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Remarks on generalized ultrafilter, dominating and reaping numbers

by

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Abstract. The following statements are the main results of the paper:

(a) $\mathrm{cf}(\mathfrak{u}) > \omega$ and $\mathrm{cf}(\mathfrak{u}_\kappa) > \omega$ for every uncountable cardinal κ where $\mathfrak{u}_\kappa$ is the generalized ultrafilter number.
(b) If $\kappa > \aleph_0$ is regular and $\mathfrak{r}_\kappa < \mathfrak{d}_\kappa$ then $\mathfrak{r}_\kappa = \mathfrak{u}_\kappa$, where $\mathfrak{r}_\kappa$ is the generalized reaping number and $\mathfrak{d}_\kappa$ is the generalized dominating number.
(c) The relations $\mathfrak{r}_\lambda < \mathfrak{d}_\lambda$ and $\mathfrak{u}_\lambda < \mathfrak{d}_\lambda$ are consistent for a strong limit singular cardinal λ.

1. Introduction. This paper deals with several cardinal characteristics of the continuum, including the reaping number and the ultrafilter number. We define them in the generalized form of $\mathfrak{r}_\kappa$ and $\mathfrak{u}_\kappa$ where κ is any infinite cardinal.

DEFINITION 1.1 (The reaping number). Let κ be an infinite cardinal and let $B \in [\kappa]^\kappa$.

($\aleph$) A set $S \in [\kappa]^\kappa$ *splits* B iff $|S \cap B| = |(\kappa - S) \cap B| = \kappa$.
($\beth$) A family $\mathcal{A} \subseteq [\kappa]^\kappa$ of sets is called an *unreaped family* iff there is no single $S \in [\kappa]^\kappa$ which splits each element of $\mathcal{A}$.
($\gimel$) The *reaping number* $\mathfrak{r}_\kappa$ is the minimal cardinality of an unreaped family in $[\kappa]^\kappa$.

An unreaped family will also be called an *unsplittable* or an $\mathfrak{r}_\kappa$-*family.* A close friend of the reaping number is the ultrafilter number. Recall that an ultrafilter $\mathscr{U}$ over κ is *uniform* iff all of its elements are of size κ, and all the ultrafilters in this paper are uniform. In particular, the definition of

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the ultrafilter number applies to uniform ultrafilters. Again, we phrase the definition in the general context.

Definition 1.2 (The ultrafilter number). Let κ be an infinite cardinal and let $\mathscr{U}$ be an ultrafilter over κ.

($\aleph$) A *base* for $\mathscr{U}$ is a collection $\mathcal{B} \subseteq \mathscr{U}$ such that for any $A \in \mathscr{U}$ there is some $B \in \mathcal{B}$ such that $B \subseteq A$.
($\beth$) The *character* of a uniform ultrafilter $\mathscr{U}$ over κ is the minimal size of a base for $\mathscr{U}$, denoted by $\mathrm{Ch}(\mathscr{U})$.
($\gimel$) The *ultrafilter number* $\mathfrak{u}_\kappa$ is the minimal size of a base for a uniform ultrafilter $\mathscr{U}$ over κ.

Lest $\kappa = \aleph_0$ we denote $\mathfrak{r}_\kappa$ by $\mathfrak{r}$ and $\mathfrak{u}_\kappa$ by $\mathfrak{u}$. Any base of an ultrafilter is unsplittable, hence $\mathfrak{r}_\kappa \leq \mathfrak{u}_\kappa$. It is easy to see that both $\mathfrak{r}_\kappa > \kappa$ and $\mathfrak{u}_\kappa > \kappa$. Our purpose in the first section is to analyze the cofinality of these characteristics.

Cardinal characteristics which may assume countable cofinality are rare, and to prove this property one needs complicated arguments. The almost disjointness number $\mathfrak{a}$ is an example, as proved in [Br03]. It is unknown whether the cofinality of $\mathfrak{r}_\kappa$ is always greater than κ, and in particular whether $\mathrm{cf}(\mathfrak{r})$ is uncountable. Additional examples are $\mathfrak{i}$ (see [Br03]) and $\mathfrak{gp}$ (see [Ga18], the possibility of countable cofinality in this case requires instances of Chang's conjecture, hence depends on the existence of large cardinals).

The question whether $\mathrm{cf}(\mathfrak{r}) = \omega$ is consistent appeared in [Mi93, Problem 3.4], though the author uses a different terminology. By adding λ many Cohen reals to a model of GCH we obtain $\mathfrak{r} = \mathfrak{u} = \mathfrak{c} = \lambda$ and hence these characteristics may be singular. We shall prove that the cofinality of $\mathfrak{u}_\kappa$ is above ω, and in some cases the same holds for $\mathfrak{r}_\kappa$: see Theorems 2.2 and 2.11. Namely, if $\kappa = \mathrm{cf}(\kappa)$ and $\mathfrak{r}_\kappa < \mathfrak{d}_\kappa$ then $\mathfrak{r}_\kappa = \mathfrak{u}_\kappa$ and hence $\mathrm{cf}(\mathfrak{r}_\kappa) > \omega$.

This brings us to the question whether $\mathfrak{r}_\kappa < \mathfrak{d}_\kappa$ is possible. If $\kappa = \aleph_0$ then the answer is yes. By adding ω_2 Miller reals to a model of the continuum hypothesis one obtains $\mathfrak{r} = \omega_1 < \omega_2 = \mathfrak{d}$. This result is due to Miller [Mi84], and a modern exposition can be found in [Ha12, p. 413]. Raghavan and Shelah [RS19] proved that if $\kappa = \mathrm{cf}(\kappa) > \beth_\omega$ then $\mathfrak{d}_\kappa \leq \mathfrak{r}_\kappa$. We shall prove in the second section that $\mathfrak{r}_\lambda < \mathfrak{d}_\lambda$ is consistent for a strong limit singular cardinal λ. This can be done with any cofinality of λ, and it holds above $\beth_\omega$ as λ is strong limit.

Cardinal characteristics at uncountable cardinals appeared in the mathematical literature a few decades ago. It seems that the first results were proved for the bounding and dominating numbers (e.g. [CS95]) and the splitting number studied by the Japanese school (e.g. [Su98]). This area became quite active in recent years. We mention here [RS17] and [BF17], among many other papers. A comprehensive background is given in [KLLS16].

Our notation is standard. We mention here the concept of the strong finite intersection property. Let κ be an infinite cardinal and $\mathscr{F} \subseteq [\kappa]^\kappa$. We say that $\mathscr{F}$ has the *strong finite intersection property* iff $|\bigcap u| = \kappa$ whenever $u \in [\mathscr{F}]^{<\omega}$. We shall use the quantifier *almost every* to mean that the set of exceptions has cardinality less than κ.

We suggest [Bl10] as an excellent background regarding cardinal characteristics. For a good background in pcf theory we suggest [AM10].

2. Cofinality. We open this section with a theorem about the cofinality of the ultrafilter number. The statement and the proof are phrased in the case of $\kappa = \aleph_0$, and possible generalizations to higher cardinals are discussed after the proof. Note that the case of $\kappa = \aleph_0$ follows from [BS99, Proposition 1.4]; we give an independent proof.

THEOREM 2.1 (The cofinality of a base). *Let $\mathscr{U}$ be a non-principal ultrafilter over ω. If* $\mathrm{Ch}(\mathscr{U}) = \mu$ *then* $\mathrm{cf}(\mu) > \omega$.

Proof. Fix an ultrafilter $\mathscr{U}$ over ω such that $\mathrm{Ch}(\mathscr{U}) = \mu$. Assume by way of contradiction that $\mathrm{cf}(\mu) = \omega$. Choose an increasing sequence $(\mu_n : n \in \omega)$ of uncountable ordinals such that $\mu = \bigcup_{n\in\omega} \mu_n$. Fix a base $\mathcal{B} = \{B_\beta : \beta \in \mu\}$ for the ultrafilter $\mathscr{U}$.

For every $\alpha \in \mu$ let $\mathcal{B}_\alpha = \{B_\beta : \beta < \alpha\}$. By our assumption toward contradiction we see that each $\mathcal{B}_\alpha$ fails to be a base for $\mathscr{U}$. Consequently, for every $\alpha \in \mu$ one can choose $y_\alpha \in [\omega]^\omega - \mathscr{U}$ such that:

(α) $u \in [\alpha]^{<\omega} \Rightarrow |\bigcap_{\beta\in u} B_\beta \cap y_\alpha| = \aleph_0$.
(β) $u \in [\alpha]^{<\omega} \Rightarrow |y_\alpha - \bigcup_{\beta\in u} y_\beta| = \aleph_0$.

Indeed, for every $\alpha \in \mu$, since $\mathcal{B}_\alpha$ does not generate $\mathscr{U}$ there is some $x_\alpha \in \mathscr{U}$ such that for every $u \in [\alpha]^{<\omega}$ it is true that $\neg(\bigcap_{\beta\in u} B_\beta \subseteq^* x_\alpha)$. Let $y_\alpha = \omega - x_\alpha$ and conclude that (α) holds. Item (β) is basically an equivalent formulation of the same statement. More precisely, if we choose an appropriate enumeration of the sets in our base then (β) follows. We shall use the formulation of (β) only once, and it will be clear that in that case it is equivalent to (α).

Using the above property we define, by induction on $n \in \omega$, a set s_n for which the following requirements are met:

(a) s_n is an infinite subset of ω.
(b) $s_n \notin \mathscr{U}$.
(c) s_n is disjoint from $\bigcup_{m<n} s_m$.
(d) $n \in \bigcup_{m\leq n} s_m$.
(e) $u \in [\mu_n]^{<\omega} \Rightarrow |\bigcap_{\beta\in u} B_\beta \cap s_n| = \aleph_0$.

The choice can be done, basically, by (α) (or (β)) above. If $n = 0$ then let $\alpha = \mu_0$ and choose y_α as guaranteed in (α) and (β). Set $s_0 = y_\alpha \cup \{0\}$.

Observe that (a), (b) are satisfied and (c) is vacuous in this case. We added zero to y_α in order to satisfy (d), and (e) is exactly (α) with respect to y_α hence also to s_0.

In the $(n+1)$st stage we choose a sufficiently large $\ell > n$ and let $\alpha = \mu_\ell$. Again, let y_α be as guaranteed in (α) and (β). Define $t_\alpha = y_\alpha - \bigcup_{m\leq n} s_m$. From (β) we infer that t_α is infinite. Now if $n+1 \in \bigcup_{m\leq n} s_m$ then let $s_{n+1} = t_\alpha$ and if $n+1 \notin \bigcup_{m\leq n} s_m$ then let $s_{n+1} = t_\alpha \cup \{n+1\}$. One can verify that all the requirements are satisfied.

For every $i \in \{0,1\}$ define $E_i = \bigcup\{s_{2n+i} : n \in \omega\}$. From (d) one can see that $E_0 \cup E_1 = \omega$, and from (c) it follows that $E_0 \cap E_1 = \emptyset$, so $\{E_0, E_1\}$ is a partition of ω. Consequently, there must be some $i \in \{0,1\}$ for which $E_i = \emptyset \bmod \mathscr{U}$.

Let $u \subseteq \mu$ be any finite set of ordinals. Pick a sufficiently large n such that $u \subseteq \mu_n < \mu_{2n}$. Apply (e) and conclude that

$$\Big|\bigcap_{\beta\in u} B_\beta \cap s_{2n}\Big| = \Big|\bigcap_{\beta\in u} B_\beta \cap s_{2n+1}\Big| = \aleph_0.$$

By the definition of E_0 and E_1 we infer that $|\bigcap_{\beta\in u} B_\beta \cap E_0| = |\bigcap_{\beta\in u} B_\beta \cap E_1| = \aleph_0$ for every $u \in [\mu]^{<\omega}$. But this means that $E_0 \neq \emptyset \bmod \mathscr{U} \wedge E_1 \neq \emptyset \pmod{\mathscr{U}}$, a contradiction. ■

The above theorem can be generalized to higher cardinals in the following manner. Assume that κ is an infinite cardinal and $\mathscr{U}$ is a uniform ultrafilter over κ. If $\mathrm{Ch}(\mathscr{U}) = \mu$ then $\mathrm{cf}(\mu) > \omega$, by the same proof. Let us phrase this observation as a proposition:

THEOREM 2.2. *If $\mathscr{U}$ is a uniform ultrafilter over κ then $\mathrm{cf}(\mathrm{Ch}(\mathscr{U})) > \omega$ and hence $\mathrm{cf}(\mathfrak{u}_\kappa) > \omega$ for every infinite cardinal κ.*

A stronger generalization seems to require some degree of completeness. For example, by assuming that κ is measurable and $\mathscr{U}$ is κ-complete, the above arguments show that if $\mathrm{Ch}(\mathscr{U}) = \mu$ then $\mathrm{cf}(\mu) > \kappa$. We believe that one can strengthen the statement about the cofinality of $\mathfrak{u}_\kappa$ even without appealing to normal ultrafilters.

QUESTION 2.3. *Is it consistent that $\mathrm{cf}(\mathfrak{u}_\kappa) \leq \kappa$ for some infinite cardinal κ?*

Back to Theorem 2.1, we know that the ultrafilter number (at any cardinal) has uncountable cofinality. Can we prove a similar theorem about $\mathfrak{r}$? It has been shown by Aubrey [Au04] that if $\mathfrak{r} < \mathfrak{d}$ then $\mathfrak{r} = \mathfrak{u}$. Hence in this case, the cofinality of the reaping number will be uncountable. Our next goal is to generalize this result to regular uncountable cardinals. We follow in the footsteps of Aubrey, with the required adaptations to the general case. Let us begin with another cardinal characteristic:

Definition 2.4 (The dominating number). Let κ be an infinite cardinal.

(ℵ) For $f, g \in {}^\kappa\kappa$ we shall say that g *dominates* f iff $\{\beta \in \kappa : f(\beta) > g(\beta)\}$ is of size less than κ. This relation will be denoted by $f \leq^* g$.
(ℶ) A family of functions $\mathcal{D} \subseteq {}^\kappa\kappa$ is called a *dominating family* iff for every $f \in {}^\kappa\kappa$ there exists $g \in \mathcal{D}$ such that $f \leq^* g$.
(ℷ) The *dominating number* $\mathfrak{d}_\kappa$ is the minimal size of a dominating family at ${}^\kappa\kappa$.

Let κ be a regular cardinal. We shall say that $\Pi = \{I^\Pi_\alpha : \alpha < \kappa\}$ is an *interval partition* of κ when each I^Π_α is a non-empty interval of the form $[\gamma_\alpha, \gamma_{\alpha+1})$, if $\alpha < \beta < \kappa$ then $\gamma_\alpha < \gamma_{\alpha+1} \leq \gamma_\beta$ so $I^\Pi_\alpha \cap I^\Pi_\beta = \emptyset$, and every ordinal of κ belongs to some I^Π_α. If Π is clear from the context then we may write I_α instead of I^Π_α.

Definition 2.5. Let $\Pi = \{I_\alpha : \alpha < \kappa\}$ be an inverval partition of κ. Assume that $\mathscr{F} \subseteq [\kappa]^\kappa$.

(ℵ) A pair (D, E) is a *nice Π-orbit* iff both D and E are unions of κ-many intervals from Π, the intervals of D are disjoint from the intervals of E, and moreover there is no interval of D which has an adjacent interval of Π in E.
(ℶ) We say that $\mathscr{F}$ is *Π-scattered* iff for every nice Π-orbit (D, E) and every $y \in \mathscr{F}$, both $y \cap D$ and $y \cap E$ are of size κ.
(ℷ) We say that Π is *$\mathscr{F}$-scattered* iff one can find a nice Π-orbit (D, E) such that $y \cap D$ and $y \cap E$ are of size κ for every $y \in \mathscr{F}$.

The following gives a simple example.

Lemma 2.6. *Let $\Pi = \{I_\alpha : \alpha < \kappa\}$ be an inverval partition of κ, and assume that $\mathscr{F} \subseteq [\kappa]^\kappa$. If every $y \in \mathscr{F}$ meets almost every interval of Π then $\mathscr{F}$ is Π-scattered.*

Suppose that Π is an interval partition and $\mathscr{F}, \mathscr{G} \subseteq [\kappa]^\kappa$. In the theorem below it is shown that if $\mathscr{F}$ is not Π-scattered and Π is not $\mathscr{G}$-scattered, then one can define an interesting $\mathfrak{r}_\kappa$-family out of $\mathscr{F}$ and $\mathscr{G}$.

Proposition 2.7. *Assume that:*

(ℵ) $\mathscr{F}, \mathscr{G} \subseteq [\kappa]^\kappa$.
(ℶ) $\Pi = \{I_\alpha : \alpha < \kappa\}$ *is an interval partition of* κ.

Then at least one of the following holds:

(a) $\mathscr{F}$ *is Π-scattered.*
(b) Π *is $\mathscr{G}$-scattered.*
(c) *There exist $y \in \mathscr{F}$ and $h \in {}^\kappa\kappa$ which is increasing and $(<\kappa)$-to-one such that $\{h[y \cap z] : z \in \mathscr{G}\}$ is an $\mathfrak{r}_\kappa$-family.*

Proof. If every $y \in \mathscr{F}$ meets almost every interval of Π then $\mathscr{F}$ is Π-scattered by Lemma 2.6, so we may assume that this is not the case and fix some $y \in \mathscr{F}$ which evades κ-many intervals of Π. We create a new interval partition $\Phi = \Phi(\Pi)$ by defining the intervals I^{Φ}_{α} using induction on $\alpha \in \kappa$. Every interval I^{Φ}_{α} will be a union of intervals from Π. This will be a union of consecutive intervals with a last element.

If $\alpha = 0$ then let $\gamma_0 \in \kappa$ be the first ordinal for which $y \cap I^{\Pi}_{\gamma_0} = \emptyset$ and let $I^{\Phi}_0 = \bigcup\{I^{\Pi}_{\beta} : \beta \leq \gamma_0\}$. At stage $\alpha + 1$ we assume that I^{Φ}_{α} is at hand and let $I^{\Pi}_{\gamma_\alpha}$ be the last interval from Π in I^{Φ}_{α}. Let $\gamma_{\alpha+1} \in \kappa$ be the first ordinal greater than γ_α such that $y \cap I^{\Pi}_{\gamma_{\alpha+1}} = \emptyset$, and define $I^{\Phi}_{\alpha+1} = \bigcup\{I^{\Pi}_{\beta} : \gamma_\alpha < \beta \leq \gamma_{\alpha+1}\}$. Notice that $I^{\Phi}_{\alpha+1}$ has a last interval from Π. Finally, assume that α is a limit ordinal and let $\gamma = \bigcup_{\beta<\alpha} \gamma_\beta < \kappa$. Let $\gamma_\alpha \in \kappa$ be the first ordinal larger than γ such that $y \cap I^{\Pi}_{\gamma_\alpha} = \emptyset$ and let $I^{\Phi}_{\alpha} = \bigcup\{I^{\Pi}_{\beta} : \gamma \leq \beta \leq \gamma_\alpha\}$.

We produce from the interval partition Φ a function $h : \kappa \to \kappa$ by letting h be constant over the intervals of Φ. Formally, $h(\delta) = \alpha$ iff $\delta \in I^{\Phi}_{\alpha}$ for every $\delta \in \kappa$. Observe that h is increasing and $({<}\kappa)$-to-one. Equipped with h, we split the rest of the proof into two cases.

Case 1: *There exists $Q \subseteq \kappa$ such that $|Q| = |\kappa - Q| = \kappa$, and for every $z \in \mathscr{G}$ it is true that $|y \cap z \cap h^{-1}[Q]| = |y \cap z \cap h^{-1}[\kappa - Q]| = \kappa$.*

In this case, Π is $\mathscr{G}$-scattered so (b) holds. To prove this, let

$$D = h^{-1}[Q] - \bigcup_{\alpha<\kappa} I^{\Pi}_{\gamma_\alpha} \quad \text{and} \quad E = h^{-1}[\kappa - Q] - \bigcup_{\alpha<\kappa} I^{\Pi}_{\gamma_\alpha}.$$

Notice that $D \cap E = \emptyset$ and there are no adjacent intervals in D, E (to reach this, we removed the intervals $I^{\Pi}_{\gamma_\alpha}$). Fix any $z \in \mathscr{G}$. One can see that

$$D \cap z = \Big(h^{-1}[Q] - \bigcup_{\alpha<\kappa} I^{\Pi}_{\gamma_\alpha}\Big) \cap z \supseteq \Big(h^{-1}[Q] - \bigcup_{\alpha<\kappa} I^{\Pi}_{\gamma_\alpha}\Big) \cap (z \cap y).$$

But $\bigcup_{\alpha<\kappa} I^{\Pi}_{\gamma_\alpha} \cap (z \cap y) = \emptyset$ by the choice of y, and hence $D \cap z = h^{-1}[Q] \cap z \cap y$. By the assumption of the present case, the size of $D \cap z$ is κ. An identical argument shows that $E \cap z$ is of size κ, upon replacing Q by $\kappa - Q$. This shows that Π is $\mathscr{G}$-scattered as claimed in (b).

Case 2: *For every $Q \subseteq \kappa$ such that $|Q| = |\kappa - Q| = \kappa$, there exists some $z \in \mathscr{G}$ such that either $|y \cap z \cap h^{-1}[Q]| < \kappa$ or $|y \cap z \cap h^{-1}[\kappa - Q]| < \kappa$.*

In this case we will try to create an unsplittable family out of $\mathscr{F}$ and $\mathscr{G}$, thus proving (c). Fix any $Q \subseteq \kappa$ and a set $z \in \mathscr{G}$ whose existence is guaranteed by the assumption of this case. If $|y \cap z \cap h^{-1}[Q]| < \kappa$ then $|Q \cap h[y \cap z]| < \kappa$ and if $|y \cap z \cap h^{-1}[\kappa - Q]| < \kappa$ then $|(\kappa - Q) \cap h[y \cap z]| < \kappa$. In any case, Q fails to split $h[y \cap z]$. Since Q was arbitrary it follows that $\{h[y \cap z] : z \in \mathscr{G}\}$ is unsplittable, so we are done. ■

Any $\mathfrak{r}_\kappa$-family of sets can be translated to a collection of functions in ${}^\kappa\kappa$ with a certain property related to splitting. This translation between sets and functions will be useful. We need the following definition:

Definition 2.8 (Big families of functions). Let κ be a regular cardinal and let $\mathcal{H} \subseteq {}^\kappa\kappa$. The family $\mathcal{H}$ will be called *big* iff for every $\mathscr{F} \subseteq [\kappa]^\kappa$ and every $g \in {}^\kappa\kappa$ such that $\mathscr{F}$ contains the set $\{\beta \in \kappa : f(\beta) \leq g(\beta)\}$ whenever $f \in \mathcal{H}$, one can find an increasing $({<}\kappa)$-to-one function $h \in {}^\kappa\kappa$ for which $\{h[y \cap z] : y, z \in \mathscr{F}\}$ is an $\mathfrak{r}_\kappa$-family.

To convert sets into functions we shall use a kind of projection. Suppose that $y \in [\kappa]^\kappa$. We define a function $p_y \in {}^\kappa\kappa$ by letting $p_y(\alpha) = \min(y \cap [\alpha, \kappa))$ for every $\alpha \in \kappa$.

Claim 2.9. *Let κ be a regular cardinal and let $\mathcal{R}$ be an $\mathfrak{r}_\kappa$-family. Then the collection $\mathcal{H} = \{p_y : y \in \mathcal{R}\}$ is big.*

Proof. Fix any $\mathfrak{r}_\kappa$-family $\mathcal{R}$. Assume toward contradiction that $\mathcal{H} = \mathcal{H}(\mathcal{R})$ is not big. By definition, there are $g \in {}^\kappa\kappa$ and $\mathscr{F} \subseteq [\kappa]^\kappa$ such that $\mathscr{F}$ is upward-closed, $\{\beta \in \kappa : p_y(\beta) \leq g(\beta)\} \in \mathscr{F}$ whenever $y \in \mathcal{R}$, and for every $({<}\kappa)$-to-one increasing $h \in {}^\kappa\kappa$, the family $\{h[y \cap z] : y, z \in \mathscr{F}\}$ is not an $\mathfrak{r}_\kappa$-family.

By induction on $\alpha \in \kappa$ we define an interval I_α as follows. For $\alpha = 0$ we simply take $I_0 = [0, 1)$. If $I_\alpha = [\gamma_\alpha, \gamma_{\alpha+1})$ has been defined then we choose $\gamma_{\alpha+2} \in \kappa$ such that $g(\delta) < \gamma_{\alpha+2}$ for all $\delta < \gamma_{\alpha+1}$, and we let $I_{\alpha+1} = [\gamma_{\alpha+1}, \gamma_{\alpha+2})$. Finally, if α is a limit ordinal and $I_\beta = [\gamma_\beta, \gamma_{\beta+1})$ has been defined for every $\beta < \alpha$ then we let $\gamma_\alpha = \bigcup_{\beta<\alpha} \gamma_{\beta+1}$. We choose $\gamma_{\alpha+1} \in \kappa$ such that $g(\delta) < \gamma_{\alpha+1}$ for all $\delta < \gamma_\alpha$, and we let $I_\alpha = [\gamma_\alpha, \gamma_{\alpha+1})$.

Let $\varPi = \{I_\alpha : \alpha \in \kappa\}$. Apply Theorem 2.7 to the triple $(\mathcal{R}, \mathscr{F}, \varPi)$ here standing for $(\mathscr{F}, \mathscr{G}, \varPi)$ there. Among the three options given in Theorem 2.7, (a) and (c) are excluded. Firstly we show that $\mathcal{R}$ cannot be $\varPi$-scattered. To this end, fix any nice $\varPi$-orbit (D, E) and let $S = D$. Notice that $E \subseteq \kappa - S$. If $\mathcal{R}$ is $\varPi$-scattered then for every $y \in \mathcal{R}$ we have $|S \cap y| = |(\kappa - S) \cap y| = \kappa$. This means that S splits $\mathcal{R}$, which is impossible since $\mathcal{R}$ is an $\mathfrak{r}_\kappa$-family. Secondly, our assumption toward contradiction (as specified in the first paragraph of the proof) says that (c) of Theorem 2.7 fails. We thus conclude that (b) of Theorem 2.7 holds, i.e. $\varPi$ is $\mathscr{F}$-scattered.

Let (D, E) be a nice $\varPi$-orbit which exemplifies this fact. Define $S = \bigcup\{I_\alpha \cup I_{\alpha+1} : I_\alpha \subseteq D\}$. Similarly, let $T = \bigcup\{I_\alpha \cup I_{\alpha+1} : I_\alpha \subseteq E\}$. Observe that $D \subseteq S$ and $E \subseteq T$ but still $S \cap T = \emptyset$ since there are no adjacent intervals in D, E. Hence $T \subseteq \kappa - S$.

Fix any $y \in \mathcal{R}$ and let $A_y = \{\beta \in \kappa : p_y(\beta) \leq g(\beta)\}$, so $A_y \in \mathscr{F}$. Since $\varPi$ is $\mathscr{F}$-scattered, $|A_y \cap D| = \kappa$. But if $\beta \in A_y \cap D$ then $p_y(\beta) \in y \cap S$ since $I_{\alpha+1} \subseteq S$ whenever $I_\alpha \subseteq D$. We conclude that $|y \cap S| = \kappa$. Similarly,

$|A_y \cap E| = \kappa$ and hence $|y \cap T| = \kappa$, which implies $|y \cap (\kappa - S)| = \kappa$. It follows that S splits all the elements of the $\mathfrak{r}_\kappa$-family $\mathcal{R}$, a contradiction. ■

We need one last concept before proving the theorem below.

Definition 2.10 (Finite domination). Let κ be a regular cardinal and $\mathcal{A} \subseteq [\kappa]^\kappa$. We say that $\mathcal{A}$ is *finitely dominating* iff for every $h \in {}^\kappa\kappa$ there is some finite collection $\{f_1, \ldots, f_n\} \subseteq \mathcal{A}$ such that $h \leq^* \max\{f_1, \ldots, f_n\}$.

If $\mathcal{A} = \{f_\alpha : \alpha < \lambda\} \subseteq [\kappa]^\kappa$ and $\lambda < \mathfrak{d}_\kappa$ then $\mathcal{A}$ is not a dominating family and moreover it is not finitely dominating since $[\lambda]^{<\omega} = \lambda$. We shall use this fact to prove the following theorem.

Theorem 2.11 (Aubrey [Au04] for $\kappa = \aleph_0$). *If κ is regular and $\mathfrak{r}_\kappa < \mathfrak{d}_\kappa$ then $\mathfrak{r}_\kappa = \mathfrak{u}_\kappa$. Consequently, $\mathfrak{r}_\kappa < \mathfrak{d}_\kappa$ implies $\mathrm{cf}(\mathfrak{r}_\kappa) > \omega$ whenever $\kappa = \mathrm{cf}(\kappa)$.*

Proof. Let $\lambda = \mathfrak{r}_\kappa < \mathfrak{d}_\kappa$. Choose an $\mathfrak{r}_\kappa$-family $\mathcal{R}$ of size λ. We shall construct an ultrafilter $\mathscr{U}$ with a base of size λ, thus proving that $\mathfrak{u}_\kappa \leq \mathfrak{r}_\kappa$. Since $\mathfrak{r}_\kappa \leq \mathfrak{u}_\kappa$ is always true we will be done.

Set $\mathcal{H} = \{p_y : y \in \mathcal{R}\}$. From Claim 2.9 we infer that $\mathcal{H}$ is big. Since $|\mathcal{H}| \leq \lambda < \mathfrak{d}_\kappa$ it is not finitely dominating. Hence we can fix a function $g \in {}^\kappa\kappa$ which is not dominated by any finite number of functions from $\mathcal{H}$. Define

$$\mathscr{B} = \big\{\{\beta \in \kappa : f(\beta) \leq g(\beta)\} : f \in \mathcal{H}\big\}.$$

Let us point out some simple properties of $\mathscr{B}$. First, this collection of sets has the strong finite intersection property. This follows from the fact that $\mathcal{H}$ is not finitely dominating. Second, the pair $(\mathscr{B}, g)$ satisfies the assumptions in the definition of a big set with respect to $\mathcal{H}$, where $\mathscr{B}$ stands for $\mathscr{F}$ in the definition. Consequently, there is an increasing $(<\kappa)$-to-one function $h \in {}^\kappa\kappa$ such that $\{h[y \cap z] : y, z \in \mathscr{B}\}$ is an $\mathfrak{r}_\kappa$-family. Finally, the cardinality of $\mathscr{B}$ is at most λ, since $|\mathcal{H}| \leq \lambda$.

Extend $\mathscr{B}$ to any ultrafilter $\mathscr{U}$ over κ and notice that a base for $\mathscr{U}$ can be obtained from the elements of $\mathscr{B}$ by taking finite intersections. It follows that this base is of size at most λ (and hence equals λ since $\lambda = \mathfrak{r}_\kappa \leq \mathfrak{u}_\kappa$). This observation concludes the proof. ■

In the main result of the next section, $\mathfrak{r}_\lambda < \mathfrak{d}_\lambda$ is forced upon a singular cardinal λ. We do not know whether $\mathfrak{r}_\lambda < \mathfrak{d}_\lambda$ implies $\mathfrak{r}_\lambda = \mathfrak{u}_\lambda$ when $\lambda > \mathrm{cf}(\lambda)$, though this is the typical case in the models of the next section. This invites the following:

Question 2.12. *Assume that $\lambda > \mathrm{cf}(\lambda)$.*

(a) *Is it consistent that $\mathfrak{r}_\lambda < \mathfrak{u}_\lambda$?*
(b) *Is it provable that $\mathfrak{r}_\lambda < \mathfrak{d}_\lambda$ implies $\mathfrak{r}_\lambda = \mathfrak{u}_\lambda$?*

We remark that the proof that $\mathfrak{r}_\kappa < \mathfrak{d}_\kappa$ implies $\mathfrak{r}_\kappa = \mathfrak{u}_\kappa$ when κ is regular translates, for the most part, to a singular cardinal λ. It seems, however, that the proof of Claim 2.9 is problematic in the case of a singular cardinal. Specifically, the intervals I_α need not form a partition of λ into λ-many intervals.

3. At singular cardinals. In this section we prove the consistency of $\mathfrak{u}_\lambda < \mathfrak{d}_\lambda$ where $\lambda > \mathrm{cf}(\lambda)$ is a strong limit cardinal. In the definition of $\mathfrak{r}_\lambda$ or $\mathfrak{u}_\lambda$ there is no difference between the regular and the singular case. But the definition of $\mathfrak{d}_\lambda$ requires some attention. If λ is singular then being of size less than λ and being bounded in λ are not the same statement. We shall use the size version, as articulated in Definition 2.4.

Assume that $\mu > \mathrm{cf}(\mu) = \theta$. The concept of $\mathfrak{d}_\mu$ relates to functions from μ into μ, but the following useful lemma shows that one can deal with functions from μ into θ. Given $f, g \in {}^\mu\theta$ we shall say that $f <^* g$ iff $\{\beta \in \mu : f(\beta) \geq g(\beta)\}$ is of size less than μ. Define $\mathfrak{d}^*_\mu$ as the minimal cardinality of a domintaing subset of ${}^\mu\theta$ with respect to $<^*$.

Lemma 3.1. *Assume that $\mu > \mathrm{cf}(\mu) = \theta$ and $\mathfrak{d}^*_\mu = \kappa$. Then $\mathfrak{d}_\mu = \kappa$ as well.*

Proof. We prove the lemma in two steps, by showing that $\mathfrak{d}_\mu \geq \kappa$ and $\mathfrak{d}_\mu \leq \kappa$. As a first step we show that $\mathfrak{d}^*_\mu = \kappa$ implies $\mathfrak{d}_\mu \geq \kappa$. Let $\mathscr{F} \subseteq {}^\mu\mu$ be of size $\tau < \kappa$. We must show that $\mathscr{F}$ is not a dominating family in ${}^\mu\mu$. Fix an increasing sequence $(\lambda_i : i \in \theta)$ of regular cardinals such that $\theta < \lambda_0$ and $\bigcup_{i\in\theta} \lambda_i = \mu$. Let $h : \mu \to \theta$ be the associated interval mapping. Explicitly, if $\alpha \in \mu$ then $h(\alpha)$ is the unique ordinal $i \in \theta$ such that $\lambda_i \leq \alpha < \lambda_{i+1}$.

For every $f \in \mathscr{F}$ we define $f' \in {}^\mu\theta$ by $f'(\beta) = h(f(\beta))$ whenever $\beta \in \mu$. Let $\mathscr{F}' = \{f' : f \in \mathscr{F}\}$, so $|\mathscr{F}'| \leq \tau < \kappa$. Since $\mathfrak{d}^*_\mu = \kappa$ there exists $g' \in {}^\mu\theta$ such that $g' \nleq^* f'$ for all $f' \in \mathscr{F}'$. Define $g \in {}^\mu\mu$ by letting $g(\beta) = \lambda_{i+1} \Leftrightarrow g'(\beta) = i$. We claim that g is not dominated by $\mathscr{F}$.

To prove this claim fix an element $f \in \mathscr{F}$ and recall that $g' \nleq^* f'$. Hence there exists a set $A \in [\mu]^\mu$ such that $\beta \in A \Rightarrow g'(\beta) > f'(\beta)$. By the definition of f' we may write $\beta \in A \Rightarrow g'(\beta) > h(f(\beta))$. This means that $g(\beta) > \lambda_{h(f(\beta))+1} > f(\beta)$ whenever $\beta \in A$, so $g \nleq^* f$ and the first direction is proved.

For the opposite direction assume that $\mathfrak{d}_\mu = \kappa$, aiming to prove that $\mathfrak{d}^*_\mu \geq \kappa$. In other words $\mathfrak{d}_\mu \leq \mathfrak{d}^*_\mu$, so $\mathfrak{d}^*_\mu = \kappa$ implies $\mathfrak{d}_\mu \leq \kappa$ as required for this direction. Let $\mathscr{G} \subseteq {}^\mu\theta$ be a family of elements of ${}^\mu\theta$ such that $|\mathscr{G}| = \tau < \kappa$. We shall see that $\mathscr{G}$ is not dominating.

For each $g \in \mathscr{G}$ we define $f_g \in {}^\mu\mu$ by letting $f_g(\beta) = \lambda_{h(g(\beta))}$. Set $\mathscr{F} = \{f_g : g \in \mathscr{G}\}$. Since $|\mathscr{F}| \leq \tau < \kappa$, we can choose $g^{\mathrm{up}} \in {}^\mu\mu$ such that $g^{\mathrm{up}} \nleq^* f$ for all $f \in \mathscr{F}$. We define a function $g \in {}^\mu\theta$ as follows. If $\beta \in \mu$ then there is a unique ordinal $i \in \theta$ for which $\lambda_i \leq g^{\rm up}(\beta) < \lambda_{i+1}$, and we let $g(\beta) = i+1$. We claim that g is not dominated by $\mathscr{G}$.

By way of contradiction assume that $g_0 \in \mathscr{G}$ and $g \leq^* g_0$. Let f_0 be f_{g_0}, and recall that $A = \{\beta \in \mu : f_0(\beta) < g^{\rm up}(\beta)\}$ is of size μ. Let $B = \{\gamma \in \mu : g(\gamma) > g_0(\gamma)\}$ and notice that $|B| < \mu$. Fix an ordinal $\gamma_0 \in \mu$ such that $B \subseteq \gamma_0$. If $\beta \in A - \gamma_0$ then $f_0(\beta) < g^{\rm up}(\beta)$ (since $\beta \in A$), which means that $\lambda_{h(g_0(\beta))} < \lambda_{h(g(\beta))}$ (by the definition of these functions), hence $g_0(\beta) < g(\beta)$. This is impossible, however, since $\beta \geq \gamma_0$, so we are done. ■

The first observation below is that $\mathfrak{d}_\lambda$ behaves nicely at singular cardinals.

Claim 3.2. *Let λ be a singular cardinal. Then $\mathfrak{d}_\lambda > \lambda$, and moreover ${\rm cf}(\mathfrak{d}_\lambda) > \lambda$.*

Proof. We spell out the easy argument for $\mathfrak{d}_\lambda > \lambda$, and it will also follow of course from the stronger statement ${\rm cf}(\mathfrak{d}_\lambda) > \lambda$. Suppose that $\{f_\alpha : \alpha \in \lambda\} \subseteq {}^\lambda\lambda$. We wish to describe a function $h : \lambda \to \lambda$ which is not dominated by any f_α. To this end, decompose λ into λ-many disjoint sets $(S_\alpha : \alpha \in \lambda)$, each of size λ. For every $\beta \in \lambda$ let $h(\beta) = f_\alpha(\beta)+1$ iff α is the unique ordinal for which $\beta \in S_\alpha$. For every $\alpha \in \lambda$ one can see that $\beta \in S_\alpha \Rightarrow f_\alpha(\beta) < h(\beta)$, so h is as required.

Assume now that $\mathfrak{d}_\lambda = \chi$, and assume toward contradiction that $\theta = {\rm cf}(\chi) < \lambda$. Let $\kappa = {\rm cf}(\lambda)$. By Lemma 3.1 we may concentrate on ${}^\lambda\kappa$, so fix a family $\mathcal{F} \subseteq {}^\lambda\kappa$ which exemplifies $\mathfrak{d}^*_\lambda = \chi$. Choose a sequence $(\mathcal{F}_i : i \in \theta)$ of disjoint sets such that $\mathcal{F} = \bigcup_{i\in\theta} \mathcal{F}_i$ and $|\mathcal{F}_i| < \chi$ for every $i \in \theta$. Decompose μ into $(A_i : i \in \theta)$ with each A_i of size μ, and fix a bijection $g_i : \mu \to A_i$ for every $i \in \theta$.

Now for each $i \in \theta$ let $\mathcal{G}_i = \{f \circ g_i : f \in \mathcal{F}_i\}$. Observe that $|\mathcal{G}_i| < \chi$ for every $i \in \theta$. For every $i \in \theta$ we have $|\mathcal{F}_i| < \chi$, and hence one can choose a function $f_i \in {}^\lambda\kappa$ such that $f_i \nleq^* f$ for all $f \in \mathcal{F}_i$. Much as in the first part of the proof, we can describe our non-dominated function g by defining its values separately over each A_i. For every $i \in \theta$ and every $\alpha \in A_i$ let $g(\alpha) = f_i(g_i^{-1}(\alpha))$. Notice that $g \in {}^\lambda\kappa$ is well defined. We claim that g is not dominated by $\mathcal{F}$, namely $g \nleq^* f$ for all $f \in \mathcal{F}$.

To prove this, fix $f \in \mathcal{F}$. Let $i \in \theta$ be the unique ordinal for which $f \in \mathcal{F}_i$. Denote the set $\{\beta \in \mu : f \circ g_i(\beta) < f_i(\beta)\}$ by B, so $|B| = \mu$. Let $C = \{g_i(\beta) : \beta \in B\}$. Observe that $C \subseteq A_i$ and $|C| = \mu$ since g_i is one-to-one. We claim that $f(\gamma) < g(\gamma)$ whenever $\gamma \in C$. Indeed, fix any $\gamma \in C$, and let $\beta \in B$ be the unique ordinal for which $\gamma = g_i(\beta)$. By the definition of B it follows that $f(\gamma) = f(g_i(\beta)) < f_i(\beta)$. Concomitantly, $g(\gamma) = f_i(g_i^{-1}(\gamma)) = f_i(\beta)$, and hence $f(\gamma) < g(\gamma)$. Since $|C| = \mu$ we conclude that $g \nleq^* f$. But f was arbitrary, so we are done. ■

To prove the main result of this section we need some control over the true cofinality of some sequences of regular cardinals. Let $(\kappa_i : i \in \theta)$ be such a sequence and let J be an ideal over θ such that $J^{\rm bd}_\theta \subseteq J$. We focus on the partial order $(\prod_{i\in\theta}\kappa_i, J)$ which results by defining $f <_J g \Leftrightarrow \{i \in \theta : g(i) \leq f(i)\} \in J$. A *scale* in $(\prod_{i\in\theta}\kappa_i, J)$ is a sequence $\bar{f} = (f_\alpha : \alpha \in \lambda)$ which is J-increasing and cofinal. Since $(\prod_{i\in\theta}\kappa_i, J)$ is only a partial order, a scale need not exist. If there is a scale and moreover λ is the minimal length of a scale then we shall say that λ is the *true cofinality* of the product. This fact is denoted by $\lambda = \mathrm{tcf}(\prod_{i\in\theta}\kappa_i, J)$. We refer the reader to [AM10] for basic facts about pcf theory.

Aiming to show that $\mathfrak{u}_\lambda < \mathfrak{d}_\lambda$ is consistent, one sequence of regular cardinals will give a (small) upper bound for $\mathfrak{u}_\lambda$ and the other sequence will give a (large) lower bound for $\mathfrak{d}_\lambda$. More precisely, for every characteristic we must force the value of the true cofinality for both the sequence of regular cardinals and the sequence of their successors. The forcing machinery for this comes from [GS12b], and we shall use the following version which is based on [GS12b, Claim 3.3]:

THEOREM 3.3. *Let λ be a supercompact cardinal. Then one can force the following statements:*

(a) $\lambda > \mathrm{cf}(\lambda) = \theta$.
(b) $\lambda < \kappa = \mathrm{cf}(\kappa) \leq 2^\lambda$.
(c) $(\lambda_i : i < \theta)$ *is an increasing sequence of strongly inaccessible cardinals,* $\theta < \lambda_0$ *and* $\lambda = \bigcup_{i\in\theta}\lambda_i$.
(d) $2^{\lambda_i} = \lambda_i^+$ *for every* $i \in \theta$.
(e) $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i, J^{\rm bd}_\theta) = \kappa$.
(f) $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i^+, J^{\rm bd}_\theta) = \kappa$.

Moreover, 2^λ can be arbitrarily large, and κ can be an arbitrarily large regular cardinal provided that $\kappa \leq 2^\lambda$.

We show now how to take care of the dominating number:

THEOREM 3.4. *Assume that λ is supercompact. Then one can force that λ is a strong limit singular cardinal, 2^λ is arbitrarily large, $\kappa = \mathrm{cf}(\kappa) \leq 2^\lambda$ is arbitrarily large above λ and $\mathfrak{d}_\lambda \geq \kappa$.*

Proof. Apply Theorem 3.3 to obtain $\theta = \mathrm{cf}(\lambda) < \lambda$, and let $(\lambda_i : i < \theta)$ be an increasing sequence of strongly inaccessible cardinals such that $\theta < \lambda_0$ and $\lambda = \bigcup_{i\in\theta}\lambda_i$ as guaranteed there. This means that $2^{\lambda_i} = \lambda_i^+$ for every $i \in \theta$, with both $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i, J^{\rm bd}_\theta) = \kappa$ and $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i^+, J^{\rm bd}_\theta) = \kappa$. Denote $J^{\rm bd}_\theta$ by J. Fix a sequence $(f_\alpha : \alpha \in \kappa)$ of functions in the product $\prod_{i\in\theta}\lambda_i$ which exemplifies $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i, J) = \kappa$. Fix also a sequence $(g_\alpha : \alpha \in \kappa)$ of functions in the product $\prod_{i\in\theta}\lambda_i^+$ which exemplifies $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i^+, J) = \kappa$.

For every $i \in \theta$ enumerate the elements of ${}^{\lambda_i}\theta$ by $\mathscr{F}_i = \{g^i_\alpha : \alpha \in \lambda_i^+\}$. Likewise, for each $i \in \theta$ fix a sequence $(h^i_\gamma : \gamma \in \lambda_i^+)$ of mappings such that every h^i_γ is a one-to-one mapping from γ into λ_i. For every $\alpha \in \kappa$, every $\sigma \in (\alpha, \kappa)$ and every $i \in \theta$, we define

$$w_{\alpha\sigma i} = \{\beta \in \lambda_i^+ : \beta < g_\alpha(i) \wedge h^i_{g_\alpha(i)}(\beta) < f_\sigma(i)\}.$$

We shall see that the collection of these sets has a very useful covering property, in the pcf sense: it covers every sequence of small sets but just over an end-segment with respect to all three parameters.

Formally, suppose that $u_i \in [\lambda_i^+]^{<\lambda_i}$ for every $i \in \theta$, and that we wish to cover the sequence $(u_i : i \in \theta)$ using the above defined sets. We claim that there is a threshold ordinal $\alpha_0 \in \kappa$ with the following property. For every $\alpha \in [\alpha_0, \kappa)$ there is an ordinal $\sigma(\alpha) \in (\alpha, \kappa)$ such that if $\sigma \in [\sigma(\alpha), \kappa)$ then for some $i_{\alpha\sigma} \in \theta$ we have $i \in [i_{\alpha\sigma}, \theta) \Rightarrow u_i \subseteq w_{\alpha\sigma i}$.

Toward proving this claim suppose that $(u_i : i \in \theta)$ is given. Define $g \in \prod_{i\in\theta} \lambda_i^+$ by $g(i) = \sup(u_i)$. Fix an ordinal $\alpha_0 \in \kappa$ so that $\alpha \in [\alpha_0, \kappa) \Rightarrow g \leq_J g_\alpha$. Now fix an ordinal $\alpha \in [\alpha_0, \kappa)$ and define $t_{\alpha i} = \sup(h^i_{g_\alpha(i)}{}'' u_i)$ for every $i \in \theta$. Denote the sequence $\langle t_{\alpha i} : i \in \theta\rangle$ by t_α, and notice that $t_\alpha \in \prod_{i\in\theta} \lambda_i$. Choose an ordinal $\sigma(\alpha) < \kappa$ such that $\alpha < \sigma(\alpha)$ and $\sigma \in [\sigma(\alpha), \kappa) \Rightarrow t_\alpha \leq_J f_\sigma$. This inequality means that there exists an ordinal $i_{\alpha\sigma} \in \theta$ such that $i \in [i_{\alpha\sigma}, \theta) \Rightarrow t_{\alpha i} = \sup(h^i_{g_\alpha(i)}{}'' u_i) < f_\sigma(i)$. By the definition of our covering sets it follows that $u_i \subseteq w_{\alpha\sigma i}$ for every $i \in [i_{\alpha\sigma}, \theta)$, as required.

Back to the main argument, for every pair of ordinals $\alpha < \sigma < \kappa$ we define a function $h_{\alpha\sigma} : \lambda \to \theta$ with the following property:

$$\text{If } \beta \in w_{\alpha\sigma i}, j \in \theta \text{ and } |(g^i_\beta)^{-1}(j)| = \lambda_i,$$
$$\text{then } |h^{-1}_{\alpha\sigma}[(j, \theta)] \cap (g^i_\beta)^{-1}(j)| = \lambda_i.$$

To choose these functions notice that one can take care of a single $j \in \theta$ and a single ordinal β, since $|(g^i_\beta)^{-1}(j)| = \lambda_i$. Now for each $\alpha \in \kappa$ and every $\sigma \in (\alpha, \kappa)$ there are only $\theta \times |w_{\alpha\sigma i}| < \lambda_i$ many pairs of the form (j, β) to take care of, so the choice of these functions is possible.

Let g be a function from λ into θ. If there is a pair of ordinals $\alpha < \sigma < \kappa$ and there are θ-many ordinals $i \in \theta$ for which $g \restriction \lambda_i \in \{g^i_\beta : \beta \in w_{\alpha\sigma i}\}$ then $h_{\alpha\sigma} \nleq^* g$. To see this, assume that $g \in {}^\lambda\theta$, let β_i be an ordinal such that $g \restriction \lambda_i = g^i_{\beta_i}$ and let $u^g_i = \{\beta_i\}$ for every $i \in \theta$. Apply the covering property of the sets $w_{\alpha\sigma i}$ to the sequence $(u^g_i : i \in \theta)$. This gives, first of all, a threshold ordinal $\alpha_0 \in \kappa$. Now for each $\alpha \in [\alpha_0, \kappa)$ there is an ordinal $\sigma(\alpha) \in (\alpha, \kappa)$ with the following property. If $\sigma \in (\sigma(\alpha), \kappa)$ then there is some $i_{\alpha\sigma} \in \theta$ such that $i \in [i_{\alpha\sigma}, \theta) \Rightarrow u^g_i \subseteq w_{\alpha\sigma i}$. But this means that $h_{\alpha\sigma} \nleq^* g$ as desired.

The above argument shows how to deal with a single function g. The nature of this argument, however, makes it applicable to any set of functions of size less than κ. Suppose that $\mathscr{G} \subseteq {}^\lambda\theta$ and $|\mathscr{G}| = \tau < \kappa$. Enumerate the elements of $\mathscr{G}$ as $\{g_\delta : \delta \in \tau\}$. For every $\delta \in \tau$ let α_δ be the threshold of α. Namely, for every $\alpha \in [\alpha_\delta, \kappa)$ there is $\sigma(\alpha) \in (\alpha, \kappa)$ such that if $\sigma \in [\sigma(\alpha), \kappa)$ then $h_{\alpha\sigma} \nleq^* g_\delta$. Observe that $\alpha = \bigcup_{\delta\in\tau} \alpha_\delta < \kappa$. Choose an ordinal $\sigma \in (\alpha, \kappa)$ such that $\sigma > \sigma(\alpha)$. It follows that $h_{\alpha\sigma} \nleq^* g_\delta$ for every $\delta \in \tau$, and hence $\mathscr{G}$ is not a dominating family. Since $\mathscr{G}$ was an arbitrary family of less than κ many functions in ${}^\lambda\theta$ we conclude that $\mathfrak{d}_\lambda \geq \kappa$, thus accomplishing the proof. ∎

It has been proved in [GS12a] that one can force that $\mathfrak{u}_\lambda = \lambda^+$ and 2^λ is arbitrarily large for some singular cardinal λ. The proof is similar to the proof in this section, but somehow from the opposite direction. Namely, under similar assumptions about the true cofinalities of sequences of regular cardinals and their successors, one can show that $\mathfrak{u}_\lambda \leq \kappa$ where κ realizes the true cofinalities. This is possible, in particular, for $\kappa = \lambda^+$. It remains to merge the result about $\mathfrak{u}_\lambda$ with the present result about $\mathfrak{d}_\lambda$. To this end, one needs two sequences of regular cardinals, with different values of true cofinalities. How to force this situation is shown in [GMS18] and [GS14].

Theorem 3.5. *Assume there exists a supercompact cardinal. Then one can force a singular cardinal* $\lambda > \mathrm{cf}(\lambda) = \theta$ *such that* $\lambda < \mathfrak{u}_\lambda \leq \kappa_0 < \kappa_1 \leq \mathfrak{d}_\lambda \leq 2^\lambda$, *and in the model obtained the gap between* $\mathfrak{u}_\lambda$ *and* $\mathfrak{d}_\lambda$ *can be arbitrarily large.*

Proof. For $\lambda > \mathrm{cf}(\lambda) = \theta$ and $\kappa = \mathrm{cf}(\kappa) \in (\lambda, 2^\lambda]$ we say that $(\lambda_i : i \in \theta)$ is κ-*qualified* iff it is an increasing sequence of regular cardinals such that $\theta < \lambda_0, \lambda = \bigcup_{i\in\theta} \lambda_i, 2^{\lambda_i} = \lambda_i^+$ for every $i \in \theta$ and $\mathrm{tcf}(\prod_{i\in\theta} \lambda_i, J) = \mathrm{tcf}(\prod_{i\in\theta} \lambda_i^+, J) = \kappa$ (where J is usually the ideal of bounded subsets of θ, but not necessarily).

Let λ be supercompact in the ground model. Let $\mathbb{P}$ be a forcing notion which makes $\lambda > \mathrm{cf}(\lambda) = \theta$ and forces the following statements:

- There exists a sequence $(\lambda_i^0 : i \in \theta)$ of measurable cardinals which is κ_0-qualified.
- There exists a sequence $(\lambda_i^1 : i \in \theta)$ of strongly inaccessible cardinals which is κ_1-qualified.

From [GS12a, Theorem 1.4] we infer that $\mathfrak{r}_\lambda \leq \mathfrak{u}_\lambda \leq \kappa_0$. From Theorem 3.4 we infer that $\mathfrak{d}_\lambda \geq \kappa_1$. Since $\lambda < \mathfrak{r}_\lambda$ and $\mathfrak{d}_\lambda \leq 2^\lambda$ are always true, we are done. ∎

The possibility to force two such qualified sequences is a version of Theorem 3.3 (see [GS14]). Similar statements appear in [GMS18], in a slightly different way. In that paper, many regular cardinals above the singular cardinal λ are realized as true cofinalities of some sequence of measurable cardinals below λ. However, if $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i, J)=\kappa$ then $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i^+, J)=\kappa^+$. For our purpose it makes no difference, since we will get $\mathfrak{u}_\lambda\leq\kappa_0^+$ and $\mathfrak{d}_\lambda\geq\kappa_1^+$, still there will be a gap between these characteristics. But there is an important difference, since in [GMS18] the GCH is kept below λ while $2^\lambda>\lambda^+$, and since λ is strong limit this means that we must work with a singular cardinal with countable cofinality. On the other hand, this theorem is very flexible in the sense that an infinite set of targets can be obtained by qualified sequences. This is the background behind the following statement.

COROLLARY 3.6. *It is consistent that* $\lambda>\mathrm{cf}(\lambda)$, $\mathfrak{u}_\lambda<\mathfrak{d}_\lambda$ *and* $\mathfrak{d}_\lambda$ *is singular.*

Proof. Let λ be supercompact in the ground model. We set $\kappa=\lambda^+$ and we choose an increasing sequence $(\kappa_j : j\in\partial)$ of regular cardinals where $\partial\geq\lambda^+, \mathrm{cf}(2^\lambda)=\partial$ in the generic extension and $\bigcup_{j\in\partial}\kappa_j=2^\lambda$. We shall say that the sequence $(\lambda_n : n\in\omega)$ is (θ,θ^+)-*qualified* iff $2^{\lambda_n}=\lambda_n^+$ for every $n\in\omega$, $\lambda=\bigcup_{n\in\omega}\lambda_n, J\supseteq J_\omega^{\mathrm{bd}}$ and $\theta=\mathrm{tcf}(\prod_{n\in\omega}\lambda_n, J)$ while $\theta^+=\mathrm{tcf}(\prod_{n\in\omega}\lambda_n^+, J)$.

We repeat the argument of the previous theorem, but now we employ [GMS18, Theorem 7]. In particular, $\lambda>\mathrm{cf}(\lambda)=\omega$ and a (κ_j,κ_j^+)-qualified sequence of measurable cardinals below λ is forced for every $j\in\partial$ as well as a sequence for (κ,κ^+). It follows from Theorem 3.4 that $\mathfrak{d}_\lambda\geq\kappa_j$ for every $j\in\partial$. Hence $\mathfrak{d}_\lambda=2^\lambda$, so $\mathfrak{u}_\lambda\leq\kappa^+<\mathfrak{d}_\lambda$ and $\mathfrak{d}_\lambda$ is singular as required. ∎

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