b(\alpha)}$ has a canonical $\mathbb{P}_\alpha$-name. This ensures that p forces that $\mathbb{P}_{\alpha+1}$ defined according to (3) is equivalent to forcing with $\mathbb{Q}_{b(\alpha)}$ over $\mathbf{V}^{\mathbb{P}_\alpha}$. By the choice of the iterand $\mathbb{Q}'_\alpha = \mathbb{Q}_{b(\alpha)}$ under the condition p, by Conclusion 4.10 the same condition $p \in \mathbb{P}_{\alpha+1} \cap G$ forces in $\mathbb{P}_{\alpha+1}$ that τ is special via the generic specialization function $f_{\mathbb{Q}'_\alpha}$ from Corollary 3.14. The property "$f_{\mathbb{Q}'_\alpha}[G_{\alpha+1}]$ specializes $\tau[G_{\alpha_1}]$" is upward absolute from $\mathbf{V}[G_{\alpha+1}]$ to $\mathbf{V}[G]$. Hence, it holds also in $\mathbf{V}[G]$. This concludes the proof that $\mathbb{P}_{\omega_2}$ forces that any Aronszajn tree is special.

In any extension by an ${}^\omega\omega$-bounding (and hence proper) forcing, the set ${}^\omega\omega \cap \mathbf{V}$ stays as a dominating family, and therefore, $\mathfrak{d} \leq |({}^\omega\omega) \cap \mathbf{V}|^{\mathbf{V}[G]} = \omega_1$. Of course, $\mathfrak{d}$ is uncountable; hence, $\mathfrak{d} = \aleph_1$ in $\mathbf{V}[G]$.

In Theorem 5.5 we prove that, for each $\mathbf{q} \in K_1$, $\mathbb{P}_\mathbf{q} \Vdash \operatorname{unif}(\mathcal{M}) = \aleph_2$. The forcing $\mathbb{P}_{\omega_2}$ given by the above definition is in K_1. Hence, in $\mathbf{V}[G]$, $2^\omega \geq \aleph_2$. By the already mentioned theorem [18, Chapter III, Theorem 4.1], $\mathbb{P}_{\omega_2} \Vdash 2^\omega \leq \aleph_2$. $\square$

5 $\mathbb{Q}_\mathbf{T}$ Makes the Ground Model Reals Meager

Let the set of reals $\mathbb{R}$ carry the usual order topology. A subset $A \subseteq \mathbb{R}$ is called *meager* if it is the union of countably many nowhere dense sets. The uniformity of the ideal of meager sets is defined as

$$\operatorname{unif}(\mathcal{M}) = \min\big\{|A| \,:\, A \subseteq \mathbb{R}, A \text{ is not meager}\big\}.$$

Moore, Hrušák, and Džamonja in [12] showed that $\diamondsuit(\mathbb{R}, \mathcal{M}, \notin)$—a strengthening of $\operatorname{unif}(\mathcal{M}) = \aleph_1$, which says that

$$(\forall \text{ Borel } F\colon {}^{\omega_1>}2 \to \text{ meager } F_\sigma)\big(\exists\langle g_\delta \,:\, \delta \in \omega_1, \delta \text{ limit}\rangle\big)$$
$$\forall x \in {}^{\omega_1}2\big\{\alpha \in \omega_1 \,:\, g_\alpha \notin F(x \restriction \alpha)\big\} \text{ is stationary}$$

—implies that there is a Souslin tree. A function $F\colon {}^{<\omega_1} \to$ meager F_σ is called *Borel* if for each infinite countable α the layer $F \restriction 2^\alpha$ is Borel in the natural topologies on 2^α and the set of F_σ-sets.

We assume that $2^{\aleph_1} = \aleph_2$, and we let $\mathbb{P}_{\omega_2}$ be a countable support iteration of $\mathbb{Q}_{\mathbf{T}_\alpha}$, with a suitable bookkeeping, so that for each $\beta < \omega_2$, each $\mathbb{P}_\beta$-name of an Aronszajn tree is named after stage β. Our forcing $\mathbb{P}_{\omega_2}$ is fairly definable; hence, the proofs sketched in [12] support the conjecture. If $\mathbb{P}_{\omega_2}$ forced $\operatorname{unif}(\mathcal{M}) = \aleph_1$, then it would also force $\diamondsuit(\mathbb{R}, \mathcal{M}, \notin)$. Here we show that $\mathbb{P}$ indeed forces $\operatorname{unif}(\mathcal{M}) = \aleph_2$.

We assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$.

Definition 5.1 Let $\mathbf{T}$ be a standard Aronszajn tree, and let $p \in \mathbb{Q}_\mathbf{T}$.

(1) We say that p is *diverse* if, for any $s \in T^p$ and for any $t_1 \neq t_2 \in \operatorname{suc}_{T^p}(s)$, the partial specializations $\nu_i = \operatorname{last}(t_i)$ are contradictory, which means that

$$\big(\exists\gamma_1 \in \operatorname{dom}(\nu_1)\big)\big(\exists\gamma_2 \in \operatorname{dom}(\nu_2)\big)$$
$$\big((\gamma_1 <_\mathbf{T} \gamma_2 \wedge \nu_1(\gamma_1) = \nu_2(\gamma_2)\big) \vee (\gamma_2 <_\mathbf{T} \gamma_1 \wedge \nu_1(\gamma_1) = \nu_2(\gamma_2)$$
$$\vee \big(\gamma_1 = \gamma_2 \wedge \nu_1(\gamma_1) \neq \nu_2(\gamma_2)\big).$$

(2) A condition $p \in \mathbb{Q}_\mathbf{T}$ is called *weakly diverse* if for any $s \in T^p$ there is some $h \in \omega$ such that, for any $t_1 \neq t_2 \in \operatorname{suc}_{T^p}(s)$ for any extensions t_1^* of t_1 and t_2^* of t_2 to level $\lg(s)+1+h$, we have that $\operatorname{last}(t_1^*)$ and $\operatorname{last}(t_2^*)$ are contradictory.

Lemma 5.2 *For a standard Aronszajn tree* $\mathbf{T}$ *the following hold.*

(1) *There is a diverse* $p \in \mathbb{Q}_\mathbf{T}$.
(2) *We assume the following.*
  (a) $p \in \mathbb{Q}_\mathbf{T}$ *is diverse.*

(b) $p \Vdash$ *"there is a unique branch* $\langle t_i : i \in [i(p), \omega)\rangle$ *of* T^p *such that*

$$\bigcup\{\text{last}(t_i) : i \in [i(p), \omega)\} \subseteq \bigcup\{\text{last}(\text{rt}(r)) : r \in G_{\mathbb{Q}_{\mathbf{T}}}\}."$$

(c) *For* $s \in T^p$ *the sequence* $\langle t_{s,\ell} : \ell \in \text{pos}(\mathbf{c}_{p,s})\rangle$ *lists* $\text{suc}_{T^p}(s)$.

(d) *The* $\mathbb{Q}_{\mathbf{T}}$*-name* $\underset{\sim}{\varrho}$ *is a name for an element of* ${}^\omega\omega$ *such that* $i \geq i(p) \rightarrow t_{i+1} = t_{i,\varrho(i)}$ *and* $\varrho(i) = 0$ *for* $i < i(p)$.

Under the assumptions (a)–(d) we have that $p \Vdash_{\mathbb{Q}_{\mathbf{T}}} \underset{\sim}{\varrho} \in {}^\omega\omega$ *is eventually different from any* $\eta \in ({}^\omega\omega)^{\mathbf{V}}$.

(3) *The set of weakly diverse* $p \in \mathbb{Q}_{\mathbf{T}}$ *is dense (not used).*

(4) *Similarly to (2) for any weakly diverse* p *there are a list of infinitely many levels and a name for an eventually different real.*

Proof (1) The condition given in Lemma 3.4 is diverse. (2) Let $\eta \in {}^\omega\omega \cap \mathbf{V}$. We let, for $n \geq i(p)$, $D_{\eta,n} = \{q \geq p : (\forall i \geq n)(\forall s \in q^{[i]})(t_{s,\eta(i)} \notin \text{suc}_{T^q}(s)\}$. It is easy to see that $\bigcup_{n \geq i} D_n$ is dense above p, and any $q \in D_n$ forces that $\underset{\sim}{\varrho}(i) \neq \eta(i)$ for $i \geq n$. □

Lemma 5.3 *Let* $p \in \mathbb{Q}_{\mathbf{T}}$ *be diverse. Then* $p \Vdash ({}^\omega 2)^{\mathbf{V}}$ *is meager.*

Proof The condition p forces that the generic real $\underset{\sim}{\varrho}$ that is constructed from p and an enumeration as in (2.c) of the previous lemma is eventually different from any real in the ground model. By Bartoszynski and Judah [2, Theorem 2.4.7] a forcing makes the ground model reals meager if and only if it adds an eventually different real. □

Definition 5.4 We have that K_1 is the class of countable support iterations $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \omega_2, \alpha \leq \omega_2\rangle$, $\mathbb{Q}_\beta = \mathbb{Q}_{\underset{\sim}{\mathbf{T}}_\beta}$, $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\omega_2}$, where $\underset{\sim}{\mathbf{T}}_\beta$ is a $\mathbb{P}_\beta$-name of a standard Aronszajn tree (as in Definition 1.1), and for every $\alpha < \omega_2$ and $\mathbb{P}_\alpha$-name $\underset{\sim}{\mathbf{T}}$ of a standard Aronszajn tree there is some $\beta \in [\alpha, \omega_2)$ such that $\mathbb{P}_\beta \Vdash$ "$\underset{\sim}{\mathbf{T}}_\beta = \underset{\sim}{\mathbf{T}}$ if $\mathbf{T}$ is an Aronszajn tree." (Note that $\mathbb{P}_\beta / \mathbb{P}_\alpha$ may add an ω_1-branch to $\mathbf{T}$.)

Theorem 5.5 *If* $\mathbf{q} \in K_1$*, then* $\mathbb{P}_{\mathbf{q}} \Vdash \text{unif}(\mathcal{M}) = \aleph_2$.

Proof It is enough to prove for $\alpha < \omega_2$ that $\mathbb{P}_q \Vdash$ "$({}^\omega 2)^{\mathbf{V}[\mathbb{P}_\alpha]}$ is meager." Let $p \in \mathbb{P}_{\mathbf{q}}$. Then there is $\beta < \omega_2$, $\beta = \alpha + i \notin \text{dom}(p)$ for some $i < \omega_1$. We let $q = p \cup \{(\beta, p_*)\}$, and $q \restriction \beta \Vdash p_* = p(\beta)$ is a diverse condition. So by [2], $q \Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$({}^\omega 2)^{\mathbf{V}[\mathbb{P}_\beta]}$ is meager." Then also $q \Vdash_{\mathbb{P}_{\mathbf{q}}}$ "$({}^\omega 2)^{\mathbf{V}[\mathbb{P}_\alpha]}$ is meager." As $\alpha < \omega_2$ and $p \in \mathbb{P}_{\mathbf{q}}$ were arbitrary, we are done. □

This concludes the proof of Theorem 1.4.

Remark 5.6 Except for the work on the halving property, all other technical steps can be performed with simple creatures, because there we never changed the value $k(\mathbf{c})$ of a creature in a condition when strengthening a condition according to the demands. So Theorem 1.4 can be proved with a slightly simpler relative of $\mathbb{Q}_{\mathbf{T}}$ in which the nodes in the conditions p are described by simple creatures.

Notes

1. This stronger form of premise (i) and hence weaker form of clause (2) is used in the de-halving Lemma 3.16.

2. This condition is used in Lemma 3.11. It is crucial for the fact that smooth conditions are dense. Only for smooth conditions do we have fusion. So the properness proof in Lemma 4.8 hinges on this clause. The existence of the finite u_t, which has the same size for all branches in a cone of T^p, is used to show that the number ℓ_1^* in Lemma 2.13 is small relative to the norm of sufficiently many creatures in a forcing condition.

3. Everything depends also on s, but we do not introduce an index s.

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Acknowledgments

Both authors were partially supported by the second author's European Research Council grant 338821. This is number 988 in the second author's list of works.

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