CHAPTER 11

Reflecting on logical dreams

Saharon Shelah

1 The independence phenomenon

The "forcing method," introduced by Cohen (1966), has produced a wealth of independence results in set theory. What about PA, that is, arithmetic? Now Shelah (2003a) described the following Dream 2.1:

> Find a "forcing method" relative to PA which shows that PA and even ZFC does not decide "reasonable" arithmetical statements, in analogy with the situation in which the known forcing method works for showing that ZFC cannot decide reasonable set theoretic questions; even showing the unprovability of various statements in bounded arithmetic (instead of PA) is formidable.

Why is it interesting to prove independence results? I can understand someone disregarding the work on cardinal arithmetic, claiming that it is not interesting, though I think he or she is wrong. If I had to act as a lawyer I could try to write an argument for him or her; I could express myself as "(s)he is wrong but consistent." But in this case it is hard for me to understand any opposition (well, from mathematicians, let us say pure mathematicians).

A mathematician can look at the work of Gödel and Cohen and say "very nice, but this does not really affect me." In fact, this was the outlook of not a few mathematicians in various fields before forcing was proved relevant to their work. A finite combinatorialist can look at all the independence results and say that they do not deal with problems he or any reasonable mathematician has been working on.

We try here to comment on Shelah (2003a), give clarifications, and also discuss what we have learnt since, finishing with a recapitulation of the main points (in Section 8). This was originally written as answers to questions by Juliette Kennedy. We thank Gregory Cherlin, Udi Hrushovski, Juliette Kennedy, Menachem Kojman and Jouko Väänänen for helpful comments and questions and Kennedy and Vāāñañen again for editorial assistance. This is paper E73 in the author's list of publications.

On pointing out that this can be translated to problems about the existence of solutions to polynomials (for Gödel, a polynomial with exponentiation, after J. Robinson–Putnam–Matiyasevich, a real polynomial) he or she will probably raise his or her head and say: "in principle it may be interesting, though I do not understand the hubris of Hilbert who thought to find an algorithm, but we are working on quite specific problems. In fact, nothing close to Ackermann functions (not to say faster growing functions) interests me; so, on the one hand the Paris–Harrington theorem and its descendants and relatives are irrelevant and, on the other hand, the polynomials mentioned above are artificial. It is very nice to know that the earth is not flat, but if I do not know anything about places far from me and if my technology is primitive, this has no real affect on me."

The dream above should change this.

2 Semi-axioms

The independence results of set theory raise the question, of what axioms could we add to the ZFC axioms in order to decide some or most of the statements left undecided by ZFC. It is a difficult question, and famously discussed by Gödel, what the criteria for new axioms should be. We have talked in this connection about "semi-axioms":

> What are our [criteria] for semi-axioms? First and most important, a semi-axiom must have many consequences, making it have a rich, deep, beautiful theory. Second, it is preferable that a semi-axiom is reasonable and "has positive measure." (Shelah 2003a, p. 214)

There is a striking difference between the first and the second criteria. Let us try to consider examples close to my heart. The axiom $\mathbf{V} = \mathbf{L}$ is an excellent example for the first criterion; there are many consequences in the works of Gödel, Jensen and others. But $\mathbf{V} = \mathbf{L}$ is not seen as being reasonable and having "positive measure," at least in the opinion of most set theorists.

Another example is the axiom "the continuum is real valued measurable," which Fremlin wrote so much about. Well, by the first criterion it is reasonably strong (though nothing like $\mathbf{V} = \mathbf{L}$) but its measure, it seemed, is not impressive. (See also, for example the universe forced in Blass and Shelah (1987).) On the other hand "the continuum is weakly inaccessible" seems to be very reasonable but has few consequences.

The so-called "California school" advocates the importance of determinacy axioms in descriptive set theory, and the relevance of such axioms being consequences of certain large cardinal axioms, and moreover, the importance of their being true. Is this a reasonable position? In Shelah (2003a) I made the following remark:

> I strongly reject the California school's position on several grounds.
>
> (a) Generally I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axioms (if there is such a thing, which I doubt) is a sufficient reason to say it is a "true axiom." In particular I do not find it compelling at all to see it as true.
> (b) The judgments of certain semi-axioms as best is based on the groups of problems you are interested in. For the California school descriptive set theory problems are central. While I agree that they are important and worth investigating, for me they are not "the center." Other groups of problems suggest different semi-axioms as best, other universes may be the nicest from a different perspective.
> (c) Even for descriptive set theory the adoption of the axioms they advocate is problematic. It makes many interesting distinctions disappear.

I still stand by (a), (b), and (c) as I stated them above.

For (b), we can consider the following two examples. For descriptive set theory, the axiom $\mathbf{V} = \mathbf{L}$ gives very definitive answers, still "the hierarchy collapses" and descriptive set theorists are not satisfied. Also the axiom AD gives very definitive answers, but ones which seem more natural, and are much more liked by descriptive set theorist. Still the knowledge that under $\mathbf{V} = \mathbf{L}$ we get other answers is illuminating.

Note that Woodin, on learning more, changed his mind on the value of the continuum, but I do not know enough to comment on this.

As to (a), note that we may well believe that such a theory ZFC + X is natural, important, elegant and will even appear in unexpected contexts *but* we may also know that another semi-axiom Y contradicting it has all these wonderful properties too.

Also, the reader may like to refer to the discussion in the sixth paragraph of Section 3.2 in Shelah (2003a). With this in mind, let us return, as an example, to another family of problems for which $\mathbf{V} = \mathbf{L}$ gives a definite answer. The problems are on Abelian groups and the answers erased distinctions. As Nunke (1977) explains nicely, the original solution of Whitehead's problem is not so satisfying. The problem as stated was whether every Whitehead group is free, where G is a Whitehead group, when it is Abelian and if H is an Abelian group extending Z with the quotient being isomorphic to G then Z is a direct summand. The solution was that if $\mathbf{V} = \mathbf{L}$ then every Whitehead group is free (as an Abelian group) while if MA (the Martin axiom) is true and CH fails

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this is not the case. Now Nunke (1977) says that really there is a family of related properties (e.g., hereditary separable) and the real problem is to sort them out; and also to find whether "$\mathbb{Q} = \mathrm{Ext}(G, \mathbb{Z})$ for some G." Indeed this was done, see Eklof and Mekler (2002). If $\mathbf{V} = \mathbf{L}$ we get that the related classes are mostly equal etc., and under some forcing axiom the answer is that they are distinct. But we may wonder whether there is an axiom which gives a different coherent picture, antithetical to the one in **L**. This is the subject and aim of Shelah (undated d) in which we force a universe **V** in which GCH holds, but for every regular uncountable λ and for every suitable uniformization property, it holds for some suitable stationary subset of λ, say with a pregiven cofinality. This answers many problems, for example in Abelian group theory, "nicely." Properties are not equivalent without good reasons. So we may look at this axiom as trying to do for this family of Abelian group problems (and hopefully more) what $\mathrm{AD}_{\mathrm{L}[\mathbb{R}]}$ does for descriptive set theory problems, though it is far from being as grand as it is.

In another direction, in some model theoretic work of the author, it appears that assuming instances of WGCH (the weak generalized continuum hypothesis, i.e., $2^\lambda < 2^{\lambda^+}$) is very helpful.[1] This motivated Baldwin (2009) to suggest that we should consider the axiom WGCH.

Concerning a large continuum, we may be even bolder.

Dream 2.1 Find a reasonable semi-axiom which not only implies that, say, many of the established cardinal invariants of the continuum are (pairwise) distinct but determines their order. For definiteness we may concentrate on Cichoń's diagram.

3 Large continuum

In Gödel's universe of constructible sets we have $2^{\aleph_0} = \aleph_1$. Remarkably, several different constructions in set theory seem to deliver $2^{\aleph_0} = \aleph_2$, most notably Martin's maximuum MM. Is this because we know how to iterate forcings only on this level, or is there some deeper reason? In Shelah (2003a), I formulated the following Dream (no. 3.4) on a large continuum:

> Develop a theory of iterated forcing for a large continuum as versatile as the one we have for $2^{\aleph_0} = \aleph_2$, and to a lesser extent, for $2^{\aleph_0} = \aleph_1$. (Shelah 2003a, Dream 3.4)

[1] See Baldwin (2009) and Shelah (2009a, 2009b).

Dream 4.5 says more or less the same thing from another perspective:

> (a) *Find a real significance for* $2^{\aleph_0} = \aleph_{753}$,
> (b) *or for* $2^{\aleph_0} = \aleph_{\omega^3} + \omega + 5$,
> (c) *show that all values of* $2^{\aleph_0}$ *which are* $> \aleph_2$ *are similar in some sense (or at least all values* $\aleph_n > \aleph_2$*, all regular* $\aleph_\alpha > \aleph_{\omega_1}$ *or whatever).*
> (Shelah 2003a, Dream 4.5)

Note that in clause (a) above we mean to ignore the too easy example (if $f\colon [\aleph_n]^{n+1} \to [\aleph_n]^{\aleph_0}$ then there is an n-free set with $n+2$ members).

Recently, Neeman has made a major advance using a new iteration. Clearly this gives the consistency of a generalization of the PFA (Proper Forcing Axiom) for large continuum (replacing $\aleph_1$ by a larger cardinal), but I am not familiar enough with it to say more about it.

Woodin, using his PMAX method, got the consistency of ZFC + maximal set of statements of the form "for every subset of $\aleph_1$ there is a subset of $\aleph_1$ such that … " It was natural to ask about replacing ZFC by ZFC + some axiom, see Shelah and Zapletal (1999). But for probably the most natural statement, CH, the answer has long eluded us.[2]

Recently this was solved. Aspero *et al.* (2013) proved that there exist sentences ψ_1 and ψ_2 which are Π_2 over the structure $(\mathscr{H}(\aleph_2), \in, \aleph_1)$ such that

1. ψ_2 can be forced by a proper forcing which does not add new reals;
2. if there exists a strongly inaccessible limit of measurable cardinals, then ψ_1 can be forced by a proper forcing which does not add new reals;
3. the conjunction of ψ_1 and ψ_2 implies that $2^{\aleph_0} = 2^{\aleph_1}$.

Even more recently Moore (2013) obtained remarkably more concrete results, for example, if CH then there is a tree with $\aleph_1$ nodes and $\aleph_1$ levels, which is a proper forcing adding no new reals, but forcing with it renders it not proper.[3] Weak progress was made in Shelah (2010, 2011a) on developing iterated forcing with large continuum to obtain results about cardinal invariants of the continuum, especially in the case that we want to force several invariants to have given values simultaneously. A different direction, so-called creature iteration, described by Kellner and Shelah (2009) is ψ. Somewhat better still in the direction of creature iteration, but still far from a real answer, is work in progress by Goldstern and Kellner on the consistency of distinguishing five cardinal invariants from each other in the

[2] Some relevant information can be found in Shelah (undated c).
[3] For complementary consistency see Shelah (2003b).

Cichoń diagram. Concerning this, more work using finite support iteration of c.c.c. forcing is being done by Brendle and coworkers.

4 Doing without the Axiom of Choice

As I said in Shelah (2003a), I am not fond of set theory with weak forms of choice and was not bothered by what have been considered unintuitive consequences of this, such as the so-called "paradoxical decomposition of the sphere" by Banach and Tarski (1924). Naturally, the Axiom of Choice, AC, is part of the standard axioms ZFC of set theory. Still, I proposed the following:

> *Develop combinatorial set theory for universes with limited amount of choice (see Shelah 2000d).* (Shelah 2003a, problem 6.16)

Clearly, AC is true. Set theory without AC has been considered as a prototypical example of a dead direction. Why should we consider this direction? There are several good reasons (in addition to a fascination with the resurrection of the dead). First, let us have intellectual honesty – failure to have a nice set theory contributed to the acceptance of AC, so if we also have a nice set theory without AC we should check this premise ("nice" means with interesting content). Second, a good motivation for proving results without AC is that it is quite reasonable to consider when we can have a definable solution to a problem, rather than a mere existence theorem. Third, and finally, I think ideology and/or good taste should not stop you from proving a good theorem.

It has long been known to me that there is no serious model theory without choice – none of the basic theorems (compactness, downward and upward Löwenheim–Skolem theorems) hold. Of course, using $\mathbf{L}[X]$ some restricted versions hold, for example, compactness for first-order theories T with a well-orderable vocabulary. This position seems consistent with considering theorems and their proofs to see how much choice was used in the "input," for example, we can ask whether the theory we study is well ordered, or how much choice was used in the proof.

Some years ago, having considered a weak version of the Axiom of Choice (Ax_4, see below), we tried to see what happens to Morley's theorem (if T is a countable theory which is categorical in one cardinal $\aleph_\alpha > \aleph_0$, then T is categorical in every $\aleph_\alpha > \aleph_0$). On checking carefully, Morley's proof gives "there is a set of reals of cardinality $\aleph_1$." If we work further and quote later results in stability, we can prove this theorem in ZF, see Shelah (2009c). Considering the place of Morley's theorem in model theory this seems quite conclusive evidence against the thesis above.

What about the Main Gap (Shelah 1990, Chapter. XII)? This seems much harder. Probably in "$\dot{I}(\lambda, T)=2^{\lambda}$" we should revise "$2^{\lambda}$".[4] What about powers which are not cardinals? When studying them, it is natural to consider *reasonable* powers: $|A|$ is called *reasonable* if A can be linearly ordered and $|A| \cdot |A| = |A|$. We can show that every countable first-order theory has a model in every reasonable power (and some first-order sentence has a model of power $|A|$ if and only if $|A|$ is a reasonable power) so we may consider categoricity in reasonable powers, but the work in Shelah (2009c) has not been continued so far. Mendick and Truss (2003) have worked on model theory with little choice but in a very different direction.

What about set theory? On the one hand some parts of set theory are not affected by the absence of choice; inner model theory is not harmed (and lower bounds on consistency strength are still interesting in this context); descriptive set theory (and $L[\mathbb{R}]$ under AD), and working out the relations between various weaker versions of choice are also unaffected. But combinatorial set theory is strongly affected: ZF alone and even ZFC + DC seems hopeless (recall that even Poincaré accepted DC).

Naturally, having pcf theory with weak choice seems interesting to me, and work seems to indicate that this direction is not empty (Shelah 1997b).

Considerably more has been done. Some researches have tried to investigate set theory under ZFC + DC + Ax_4, see below. Others have tried to assume ZFC + DC + AC_X where X is $\mathcal{P}(\kappa)$ or essentially $\mathcal{P}(\mathcal{P}(\kappa))$, and investigate ${}^{\kappa}\lambda$, and even consider RGCH (discussed below). Note that after Shelah (1994b), a presentation of pcf was included in Holz *et al.* (1999) and Abraham and Magidor (2010). There are also several application, see for example Rinot (2007). I have also continued this work (e.g., Shelah 1993a, 1996, 1997a, 2000a, 2000e, 2007, undated f).

Gitik has many additional works on forcing, in particular proving the consistency of the failure of $(\mathrm{WH})_2$, (proving that there may be $\aleph_1$ cardinals of cofinality $\aleph_0$ whose pp is above their supremum). But the parallel major result of Shelah (1997b) remains open. The question is: can there be a strong limit singular λ of cofinality κ and increasing continuous sequence $\langle \mu_i : i < k \rangle$ with limit λ and stationary subsets S_1, S_2 of κ such that $(\prod_i \in S_l \mu_i^{+l}, < J_k^{\mathrm{bd}})$ has true cofinality λ^{+l}?

Close to my heart is the RGCH, the revised generalized continuum hypothesis (see Shelah (2000b, 2002, 2006) which give more accessible proof). It asserts that we should refine the definition of power to $\lambda^{[\kappa]}$ such that it is not monotonic in κ (still $\lambda^{\kappa} = \lambda$ if and only if $(\forall \theta)\ \theta = \mathrm{cf}(\theta) \leq \kappa \rightarrow \lambda^{[\kappa]} = \lambda$).

[4] For still more on dichotomies related to "$\dot{I}(\aleph_{\alpha}, T) \geq |\alpha|$" see Shelah (1990).

Our reward is now that for every λ for most $\kappa \ll \lambda$ we have $\lambda^{[\kappa]} = \lambda$; recall that $\lambda^{[\kappa]} = \min\{|\mathcal{P}|$: every subset of λ of cardinality κ is the union of $<\kappa$ members of $\mathcal{P}\}$. More specifically, the theorem says that for every $\lambda \geq \beth_\omega$ for every large enough $\kappa < \beth_\omega$ we have $\lambda^{[\kappa]} = \lambda$. Kojman and Soukup have used this result, and Kojman has written on the history of singular cardinals.

Now in Shelah (2011b, undated j), an attempt is made to find parallels to pcf theorems under weak choice. The pcf theorem is generalized by replacing "cofinal sequence of members of $\Pi\mathfrak{a}$ modulo a filter" by "a sequence of subsets of the product which is increasing and cofinal." In Shelah (undated i) it is shown that ${}^\kappa\lambda$ can be decomposed to a few well-ordered sets, where few means not depending on λ, only on κ. Larson and Shelah (2009) showed that a theorem on partition of a stationary set into many stationary subsets can be generalized.

Let us return to Ax_4. Shelah (undated i) suggests a direction orthogonal to $\mathbf{L}[\mathbb{R}]$, recalling that in $\mathbf{L}[\mathbb{R}]$ the only reason for non-well-orderability is $\mathbb{R}$. The condition suggested is Ax_4: "$[\lambda]^{\aleph_0}$ is well orderable for every cardinal λ," and also weaker relatives. The thesis is that in this context "set theory is not so far from normal," the only reasons for non-well-orderability are the "$\mathcal{P}(\kappa)$ not well ordered" for regular κ or sometimes $\mathcal{P}(\mathcal{P}(\kappa))$. In what way does Shelah (undated i) show that ZFC + DC + Ax_4 is "nice"? For example, (provably in it) there is a proper class of regular successor cardinals, and $[\lambda]^\kappa$ can be almost well ordered, modulo $\mathcal{P}(\mathcal{P}(\kappa))$, see above. So still we can have successor cardinals which are singular, but not too many. In fact, it suggests the following.

Thesis ZF + DC + Ax_4 is an interesting set theory to investigate.

A relevant universe is the following variant of Easton's model: for regular $\kappa > \aleph_0$ we add many subsets but not a well ordering. We may try to investigate ZF + DC + Ax_4 more systematically. This is the aim of the forthcoming Shelah (undated h), which in particular gives an effective version of the pcf theorem for any set a of regular cardinals that are large enough ($\geq \theta$ which is about the size of hrtg$(\mathcal{P}(\mathcal{P}(\mathfrak{a})))$, the Hartog number of $\mathcal{P}(\mathcal{P}(\mathfrak{a})))$. This restriction is a serious loss. But the assumption "$\mathfrak{a}$ is a set of *regular* cardinals" is not so natural in the present context. However, we can correct this: $\mathfrak{a}$ can be a set of just limit ordinals, possibly each of cofinality $\aleph_0$, but all is done modulo the ideal:

$$\text{cf}-\text{id}_\theta(\mathfrak{a}) = \{\mathfrak{b} \subseteq \mathfrak{a} : \text{there is a set } u \subseteq \sup(\mathfrak{a}) \text{ of cardinality } < \theta \text{ such that } \forall \delta \in \mathfrak{b}(\delta = \sup(u \cap .\delta))\}.$$

5 Model theory

The so-called Main Gap theorem is for countable first-order theories. In Shelah (2003a) the following proposal is made:

> Prove a form of the main gap for $\psi \in \mathbb{L}_{\lambda+,\aleph_0}$ (or just $\mathbb{L}_{\aleph_1,\aleph_0}$); i.e., for every such ψ either $\dot{I}(\lambda, \psi) > \lambda$ (where $\dot{I}(\lambda, \psi) = |\{M/\cong : M \models \psi, |M| = \lambda\}|$) for every λ large enough or there is an ordinal γ such that for every ordinal α, $\dot{I}(\aleph_\alpha, \psi) \leq \beth_\gamma(|\alpha|)$. (Shelah 2003a, Question 6.9)

Before trying to solve the Main Gap question it is natural to try to understand categoricity. The two (quite long) volumes, Shelah (2009a, 2009b), are a step in this direction. See in particular the introduction of Shelah (2009a); the works are in the context of AEC (abstract elementary classes).

While it is natural to repeat history and concentrate on the case of categoricity, can we say something *without* categoricity? In Shelah (undated a) it is suggested that we concentrate on cardinals $\lambda = \beth_\lambda$, i.e., fix points of the Beth sequence, which are of cofinality $\aleph_0$. The results are of the form: few models up to isomorphism implies every model in many such cardinals has arbitrarily large extensions.

Another direction is classifying theories T by the existence of non-isomorphic models of T of cardinality λ which are very equivalent by suitable long EF games (see Hyttinen and Tuuri 1991; Hyttinen *et al.* 1993; Hyttinen and Shelah 1994; Shelah 2008; and works in preparation). Hyttinen *et al.* (2013) have gone in a still different direction.

6 New logics

The model theory of first-order logic is relatively well understood, and inroads have been made into the model theory of various generalized quantifiers and infinitary logics. Still the following proposal seems timely:

> Find a new logic with good model theory (like compactness, completeness theorem, interpolation and those from 6.12) and strong expressive power preferably concerning other parts of mathematics (see Shelah (undated c), possibly specifically derive for them). (Shelah 2003a, Question 6.13)

On new model theoretically interesting logics, a step ahead seems the infinitary logic of Shelah (2012), which is between $\mathbb{L}_{<\kappa,\aleph_0} (= \cup_{\lambda<\kappa}\mathbb{L}_{\lambda,\aleph_0})$ and $\mathbb{L}_{<\kappa,<\kappa}$ $(=\cup_{\lambda<\kappa} \mathbb{L}_{\lambda,\lambda})$ for $\kappa = \beth_\kappa$, and which seems susceptible to

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model theoretic treatment. For example, in these logics we cannot define well ordering (in a strong sense), we have addition of theories and product of two (all this like $\mathbb{L}_{<\kappa,\aleph_0}$ when $\kappa = \beth_\kappa$) but we also have the (long sought for) interpolation and, last but not least, a characterization, as Lindström had the celebrated characterization of first-order logic.

In other contexts we have explained that while advanced model theory, say classification theory, can be pursued for $\mathbb{L}_{\lambda,\aleph_0}$, this is not the case for $\mathbb{L}_{\aleph_1,\aleph_1}$ and stronger logics, because properties like categoricity are very sensitive to set theory. Note however that for example, monadic second-order logic on linear orders is not so affected. Still, it seemed to me that the arguments were quite conclusive. Having explained it (categoricity is sensitive to set theory) in a class, second thoughts have arisen. We have in mind the logic $\mathbb{L}_{\theta,\theta}$, with θ strongly compact. In Shelah (undated e) we have generalized the characterization of elementary equivalence in terms of isomorphic ultra-limits for the logic introduced in Shelah (2012), but the main point is generalizing the characterization of theories T such that for some theory T_1 in the logic $\mathbb{L}_{\theta,\theta}$ extending T the following holds: if M_1 is a model of T_1 then the τ_T -reduct of M_1 is saturated (reasonably defined). This seems to show that advanced model theory (e.g., classification theory) for the logic $\mathbb{L}_{\theta,\theta}$, θ a compact cardinal, is not empty. However, for categoricity the picture is opaque and quite different than the first-order case, for example, some categorical theories have models with long orders.

In another direction it seems interesting to find similar logics which would still have the upward Löwenheim–Skolem theorem. It seems best to find a maximal logic $\subseteq \mathbb{L}_{<\kappa,<\kappa}$ which is "nice" and satisfies the upward Löwenheim–Skolem theorem. It seems reasonable to base such a logic on Ehrenfeucht–Mostowski models. The proof in Shelah (undated a) seems relevant.

Also the question of the existence of a compact proper extension of first-order logic with interpolation is still unsolved.

As for compact logics related to algebra, the old papers Shelah (1994a, undated b) and the history given there, offer avenues forward but this has not materialized yet. A later paper with similar content is the unfinished Shelah (undated c), dealing with generalized quantifiers Q for which $\mathbb{L}(Q)$ is compact, expressing interesting properties. These works generally deal with (quantifying on automorphisms of) Boolean algebras, ordered fields and some triangle-free graphs. There is an undercurrent of connection to instability theory but no general theory has emerged so far. Also it is reasonable to assume that on finding such

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logics with quantifiers related to some specific mathematical field, proving compactness will have applications to such a field.

7 Abstract elementary classes

Is there any such thing as a pure semantic proof in model theory? Baldwin (2013) says about the author's Presentation Theorem that, passing through the syntax, we obtain a purely semantic theorem in the sense that we are able to deduce what he calls "purely semantical conclusions." We prefer to phrase it by saying that we can do model theory, i.e., investigate classes of models, without going through logic. We suppose this is the same (this means looking at AEC, see below).

The original aim in the investigation of abstract elementary classes (AEC), in Shelah (1975), was that we could strengthen the logic, but it was not clear how much. Originally it was $\mathbb{L}(Q)$, and then the infinitary version $\mathbb{L}_{\aleph_1,\aleph_0}(Q)$. But there was no maximal logic. So rather than trying to find the maximal logic, which seemed difficult, we wrote down the semantic assumptions that we needed. This became what is known as the study of AECs.

Type, in particular complete type over a model, is a central notion. Traditionally a type is defined as a set of formulas. However, passing from the syntactical approach to the semantical one in Shelah (1987b) (in 1986) the orbital type was introduced (Grossberg and Baldwin prefer to call this the Galois type).

Formulas and types are different notions; in AEC (orbital) types remain central but the parallel to formulas may, but so far has not, play a significant role.

In the 1960s it become clear that saturation is better than universal homogeneity because we can deal with one element at a time. Hence model homogeneity should be replaced by sequence homogeneity; this is still true for elementary classes. In Shelah (1987a) this problem was circumvented using enough stability (that is, definability of types), and so-called materializing a type (rather than realizing it). In Shelah (1987b) we succeeded in getting the best of the two worlds; well, when we have amalgamation, otherwise model homogeneous makes no sense. That is, we showed that model homogeneity is equivalent to saturation. Saturation means that we have to realize types of single elements, as in saturation, recalling that the type of an α-tuple is not determined by its restriction to finite sub-tuples. Moreover, these are not arbitrary types, just (complete) types over sub-models, in the right abstract sense.

8 Conclusion

One cannot emphasize enough the interest in trying to find a forcing-like method for Peano arithmetic with the potential of leading to proofs of independence from PA of even very "reasonable" arithmetical statements.

For set theory we have forcing, but it has led to the question how to decide what kind of new axioms we should add to ZFC in order to decide some or most of the statements left undecided by ZFC. In the face of the difficulty of this question, what we call "semi-axioms" suggest themselves. These are new axioms which have many consequences and lead to a deep and beautiful theory, but at the same time are reasonable and have "positive measure." The axiom $\mathbf{V} = \mathbf{L}$ would fulfil the first criterion but fails miserably on the second. On the other hand, the axiom "the continuum is weakly inaccessible" would be reasonable but has only few consequences.

The "California school," based on determinacy axioms, is to me unsatisfactory for several reasons that we have discussed above. The main point is that the fact that a statement, such as a determinacy axiom, solves everything really nicely is not a sufficient reason to call the statement a "true axiom." I also consider that the California school gives too much weight to success in descriptive set theory. While descriptive set theory is surely important, for me it is not "the center."

Other problems, arising perhaps from model theory or algebra, suggest different semi-axioms and universes that may be the nicest from the corresponding perspective. One such is WGCH (the weak generalized continuum hypothesis, i.e., $2^\lambda < 2^{\lambda^+}$).

Another potentially interesting direction is large continuum. I dream about a reasonable semi-axiom which implies that many of the established cardinal invariants of the continuum are (pairwise) distinct, and also determines their order. Understanding large continuum seems to require a theory of iterated forcing for a large continuum as versatile as the one we have for $2^{\aleph_0} = \aleph_2$ (or for $2^{\aleph_0} = \aleph_1$). In this connection I ask what would be the real significance of $2^{\aleph_0} = \aleph_{753}$, or of $2^{\aleph_0} = \aleph_{\omega^3+\omega+5}$, or is it so that all values of $2^{\aleph_0}$ which are $> \aleph_2$ are similar in some sense.

Despite some people's misgivings about the so-called unintuitive consequences, the Axiom of Choice, AC, is part of the standard axioms ZFC of set theory. We have proposed developing combinatorial set theory for universes with limited amount of choice, and here we comment more, particularly on further developments concerning this proposal. I have previously maintained that there is no serious model theory without

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choice, but my mind has changed. In particular we now have a proof, discussed above, of Morley's theorem in ZF. The Main Gap theorem seems much harder. In this direction we suggest studying "reasonable powers" as analogs of cardinals, and considering categoricity in reasonable powers.

In combinatorial set theory, considering pcf theory is interesting to me. In this respect we suggest focusing on the RGCH, the revised generalized continuum hypothesis which gives, for example, that for every $\lambda \geq \beth_\omega$ for every large enough $\kappa < \beth_\omega$ we have $\lambda^{[\kappa]} = \lambda$. This also applies to set theory with weak choice, and there have been some advances. Some generalize the pcf theorem by replacing "cofinal sequence of members of $\prod \mathfrak{a}$ modulo a filter" by "a sequence of subsets of the product which is increasing and cofinal."

Continuing on the topic of set theory without full choice, we suggest Ax_4, which says that "$[\lambda]^{\aleph_0}$ is well orderable for every cardinal λ." The thesis is that in this context "set theory is not so far from normal," the only reasons for non-well-orderability are the "$\mathcal{P}(\kappa)$ not well ordered" for regular κ or sometimes $\mathcal{P}(\mathcal{P}\kappa)$). A relevant universe is the following variant of Easton's model: for regular $\kappa > \aleph_0$ we add many subsets but not a well ordering. We suggest investigating ZF + DC + Ax_4 more systematically.

The so-called Main Gap theorem was for first-order theories, but we propose to prove a form of it for $\psi \in \mathbb{L}_{\lambda+,\omega}$. The abstract elementary classes (AEC) provide a natural framework for the study of such sentences. The two volumes, Shelah (2009a, 2009b), make advances on categoricity in abstract elementary classes with the long term aim of proving a Main Gap theorem in this context.

Another approach to classifying theories, where recent progress has taken place, is by means of existence of winning strategies in transfinite EF games between large models of the theory.

It was proposed in Shelah (2003a) to find a new logic with good model theory (like compactness, completeness theorem, interpolation and those from 6.12) and strong expressive power preferably concerning other areas of mathematics. One interesting candidate has emerged recently, namely a logic between $\mathbb{L}_{<\kappa,\aleph_0} (= \cup_{\lambda<\kappa} \mathbb{L}_{\lambda,\aleph_0})$ and $\mathbb{L}_{<\kappa,<\kappa}$ $(=\cup_{\lambda<\kappa} \mathbb{L}_{\lambda,\lambda})$ for $\kappa = \beth_\kappa$, with interpolation and a kind of Lindström characterization. This logic is under further investigation.

Despite earlier reservations on having any model theory for $\mathbb{L}_{\aleph_1,\aleph_1}$ and stronger logics, we have started to study $\mathbb{L}_{\theta,\theta}$, with θ strongly compact with some reasonable initial results. For example, we generalize the characterization of elementary classes such that for some elementary class in a larger vocabulary all the reducts to the original vocabulary are in the original class and are saturated. Also, for the new logic mentioned above, we get a characterization of the elementary equivalence in terms of isomorphic ultralimits. This seems to show that advanced model theory (e.g., classification theory) for the logic $\mathbb{L}_{\theta,\theta}$, θ a compact cardinal, is possible.

The question of the existence of a compact proper extension of first-order logic with interpolation is still unsolved.

This chapter ends with a discussion on the question to what extent abstract elementary classes lead to purely semantic proofs in model theory. We claim that we can do model theory, i.e., investigate model classes, without going through logic.

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