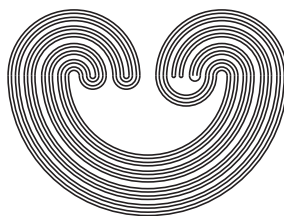


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## REMARKS ON $\lambda$ -COLLECTIONWISE HAUSDORFF SPACES

by

SAHARAN SHELAH

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**Mail:** Topology Proceedings  
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Auburn University, Alabama 36849, USA

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## REMARKS ON $\lambda$ -COLLECTIONWISE HAUSDORFF SPACES

Saharon Shelah<sup>1</sup>

The purpose of this note is to answer questions raised by Fleissner in [F]. Explicitly, our results are

*Theorem 1.* Let  $\Sigma$  be the statement "there is a locally countable, locally compact, normal Moore space which is  $\leq\omega_1$ -collectionwise Hausdorff but not  $\leq\omega_2$ -collectionwise Hausdorff."  $\Sigma$  is consistent with ZFC (the usual axioms for set theory). Moreover, both  $\Sigma + \text{not CH}$  and  $\Sigma + \text{CH}$  are consistent with ZFC.

*Theorem 2.* Let  $M$  be a model of set theory obtained by using Levy forcing to collapse a weakly compact cardinal to  $\omega_2$ . In  $M$ , let  $X$  be a locally countable space. Then  $X$  is  $\leq\omega_2$ -collectionwise Hausdorff if  $X$  is  $<\omega_2$ -collectionwise Hausdorff.

There are variations on Theorem 2. We may replace "locally countable" with "first countable and locally of cardinality  $\leq\omega_1$ ." Also, if we collapse a supercompact cardinal (rather than a merely weakly compact cardinal), we may strengthen the conclusion to  $X$  is collectionwise Hausdorff.

A subset  $Y$  of a topological space  $X$  is called closed, discrete if every point of  $X$  has a neighborhood containing

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at most one point of  $Y$ . A closed discrete set  $Y = \{y_i : i \in I\}$  can be screened if there is a family of disjoint open sets  $\{U_i : i \in I\}$  such that  $U_i \cap Y = \{y_i\}$ . A space  $X$  is called collectionwise Hausdorff if every closed discrete subset of  $X$  can be screened.  $X$  is  $\langle \lambda$ -collectionwise Hausdorff if every closed discrete subset of cardinality  $\langle \lambda$  can be screened;  $\leq \lambda$ -collectionwise Hausdorff is defined similarly.

On the situation of Theorem 1 for  $\Sigma + \text{GCH}$ , see [S'].

### 1. Proof of Theorem 1

For concreteness, let us start with a model of  $V = L$ . Then by Jensen's work [J], there is a subset  $E$  of  $\omega_2$  such that

- a)  $\alpha \in E$  implies  $\text{cf } \alpha = \omega$
- b)  $E$  is stationary in  $\omega_2$
- c)  $E \cap \delta$  is not stationary in  $\delta$  for any  $\delta < \omega_2$ .

For each  $\alpha \in E$ , choose  $\eta_\alpha : \omega \rightarrow \alpha$  to be strictly increasing with range  $\eta_\alpha$  cofinal in  $\alpha$ . Set  $I = \{\eta_\alpha \mid m : \alpha \in E \ m \in \omega\}$ . The point set of our space is  $E \cup I$ . (Note that  $E$  and  $I$  are disjoint). Points of  $I$  are isolated; the  $n$ th neighborhood of  $\alpha \in E$ ,  $B(\alpha, n)$ , is  $\{\alpha\} \cup \{\eta_\alpha \mid m : m > n\}$ . Where we consider  $E$  as a subset of  $X$  we will call it  $Y$ ; it is easy to check that  $Y$  is closed, discrete.

From b) and the Pressing Down Lemma it quickly follows that  $Y$  can not be screened, and so  $X$  is not  $\leq \omega_2$ -collectionwise Hausdorff. It is straightforward to prove by induction on  $\rho < \omega_2$  that  $\{\alpha \in Y : \alpha < \rho\}$  can be screened. Thus  $X$  is  $\leq \omega_1$ -collectionwise Hausdorff. Clearly  $X$  is locally countable, locally compact Moore space. This space has been described

in  $[F']$ .

In  $L[X]$  is not normal, so our plan is to extend to a model in which  $X$  is normal. Of course, we must check that a), b), and c) are preserved. First all a) and c) assert is that certain sets exist, and so are preserved by any extension. Further we want  $\omega_2$  in the ground model to remain  $\omega_2$  in the extension so that c) has its intended meaning. This will happen because our two extensions are ccc, and  $\omega_2$ -cc and  $\omega$ -Baire, respectively. Finally we note that b) is preserved by an  $\omega_2$ -cc extension. To see this, note that the  $\alpha^{\text{th}}$  element of  $C^\circ$ , a club set in the extension, is contained in a set in the ground model of cardinality  $\omega_1$ . Using this fact, we can find point that are limits of elements of  $C^\circ$  whatever  $C^\circ$  is. Thus we can find for every club set  $C^\circ$  in the extension a club set  $C$  in the ground model satisfying  $C \subset C^\circ$ . We conclude that a set stationary in  $\omega_2$  in the ground model remains stationary in  $\omega_2$  in the extension. Our extensions will preserve a), b), and c), and  $X$  will remain  $\leq \omega_1$ -collectionwise Hausdorff and not  $\leq \omega_2$ -collectionwise Hausdorff.

The first extension is the Solovay-Tennenbaum extension forcing Martin's Axiom and  $c > \omega_2$ , a ccc extension [ST]. We aim at showing that  $X$  is normal in this model. It is sufficient to show that disjoint subsets  $H$  and  $K$  of  $Y$  can be separated by disjoint open sets. Define  $P$  to be the set of pairs  $(u,v)$  satisfying

$$d) u \cap v = \emptyset$$

e)  $u$  (respectively,  $v$ ) is the union of finitely many basic open sets  $B(\alpha, n)$  with  $\alpha \in H$  (respectively,  $\alpha \in K$ ).

Define  $(u,v) \leq (u',v')$  if  $u \supseteq u'$  and  $v \supseteq v'$ . That this partial order has ccc follows quickly from the Delta System Lemma. Since  $c > \omega_2 = \text{card } H \cup K$ , by Martin's Axiom we can define disjoint open sets  $U$  and  $V$  separating  $H$  and  $K$ .

We have shown that  $\Sigma + \text{not CH}$  is consistent. To show that  $\Sigma + \text{CH}$  is consistent, we need a Martin's Axiom-like extension which adds no subsets of  $\omega$ . Analogues of Martin's Axiom have been shown consistent and investigated [T], [S], but they are not applicable in this situation because we need a notion of forcing which is not countably closed. Our plan is to make  $X$  normal by an extension which is not countably closed, but is "sufficiently" countably closed.

Let  $H$  and  $K$  be disjoint subsets of  $Y$ . We will define a partial order  $P(H,K)$  of pairs  $(u,v)$  parallel to the order  $P$  used above with Martin's Axiom. Requirement d) remains the same, but in order to make  $P(H,K)$  sufficiently countably closed, we change e) to

e')  $u$  (respectively,  $v$ ) is the union of countably many basic open sets  $B(\alpha,n)$  where  $\alpha \in H$  (respectively,  $\alpha \in K$ ).

Now a new problem arises. It can happen that there is a  $y \in \text{closure } u \cap K$ . If this occurs,  $(u,v)$  cannot be extended to  $(u',v')$  with  $y \in v$ , and so the generic filter need not define a separation of  $H$  and  $K$ . To prevent this we add, defining  $s((u,v))$  to be  $[\text{closure } (u \cup v)] \cap Y$ ,

e")  $s((u,v)) \subseteq u \cup v$ .

We now define  $P(H,K)$  to be the set of pairs  $(u,v)$  satisfying d), e') and e").  $P(H,K)$  is not countably closed, for the "union" of a countable sequence of elements of  $P(H,K)$  will

satisfy d) and e'), but might not satisfy e"). A sufficient condition for  $(u,v)$  to satisfy e") is that  $s((u,v))$  be closed in  $E$  (with the topology inherited from  $\omega_2$  with the order topology). Lemma 1, below will give us a way to insure that certain countable sequences of elements of  $P(H,K)$  will have an infimum. (In our application, the  $A_\alpha$ 's will be  $s((u,v))$ 's).

Call a well ordered sequence of sets,  $\{A_\alpha: \alpha < \rho\}$ , continuous and increasing if

$$f) \alpha < \beta \text{ implies } A_\alpha \subseteq A_\beta$$

$$g) \delta \text{ a limit ordinal implies } A_\delta = \cup\{A_\alpha: \alpha < \delta\}$$

*Lemma 1. Suppose that  $E$  satisfies a), b), c);  $\nu$  is an ordinal less than  $\omega_2$ ; and  $\{A_\alpha: \alpha < \omega_1\}$  is a continuous increasing sequence of countable sets with  $\cup\{A_\alpha: \alpha < \omega_1\} = E \cap \nu$ . Give  $E \cap \nu$  the topology inherited from  $\omega_2$  with the order topology. Then*

*$\{\alpha: A_\alpha \text{ is closed in } E \cap \nu\}$  contains a club set.*

*Proof.* We prove the lemma by induction on  $\nu$ . For  $\nu < \omega_1$  or  $\nu$  a successor ordinal, the induction step is trivial.

**Case 1:**  $\nu$  is a limit ordinal of cofinality  $\omega$ . Let  $\nu_n$ ,  $n < \omega$ , be increasing and cofinal in  $\nu$ . By induction hypothesis,  $\{\alpha: A_\alpha \cap \nu_n \text{ is closed in } E \cap \nu_n\}$  contains a club set. Then  $\{\alpha: A_\alpha \text{ is closed in } E \cap \nu\} = \cap\{\alpha: A_\alpha \cap \nu_n \text{ is closed in } E \cap \nu_n\}$  contains a club set.

**Case 2:**  $\nu$  is a limit ordinal of cofinality  $\omega_1$ . By c) we can find  $\{\nu_\alpha: \alpha < \omega_1\}$  continuous, increasing, cofinal in  $\nu$ , and disjoint from  $E$ . If  $A_\alpha \cap \nu$  is not closed in  $E \cap \nu$ , define  $h(\alpha)$  to be the least ordinal such that  $A_\alpha \cap \nu_{h(\alpha)}$  is

not closed in  $E \cap v$ . Using the regularity of  $\omega_1$ , the hypothesis that each  $A_\alpha$  is countable, and f), we can find a club set  $C$  of limit ordinals such that if  $\gamma \in C$  and  $\alpha < \gamma$ , then  $A_\alpha \subseteq v_\gamma$  and  $h(\alpha)$  is either undefined or less than  $\gamma$ . Using g), for  $\gamma \in C$ ,  $A_\gamma \subseteq v_\gamma$ , hence any limit point of  $A_\gamma$  in  $E$  is less than  $v_\gamma$  (not equal to  $v_\gamma \notin E$ ). Hence  $h(\gamma)$  is either undefined or  $h(\gamma) < \gamma$ .

If  $h$  presses down on a stationary set, then by the Pressing Down Lemma  $h(\alpha) = \beta$  for some  $\beta$  and stationarily many  $\alpha$ 's. Then the lemma fails for  $v_\beta$  and  $\{A_\alpha \cap v_\beta : \alpha < \omega_1\}$ , contradicting the inductive hypothesis.

We now define our desired forcing,  $P_{\omega_3}$ , by inductively defining notions of forcing  $P_\beta$ ,  $\beta \leq \omega_3$ . Simultaneously, we will show that  $P_\beta$  is  $\omega_2$ -cc and  $\omega$ -Baire (i.e. adds no  $\omega$ -sequences of ordinals), so that we may require j) and k) below. Explicitly, by induction on  $\beta \leq \omega_3$ , we define  $P_\beta$  to be the set of  $p$  satisfying

- h)  $p$  is a function with domain  $\beta$
- i)  $p(\alpha) \in P(H_\alpha, K_\alpha)$  where  $H_\alpha, K_\alpha$  are terms for disjoint subsets of  $Y$  in the forcing language for  $P_\alpha$
- j)  $\{(H_\alpha, K_\alpha) : \alpha < \omega_3\}$  enumerates all terms for disjoint subsets of  $Y$  in the language for  $P_{\omega_3}$
- k)  $p(\alpha) \in L$  (the ground model) (i.e. it is not a term for an element of  $P(H_\alpha, K_\alpha)$ , it is an element of  $P(H_\alpha, K_\alpha)$ ).
- l)  $p(\alpha) = (\emptyset, \emptyset)$  for all but countably many  $\alpha$ 's
- m)  $p \leq q$  if  $p(\alpha) \supseteq q(\alpha)$  for all  $\alpha < \beta$ .

That  $P$  has  $\omega_2$ -cc follows from the continuum hypothesis, l), and the Delta System Lemma. So j) is possible.

Aiming towards showing that  $P_\beta$  is  $\omega$ -Baire, let  $\{D_n: n \in \omega\}$  be a countable set of dense open subsets of  $P_\beta$ , and  $p$  an arbitrary element of  $P_\beta$ . Let  $\underline{N}$  be a structure containing everything relevant, e.g.  $\underline{N} = \langle V_{\omega_4}, \in P_\beta, \Vdash_{P_\beta}, E, \beta, \{D_n: n \in \omega\} \rangle$ . Let  $\underline{N}_\rho$ ,  $\rho < \omega_1$ , be a continuous increasing sequence of countable elementary submodels of  $\underline{N}$  satisfying  $\underline{N}_\rho \in N_{\rho+1}$ . Set  $\omega_2 \cap \cup\{N_\rho: \rho < \omega_1\} = \nu$ , an ordinal less than  $\omega_2$ . Applying Lemma 1 to  $E$ ,  $\nu$ ,  $\{E \cap N_\rho: \rho < \omega_1\}$ , we can find  $\rho_n$ ,  $n \in \omega$ ,  $\sup\{\rho_n: n \in \omega\} = \rho$ , such that

n)  $E \cap N_\rho$  is closed in  $E \cap \nu$ .

We define a sequence  $\{p_n: n \in \omega\}$  of forcing conditions satisfying

o)  $p_0 = p$ ,  $p_{n+1} < p_n$

p)  $p_{n+1} \in D_n \cap N_{\rho_n}$

q)  $s(p_{n+1}(\alpha)) \supseteq N_{\rho_n} \cap E$ , when  $p_{n+1}(\alpha) \neq (\phi, \phi)$

Define  $q$  with domain  $\beta$  by  $q(\alpha) = \cup\{p_n(\alpha); n \in \omega\}$ .

Clearly  $q$  satisfies h), k), and l), and  $q$  satisfies i) by n) and q), so  $q \in P_\beta$ . We have found  $q$ ,  $q < p$  and  $q \in \cap\{D_n: n \in \omega\}$  and may conclude that  $P_\beta$ ,  $\beta \leq \omega_3$ , is  $\omega$ -Baire. This completes the simultaneous definition of  $P_\beta$  and verification of  $\omega_2$ -cc and  $\omega$ -Baire.

In the extension by  $P_{\omega_3}$ ,  $X$  is normal. For it is sufficient to consider disjoint  $H$  and  $K$  subsets of  $Y$ , and by j) there is a generic pair of open sets separating them. The Continuum Hypothesis is preserved by the extension because it is  $\omega$ -Baire.

## 2. Proof of Theorem 2

We imitate Baumgartner [B]. Let  $\kappa$  be weakly compact in



$M$ , the ground model, and let  $P(\kappa, \omega_2)$  be the Levy forcing collapsing  $\kappa$  to  $\omega_2$ . Let  $X^\circ$  be the name of a locally countable,  $\leq_{\omega_1}$ -collectionwise Hausdorff space with  $\{y_\alpha : \alpha < \kappa\}$  a closed discrete subset of  $X^\circ$  that can not be screened. We may assume that  $X^\circ \in V_\kappa$ , by  $\Pi_1^1$  indescribability, there is a  $\lambda < \kappa$  with the same properties. Explicitly,  $X^\circ \cap V_\lambda$  is the name in the language for  $P(\lambda, \omega_2)$  of a locally countable,  $\leq_{\omega_1}$ -collectionwise Hausdorff space with  $\{y_\alpha : \alpha < \lambda\}$  a closed discrete subset of  $X^\circ \cap V_\lambda$  that can not be screened.

Let  $G$  be an  $M$ -generic ultrafilter on  $P(\lambda, \omega_2)$ . We will work in  $M^1 = M[G]$ , where  $\omega_2 = \lambda$ ,  $X$  is  $\leq_{\omega_1}$ -collectionwise Hausdorff, and  $Y = \{y_\alpha : \alpha < \lambda\}$  witnesses that  $X$  is not  $\leq_{\omega_2}$ -collectionwise Hausdorff. For each  $\alpha < \kappa$  we choose a countable neighborhood  $B_\alpha$  of  $y_\alpha$ , fixed throughout this section. Set  $W_\beta = \cup\{B_\alpha : \alpha < \beta\}$ .

*Lemma 2.* *There are  $S, h$  such that*

- 1)  $S$  is a stationary subset of  $\omega_2$
- 2)  $\delta \in S$  implies that  $\text{cf } \delta = \omega$
- 3)  $h: S \rightarrow \omega_2$ ,  $h(\delta) \geq \delta$
- 4)  $y_{h(\delta)} \in \text{closure } W_\delta$ .

*Proof.* It suffices to find a set  $S$  satisfying 1), 2) and

- 5) for  $\delta \in S$ ,  $\text{closure } W_\delta \cap \{y_\alpha : \delta \leq \alpha < \omega_2\} \neq \emptyset$ .

Aiming for a contradiction, we assume that there is no such set  $S$ . Specifically, we assume that there is a club set  $C$  such that for  $\delta \in C_0 = \{\delta \in C : \text{cf } \delta = \omega\}$ ,  $\text{closure } W_\delta \cap \{y_\alpha : \delta \leq \alpha < \omega_2\} = \emptyset$ . Let  $C'$  be the set of limit points of  $C_0$ ;  $C'$  is a club set. We claim

- 6) for  $\delta \in C'$ ,  $\text{closure } W_\delta \cap \{y_\alpha : \delta \leq \alpha < \omega_2\} = \emptyset$ .

There are two cases. First, if  $\delta \in C'$ , cf  $\delta = \omega$ , 6) holds because  $\delta \in C_0$ . Second, if  $\delta \in C'$ , cf  $\delta > \omega$  we show 6) using the fact that  $X$  is locally countable. If there were  $y \in \text{closure } W_\delta \cap \{y_\alpha: \delta \leq \alpha < \omega_2\}$ , then  $y \in \text{closure } W_\gamma$  for cofinally many  $\gamma$  in  $\delta$ , in particular for some  $\gamma \in C_0$ , contradiction.

Let  $\langle \gamma(v): v < \omega_2 \rangle$  be the natural, monotone increasing enumeration of  $C'$ . Define  $U_v = W_{\gamma(v+1)}\text{-closure } W_{\gamma(v)}$ ; set  $\mathcal{U} = \{U_v: v < \omega_2\}$ . By definition,  $\mathcal{U}$  is a disjoint family of open sets, each containing at most  $\omega_1$  points of  $Y$ . By 6)  $\mathcal{U}$  covers  $Y$ . Using that  $X$  is  $\leq \omega_1$ -collectionwise Hausdorff, we can improve  $\mathcal{U}$  to screen  $Y$ . This contradiction establishes Lemma 2.

Note that  $P(\kappa, \omega_2) = P(\lambda, \omega_2) \oplus P'$ , where  $P'$  is countably closed. Our goal is to show that  $P'$  does not add a screening of  $Y$ . Since in the extension  $Y = \{y_\alpha: \alpha < \lambda\}$  has cardinality  $\omega_1$ , we will have shown that  $X^0$  is not  $< \omega_2$ -collectionwise Hausdorff, a contradiction. Towards this goal, suppose that  $p \in P'$  forces that  $\{V_\alpha: \alpha < \lambda\}$  screens  $Y$ .

Working in  $M^1$ , let  $\underline{N} = \langle V_{\kappa+\omega}, P', \Vdash_{P'}, p, \{y_\alpha: \alpha < \kappa\}, X^0, \{B_\alpha: \alpha < \kappa\} \rangle$ . Define a continuous increasing sequence  $\underline{N}_\rho$ ,  $\rho < \omega_2$ , of elementary submodels of  $\underline{N}$  satisfying  $\omega_1 \subset N_0$ ,  $\text{card } N_\rho = \omega_1$ ,  $W_\rho \subset N_\rho$ . Set  $\delta_\rho = N_\rho \cap \lambda$ . Then  $\{\delta \in \lambda: \delta = \delta_\rho\}$  is a club set in  $\lambda$ , so there is such a  $\delta$  in  $S$ . Let  $B_{h(\delta)} \cap N_\delta = \{z_n: n \in \omega\}$ .

We define a sequence  $p_n$ ,  $n \in \omega$ , of forcing conditions as follows. Set  $p_0 = p$ ; let  $p_{n+1} \in N_\delta$  decide  $z_n$ --either  $z_n \notin U\{V_\alpha: \alpha < \lambda\}$  or  $z_n \in V_\alpha$  for some specific  $\alpha$ . The point is that this specific  $\alpha$  must be in  $N_\delta$ , and thus can not be  $h(\delta)$ . Set  $q = U\{p_n: n \in \omega\}$ ;  $q$  might not be in  $N_\delta$ , but  $q$  is

in  $P'$ . Let  $q' \supseteq q$  choose  $V_{h(\delta)}$ . Because  $P'$  is countably closed,  $V_{h(\delta)} \cap B_{h(\delta)} \in M'$ . By our choice of  $p_n$ 's  $V_{h(\delta)} \cap B_{h(\delta)} \cap N_\delta = \emptyset$ . As  $W_\delta \subset N_\delta$ ,  $V_{h(\delta)} \cap B_{h(\delta)}$  is an open neighborhood of  $y_{h(\delta)}$  demonstrating that  $y_{h(\delta)} \notin \text{closure } W_\delta$ . We chose  $\delta \in S$ , so this contradicts 4). This contradiction completes the proof of Theorem 2.

The proofs of the variants of Theorem 2 are parallel and so omitted.

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The Hebrew University  
Jerusalem