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GENERALIZING RANDOM REAL FORCING FOR INACCESSIBLE CARDINALS

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ABSTRACT

The two classical parallel concepts of "small" sets of the real line are meagre sets and null sets. Those are equivalent to Cohen forcing and Random Real forcing for ${}^{\mathbb{N}}\mathbb{N}$; in spite of this similarity, the Cohen forcing and Random Real forcing have very different shapes. One of these differences is in the fact that the Cohen forcing has an easy natural generalization for ${}^{\lambda}2$ for regular $\lambda > \aleph_0$, corresponding to an extension for the meagre sets, while the Random Real forcing didn't seem to have a natural generalization, as Lebesgue measure doesn't have a generalization for space 2^{λ} while $\lambda > \aleph_0$. The work [6] found a forcing resembling the properties of Random Real forcing for 2^{λ} while λ is a weakly compact cardinal. Here we describe, with additional assumptions, such a forcing for 2^{λ} while λ is just an Inaccessible Cardinal; this forcing is strategically $< \lambda$-complete and satisfies the λ^+- c.c. hence preserves cardinals and cofinalities, however unlike Cohen forcing, does not add an undominated real.

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Introduction

There are two classical ways of defining what is a small set of the real line ${}^{\omega}2$; the topological definition of a small set is a **meagre** set, which is a countable union of nowhere dense sets. The second definition uses measure and defines a set to be small if it is a **null** set, which means that it has Lebesgue measure zero.

Both the collection of meagre sets and the collection of null sets are ideals in the set ${}^{\omega}2$; the forcing modulo the ideal of meagre sets is the **Cohen Forcing** while the forcing modulo the ideal of null sets is **Random Real forcing** [5].

Looking at λ-reals for $\lambda > \aleph_0$, so elements of the set

$$^{\lambda}2 = \{\eta : \eta \text{ is a sequence of 0's and 1's of length } \lambda\},$$

there is a natural extension to a Cohen Forcing; that would be a forcing modulo sets that are λ-meagre [3]. Unlike this case, Lebesgue measure has no natural extension in ${}^{\lambda}2$ for regular cardinals $\lambda > \aleph_0$, thus there is no generalization of Random Real forcing for those cardinals.

An important and useful property of Random Real forcing is not adding a function that is undominated; recall that Cohen Forcing adds $f : \lambda \to \lambda$ not smaller (meaning, modulo finite set) than any real in the ground model (where λ-reals here are functions $\lambda \to \lambda$, i.e., members of ${}^{\lambda}\lambda$). However, Random Real forcing has the property that every "new" real (i.e., every element of ${}^{\omega}\omega$) is bounded by a real in the ground model. One of the uses of this property is for cardinal invariants; the bounding number $\mathfrak{d}$ [2] does not change after forcing with Random Real forcing.

In the paper [6], the second author described a generalization of the null ideal (meaning, the ideal of Lebesgue measure zero sets) for a weakly compact cardinal λ; that was done by constructing a forcing that has the properties of Random Real forcing in 2^{λ} for a weakly compact λ; this result is surprising since there is no clear similarity in the definition of the forcing in [6] and Random Real forcing.

By "having the properties of Random Real forcing" we mean a forcing for which: (1) the λ^{+}-chain condition holds and (2) the forcing is strategically $< \lambda$-complete; by those conditions it follows that the forcing preserves cardinals and cofinalities when $\lambda = \lambda^{<\lambda}$. Moreover, any new real added in the forcing shall be bounded by a real in the ground model, that will be condition (3): the forcing is λ-bounding. An additional important property is symmetry, but it fails by [6].

The purpose of this work is to find a forcing as in [6] for Mahlo, and even any inaccessible cardinal (therefore may be smaller than the first weakly compact cardinal). In section 2 we shall describe a construction for which the properties of Random Real forcing (Definition 27) hold for any inaccessible and in particular a **Mahlo cardinal**; those are cardinals whose existence is a weaker condition than the existence of a weakly compact cardinal [4]. However, compared to [6] we need some parameter $X \subseteq \lambda$ so the definition is not "pure" as in [6].

An additional difference from [6] is that the large cardinal property is not enough. We shall assume the existence of a stationary set that reflects only in inaccessibles and has a diamond sequence. Note that this demand can be obtained by an easy forcing [1], and if $V = L$ this is equivalent to not being weakly compact. For a Mahlo cardinal there is a stationary set of inaccessible cardinals below it, so in particular this set reflects only in inaccessibles and then we still need to assume the existence of a diamond sequence for it. In [6], the main use of the weak compactness was by reflecting a maximal antichain of conditions to a maximal antichain in a corresponding forcing for a smaller cardinal; the purpose of the diamond sequence here will be to overcome this inability.

Furthermore, for convenience we shall assume that the conditions of the desired forcing are trees that are pruned only in levels of the stationary set (we demand the stationary set only to contain limit ordinals). However, it is possible to allow pruning in successor levels; e.g., as long as the pruning is only of a bounded set, when we use a tree with splitting to $\theta_\epsilon = cf(\theta_\epsilon) \in [|\epsilon|^+, \lambda)$.

We may like to make our forcing $< \lambda$-complete (rather than strategically $< \lambda$-complete); this is not clear.

This work is a part of what was promised in [6]; the ideas of the construction were stated in Rutgers in 2011.

We intend to deal later with accessible $\lambda = \lambda^{<\lambda} > \aleph_0$ (under reasonable conditions); also we can use an $|\epsilon|^+$-complete D_ϵ filter on θ_ϵ (or D_η on $suc_{\boldsymbol{T}}(\eta)$ when $\lg(\eta) = \epsilon$, as in [6], or Remark 17 below).

We also continue [6] in [7] in work with T. Baumhauer and M. Goldstern [9] and in [8].

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NOTATION.

(1) We use $\alpha, \beta, \gamma, \delta, \epsilon$ to denote ordinals.
(2) We use λ, μ, κ to denote cardinals.
(3) ${}^{B}A$ denotes the set of functions from B to A.

1. Preliminaries

The paper [6] showed a method of finding, for a weakly compact cardinal λ, a forcing that generalizes the properties of a Random Real forcing for $\aleph_0$. In section 2 we add the assumption of a diamond principle and then see a similar forcing that generalizes the same properties for inaccessible cardinals, with the assumption that there exists a stationary set that reflects only in inaccessibles; so in particular for Mahlo cardinals it follows. Here we show present general definitions that will be used throughout this paper.

Definition 1: A forcing **resembling Random Real** forcing for a regular cardinal $\lambda = \lambda^{<\lambda}$ will be a forcing for which the following conditions hold:

(1) The forcing is not trivial and the λ^+-chain condition holds.
(2) The forcing is $< \lambda$-strategically complete.
(3) The forcing is λ-bounding.
(4) The forcing does not add λ-Cohen reals (follows from (3)).

This definition reflects the properties of Random Real forcing in the case of $\lambda = \aleph_0$.

Remark 2: At this point we shall ignore another desired property, symmetry. This property states that for all η_1, η_2 and a model M: η_1 is generic over M and η_2 is generic over $M[\eta_1]$ if and only if η_2 is generic over M and η_1 is generic over $M[\eta_2]$. By [6] it fails.

Now we can define the terms used for Definition 1:

Definition 3: Let α be an ordinal.

(1) For a forcing notion $\mathbb{P}$ and condition $p \in \mathbb{P}$, we define a game $\Game_\alpha(p, \mathbb{P})$ as follows. A play of the game has α moves and for $\beta < \alpha$ first the player COM chooses a condition $p_\beta \in \mathbb{P}$ such that:
  (a) $p \leq p_\beta$.
  (b) For all $\gamma < \beta$ it holds that $q_\gamma \leq p_\beta$.

Next, the player INC plays and chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq q_\beta$.

The player COM **wins** the game if she survived; i.e., had a legal move for all $\beta < \alpha$.

(2) A forcing $\mathbb{P}$ is said to be **strategically complete** in α (or α-strategically complete) if for all $p \in \mathbb{P}$ it holds that in the game $\Game_\alpha(p, \mathbb{P})$ between players COM and INC, player COM has a winning strategy.

Definition 4: A forcing $\mathbb{P}$ is $< \lambda$**-strategically complete** if it is α-strategically complete for all $\alpha < \lambda$.

Definition 5: For a cardinal λ, a forcing $\mathbb{P}$ will be called λ**-bounding** when the following holds:

$$\Vdash_{\mathbb{P}} (\forall f : \lambda \to \lambda)\ ((\exists g \in ({}^\lambda\lambda)^V) : ((\forall \alpha < \lambda)(f(\alpha) \leq g(\alpha)))).$$

Definition 6: A set of ordinals S will be called **tenuous** (or "nowhere stationary" as in [6]) if for each ordinal δ of uncountable cofinality, the set $S \restriction \delta$ is not a stationary set in δ.

Definition 7: Let λ be a cardinal and $S \subseteq \lambda$ a stationary set of λ. Then S is said to be **non-reflecting** when for each ordinal $\delta < \lambda$ of cofinality $> \aleph_0$ the set $S \restriction \delta$ is not stationary in δ.

Remark 8: Let λ be a cardinal and let S_* be a non-reflecting stationary subset of λ. Then the set $S \subseteq S_*$ is tenuous if and only if S is not stationary.

Claim 9: *Let λ be a cardinal and S_* be a non-reflecting stationary subset of λ.*

(1) *If $\bar{S} = \langle S_i : i < i(*)\rangle$ is such that for all $i < i(*)$, $S_i \subseteq \lambda$ is a non-stationary with $i(*) < cf(\lambda)$, then $S = \bigcup_{i<i(*)} S_i$ is not stationary.*

(2) *If $\bar{S} = \langle S_i : i < i(*)\rangle$ is such that for all $i < i(*)$, $S_i \subseteq S_*$ is a tenuous set with $i(*) < cf(\lambda)$, then $S = \bigcup_{i<i(*)} S_i$ is tenuous.*

Proof. We see that:

(1) For each $i < i(*)$, there is a club E_i such that $S_i \cap E_i = \emptyset$ (as S_i is not stationary), so let $E = \bigcap_{i<i(*)} E_i$; E is a club in λ, as the intersection of $i(*) < cf(\lambda)$ clubs. In addition, $S \cap E = \emptyset$, thus S is not stationary.

(2) From clause (1), S is not stationary. In addition, for each $\alpha < \lambda$, $S_* \cap \alpha$ is non-stationary, hence so is $S \cap \alpha$, as a subset of it. ∎

2. New λ-real for inaccessible cardinal λ

To find a forcing resembling Random Real forcing for a Mahlo Cardinal, we need to add an additional assumption to those of the weakly compact cardinal case in [6]; the new assumption will be a diamond sequence indexed on a stationary set of inaccessible cardinals (a stationary set of inaccessibles exists for a Mahlo Cardinal). For the more general case of any Inaccessible Cardinal, there is still a need to assume the existence of a diamond sequence; however, here it will be indexed on a stationary set that only reflects in inaccessible cardinals. Those two cases are unified here, dealing with an Inaccessible Cardinal with a stationary set that only reflects in inaccessible cardinals; a Mahlo Cardinal will be a special case of this.

2.1. Useful Definitions.

Definition 10: A **good structure** $\mathfrak{r}$ contains:

(1) An inaccessible cardinal $\lambda = \lambda_{\mathfrak{r}}$.
(2) A stationary set $S_* = S_*^{\mathfrak{r}} \subseteq \lambda$ of strong limit cardinals, such that if $S_* \cap \delta$ is stationary in δ then δ is inaccessible.
(3) An increasing sequence of cardinals $\bar{\theta} = \bar{\theta}_{\mathfrak{r}} = \langle \theta_\epsilon : \epsilon < \lambda \rangle$ such that for all $\epsilon < \lambda$: $2 \leq \theta_\epsilon < \lambda$; and if $\epsilon \in S_*$, then for all $\zeta < \epsilon$, $\theta_\zeta < \epsilon$.
(4) We assume the diamond principle for S_*, $\diamondsuit_{S_*}$, and let $\bar{X} = \bar{X}_{\mathfrak{r}}$ be a sequence witnessing it, i.e., $\bar{X} = \langle X_\delta : \delta \in S_* \rangle$; $X_\delta \subseteq \mathscr{H}(\lambda)$.

Remark 11: Observe that:

(1) For λ Mahlo there is a stationary set $S_* \subseteq \lambda$ that only contains inaccessible cardinals, thus in particular its reflection will only be in inaccessible cardinals.
(2) For λ inaccessible that isn't Mahlo, a non-reflecting stationary set can be added by a forcing that uses initial segments as in [1].
(3) It is possible to assume that S_* is a set of just limit ordinals (maybe not strong limit) and the only difference will be that for all $\delta \in S_*$ the forcing $\mathbb{Q}_\delta$ (to be defined later) will have the $|\boldsymbol{T}_{<\delta}|^+$-chain condition rather than the δ^+-chain condition as we have here; however, the forcing $\mathbb{Q}_\lambda$ will still have the λ^+-chain condition.

(4) Concerning Definition 10(4), the standard phrasing of $\diamondsuit_S$ is "there is $\langle A_\alpha : \alpha \in S\rangle$, $A_\alpha \subseteq \alpha$ such that for every $A \subseteq \alpha$ the set $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is a stationary subset of λ". However, given a sequence as above, and h an one-to-one function from λ onto $\mathscr{H}(\lambda)$ (they are of the same cardinality because $\lambda = \lambda^{<\lambda}$ follows from "λ is inaccessible"), let

$$E = \{\mu < \lambda : h \text{ maps } \mathscr{H}(\mu) \text{ onto } \mu\}.$$

It is a club of λ because $\mu < \lambda \Rightarrow 2^\mu < \lambda$. Lastly, let $X_\delta \subseteq \delta$ be

$$\{\alpha < \delta : h(\alpha) \in A_\delta\};$$

it is easily as required. (Inversely, starting from the X_δ's we can choose suitable A_δ's.)

Remark 12: When S_* is non-reflecting, the proofs are simpler.

Next, the forcing will be defined in several steps; those will be tree forcings for each $\delta \in S_* \cup \{\lambda\}$. First, we shall define the "biggest" forcing $\mathbb{Q}^0_\delta$; later we will define two additional forcings $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$. For each of those forcings the forcing relation will be of inverse inclusion.

Definition 13: Given a good structure $\mathfrak{r}$, we shall define for each $\alpha \leq \lambda$ the collection of vertices of level α:

$$\boldsymbol{T}_\alpha = \prod_{\epsilon < \alpha} \theta_\epsilon;$$

for $\alpha \leq \lambda$ we will define the complete tree up to α to be the union of those sets:

$$\boldsymbol{T}_{<\alpha} = \bigcup \{\boldsymbol{T}_\beta : \beta < \alpha\}.$$

Remark 14: We assume we have a good structure $\mathfrak{r}$ until the end of section 2.

Convention: (1) For all $\delta_1 < \delta \leq \lambda$ and $\nu \in \boldsymbol{T}_\delta$ let $\nu \restriction \delta_1$ the restriction of ν to δ_1.

(2) For each $\delta \in S_* \cup \{\lambda\}$ and a set $u \subseteq \boldsymbol{T}_{<\delta}$ we write

$$\lim_\delta(u) = \{\nu \in \boldsymbol{T}_\delta : \forall \alpha < \delta, \nu \restriction \alpha \in u\}.$$

(3) For all $\delta \leq \lambda$ and a set $u \subseteq \boldsymbol{T}_{<\delta}$, for $\delta_1 < \delta$ we shall write

$$u \restriction \delta_1 = u \cap \boldsymbol{T}_{<\delta_1}.$$

(4) Assume $\alpha < \delta$ and $u \subseteq \boldsymbol{T}_{<\alpha}$ is a tree: a non-empty set closed under taking initial segments. Let $\eta \in u$ be some node; we write

$$u^{[\eta]} = \{\nu \in u : \eta \trianglelefteq \nu \vee \nu \triangleleft \eta\}.$$

Definition 15: We can now define the forcing $\mathbb{Q}^0_\delta$ for each $\delta \in S_* \cup \{\lambda\}$.

(1) A condition in the forcing will be a tree $p \subseteq \boldsymbol{T}_{<\delta}$, such that:
  (a) There is a trunk $tr(p)$; this is the unique element $\eta \in p$ with the following properties:
    (i) For all $\nu \in p$, $\nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu$.
    (ii) For every η' with the property (1)(a)(i), we have that $\eta' \trianglelefteq \eta$.
  (b) For each $\eta \in p$ there is a $\nu \in \lim_\delta(p)$ with $\eta \triangleleft \nu$.
  (c) For $tr(p) \trianglelefteq \eta \in p$, $\{j \in \theta_{\lg(\eta)} : \eta^\frown\langle j\rangle \in p\} = \theta_{\lg(\eta)}$.
  (d) The set $S_p = \{\delta_1 \in (\lg(tr(p)), \delta) : \lim_{\delta_1}(p \restriction \delta_1) \not\subseteq p\}$ is a tenuous subset of S_*; we call this set the witness set.

(2) For all $p, q \in \mathbb{Q}^0_\delta$ we say that $p \leq q$ if and only if $p \supseteq q$.

Remark 16: We can think of a tree $p \in \mathbb{Q}^0_\delta$ for $\delta \in S_* \cup \{\lambda\}$ as a complete tree from the level $\lg(tr(p))$ up that we are pruning: in successor levels we are not allowed to prune. On limit levels we are allowed to prune the tree only if the level is an ordinal in S_p, so in most limit levels we take all the limits while in stages in S_p we are allowed to cut as much as we want as long as (1)(b) holds; hence there will be a continuation to each node in each level higher than its length.

Remark 17: An alternative definition can be such that in successor levels there might be prunings, as long as those are not too big, that is, $\{j \in \theta_{\lg(\eta)} : \eta^\frown\langle j\rangle \notin p\}$ is bounded in $\theta_{\lg(\eta)}$ (when $cf(\theta_{\lg(\eta)}) > \lg(\eta)$) or even belong to D_η, a $|\lg(\eta)|^+$-complete filter on $\theta_{\lg(\eta)}$—not a serious difference.

Claim 18: *For all $\delta \in S_* \cup \{\lambda\}$, the forcing $\mathbb{Q}^0_\delta$ has the following properties:*

(1) *The whole tree $\boldsymbol{T}_{<\delta} \in \mathbb{Q}^0_\delta$ and is weaker than any other condition in the forcing $\mathbb{Q}^0_\delta$.*
(2) *If $p \in \mathbb{Q}^0_\delta$ and $\eta \in p$, then $p^{[\eta]} \in \mathbb{Q}^0_\delta$ and $p \leq_{\mathbb{Q}^0_\delta} p^{[\eta]}$.*
(3) *Let $\epsilon < \delta$; then the set $\{(\boldsymbol{T}_{<\delta})^{[\eta]} : \eta \in \boldsymbol{T}_\epsilon\}$ is a maximal antichain of the forcing $\mathbb{Q}^0_\delta$.*
(4) *Let $p \in \mathbb{Q}^0_\delta$ and $\epsilon < \delta$; then $\{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\epsilon\}$ is a maximal antichain above p (if $\epsilon \leq \lg(tr(p))$, $p = p^{[\eta]}$ and this set is a singleton).*

Proof. Let $\delta \in S_* \cup \{\lambda\}$:

(1) Trivial.
(2) Trivial.

(3) Let $\epsilon < \delta$; the set is an antichain since for any $\eta \neq \nu \in \boldsymbol{T}_\epsilon$, clearly $(\boldsymbol{T}_{<\delta})^{[\eta]}$ and $(\boldsymbol{T}_{<\delta})^{[\nu]}$ are not compatible. Let $p \in \mathbb{Q}^0_\delta$ and let $\eta \in p \cap \boldsymbol{T}_\epsilon$ be a node. There exists such a node recalling clauses (1)(a) and (1)(b) of Definition 15. Then $p^{[\eta]} \in \mathbb{Q}^0_\delta$ by the previous clause; $p \leq_{\mathbb{Q}^0_\delta} p^{[\eta]}$ and clearly $(\boldsymbol{T}_{<\delta})^{[\eta]} \leq_{\mathbb{Q}^0_\delta} p^{[\eta]}$; thus $\{(\boldsymbol{T}_{<\delta})^{[\eta]} : \eta \in \boldsymbol{T}_\epsilon\}$ is a maximal antichain in $\mathbb{Q}^0_\delta$.

(4) Similar. ∎

Next, define a structure that will fulfill the roll of using the diamond principle; the structure will be a collection of objects that contain elements that are antichains with additional properties. In the weakly compact case [6] there was an important role for the maximal antichains; in the proof of the λ-bounding of the forcing there was a maximal antichain that reflected to an antichain in the forcing corresponding to a smaller cardinal.

In the inaccessible case with which we are dealing here, we will have to use diamond to gain a similar property. Each element is an antichain in the forcing $\mathbb{Q}^0_\delta$.

Definition 19: For any ordinal $\delta \in S_* \cup \{\lambda\}$, Ξ_δ will be the collection of objects $\bar{q}$, for which the following conditions hold:

(1) $\bar{q} = \langle q_\eta : \eta \in \Lambda \rangle$,

(2) $\Lambda \subseteq \boldsymbol{T}_{<\delta}$,

(3) for each $\eta \in \Lambda$, $q_\eta \in \mathbb{Q}^0_\delta$ and $\eta = tr(q_\eta)$,

(4) if $\eta, \nu \in \Lambda$ and $\eta \neq \nu$ then $\eta = tr(q_\eta) \notin q_\nu \vee \nu = tr(q_\nu) \notin q_\eta$,

(5) the union of all the conditions from the set will be an element in the forcing $r^*_{\bar{q}} = \{\rho \in \boldsymbol{T}_{<\delta} : (\exists \eta \in \Lambda)(\rho \in q_\eta)\} \in \mathbb{Q}^0_\delta$.

Definition 20: For all $\delta \in S_* \cup \{\lambda\}$ and $\bar{q} \in \Xi_\delta$, we define the **coder** $X_{\bar{q}}$:

$$X_{\bar{q}} = \{(\eta, \nu) : (\eta \in \Lambda) \wedge (\nu \in q_\eta)\} \subseteq \mathscr{H}(\delta).$$

Definition 21: Let $\delta \in S_*$; we call δ **weakly successful** when there exists $\bar{q} \in \Xi_\delta$ with $X_{\bar{q}} = X_\delta$, recalling X_δ is from the good structure $\mathfrak{r}$ defined in clause (4) of Definition 10.

Claim 22: *For a weakly successful $\delta \in S_*$, the $\bar{q}$ of Definition 21 is unique.*

Proof. Observe that the coder $X_{\bar{q}}$ has all the information on $\bar{q}$, therefore such a $\bar{q}$ must be unique. ∎

Definition 23: (1) For a weakly successful $\delta \in S_*$:

- look at the unique sequence $\bar{q} = \langle q_\eta : \eta \in \Lambda \rangle$ for which $X_{\bar{q}} = X_\delta$ and write $\Lambda^*_\delta = \Lambda$; for all $\eta \in \Lambda^*_\delta$ let $q^*_{\delta,\eta} = q_\eta$ and lastly

$$\bar{q}^*_\delta = \langle q^*_{\delta,\eta} : \eta \in \Lambda^*_\delta \rangle,$$

- let $r^*_\delta = r^*_{\bar{q}^*_\delta} = \{\rho \in \boldsymbol{T}_{<\delta} : (\exists \nu \in \Lambda^*_\delta)(\rho \in q^*_{\delta,\nu})\}$,
- finally, for each $\eta \in \Lambda^*_\delta$ and $\nu \in \boldsymbol{T}_{<\delta}$ with $\eta \trianglelefteq \nu \in q^*_{\delta,\eta}$, let $q^*_{\delta,\nu} = (q^*_{\delta,\eta})^{[\nu]}$.

(2) For $\delta \in S_*$ which is not weakly successful, let $r^*_\delta = \boldsymbol{T}_{<\delta}$, and for all $\eta \in r^*_\delta$, $q^*_{\delta,\eta} = (r^*_\delta)^{[\eta]}$.

Now we can use the $\bar{X}$, being a diamond sequence:

Claim 24: *For every $\bar{q} = \langle q_\eta : \eta \in \Lambda \rangle \in \Xi_\lambda$ there is a stationary set of weakly successful $\delta \in S_*$ for which*

$$\bar{q}^*_\delta = \langle q_\eta \cap \boldsymbol{T}_{<\delta} : \eta \in \Lambda \cap \boldsymbol{T}_{<\delta} \rangle.$$

Proof. Recall that $\bar{X}$ is a diamond sequence, therefore for the set $X_{\bar{q}}$, there is a stationary set of $\delta \in S_*$ for which $X_\delta = X_{\bar{q}} \cap \mathscr{H}(\delta) = X_{\bar{q}} \cap (\boldsymbol{T}_{<\delta} \times \boldsymbol{T}_{<\delta})$, from the definition of the coder, and as X_δ is the coder of $\bar{q}^*_\delta$ the conclusion follows: $\bar{q}^*_\delta = \langle q_\eta \cap \boldsymbol{T}_{<\delta} : \eta \in \Lambda \cap \boldsymbol{T}_{<\delta} \rangle$, where $\bar{q}^*_\delta \in \Xi_\delta$ witness that δ is weakly successful. ∎

2.2. Defining the Main Forcing.

Remark 25: Below, the main forcing will be defined, however prior to the definition we would like to state the properties that this forcing is expected to have; this remark is meant to describe the general structure of the forcings $\mathbb{Q}'_\delta$ and $\mathbb{Q}_\delta$ for each $\delta \in S_* \cup \{\lambda\}$.

(1) We would like those forcings to be subforcings of $\mathbb{Q}^0_\delta$ (but not necessarily complete subforcings), where $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$.
(2) For a condition $p \in \mathbb{Q}_\delta$ and a node $\eta \in p$, we have $p^{[\eta]} \in \mathbb{Q}_\delta$; the same holds for $\mathbb{Q}'_\delta$.
(3) The complete tree $\boldsymbol{T}_{<\delta}$ belongs to $\mathbb{Q}_\delta$; so in particular it belongs to $\mathbb{Q}'_\delta$.

Remark 26: We are now ready to finally define the desired forcings; the following is the main definition of the forcing. Pedantically, the induction in Definition 27 should be carried together with the proof of Claim 29.

Definition 27: The definition is inductive on δ; we will define the subforcings of $\mathbb{Q}^0_\delta$: $\mathbb{Q}_\delta$ and $\mathbb{Q}'_\delta$ for all $\delta \in S_* \cup \{\lambda\}$; in addition we will define the term successful for ordinals $\delta \in S_*$, and for each $\eta \in \boldsymbol{T}_{<\delta}$ and a tenuous $S \subseteq S_* \cap \delta$ we will define $p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$, so we have to verify this, see Claim 29 below.

(1) $\forall p \in \mathbb{Q}^0_\delta, \quad p \in \mathbb{Q}'_\delta$ iff

$$\forall \delta_1 \in \delta \cap S_*,\ \lg(tr(p)) < \delta_1 \Rightarrow p \restriction \delta_1 \in \mathbb{Q}_{\delta_1},$$

so the forcing $\mathbb{Q}'_\delta$ is derived from the forcings $\mathbb{Q}_{\delta_1}$ for $\delta_1 \in \delta \cap S_*$.

(2) $\forall p \in \mathbb{Q}^0_\delta, \quad p \in \mathbb{Q}_\delta$ iff $p = p^*_{\eta,\delta,S}$ for some η, S satisfying $\eta \in \boldsymbol{T}_{<\delta}$, $S \subseteq S_* \cap \delta$ is tenuous; such $p^*_{\eta,\delta,S}$ is defined below in clause (4).

(3) We call $\delta < \lambda$ **successful** when it is weakly successful and in addition $r^*_\delta, q^*_{\delta,\eta} \in \mathbb{Q}'_\delta$ for all $\eta \in \Lambda^*_\delta$.
Explanation: The successful ordinals represent the levels in which there will be a special pruning, determined by the diamond condition, so there is a "control" on the conditions defined uniquely, and in relation to $r^*_{\delta_1}$ of the corresponding levels δ_1.

(4) We assume that the forcing notions $\mathbb{Q}'_{\delta'}, \mathbb{Q}_{\delta'}$ are defined for all $\delta' \in S_* \cap \delta$; the condition $p^*_{\eta',\delta',S'}$ is defined for all $\eta' \in \boldsymbol{T}_{<\delta'}$ and tenuous $S' \subseteq S_* \cap \delta'$; we shall define $p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$ for η, S as above in the following way:
   (a) If $\sup(S) \leq \lg(\eta)$ then
   $$p^*_{\eta,\delta,S} = \boldsymbol{T}^{[\eta]}_{<\delta}.$$
   (b) If $\sup(S) > \lg(\eta)$ and S has no maximal element, then for each $\nu \in \boldsymbol{T}_{<\delta}$, $\nu \in p^*_{\eta,\delta,S}$ if and only if one of the following conditions holds:
      (i) $\nu \trianglelefteq \eta$,
      (ii) $\eta \triangleleft \nu$ and there exists $\lg(\nu) < \delta_1 \in S$ such that $\nu \in p^*_{\eta,\delta_1,S\cap\delta_1}$,
      (iii) $\eta \triangleleft \nu$, $\lg(\nu) \geq \sup(S)$ and for all $\delta_1 \in S \setminus (\lg(\eta)+1)$ and $\zeta < \delta_1$ we have $\nu \restriction \zeta \in p^*_{\eta,\delta_1,S\cap\delta_1}$.
   (c) If $\sup(S) > \lg(\eta)$ and S has a last element $\delta_1 < \delta$, such that δ_1 is not successful, then for each $\nu \in \boldsymbol{T}_{<\delta}$ we have $\nu \in p^*_{\eta,\delta,S}$ if and only if one of the followings holds:
      (i) $\lg(\nu) < \delta_1 \wedge \nu \in p^*_{\eta,\delta_1,S\cap\delta_1}$,
      (ii) $\lg(\nu) \geq \delta_1 \wedge \nu \restriction \delta_1 \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$ recalling Definition 23(2).
   (d) If $\sup(S) > \lg(\eta)$ and S has a last element $\delta_1 < \delta$, such that δ_1 is successful, then for each $\nu \in \boldsymbol{T}_{<\delta}$ we have $\nu \in p^*_{\eta,\delta,S}$ if and only if one of the following holds:

(i) $\lg(\nu) < \delta_1 \wedge \nu \in p^*_{\eta,\delta_1,S\cap\delta_1}$,
(ii) $\lg(\nu) > \delta_1 \wedge \nu \restriction \delta_1 \in p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1}$ according to the definition of $p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1}$ in the following clause:
(iii) On the level of the last element of S, the process is more interesting; for $\lg(\nu) = \delta_1$:
  (A) if $\nu \notin \lim_{\delta_1}(r^*_{\delta_1})$ then
$$\nu \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1}),$$
  (B) if $\nu \in \lim_{\delta_1}(r^*_{\delta_1})$ then
$$\nu \in \Big(\bigcup\{\lim_{\delta_1}(q^*_{\delta_1,\eta'}) : \eta' \in \Lambda^*_{\delta_1}\}\Big) \cap \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1}).$$

Explanation:

(0) Why do we arrive at this definition? In [6], we start with $p = (\boldsymbol{T}_\lambda)^{[\eta]}$, as in the λ-Cohen forcing, but add the following pruning: for each condition for some tenuous subset $S \subseteq S_*$ consisting of inaccessible cardinals, for each $\delta \in S$ we have a set Λ_δ of $\leq \delta$ maximal anti-chains of $\mathbb{Q}_\delta$. The pruning is: for each $\delta \in S$ we omit the η of level δ avoiding at least one of those maximal antichains (and all proper initial segments of which are in the condition). How is this useful in proving the ${}^\lambda\lambda$ case? In [6] it is done by having reflection of the property of being a maximal antichain; this naturally requires reflecting "a set of conditions is a maximal antichain in the forcing". Naturally each maximal antichain in Λ_δ comes from starting with a maximal antichain in the whole forcing representing a name of an ordinal $< \lambda$, and demanding that a condition from the maximal antichain with trunk of length $< \delta$ will be in the generic sets; i.e., our condition forces this. How can we do this without the weak compactness assumption? By the diamond on S_*, we guess a "poor man maximal antichain", those are the $\bar{q}^*_\delta$-s. They look like an approximation to a maximal antichain inside r^*_δ, but usually are far from being a maximal antichain. So we intend to make them a maximal antichain above r^*_δ by "decree"; or you may say by definition. This of course changes the proof in several ways.

(1) For the level $\delta \in S_* \cup \{\lambda\}$, the idea is that the main forcing $\mathbb{Q}_\delta$ is a full tree that is only being pruned in levels that are in the matching tenuous set; the idea is to have enough "thickness" to achieve the required completeness and more. In addition, the conditions are unique relative

to the tenuous set and the trunk, and by their definition closely related to the diamond sequence, intuitively, that is needed for the forcing to be bounding.

(2) Note that we gave a definition of $p^*_{\eta,\delta,S}$ in Definition 27(4); it will be defined as a subset of $\boldsymbol{T}_{<\delta}$, but it is really a member of $\mathbb{Q}^0_\delta$ as is proved in Claim 29 Below.

Definition 28: For $\delta \in S_* \cup \{\lambda\}$, define $\underset{\sim}{\eta}_\delta$ to be a $\mathbb{Q}_\delta$-name:

$$\underset{\sim}{\eta} = \bigcup \{tr(p) : p \in \underset{\sim}{G}_{\mathbb{Q}_\delta}\}$$

where $\underset{\sim}{G}_{\mathbb{Q}_\delta}$ is a canonical $\mathbb{Q}_\delta$- name for the generic filter. If δ is clear from the context; we may write $\underset{\sim}{\eta}$ instead of $\underset{\sim}{\eta}_\delta$.

Claim 29: *For all $\delta \in S_* \cup \{\lambda\}$, $\eta \in \boldsymbol{T}_{<\delta}$ and tenuous $S \subseteq S_* \cap \delta$, if $p = p^*_{\eta,\delta,S}$ then:*

(1) *If $\delta_0 \in S_* \cap \delta$ is such that $\eta \in \boldsymbol{T}_{<\delta_0}$ then $p^*_{\eta,\delta,S} \restriction \delta_0 = p^*_{\eta,\delta_0,S\cap\delta_0}$.*
(2) *η is the trunk of $p^*_{\eta,\delta,S}$.*
(3) *$p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$, moreover $p^*_{\eta,\delta,S} \in \mathbb{Q}'_\delta$.*
(4) *The tenuous set S contains the set of pruning levels corresponding to the condition $p = p^*_{\eta,\delta,S}$, that is $S_p \subseteq S$.*
(5) *In addition, $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$ and $\mathbb{Q}_\lambda = \mathbb{Q}'_\lambda$.*

Proof. We prove all statements of the claim by simultaneous induction on the ordinals $\delta \in S_* \cup \{\lambda\}$, assume that the claim is true for $\mathbb{Q}_{\delta'}$, so for all conditions $p^*_{\eta',\delta',S'}$ where $\delta' \in \delta \cap S_*$, $\eta' \in \boldsymbol{T}_{<\delta'}$ and $S' \subseteq S_* \cap \delta'$; we will now prove it for $p = p^*_{\eta,\delta,S}$ where $\eta \in \boldsymbol{T}_{<\delta}$ and $S \subseteq S_* \cap \delta$:

(1) Assume that $\delta_0 \in S_* \cap \delta$ is such that $\eta \in \boldsymbol{T}_{<\delta_0}$.

(a) First, for $\delta_0 \in S$, look at the different cases in the definition of $p^*_{\eta,\delta,S}$:

(i) For case (4)(a),

$$p^*_{\eta,\delta,S} \restriction \delta_0 = ((\boldsymbol{T}_{<\delta})^{[\eta]}) \restriction \delta_0 = (\boldsymbol{T}_{<\delta_0})^{[\eta]} = p^*_{\eta,\delta_0,S\cap\delta_0}.$$

(ii) For case (4)(b), recalling $\lg(\eta) < \delta_0$, the initial segments of η are clearly both in $p^*_{\eta,\delta,S} \restriction \delta_0$ and in $p^*_{\eta,\delta_0,S\cap\delta_0}$; for all $\eta \trianglelefteq \nu \in \boldsymbol{T}_{<\delta_0}$ we have by clause (4)(b)(ii) of the definition that

$$\nu \in p^*_{\eta,\delta,S} \Longleftrightarrow \nu \in p^*_{\eta,\delta_0,S\cap\delta_0}.$$

(iii) For cases (4)(c) and (4)(d), for each $\nu \in \boldsymbol{T}_{<\delta_0}$ the relevant clauses are (i) of (4)(c) and (i) of (4)(d). Those clauses trivially imply $\nu \in p^*_{\eta,\delta,S} \Longleftrightarrow \nu \in p^*_{\eta,\delta_0,S\cap,\delta_0}$.

(b) Next, take any $\delta_0 \in S_* \cap \delta \setminus S$.
   (i) For $\delta_0 < \sup(S)$, there is $\delta_0 < \delta' \in S$; by the induction hypothesis $p^*_{\eta,\delta',S\cap\delta'} \restriction \delta_0 = p^*_{\eta,\delta_0,S\cap\delta_0}$. By clause (1)(a) $p^*_{\eta,\delta,S} \restriction \delta' = p^*_{\eta,\delta',S\cap\delta'}$, thus $p^*_{\eta,\delta,S} \restriction \delta_0 = p^*_{\eta,\delta_0,S\cap\delta_0}$ follows.
   (ii) For $\delta_0 \geq \sup(S)$, hence $\forall\delta' \in S, \delta_0 > \delta'$ (else it is in the case of the previous clause), clearly $p^*_{\eta,\delta,S} \restriction \delta_0 \subseteq p^*_{\eta,\delta_0,S\cap\delta_0}$; to prove the other inclusion let $\nu \in p^*_{\eta,\delta_0,S\cap\delta_0}$.
      (A) If for some $\delta' \in S \cap \delta_0$, $\lg(\nu) < \delta'$, then by the induction assumption $\nu \in p^*_{\eta,\delta',S\cap\delta'}$, which by the previous clause implies $\nu \in p^*_{\eta,\delta,S}$.
      (B) If for all $\delta' \in S \cap \delta_0$, $\lg(\nu) \geq \delta'$ and $\delta_1 = \lg(\nu)$ is the last element of S, in both relevant cases of the definition ((4)(c), (4)(d)) the level δ_1 of the condition is determined by the previous levels of S and by whether or not it is successful, so $p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1} = p^*_{\eta,\delta_0,S} \cap \boldsymbol{T}_{\delta_1}$, in particular $\nu \in p^*_{\eta,\delta,S}$.
      (C) If for all $\delta' \in S \cap \delta_0$ we have $\lg(\nu) \geq \delta'$ and $\lg(\nu) \notin S$, then the level $\lg(\nu)$ is determined entirely by the restrictions to previous levels, so we are done.

(2) For all $\nu \in p^*_{\eta,\delta.S}$, first we will show that $\nu \trianglelefteq \eta$ or $\eta \triangleleft \nu$. The proof splits into cases according to the cases in Definition 27(4):

(a) For case (4)(a) it is clear.
(b) For case (4)(b) it is also clear from the definition and the induction hypothesis.
(c) For case (4)(c):
   (i) For (4)(c)(i) by the induction hypothesis.
   (ii) For (4)(c)(ii) also by the induction hypothesis—each such ν has $\eta \triangleleft \nu$.
(d) For case (4)(d):
   (i) For ν chosen in clause (4)(d)(i) we have $\eta \triangleleft \nu$ or $\nu \trianglelefteq \eta$ by the induction hypothesis.
   (ii) For ν chosen in clause (4)(c)(ii) we have $\eta \triangleleft \nu$, again using the induction hypothesis.
   (iii) For a node ν chosen in clause (4)(c)(iii), since $\nu \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$ and by the induction hypothesis, $\eta \trianglelefteq \nu$.

Second, it remains to prove that η is the maximal node for which each other branch is an extension or an initial segment of it.

In case (4)(a) it is clear; in cases (4)(b) and (4)(c) it follows from the induction hypothesis; the node η is the trunk of the condition $p^*_{\eta,\delta_1,S\cap\delta_1}$ for each $\delta_1 \in \delta\cap S_*$ and so it has θ_ϵ extensions to the level of the trunk $+1$; those extensions will be in the new condition $p^*_{\eta,\delta,S}$, thus η will be a trunk there as well. For case (4)(d) recall that δ_1 is a limit cardinal $> \lg(\eta)$. We can use the induction hypothesis again observing that before the δ_1-th level there are no new prunings that didn't exist in $p^*_{\eta,\delta_1,S\cap\delta_1}$, i.e., $p^*_{\eta,\delta,S} \restriction \delta_1 = p^*_{\eta,\delta_1,S\cap\delta_1}$, therefore if there were a different trunk containing η, it would have been a trunk of $p^*_{\eta,\delta_1,S\cap\delta_1}$ as well—a contradiction.

(3) Using induction, first we will show that $p^*_{\eta,\delta,S} \in \mathbb{Q}^0_\delta$, checking the clauses in Definition 15:

(a) For clause (1)(a), $p^*_{\eta,\delta,S}$ is a tree (follows directly from the induction hypothesis) and it has a trunk η by part (2) of this claim.

(b) To show clause (1)(b), let $\nu \in p^*_{\eta,\delta,S}$. Assume $\eta \trianglelefteq \nu$ (the case of $\nu \triangleleft \eta$ follows from the case $\nu = \eta$) to show that there is an extension of ν to the level δ:

  (i) In case (4)(a) of Definition 27, let $\nu \trianglelefteq \nu' \in \boldsymbol{T}_\delta$; then also $\eta \trianglelefteq \nu'$ thus $\nu' \in \lim_\delta(p^*_{\eta,\delta,S})$.

  (ii) In case (4)(b) of Definition 27, as S is tenuous with no last element, if $\lg(\nu) < \sup(S)$ there is a closed unbounded subset C of $\sup(S)$ with $\min(C) > \lg(\nu)$, such that $\alpha \in C \Rightarrow [\alpha = \sup(C \cap \alpha) \Leftrightarrow \alpha \notin S]$. Let $\langle \alpha_i : i < \zeta \rangle$ list C in increasing order.
  We choose $\nu_i \in p^*_{\eta,\alpha_{i+1},S\cap\alpha_{i+1}}$, $\triangleleft$-increasing, $\nu \triangleleft \nu_i$ and $\lg(\nu_i) = \alpha_i$; this is easy and let $\varrho = \bigcup_{i<\zeta} \nu_i$. Now if $\lg(\varrho) = \sup(S) = \delta$, then $\varrho \in \lim_\delta(p^*_{\eta,\delta,S})$ and we are done. So assume $\lg(\varrho) = \sup(S) < \delta$, clearly $\varrho \in p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\lg(\varrho)}$ hence $(p^*_{\eta,\delta,S})^{[\varrho]} = (\boldsymbol{T}_{<\delta})^{[\varrho]}$ so the derived conclusion is clear. Finally, if $\lg(\nu) \geq \sup(S)$ then every $\nu \trianglelefteq \nu'$ with $\lg(\nu') = \delta$ has the property that for all $\delta_1 \in S \setminus (\lg(\eta) + 1)$, $\zeta < \delta_1$, we have $\nu' \restriction \zeta = \nu \restriction \zeta \in p^*_{\eta,\delta_1,S\cap\delta_1}$ and so $\nu' \in \lim_\delta(p^*_{\eta,\delta,S})$.

  (iii) In case (4)(c) of definition 27, $\delta_1 = \max(S)$. First assume that $\nu \in p^*_{\eta,\delta,S}$ satisfies $\lg(\nu) < \max(S) = \delta_1$. Then $\nu \in p^*_{\eta,\delta_1,S\cap\delta_1}$ and, by the induction hypothesis, there is some $\nu' \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$ with $\nu \trianglelefteq \nu'$; by the definition we also have that $\nu' \in p^*_{\eta,\delta,S}$. Thus, it is left to prove the claim for any $\nu \in p^*_{\eta,\delta,S}$ such that $\lg(\nu) \geq \delta_1$; clearly for any extension $\nu \trianglelefteq \nu' \in \lim_\delta(p^*_{\eta,\delta,S})$, $\nu' \restriction \delta_1 \in p^*_{\eta,\delta,S}$ and so $\nu' \restriction \xi \in p^*_{\eta,\delta,S}$ for all $\delta_1 \leq \xi < \delta$.

(iv) In case (4)(d) of Definition 27, $\nu \in p^*_{\eta,\delta,S}$, $\delta_1 = \max(S)$ and δ_1 is successful; let $\beta = \lg(\nu)$:

(A) First assume $\beta < \delta_1$. By the induction hypothesis there is a node $\nu \trianglelefteq \nu'$, $\nu' \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$. Now the proof splits into cases:
Case 1: If $\nu' \notin \lim_{\delta_1}(r^*_{\delta_1})$, then $\nu' \in p^*_{\eta,\delta,S}$, hence we have reduced the problem to the case $\beta = \delta_1$ dealt with below.
Case 2: If $\nu' \in \lim_{\delta_1}(r^*_{\delta_1})$ we still know that $\eta \in \nu'$, hence $\eta \in r^*_{\delta_1}$, hence for some $\varrho \in \Lambda^*_{\delta_1}$ we have $\eta \in q^*_{\delta_1,\varrho}$, so there is $\nu'' \in \lim_{\delta_1}(q^*_{\delta_1,\eta})$ and we continue with $\beta = \delta_1$ below.

(B) Second, assume $\beta \geq \delta_1$. Now every possible extension is being chosen after the level of height δ_1, so by the previous clause certainly there is an element in the β level by clause (4)(c)(ii).

(c) In successor levels all the extensions are taken, as defined in $\mathbb{Q}^0_\delta$.

(d) The set S is tenuous and $S_p \subseteq S$ by the next clause, so S_p (the set of the levels with the prunes) is also tenuous.

Now we can see that $p^*_{\eta,\delta,S} \in \mathbb{Q}'_\delta$:

- Let δ' be $\lg(tr(p)) < \delta' \in S_*$; observe that in all the cases of the definition, $p^*_{\eta,\delta,S} \restriction \delta' = p^*_{\eta,\delta',S\cap\delta'} \in \mathbb{Q}_{\delta'}$ and so we are done.

(4) Looking at the definition, case (4)(a) is trivial. For case (4)(b) we will have that

$$S_{p^*_{\eta,\delta,S}} = \bigcap_{\delta' \in S} S_{p^*_{\eta,\delta',S\cap\delta'}}$$

so by induction $S_{p^*_{\eta,\delta,S}} \subseteq S$. In case (4)(c), $S_{p^*_{\eta,\delta.S}} = S_{p^*_{\eta,\delta_1,S\cap\delta_1}}$, and in case (4)(d), $S_{p^*_{\eta,\delta.S}} = S_{p^*_{\eta,\delta_1,S\cap\delta_1}}$ or $S_{p^*_{\eta,\delta.S}} = S_{p^*_{\eta,\delta_1,S\cap\delta_1}} \cup \{\delta_1\}$. Using the induction hypothesis and as $\delta_1 \in S$, we are done.

(5) Reading Definition 27, clearly $\mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$ and $\mathbb{Q}_\delta \subseteq \mathbb{Q}'_\delta$ follows by clause (3) because

$$\mathbb{Q}_\delta = \{p^*_{\eta,\delta,S} : \eta \in \boldsymbol{T}_{<\delta} \text{ and } S = S_* \cap \delta \text{ is tenuous}\},$$

so caluse (5) follows by clause (3). So clause (5) indeed holds.

To show $\mathbb{Q}_\lambda = \mathbb{Q}'_\lambda$, assume by contradiction that there is $p \in \mathbb{Q}'_\lambda \setminus \mathbb{Q}_\lambda$, so for all $\delta \in S_*$, $p \restriction \delta \in \mathbb{Q}_\delta$. Let $S = \bigcup_\delta \in S_* S_{p\restriction\delta}$. If S has a last element, then for some $\delta_* \in S_*$, $S = S_{p\restriction\delta_*}$ and so

$$p = \{\nu \in \boldsymbol{T}_{<\lambda} : \nu \in p \restriction \delta_* \vee \nu \restriction \delta_* \in \lim_{\delta_*}(p)\},$$

as $\max(S) < \delta$, and by clauses (c) and (d) of Definition 27(4), $p \in \mathbb{Q}_\lambda$ follows. Otherwise S has no last element; $\nu \in p$ iff for each $\delta \in S$ either $\nu \in p \restriction \delta$ or $\lg(\nu) \geq \delta$ and $\forall \zeta < \delta$, $\nu \restriction \zeta \in p \restriction \delta$, and so by clause (b) of Definition 27(4), $p \in \mathbb{Q}_\lambda$. ∎

Claim 30: *Let $\delta \in S_* \cup \{\lambda\}$.*

(1) *Let $p, q \in \mathbb{Q}^0_\delta$; if p, q are compatible then $p \cap q \in \mathbb{Q}^0_\delta$.*
(2) *Let $p, q \in \mathbb{Q}'_\delta$; then p, q are compatible if and only if $p \cap q \in \mathbb{Q}'_\delta$.*
(3) *Let $p, q \in \mathbb{Q}_\delta$; then p, q are compatible if and only if $p \cap q \in \mathbb{Q}_\delta$.*
(4) *Let $p, q \in \mathbb{Q}'_\delta$; then p, q are compatible if and only if $tr(p) \in q \wedge tr(q) \in p$.*
(5) *Let $p, q \in \mathbb{Q}_\delta$; then p, q are compatible if and only if $tr(p) \in q \wedge tr(q) \in p$.*

Proof. In fact we have already seen the existence of most of the statements in this claim. Observe:

(1) If p and q are compatible, let $r \in \mathbb{Q}^0_\delta$ be such that $r \subseteq p, q$; then $tr(p), tr(q) \trianglelefteq tr(r)$. Now, $r \subseteq p \cap q$; assume wlog $tr(p) \triangleleft tr(q) = \eta$; then η will be the trunk of $p \cap q$. For each $\eta \in p \cap q$, the sets $\{\nu \in \lim_\delta(p) : \eta \triangleleft \nu\}$ and $\{\nu \in \lim_\delta(q) : \eta \triangleleft \nu\}$ must have a non-empty intersection as S_p, S_q are tenuous. For all $\eta \in p \cap q$,

$$\{j \in \theta_{\lg(\eta)} : \eta^\frown\langle j\rangle \in p\} = \{j \in \theta_{\lg(\eta)} : \eta^\frown\langle j\rangle \in q\} = \theta_{\lg(\eta)}$$

and so $\{j \in \theta_{\lg(\eta)} : \eta^\frown\langle j\rangle \in p \cap q\}$. Finally, as S_p, S_q are tenuous, so is $S_{p\cap q} \subseteq S_p \cup S_q$ (by Claim 9), thus, $p \cap q \in \mathbb{Q}^0_\delta$.

(2) This clause and the following are shown by simultaneous induction: considering the forcing $\mathbb{Q}'_\delta$, if p and q are compatible there exists a condition $r \in \mathbb{Q}'_\delta$: $r \subseteq p, q$, thus

$$tr(p),\ tr(q) \trianglelefteq tr(r).$$

Let $\delta_1 \in \delta \cap S_*, \lg(tr(p)) < \delta_1$, as $p \restriction \delta_1, q \restriction \delta_1, r \restriction \delta_1 \in \mathbb{Q}_{\delta_1}$ and $r \restriction \delta_1 \subseteq p \restriction \delta_1, q \restriction \delta_1$ and the following clause's induction assumption. We conclude that $p \cap q \restriction \delta_1 \in \mathbb{Q}_{\delta_1}$ and $p \cap q \in \mathbb{Q}'_\delta$ follows; the other direction is trivial.

(3) We use induction. Considering the forcing $\mathbb{Q}_\delta$, if p and q are compatible there exists a condition $r \in \mathbb{Q}_\delta$: $r \subseteq p, q$, thus $tr(p), tr(q) \trianglelefteq tr(r)$. Assume wlog $tr(p) \triangleleft tr(q) = \eta$ and let $S = S_p \cup S_q$. For $\nu \in \boldsymbol{T}_{<\delta'}$ with $\delta' \in S$, $\nu \in p \cap q \iff \nu \in \lim_{\delta'}(p \restriction \delta')$ and $\nu \in \lim_{\delta'}(q \restriction \delta')$, which by the induction hypothesis implies $\nu \in \lim_{\delta'}(p \cap q \restriction \delta')$. In addition, one of the following holds:
   (a) δ' is not successful,
   (b) δ' is successful and $\nu \notin \lim_{\delta'}(r^*_{\delta'})$,
   (c) δ' is successful and $\nu \in \lim_{\delta'}(r^*_{\delta'}) \cap (\bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\})$.

   There are no additional prunings, therefore $p \cap q = p^*_{\eta,\delta,S}$.

(4) This clause and the next one are shown by simultaneous induction on δ. Consider the forcing $\mathbb{Q}'_\delta$.
   - For the first direction, assume p and q are compatible; thus there exists a condition $r \in \mathbb{Q}'_\delta$: $r \subseteq p, q$. In particular, $tr(p), tr(q) \trianglelefteq tr(r)$, thus $tr(p), tr(q) \in r \subseteq p \cap q$.
   - For the other direction, assume $tr(p) \in q \wedge tr(q) \in p$; let $r = p \cap q$ and by the previous clause $r \in \mathbb{Q}^0_\delta$. In particular,

$$\lg(tr(p)),\ \lg(tr(q)) < \delta_1$$

     and since $p, q \in \mathbb{Q}'_\delta$ it implies that $p \restriction \delta_1, q \restriction \delta_1 \in \mathbb{Q}_{\delta_1}$. We can use the induction hypothesis to conclude that

$$(p \restriction \delta_1) \cap (q \restriction \delta_1) = r \restriction \delta_1 \in \mathbb{Q}_{\delta_1};$$

     therefore indeed $r \in \mathbb{Q}'_\delta$.

(5) Considering the forcing $\mathbb{Q}_\delta$, assume it holds for $\mathbb{Q}_{\delta_1}$ with $\delta_1 < \delta$:
   - For the first direction, assume that p and q are compatible; thus there exists $r \in \mathbb{Q}_\delta$: $r \subseteq p, q$. In particular, $tr(p), tr(q) \trianglelefteq tr(r)$ thus $tr(p), tr(q) \in r \subseteq p \cap q$.
   - For the other direction, assume $tr(p) \in q \wedge tr(q) \in p$ and remember that for some nodes $\eta_1, \eta_2 \in \boldsymbol{T}_{<\delta}$ and tenuous sets $S_1, S_2 \subseteq S_* \cap \delta$, the conditions are in fact

$$p = p^*_{\eta_1,\delta,S_1}, \quad q = p^*_{\eta_2,\delta,S_2}.$$

     Recall the assumption and assume by symmetry that $\eta_1 \trianglelefteq \eta_2$. Let $S = S_1 \cup S_2$ and we will show that $p^*_{\eta_2,\delta,S} \subseteq p \cap q$; this is indeed a condition in the forcing $\mathbb{Q}_\delta$, looking at Definition 27. Let $\nu \in p^*_{\eta_2,\delta,S}$; the possibilities by clause (4) are:

— If S has no last element:

* If $\nu \trianglelefteq \eta_2$ then $\nu \in q$; as $\eta_2 \in p$ it follows that $\nu \in p \cap q$.

* If for some $\delta_1 \in S$, $\nu \in p^*_{\eta_2,\delta_1,S\cap\delta_1}$, then by the induction assumption $p^*_{\eta_2,\delta_1,S\cap\delta_1} \subseteq p \cap q \cap \boldsymbol{T}_{<\delta_1}$ so $\nu \in p \cap q$.

* If $\forall\delta_1 \in S \forall\zeta < \delta_1 : \nu \restriction \zeta \in p^*_{\eta_2,\delta_1,S\cap\delta_1}$, by the induction assumption $\forall\delta_1 \in S\forall\zeta < \delta_1 : \nu \restriction \zeta \in p \cap q$. If S_1 or S_2 had a last element, it would be below $\sup(S)$ and in all the construction possibilities it can be seen that this implies $\nu \in p$ and $\nu \in q$.

— If S has a last element δ_1, which is not successful:

* If $\nu \in p^*_{\eta_2,\delta_1,S\cap\delta_1}$, then by the we have induction assumption $p^*_{\eta_2,\delta_1,S\cap\delta_1} \subseteq p \cap q \cap \boldsymbol{T}_{<\delta_1}$ so $\nu \in p \cap q$.

* If $\nu \restriction \delta_1 \in \lim_{\delta_1}(p^*_{\eta_2,\delta_1,S\cap\delta_1})$, then by the induction assumption $p^*_{\eta_2,\delta_1,S\cap\delta_1} \subseteq p \cap q \cap \boldsymbol{T}_{<\delta_1}$, so $\nu \restriction \delta_1 \in p \cap q$. For each one of S_1, S_2, if it doesn't contain δ_1 then ν belongs to the matching condition (p or q), while if it does contain δ_1, as δ_1 is not successful, the matching condition, say p, will have that $\nu \restriction \delta_1 \in \lim_{\delta_1}(p) \Rightarrow \nu \in p$.

— If S has a last element δ_1, which is successful:

* If $\nu \in p^*_{\eta_2,\delta_1,S\cap\delta_1}$, then by the induction assumption we have $p^*_{\eta_2,\delta_1,S\cap\delta_1} \subseteq p \cap q \cap \boldsymbol{T}_{<\delta_1}$ so $\nu \in p \cap q$.

* If $\nu \in \lim_{\delta_1}(p^*_{\eta_2,\delta_1,S\cap\delta_1})$, then by the induction assumption $\nu \in \lim_{\delta_1}(p \cap q \cap \boldsymbol{T}_{<\delta_1})$:

  · In case $\nu \notin \lim_{\delta_1}(r^*_{\delta_1})$, for each one of p, q, if S_1 or S_2 has δ_1 as their last element, $\nu \in p$ or $\nu \in q$ accordingly. Else the corresponding S_1 or S_2 has all its elements below δ_1 and so by the possibilities in Definition 27(4), $\nu \in p \cap q$.

  · Else $\nu \in \lim_{\delta_1}(r^*_{\delta_1})$, then $(*)$ $\nu \in \bigcup\{\lim_{\delta_1}(q^*_{\delta_1,\eta'}) : \eta' \in \Lambda^*_{\delta_1}\}$; for each one of p, q, if S_1 or S_2 has δ_1 as their last element, then since $\nu \in \lim_{\delta_1}(p \cap q)$ and by $(*)$, ν in contained in the corresponding condition. Else the corresponding S_1 or S_2 has all its elements below δ_1 and so by all the possibilities in Definition 27(4), $\nu \in p \cap q$.

* If $\nu \restriction \delta_1 \in p^*_{\eta_2,\delta,S} \cap \boldsymbol{T}_{\delta_1}$, since $p^*_{\eta_2,\delta,S} \cap \boldsymbol{T}_{\delta_1} \subseteq p \cap q \cap \boldsymbol{T}_{\delta_1}$, $\nu \restriction \delta_1 \in p \cap q$ and so $\nu \in p \cap q$, for any possibility for the construction of p and q. ∎

Recall the required properties of the forcings discussed in Remark 25 ;the first was shown in Claim 29(5) and the rest are proven below:

CLAIM 31: *Let $\delta \in S_* \cup \{\lambda\}$.*

(1) *For a condition $p = p^*_{\eta,\delta,S} \in \mathbb{Q}_\delta$ and a node $\nu \in p$, we have $p \leq_{\mathbb{Q}_\delta} p^{[\nu]} \in \mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \max\{tr(p), \nu\}$. If $\eta \lhd \nu$ then also $p^{[\nu]} = p^*_{\nu,\delta,S}$ holds; the same is true for $\mathbb{Q}'_\delta \subseteq \mathbb{Q}^0_\delta$.*

(2) *We have $\boldsymbol{T}_{<\delta} \in \mathbb{Q}_\delta$ and $\boldsymbol{T}_{<\delta} \in \mathbb{Q}'_\delta$, therefore $\boldsymbol{T}_{<\delta}$ is the minimal condition of $\mathbb{Q}_\delta$ and $\mathbb{Q}'_\delta$.*

Proof. For $\delta \in S_* \cup \{\lambda\}$:

(1) Assume $p = p^*_{\eta,\delta,S} \in \mathbb{Q}_\delta$ and let $\nu \in p$.

- If $\nu \trianglelefteq \eta$ then $p^{[\nu]} = p \in \mathbb{Q}_\delta$; in particular $tr(p^{[\nu]}) = \eta$.
- Else, $\eta \lhd \nu$. In that case $p^{[\nu]} = p^*_{\nu,\delta,S}$, we will show that using induction, looking at the clauses of Definition 27(4):
  (a) If, as in case (4)(a), $p = \boldsymbol{T}^{[\eta]}_{<\delta}$, then $p^{[\nu]} = \boldsymbol{T}^{[\nu]}_{\delta}$, which is in fact $p^*_{\nu,\delta,\emptyset}$ and thus belongs to $\mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \nu$.
  (b) If, as in case (4)(b), there is a $\lg(\eta) < \delta' \in S$ with $\nu \in p^*_{\eta,\delta',S\cap\delta'}$, then by the induction hypothesis $p^{[\nu]} \cap \boldsymbol{T}_{<\delta'} \in \mathbb{Q}_{\delta'}$. In addition, $p^{[\nu]} = p^*_{\nu,\delta,S}$ and therefore belongs to $\mathbb{Q}_\delta$. However,

  $$(\forall \eta_1, \eta_2 \in \boldsymbol{T}_{<\delta})(\eta_1 \lhd \eta_2 \in p^*_{\eta_1,\delta,S} \Rightarrow p^*_{\eta_1,\delta,S} \leq_{\mathbb{Q}_\delta} p^*_{\eta_2,\delta,S})$$

  hence we are done. In the case $\lg(\nu) \geq \sup(S)$, $p^{[\nu]} = \boldsymbol{T}^{[\nu]}_{<\delta} = p^*_{\nu,\delta,S}$.
  (c) If, as in cases (4)(c) and (4)(d), S has a last element $\delta_1 < \delta$ such that $\lg(\eta) < \delta_1 \in S$, then if $\delta_1 \leq \lg(\nu)$, $p^{[\nu]} = \boldsymbol{T}^{[\nu]}_{<\delta} = p^*_{\nu,\delta,S}$.
    (i) In case (4)(c) (δ_1 is not successful):
      (A) If $\lg(\nu) < \delta_1$, then $p^{[\nu]}$ contains all the nodes of the shape $\nu' \in p^*_{\eta,\delta,S}$ such that: (1) $\nu' \lhd \nu$, (2) $\nu \trianglelefteq \nu'$ and $\lg(\nu') < \delta_1$ and $\nu' \in p^*_{\eta,\delta_1,S\cap\delta_1}$ or (3) $\nu \trianglelefteq \nu'$ and $\lg(\nu') \geq \delta_1$ and $\nu' \restriction \delta_1 \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$. By induction we have that $p^{*[\nu]}_{\eta,\delta_1,S\cap\delta_1} = p^*_{\nu,\delta_1,S\cap\delta_!}$ therefore $p^{[\nu]} = p^*_{\nu,\delta,S} \in \mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \nu$.
      (B) If $\lg(\nu) \geq \delta_1$, then $p^{[\nu]}$ contains all the nodes of the shape $\nu' \in p^*_{\eta,\delta,S}$ such that: (1) $\nu' \lhd \nu$ or (2) $\nu \trianglelefteq \nu'$ (then $\lg(\nu') \geq \delta_1$) and $\nu' \restriction \delta_1 = \nu \restriction \delta_1 \in \lim_{\delta_1}(p^*_{\eta,\delta_1,S\cap\delta_1})$, so in fact any ν' with $\nu \trianglelefteq \nu'$ is in that group. Clearly $p^{[\nu]} = p^*_{\nu,\delta,S} \in \mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \nu$.

(ii) In case (4)(d) (δ_1 is successful):

(A) If $\lg(\nu) < \delta_1$, then $p^{[\nu]}$ contains all the nodes of the shape $\nu' \in p^*_{\eta,\delta,S}$ such that (1) $\nu' \triangleleft \nu$, (2) $\nu \trianglelefteq \nu'$ and $\lg(\nu') < \delta_1$ and $\nu' \in p^*_{\eta,\delta_1,S\cap\delta_1}$, (3) $\nu \triangleleft \nu'$ and $\lg(\nu') = \delta_1$ and

$$\nu' \in \lim_{\delta_1}(p^*_{\nu,\delta_1,S\cap\delta_1}) \cap \lim_{\delta_1}(r^*_{\delta_1}) \cap \Big(\bigcup_{\eta'\in\Lambda^*_{\delta_1}} \lim_{\delta_1}(q^*_{\delta_1,\eta'})\Big)$$

$$\text{or } \nu' \in \lim_{\delta_1}(p^*_{\nu,\delta_1,S\cap\delta_1}) \setminus \lim_{\delta_1}(r^*_{\delta_1})$$

or (4) $\nu \triangleleft \nu'$, $\lg(\nu') > \delta_1$ and $\nu' \restriction \delta_1 \in p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1}$. By induction, $p^{*[\nu]}_{\eta,\delta_1,S\cap\delta_1} = p^*_{\nu,\delta_1,S\cap\delta_1}$, and since $r^*_{\delta_1}$ and $\forall\eta' \in \Lambda^*_{\delta_1} : q^*_{\delta_1,\eta'}$ do not depend on the trunk, we see that

$$p^{*[\nu]}_{\eta,\delta,S} \restriction \delta_1 = p^*_{\nu,\delta,S} \restriction \delta_1 \quad \text{and} \quad p^{*[\nu]}_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1} = p^*_{\nu,\delta,S} \cap \boldsymbol{T}_{\delta_1},$$

thus the equality follows also for $\nu \triangleleft \nu'$ such that

$$\lg(\nu') > \delta_1$$

and $p^{*[\nu]}_{\eta,\delta,S} = p^*_{\nu,\delta,S} \in \mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \nu$.

(B) If $\lg(\nu) \geq \delta_1$, then $p^{[\nu]}$ contains all the nodes of the shape $\nu' \in p^*_{\eta,\delta,S}$ such that (1) $\nu' \trianglelefteq \nu$ or (2) $\nu \triangleleft \nu'$, $\lg(\nu') > \delta_1$ and $\nu' \restriction \delta_1 \in p^*_{\eta,\delta,S} \cap \boldsymbol{T}_{\delta_1}$, which, since $\nu' \restriction \delta_1 = \nu \restriction \delta_1$, implies $p^{*[\nu]}_{\eta,\delta,S} = p^*_{\nu,\delta,S} \in \mathbb{Q}_\delta$ and $tr(p^{[\nu]}) = \nu$.

In particular, $tr(p^{[\nu]}) = \max\{\eta,\nu\}$.

We have finished showing that $\nu \in p \in \mathbb{Q}_\delta \Rightarrow p^{[\nu]} \in \mathbb{Q}_\delta$. What about $\mathbb{Q}'_\delta$?

Let $p \in \mathbb{Q}'_\delta$. Then for each $\delta' \in \delta \cap S_*$ we have that $p \restriction \delta' \in \mathbb{Q}_{\delta'}$. Next, observe that $q = p^{[\nu]}$ for $\eta \trianglelefteq \nu \in p$; then for all $\lg(\nu) \leq \delta' \in \delta \cap S_*$, $q \restriction \delta' = (p \restriction \delta')^{[\nu]}$. Observe that $p \restriction \delta' \in \mathbb{Q}_{\delta'}$ and by the first part of this clause also $(p \restriction \delta')^{[\nu]} \in \mathbb{Q}_{\delta'}$. For $\nu \trianglelefteq \eta$, $p^{[\nu]} = p \in \mathbb{Q}'_\delta$, and in particular $tr(p^{[\nu]}) = \eta$; so indeed

$$tr(p^{[\nu]}) = \max\{\eta,\nu\}.$$

By the definition of $p^{[\nu]}$, $p^{[\nu]} \subseteq p$, and since the order of both forcing $\mathbb{Q}_\delta$ and $\mathbb{Q}'_\delta$ is inverse inclusion, and by what we just showed if $p \in \mathbb{Q}_\delta$, then $p \leq_{\mathbb{Q}_\delta} p^{[\nu]}$, and if $p \in \mathbb{Q}'_\delta$ then $p \leq_{\mathbb{Q}'_\delta} p^{[\nu]}$.

(2) $\boldsymbol{T}_{<\delta} = p^*_{\langle\rangle,\delta,\emptyset}$, so trivially it belongs to $\mathbb{Q}_\delta$. Then by the first clause of this claim it follows that $\boldsymbol{T}_{<\delta} \in \mathbb{Q}'_\delta$ and we are done. ∎

Lemma 32: *If $p \in \mathbb{Q}_\delta$ and $\lg(tr(p)) < \alpha < \delta$, then $\{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\alpha\}$ is a maximal antichain of $\mathbb{Q}_\delta$ above p; the same holds for $\mathbb{Q}'_\delta$.*

Proof. Let $\eta, \nu \in p \cap \boldsymbol{T}_\alpha$ be different; then $\eta \notin p^{[\nu]}$ and $\nu \notin p^{[\eta]}$. Recalling clause (5) of Claim 30 it follows that $p^{[\eta]}, p^{[\nu]}$ are incompatible and so the set $\{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\alpha\}$ is an antichain in $\mathbb{Q}_\delta$ above p. In addition, let $p \leq_{\mathbb{Q}_\delta} q \in \mathbb{Q}_\delta$ and let $\eta_0 \in q \cap \boldsymbol{T}_\alpha \subseteq p \cap \boldsymbol{T}_\alpha$; then $p^{[\eta_0]}$ is compatible with q: their common upper bound is $q^{[\eta_0]}$ and this is in $\mathbb{Q}_\delta$ by what we just showed. Clearly $p^{[\eta_0]} \in \{p^{[\eta]} : \eta \in p \cap \boldsymbol{T}_\alpha\}$ so this set is indeed a maximal antichain. The proof for $\mathbb{Q}'_\delta$ is identical. ∎

Corollary 33: *Let $\delta \in S_*$. If δ is successful then the set $\bar{q}^*_\delta = \langle q^*_{\delta,\eta} : \eta \in \Lambda^*_\delta \rangle$ is an antichain of $\mathbb{Q}'_\delta$ above r^*_δ.*

Proof. Recall that if $\eta \neq \nu \in \Lambda^*_\delta$ then $\eta \notin q^*_{\delta,\nu} \vee \nu \notin q^*_{\delta,\eta}$ (see Definition 19) and recall Claim 30(4). ∎

2.3. Properties of the Forcing.

Claim 34: *Let $\delta \in S_*$ be such that $S_* \cap \delta$ is non-stationary in δ. Then the forcing $\mathbb{Q}_\delta$ is strategically complete in $cf(\delta)$.*

Remark 35: Remember that if $\delta \in S_*$ is not inaccessible, then $S_* \cap \delta$ is not stationary in δ. Also, if $\alpha < \lambda$ and $\delta = \min(S_* \setminus (\alpha + 1))$, then $S_* \cap \delta$ is not stationary.

Proof. First, assume that this holds for each $\delta_0 < \delta$. Now, there is a club E of δ such that $E \cap S_* = \emptyset$. Let $p \in \mathbb{Q}_\delta$ and $\alpha = cf(\delta)$. We shall play the game $\Game_\alpha(p, \mathbb{Q}_\delta)$, determining a strategy for COM;

(1) At the first step player COM will choose a condition $p_0 \geq p$ and after INC chose q_0, COM chooses a club E_0 of δ disjoint to S_{q_0}.

(2) In successor step $i+1 < \alpha$: look at the condition q_i that player INC chose in the i-th step; let $\beta_i = \lg(tr(q_i))$. In addition let $\gamma_i = \min(E \setminus (\beta_i + 1))$. Now choose some $\eta_{i+1} \in q_i \cap \boldsymbol{T}_{\gamma_i}$; player COM will choose $p_{i+1} = (q_i)^{[\eta_{i+1}]}$; this is a condition of the forcing $\mathbb{Q}_\delta$ by Claim 30. Observe that

$$tr(q_i) \trianglelefteq \eta_{i+1}, \quad q_i \leq_{\mathbb{Q}_\delta} p_{i+1},$$

and by the choice player COM made, she forced player INC to have

$$\eta_{i+1} \trianglelefteq tr(q_{i+1}).$$

(3) In limit step $i(*) < \alpha$: player COM will choose $p_{i(*)} = \bigcap_{i<i(*)} q_i$; let

$$S_{i(*)} = \bigcup_{i<i(*)} S_{q_i} \setminus \lg(\nu_{i(*)}) \quad \text{and} \quad \nu_{i(*)} = \bigcup_{i<i(*)} tr(q_i).$$

(a) The node $\nu_{i(*)}$ belongs to all the conditions that player INC had chosen in the steps $i < i(*)$: observe that

$$\delta' = \sup\{\beta_i : i < i(*)\} = \sup\{\gamma_i : i < i(*)\}$$

but $\gamma_i \in E$, hence $\delta' \in E$. Since E is a club disjoint to S_i there is no pruning in the level δ', in particular $\nu_{i(*)}$ is not being pruned. Thus $\nu_{i(*)} \in q_i$ for all $i < i(*)$.

(b) It remains to show that $p_{i(*)}$ is indeed a condition in the forcing and in fact $p_{i(*)} = p^*_{\nu_{i(*)},\delta,S_{i(*)}}$; first observe that $cf(\delta') = cf(i(*))$. Next:

(i) For each node $\nu' \in p_{i(*)}$ such that $\lg(\nu') < \lg(\nu_{i(*)})$ there is $i < i(*)$ such that $\lg(\nu') < \lg(tr(q_i))$ and, as $p_{i(*)}$ is the intersection, we get $\nu' \lhd tr(q_i)$ and so $\nu' \trianglelefteq \bigcup_{i<i(*)} tr(q_i)$. Thus $\bigcup_{i<i(*)} tr(q_i)$ is a node such that there is no splitting before it in $p_{i(*)}$. However in each level above this node there are splittings as those splittings exist for each q_i, and they are "full"; see Definition 15(1)(c) and Definition 27. In addition, for each $i < j < i(*)$ any splitting in the tree q_j exists in the tree q_i as well: this is an increasing sequence of conditions and $q_j \subseteq q_i$. It follows that $p_{i(*)}$ is a tree with trunk $\nu_{i(*)}$ (if we use the filters D_ϵ for $\epsilon < \lambda$, this is somewhat more delicate, still OK).

(ii) The set $S_{i(*)}$ is tenuous: the set $S_* \cap \delta$ is non-stationary by the claim assumption, if $\epsilon \leq \lg(\nu_{i(*)})$ then $S_{i(*)} \cap \epsilon = \emptyset$ so non-stationary. For all $\epsilon \in (\lg(\nu_{i(*)}), \delta)$, if S_* doesn't reflect in ϵ, then $S_{i(*)} \restriction \epsilon \subseteq S_*$ is non-stationary in ϵ by Claim 9(1); if S_* reflects to ϵ, then ϵ is inaccessible and thus $S_{i(*)} \restriction \epsilon$ is a union of $i(*)$ sets, non-stationary in ϵ, so by Claim 9(1) $S_{i(*)} \restriction \epsilon$ is non-stationary. Putting everything together $S_{i(*)}$ is tenuous.

For all $i < i(*)$, we shall see that $p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i$; as $tr(q_i) \trianglelefteq \nu_{i(*)}$, observe that $q_i^{[\nu_{i(*)}]} \subseteq q_i$ by a previous lemma. Also, $q_i^{[\nu_{i(*)}]}$ and $p^*_{\nu_{i(*)},\delta,S_{i(*)}}$ have the same trunk, with the first having a smaller sationary set:

$$S_{q_i^{[\nu_{i(*)}]}} \subseteq S_{i(*)};$$

recalling Claim 30(5), we get $p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i^{[\nu_{i(*)}]}$, thus

$$p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i \quad \text{and} \quad p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq p_{i(*)}.$$

We need to also see that $p_{i(*)} \subseteq p^*_{\nu_{i(*)},\delta,S_{i(*)}}$; assume this doesn't hold. Then, for some $\nu' \in p_{i(*)}$, $\nu' \notin p^*_{\nu_{i(*)},\delta,S_{i(*)}}$. Let δ' be the minimal such that $\nu' \restriction \delta' \notin p^*_{\nu_{i(*)},\delta,S_{i(*)}}$; recalling Definition 4, neccessarily

$$\nu' \restriction \delta' \in \lim_{\delta'}(p^*_{\nu_{i(*)},\delta',S_{i(*)}\cap\delta'}) \quad \text{and} \quad \nu' \restriction \delta' \in \lim_{\delta'}(r^*_{\delta'}) \setminus \Big(\bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\}\Big).$$

As $\delta' \in S_{i(*)}$, there exists $i < i(*)$ such that $\delta' \in S_{q_i}$; since for all $\delta'' < \delta'$, $\nu' \restriction \delta'' \in p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i$ so $\nu' \restriction \delta' \in \lim_{\delta'}(q_i)$, and by the construction of q_i, as $\nu' \restriction \delta' \in \lim_{\delta'}(r^*_{\delta'}) \setminus (\bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\})$ it follows that $\nu' \restriction \delta' \notin q_i$, a contradiction to the assumption $\nu' \in p_{i(*)} \subseteq q_i$.

We have shown that $p^*_{\nu_{i(*)},\delta,S_{i(*)}} = p_{i(*)}$. In addition, for all $i < i(*)$, $q_i \leq_{\mathbb{Q}_\lambda} p_{i(*)}$ so easily $p_{i(*)}$ is the smallest supremum of those conditions. ∎

THEOREM 36: *If $\delta \in S_* \cup \{\lambda\}$ is inaccessible, then the forcing $\mathbb{Q}_\delta$ is strategically complete in δ.*

Proof. For $\delta \in S_* \cup \{\lambda\}$ and $p \in \mathbb{Q}_\delta$, we shall play the game $\eth_\delta(p, \mathbb{Q}_\delta)$. We shall construct inductively the sequence $\langle p_i, q_i, E_i : i < \delta\rangle$ where p_i is the i-th move of player COM, q_i is the i-th move of player INC, E_i is a club in δ chosen by COM after INC plays his i-th move; it shall be disjoint to S_{q_i}. Assume that for all $i' < j' < i$, $E_{j'} \subseteq E_{i'}$; this will prove the desired condition.

(1) At the first step player COM will choose a condition $p_0 \geq p$ and, after INC chose q_0, COM chooses a club E_0 of δ disjoint to S_{q_0}.

(2) In successor step $i+1 < \delta$: look at the condition q_i that player INC chose in the i-th step; E_i is a club disjoint to S_{q_i} s.t. $E_i \subseteq \bigcap_{j<i} E_j$ (this club was defined in step i). Let $\beta_i = \lg(tr(q_i))$ and let $\gamma_i = \min(E_i \setminus (\beta_i + 1))$. Next, for some node $\eta_{i+1} \in q_i \cap \boldsymbol{T}_{\gamma_i}$, player COM will choose $p_{i+1} = (q_i)^{[\eta_{i+1}]}$; this is a condition of the forcing $\mathbb{Q}_\delta$ by Claim 30. Observe that

$$tr(q_i) \trianglelefteq \eta_{i+1}, \quad q_i \leq_{\mathbb{Q}_\delta} p_{i+1},$$

and by the choice player COM made she forced player INC to have $\eta_{i+1} \trianglelefteq tr(q_{i+1})$. Finally, after INC will play his $i+1$-th turn, COM will let E_{i+1} be a club: $E_{i+1} \subseteq E_i \setminus S_{q_{i+1}}$; this is possible as E_i is a club of δ, and $S_{q_{i+1}}$ is tenuous.

(3) In limit step $i(*) < \delta$: player COM will choose $p_{i(*)} = \bigcap_{i<i(*)} q_i$. Let $S_{i(*)} = \bigcup_{i<i(*)} S_{q_i} \setminus \lg(\nu_{i(*)})$ where

$$\nu_{i(*)} = \bigcup_{i<i(*)} tr(q_i) \quad \text{and} \quad E_{i(*)} = \bigcap_{i<i(*)} E_i.$$

Observe that $\lg(\nu_{i(*)}) < \delta$ since δ is inacessible; in addition, $E_{i(*)}$ is a club in δ as an intersection of $i(*) < \delta$ clubs.

(a) The node $\nu_{i(*)}$ belongs to all the conditions that player INC had chosen in the steps $i < i(*)$: observe that $\delta' = \sup\{\beta_i : i < i(*)\} = \sup\{\gamma_i : i < i(*)\}$, as $tr(q_i) \in q_i$ for $i < i(*)$ is $\triangleleft$-increasing, q_i decreasing clearly

$$\{\nu_{i(*)} \restriction \beta : \beta < \delta'\} = \{tr(q_i) \restriction \beta : i < i(*), \beta < \lg(tr(q_i))\} \subseteq \{q_i : i < i(*)\}.$$

Since $E_{i(*)}$ is a club that is a decreasing intersection of clubs, note $i < i(*) \Rightarrow E_i \cap S_{q_i} = \emptyset \Rightarrow E_{i(*)} \cap S_{q_i} = \emptyset$, there are no prunings in the level δ', in particular $\nu_{i(*)}$ is not being pruned. Thus $\nu_{i(*)} \in q_i$ for all $i < i(*)$.

(b) It remains to show that $p_{i(*)}$ is indeed a condition in the forcing. First observe that $cf(\delta') = cf(i(*))$ and $\delta' \geq i(*)$. Next:

(i) For each node $\nu' \in p_{i(*)}$ such that $\lg(\nu') < \delta'$ there is $i < i(*)$ such that $\lg(\nu') < \lg(tr(q_i))$ and, as $p_{i(*)}$ is the intersection, we get $\nu' \triangleleft tr(q_i)$ and so $\nu' \trianglelefteq \bigcup_{i<i(*)} tr(q_i)$. We get that $\bigcup_{i<i(*)} tr(q_i)$ is a node such that there is no splitting before it in $p_{i(*)}$. However, in each level above it (in the $\trianglelefteq$ sense) there are splittings as there are such splittings for each q_i. In addition, for each $i < j < i(*)$ any splitting in the tree q_j exists in the tree q_i as well: this is an increasing sequence of conditions and $q_j \subseteq q_i$. It follows that $p_{i(*)}$ is a tree with trunk $\nu_{i(*)}$.

(ii) The set $S_{i(*)}$ is tenuous: as a union of $i(*) < \delta = cf(\delta)$ non-stationary sets, $S_{i(*)}$ is non-stationary in δ by Claim 9(1). For all $\epsilon < \delta$, if $\epsilon < \delta'$ then $S_{i(*)} \cap \epsilon = \emptyset$, so this is trivial, hence assume $\epsilon > \delta'$; hence $\epsilon > i(*)$. If S_* doesn't reflect to ϵ, then $S_{i(*)} \restriction \epsilon \subseteq S_*$ is non-stationary in ϵ by Claim 9(1); if S_* reflects to ϵ, then ϵ is inaccessible and thus $S_{i(*)} \restriction \epsilon$ is a union of $i(*)$ sets, non-stationary in ϵ and, recalling that $\epsilon > i(*)$, so by Claim 9(1) $S_{i(*)} \restriction \epsilon$ is non-stationary; putting everything together $S_{i(*)}$ is tenuous.

For all $i < i(*)$, we shall see that $p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i$; as $tr(q_i) \trianglelefteq \nu_{i(*)}$, observe that $q_i^{[\nu_{i(*)}]} \subseteq q_i$ by a previous lemma. Also, $q_i^{[\nu_{i(*)}]}$ and $p^*_{\nu_{i(*)},\delta,S_{i(*)}}$ have the same trunk, with the first having a smaller sationary set: $S_{q_i} \subseteq S_{i(*)}$. Recalling Claim 30(5) we get $p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i^{[\nu_{i(*)}]}$, thus $p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i$ and $p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq p_{i(*)}$.

We need to also see that

$$p_{i(*)} \subseteq p^*_{\nu_{i(*)},\delta,S_{i(*)}}.$$

Assume this doesn't hold. Then, for some $\nu' \in p_{i(*)}$, $\nu' \notin p^*_{\nu_{i(*)},\delta,S_{i(*)}}$. Let δ' be the minimal such that $\nu' \restriction \delta' \notin p^*_{\nu_{i(*)},\delta,S_{i(*)}}$. Recalling Definition 4, neccessarily $\nu' \restriction \delta' \in \lim_{\delta'}(p^*_{\nu_{i(*)},\delta',S_{i(*)}\cap\delta'})$ and $\nu' \restriction \delta' \in \lim_{\delta'}(r^*_{\delta'}) \setminus (\bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\})$. As $\delta' \in S_{i(*)}$, there exists $i < i(*)$ such that $\delta' \in S_{q_i}$; since for all $\delta'' < \delta'$, $\nu' \restriction \delta'' \in p^*_{\nu_{i(*)},\delta,S_{i(*)}} \subseteq q_i$, so $\nu' \restriction \delta' \in \lim_{\delta'}(q_i)$, and by the construction of q_i as $\nu' \restriction \delta' \in \lim_{\delta'}(r^*_{\delta'}) \setminus (\bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\})$, it follows that $\nu' \restriction \delta' \notin q_i$, a contradiction to the assumption $\nu' \in p_{i(*)} \subseteq q_i$.

Finally, we have that

$$p^*_{\nu_{i(*)},\delta,S_{i(*)}} = p_{i(*)},$$

In addition, for all $i < i(*)$, $q_i \leq_{\mathbb{Q}_\lambda} p_{i(*)}$, so easily $p_{i(*)}$ is the smallest supremum of those conditions.

Finally, we can see that player COM has a legal move for each $i < \delta$, thus the forcing $\mathbb{Q}_\delta$ is strategically complete in α. ∎

By Claim 34 and Theorem 36:

Corollary 37: *For all $\delta \in S_* \cup \{\lambda\}$, the forcing $\mathbb{Q}_\delta$ is strategically complete in $cf(\delta)$.*

Theorem 38: *If $\delta \in S_* \cup \{\lambda\}$, then the δ^+-chain condition holds for the forcing $\mathbb{Q}_\delta$.*

Proof. Let $\mathcal{A} \subseteq \mathbb{Q}_\delta$ be an antichain. Then for all $p, q \in \mathcal{A}$, by Claim 30(5), $tr(p) \neq tr(q) \in \boldsymbol{T}_{<\delta}$. Recalling the definition of the good structure $\mathfrak{r}$ we have that for each $\zeta < \delta$, $\theta_\zeta < \delta$, and as δ is a strong limit, $|\bigcup_{\epsilon<\delta}\Pi_{i<\epsilon}\theta_i| = |\boldsymbol{T}_{<\delta}| \leq \delta$; in particular, for any antichain $\mathcal{A} \subseteq \mathbb{Q}_\delta$, $|\mathcal{A}| \leq \delta$. ∎

Corollary 39: *By Corollary 37 and Theorem 38, the forcing $\mathbb{Q}_\lambda$ is $\leq \lambda$-strategically complete and the λ^+-chain condition holds for it.*

Theorem 40: *If λ is an inaccessible cardinal, then the forcing $\mathbb{Q}_\lambda$ is λ-bounding.*

Proof. Let $p_* \in \mathbb{Q}_\lambda$ and $\underset{\sim}{\tau}$ be a $\mathbb{Q}_\lambda$-name for a function from λ to λ. We would like to have a condition $q \geq_{\mathbb{Q}_\lambda} p_*$, $q \in \mathbb{Q}_\lambda$ and a function $g : \lambda \to \lambda$ such that $q \Vdash_{\mathbb{Q}_\lambda}$ "$\underset{\sim}{\tau} \leq g$". In this proof we denote $\leq$ instead of $\leq_{\mathbb{Q}_\lambda}$ when comparing forcing conditions.

We will find a sequence $\langle p_\epsilon, S_\epsilon, E_\epsilon, \alpha_\epsilon \rangle$ for each $\epsilon < \lambda$ such that:

(1) $p_0 = p_*$,
(2) $p_\epsilon = p^*_{\varrho,\lambda,S_\epsilon}$ for $\varrho = tr(p_*)$,
(3) the sequence $\langle p_\zeta : \zeta \leq \epsilon \rangle$ is increasing and continuous,
(4) E_ϵ is a club disjoint to S_ϵ,
(5) the sequence $\langle E_\epsilon : \epsilon < \lambda \rangle$ is decreasing,
(6) for $\epsilon = \zeta + 1 < \lambda$, we have that $\alpha_\epsilon \in E_\zeta$ and $\alpha_\epsilon \in S_* \setminus (S_\zeta \setminus (\alpha_\zeta + 1))$,
(7) for a limit $\epsilon < \lambda$, $\alpha_\epsilon \in E_\epsilon$,
(8) the sequence $\langle \alpha_\zeta : \zeta \leq \epsilon \rangle$ will be increasing continuous, consisting of ordinals greater than $\lg(\varrho)$,
(9) for $\zeta < \epsilon < \lambda$, $S_\zeta \cap (\alpha_\zeta + 1) = S_\epsilon \cap (\alpha_\zeta + 1)$,
(10) for $\epsilon = \zeta + 1$, the ordinal α_ϵ represents a level in which, in the corresponding tree, the value of the function in ζ will be determined, that is:
  (a) for all $\nu \in p_\epsilon \cap \boldsymbol{T}_{\alpha_\epsilon}$, $p_\epsilon^{[\nu]}$ forces a value for $\underset{\sim}{\tau}(\zeta)$,
  (b) $p_\epsilon \Vdash_{\mathbb{Q}_\lambda}$ "$\underset{\sim}{\tau}(\zeta) \in u_\zeta$" where $u_\zeta \subseteq \lambda$ of cardinality $< \lambda$.

Next we see that this construction is possible, by induction:

— For the basis $\epsilon = 0$:

We have that $p_0 = p_*$, $\alpha_0 = \lg(\varrho)$, so (1) holds; S_ϵ is the tenuous set corresponding to p and let E_ϵ be a club in λ disjoint to S_ϵ (as S_ϵ is tenuous).

— For $\epsilon < \lambda$ limit:

Start with the set S_ϵ: let

$$S_\epsilon = \bigcup_{\zeta < \epsilon} S_\zeta \subseteq S_*.$$

Then it is easy to see that clause (9) holds (by the induction hypothesis). Let also $\alpha_\epsilon = \bigcup_{\zeta<\epsilon} \alpha_\zeta$ and $E_\epsilon = \bigcap_{\zeta<\epsilon} E_\zeta$, and observe that E_ϵ is a club disjoint to S_ϵ, so clauses (4) and (5) hold.

Now we will show that the set S_ϵ is indeed tenuous: first, the set S_ϵ is non-stationary in λ as a union of $\epsilon < \lambda = cf(\lambda)$ sets that are non-stationary in λ and, by Remark 8, when S_* is non-reflecting, S_ϵ is also tenuous, but we have to prove it in general.

Next, let $\gamma < \lambda$ be an ordinal of uncountable cofinality and look at $S_\epsilon \restriction \gamma$:

If there exists $\zeta < \epsilon$ for which $\gamma < \alpha_\zeta$, then as $S_\epsilon \cap (\alpha_\zeta + 1) = S_\zeta \cap (\alpha_\zeta + 1)$ it follows that

$$S_\epsilon \cap \gamma = S_\zeta \cap \gamma,$$

and since S_ζ is tenuous this set is non-stationary.

For $\gamma = \alpha_\epsilon$, first observe that by the definition of E_ϵ as the limit of the clubs $\langle E_\zeta : \zeta < \epsilon \rangle$, and since the sequence of clubs is decreasing, and by (6) of the induction hypothesis we have that

$$\alpha_\epsilon \in \bigcap_{\zeta<\epsilon} E_\zeta = E_\epsilon,$$

this was clause (7), and so $\alpha_\epsilon \notin S_\epsilon$.

- When α_ϵ is regular (and thus inaccessible): by (8) in the induction hypothesis, the set $\{\alpha_\zeta : \zeta \text{ is a limit ordinal } < \epsilon\}$ is a club of α_ϵ. In addition, by clause (7) in the induction hypothesis, for all $\zeta < \epsilon$ limit, $\alpha_\zeta \notin S_\zeta$, and by clause (9) in the induction hypothesis, for every $\zeta < \xi < \epsilon$, $\alpha_\zeta \notin S_\xi$ and therefore $\alpha_\zeta \notin S_\epsilon$ and this club is disjoint to $S_\epsilon \restriction \alpha_\epsilon$, so this is not a stationary set.
- When α_ϵ is singular, the set S_* doesn't reflect to α_ϵ by definition, so $S_* \restriction \alpha_\epsilon$ is a non-stationary set and, in particular, $S_\epsilon \restriction \alpha_\epsilon \subseteq S_* \restriction \alpha_\epsilon$ is not a stationary set by (8).

Lastly, for $\gamma > \alpha_\epsilon$:

- If $cf(\gamma) > \epsilon$, then for all $\zeta < \epsilon$ we have that $S_\zeta \restriction \gamma$ is a non-stationary set from clause (2) of the induction hypothesis, so there is a club of γ disjoint to it, call it C_ζ. Letting

  $$C_\epsilon = \bigcap_{\zeta<\epsilon} C_\zeta,$$

  this is a club as the intersection of ϵ clubs, disjoint to S_ϵ by its definition, so $S_\epsilon \restriction \gamma$ is non-stationary.
- Otherwise, if $\gamma > \epsilon \geq cf(\gamma)$, in particular, it follows that γ is singular, thus S_* doesn't reflect to γ and so also $S_\epsilon \subseteq S_*$ using Remark 8.

Let $p_\epsilon = p^*_{\varrho,\lambda,S_\epsilon}$ so clause (2) holds. Moreover, $p_\epsilon \subseteq \bigcap_{\zeta<\epsilon} p^*_{\varrho,\lambda,S_\zeta}$. Why? Assume there is some $\nu' \in \bigcap_{\zeta<\epsilon} p^*_{\varrho,\lambda,S_\zeta} \setminus p_\epsilon$; as $\varrho \trianglelefteq \nu'$ there is some minimal δ' for which $\nu' \restriction \delta' \notin p_\epsilon$. Then $\nu' \restriction \delta' \in \lim_{\delta'}(p_\epsilon \cap \boldsymbol{T}_{<\delta'})$ and by Definition 4 necessarily $\nu' \restriction \delta' \in \lim_{\delta'}(r^*_{\delta'}) \setminus (\bigcup\{\lim_{\delta'}(q^*_{\delta',\eta'}) : \eta' \in \Lambda^*_{\delta'}\})$. Since for some $\zeta < \epsilon$, $\delta' \in S_\zeta$, it follows that $\nu' \restriction \delta' \notin p^*_{\varrho,\lambda,S_\zeta}$ and so $\nu' \notin p^*_{\varrho,\lambda,S_\zeta}$, a contradiction. Thus,

$$p_\epsilon = \bigcap_{\zeta<\epsilon} p^*_{\varrho,\lambda,S_\zeta}$$

and (3) holds.

— For $\epsilon = \zeta + 1$:

This is the main case, as here we deal with clause (10) that is responsible for determining the values of the function.

Define the following set:

$$\mathscr{I}_\epsilon = \{r \in \mathbb{Q}_\lambda : r \text{ forces a value on } \underset{\sim}{\tau}(\zeta) \wedge p_\zeta \leq_{\mathbb{Q}_\lambda} r \wedge \lg(tr(r)) > \alpha_\zeta\}$$

and observe:

(a) This set is dense above p_ζ: for all $p \in \mathbb{Q}_\lambda$ with $p_\zeta \leq p$, we will find a condition r stronger than p that forces a value on $\underset{\sim}{\tau}(\zeta)$ and, if $\lg(tr(r)) > \alpha_\zeta$ doesn't hold, we can extend r to a stronger condition with long enough trunk.

(b) The set is open: for all $q \in \mathscr{I}_\epsilon$ and $r \geq q$, q forces a value on $\underset{\sim}{\tau}(\zeta)$ and therefore, so does r,

$$\lg(tr(r)) \geq \lg(tr(q)) > \alpha_\zeta$$

and of course $p_\zeta \leq q \leq r$.

Now define a set

$$\Lambda_\epsilon = \{tr(r) : r \in \mathscr{I}_\epsilon\}$$

and for every $\eta \in \Lambda_\epsilon$ choose some $q_{\epsilon,\eta} \in \{r \in \mathscr{I}_\epsilon : tr(r) = \eta\}$.

Choose a set $\Lambda^1_\epsilon \subseteq \Lambda_\epsilon$ that is maximal under the restriction that for any different $\eta, \nu \in \Lambda^1_\epsilon$, $\nu \notin q_{\epsilon,\eta} \vee \eta \notin q_{\epsilon,\nu}$; let

$$\bar{q}_\epsilon = \langle q_{\epsilon,\eta} : \eta \in \Lambda^1_\epsilon\rangle.$$

- Observe that the sequence $\bar{q}_\epsilon = \langle q_{\epsilon,\eta} : \eta \in \Lambda^1_\epsilon\rangle \in \Xi_\lambda$ because:
  (1) $\Lambda^1_\epsilon \subseteq \boldsymbol{T}_{<\lambda}$.
  (2) For all $\eta \in \Lambda^1_\epsilon$ we have $q_{\epsilon,\eta} \in \mathbb{Q}_\lambda \subseteq \mathbb{Q}^0_\lambda$ and $tr(q_{\epsilon,\eta}) = \eta$.
  (3) If $\eta, \nu \in \Lambda^1_\epsilon$ are different, then by the definition of Λ^1_ϵ,

$$tr(q_{\epsilon,\nu}) = \nu \notin q_{\epsilon,\eta} \vee tr(q_{\epsilon,\eta}) = \eta \notin q_{\epsilon,\nu}.$$

  (4) $r^*_{\bar{q}_\epsilon} = \{\rho \in \boldsymbol{T}_{<\lambda} : (\exists \eta \in \Lambda^1_\epsilon)(\rho \in q_{\epsilon,\eta})\} = p_\zeta$; in particular, it belongs to $\mathbb{Q}_\lambda \subseteq \mathbb{Q}^0_\lambda$. Observe that for all $\eta \in \Lambda^1_\epsilon$, $q_{\epsilon,\eta} \subseteq p_\zeta$ and so $r^*_{\bar{q}_\epsilon} \subseteq p_\zeta$. Assume via a contradiction that $\nu \in p_\zeta \setminus r^*_{\bar{q}_\epsilon}$; then there is $p^{[\nu]}_\zeta \leq_{\mathbb{Q}_\lambda} q$ that forces a value for $\underset{\sim}{\tau}(\zeta)$ and its trunk is longer

than α_ζ, so $q \in \mathscr{J}_\epsilon$ and $tr(q) \in \Lambda_\epsilon$. If $tr(q) \in \Lambda^1_\epsilon$ we get $tr(q) \in r^*_{\bar{q}_\epsilon}$, a contradiction to the assumption; hence there is $\nu' \in \Lambda^1_\epsilon$ such that $tr(q) \in q_{\epsilon,\nu'} \vee tr(q_{\epsilon,\nu'}) \in q$ so again we get $tr(q) \in r^*_{\bar{q}_\epsilon}$, but the later conjunct contradicts the choice of ν and $\nu \in r^*_{\bar{q}_\epsilon}$, a contradiction.

For all $\eta \in \Lambda^1_\epsilon$, $q_{\epsilon,\eta}$ forces a value on $\underset{\sim}{\tau}(\zeta)$; call this value $\gamma_{\epsilon,\eta}$. In addition, let C_η be a club disjoint to $S_{q_{\epsilon,\eta}}$.

First, define an approximation for the club E_ϵ.

$$E'_\epsilon = \{\delta \in E_\zeta : \delta > \alpha_\zeta \text{ is a limit ordinal such that } \nu' \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta} \to \delta \in C_{\nu'} \\ \text{and } \nu \in p_\zeta \cap \boldsymbol{T}_{<\delta} \to \nu \in q_{\epsilon,\eta} \text{ for some } \eta \in \boldsymbol{T}_{<\delta} \cap \Lambda^1_\epsilon)\}.$$

The set E'_ϵ is a club in λ:

- Closed—for every increasing sequence of ordinals $\langle \delta_i : i < \zeta^* \rangle$ such that for all $i < \zeta^*$, $\delta_i \in E'_\epsilon$ and $\zeta^* < \lambda$, their limit $\delta = \lim_{i<\zeta^*} \delta_i$ is of course a limit ordinal and belongs to E_ζ. In addition, for all $\nu' \in \Lambda^1_\epsilon$ with $\lg(\nu') < \delta$ there is $j_0 < \zeta^*$ such that for all $j_0 < j < \zeta^*$ we have $\lg(\nu') < \delta_j$ (as δ is defined to be the limit of those). Then $\delta_j \in C_{\nu'}$, and since $C_{\nu'}$ is a club it follows that $\delta \in C_{\nu'}$, as the limit of $\langle \delta_j : j_0 < j < \zeta^* \rangle$.

  Lastly, if $\nu \in p_\zeta \cap \boldsymbol{T}_{<\delta}$ then $\lg(\nu) < \delta$, hence for some $i < \zeta^*$, $\lg(\nu) < \delta_i$, hence $\nu \in \boldsymbol{T}_{<\delta_i}$, hence $\nu \in p_\zeta \cap \boldsymbol{T}_{<\delta}$. As $\delta_i \in E'_\epsilon$ necessarily there is $\eta \in \boldsymbol{T}_{<\delta_i} \cap \Lambda^1_\epsilon$ such that $\nu \in q_{\epsilon,\eta}$, but clearly $\eta \in \boldsymbol{T}_{<\delta} \cap \Lambda^1_\epsilon$ so we are done.
- Unbounded—otherwise, the set E'_ϵ was bounded by some $\xi < \lambda$; then for every limit $\xi < \delta \in E_\zeta$, $\delta \notin E'_\epsilon$ so either (1) $\exists \nu' \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}$ such that $\delta \notin C_{\nu'}$ o r (2) $(\exists \nu \in p_\zeta \cap \boldsymbol{T}_{<\delta})(\forall \eta \in \boldsymbol{T}_{<\delta} \cap \Lambda^1_\epsilon)(\nu \notin q_{\epsilon,\eta})$. As $E_\zeta \setminus (\xi+1)$ is stationary, for some $W \subseteq E_\zeta \setminus (\xi+1)$ stationary in λ, for all $\delta \in W$ the same case occurs. If it is by case (2), as $|\boldsymbol{T}_{<\alpha}| < \lambda$ for $\alpha < \lambda$, by Fodor's lemma there is a stationary set $W_2 \subseteq E_\zeta \setminus (\xi+1)$ such that for all $\delta \in W_2$ we can choose the same $\nu \in p_\zeta \cap \boldsymbol{T}_{<\delta}$—a contradiction. Thus (2) is impossible and if it is by case (1) so $\delta \notin \bigcap'_\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta} C_{\nu'} \subseteq \bigcap'_\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\xi} C_{\nu'} = C$ as $|\Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}| < \lambda$. For any $\xi < \delta \in E_\zeta$ we get $\{\delta \in (\xi, \lambda) \cap E_\zeta : \delta \text{ is a limit ordinal}\} \cap C = \emptyset$; however, C is a club as the intersection of less than λ clubs—a contradiction.

Define the level.

We would like to have an ordinal δ for which the following properties hold:

(a) $\delta \in E'_\epsilon \cap S_*$,
(b) $\alpha_\zeta < \delta$ (follows from (a)),
(c) $r^*_\delta = p_\zeta \cap \boldsymbol{T}_{<\delta}$,
(d) $\bar{q}^*_\delta = \langle q_{\epsilon,\eta} \cap \boldsymbol{T}_{<\delta} : \eta \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta} \rangle$.

An ordinal with those properties exists:

First, by Claim 24 there is a stationary set of $\delta \in S_*$ such that clause (d) holds and call it S^+; as E'_ϵ is a club, we get that $S^+ \cap E'_\epsilon$ is stationary. Observe that for all $\delta \in S^+ \cap E'_\epsilon$ from clause (d) it follows that

$$r^*_\delta = \bigcup_{\eta \in \Lambda^*_\delta} q^*_\eta = \bigcup_{\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}} q_{\epsilon,\nu} \cap \boldsymbol{T}_{<\delta};$$

in addition, by the definition of E'_ϵ we have that

$$p_\zeta \cap \boldsymbol{T}_{<\delta} = \bigcup_{\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}} q_{\epsilon,\nu} \cap \boldsymbol{T}_{<\delta},$$

so for all $\delta \in S^+ \cap E'_\epsilon$ clause (c) holds. As this set is not empty (as a stationary set) there is such δ, and we are done.

Let $\alpha_\epsilon = \delta$. Observe that in particular it follows that

$$\Lambda^1_\epsilon \cap \boldsymbol{T}_{<\alpha_\epsilon} = \Lambda^*_{\alpha_\epsilon}.$$

We can now let $E_\epsilon = E'_\epsilon \setminus (\alpha_\epsilon + 1)$. Notice that also E_ϵ is a club in λ.

Define the tenuous set of p_ϵ.

First, in the α_ϵ-th level we define the set of all the limits formed from the conditions of $\bar{q}^*_{\alpha_\epsilon}$:

$$\Lambda^2_\epsilon = p_\zeta \cap \boldsymbol{T}_{\alpha_\epsilon} \cap \Big(\bigcup \{\lim(q^*_{\alpha_\epsilon,\nu}) : \nu \in \Lambda^*_{\alpha_\epsilon}\} \Big).$$

For $\eta \in \Lambda^2_\epsilon$, by the definition above and the definition of the level there is unique $\nu \in \Lambda^*_{\alpha_\epsilon}$ with $\eta \in \lim(q^*_{\alpha_\epsilon,\nu})$, as $q_{\epsilon,\nu} \cap \boldsymbol{T}_{<\alpha_\epsilon} = q^*_{\alpha_\epsilon,\nu}$ and, recalling Definition 4, the fact that $\eta \in \lim(q^*_{\alpha_\epsilon,\nu})$ also implies $\eta \in q_{\epsilon,\nu}$; let $r_\eta := (q_{\epsilon,\nu})^{[\eta]}$.

Now, define

$$S^1_\epsilon = \bigcup \{S_{r_\eta} \setminus (\alpha_\epsilon + 1) : \eta \in \Lambda^2_\epsilon\}.$$

Observe that for every $\eta \in \Lambda^2_\epsilon$, $S_{r_\eta} \subseteq S_{q_{\epsilon,\nu}}$ for some $\nu \in \Lambda^*_{\alpha_\epsilon} \subseteq \boldsymbol{T}_{<\alpha_\epsilon}$ (follows from $r_\eta = (q_{\epsilon,\nu})^{[\eta]}$ and Claim 30 (1)). Thus

$$S^1_\epsilon \subseteq \bigcup \{S_{q_{\epsilon,\nu}} : \nu \in \Lambda^*_{\alpha_\epsilon}\}$$

and this is a union of $\leq |\boldsymbol{T}_{<\alpha_\epsilon}| \leq \alpha_\epsilon$ sets, each one is a tenuous subset of $S_* \setminus (\alpha_\epsilon + 1)$ and, in particular, non-stationary in λ. So their union will be the union of $\leq \alpha_\epsilon < \lambda$ (as λ is inaccessible) non-stationary sets, and as $\lambda = cf(\lambda)$ and by Claim 9 it follows that S^1_ϵ is a non-stationary subset of λ.

Next, let $\alpha_\epsilon < \delta < \lambda$:

- If δ is an inaccessible cardinal in S_*, we want to show that $S^1_\epsilon \restriction \delta$ is non-stationary in δ: as $2^{\alpha_\epsilon} < \delta$ (by inaccessibility of δ) and since for all $\eta \in \Lambda^2_\epsilon$ the set S_{r_η} is tenuous, in particular $S_{r_\eta} \restriction \delta$ is non-stationary so S^1_ϵ is the union of $< \delta = cf(\delta)$ non-stationary sets and by Claim 9 it is not stationary.
- Else, in particular S_* does not reflect to δ, then the set $S_* \restriction \delta$ is non-stationary in δ and so also in $S^1_\epsilon \restriction \delta$ by (8).

This shows that S^1_ϵ is tenuous and therefore $S_\epsilon = S_\zeta \cup \{\alpha_\epsilon\} \cup S^1_\epsilon$ that is also tenuous.

Moreover, we can see that E_ϵ is disjoint to $S_\zeta \cup \{\alpha_\epsilon\}$ as a subset of $E_\zeta \setminus (\alpha_\epsilon + 1)$ and by the induction hypothesis; in addition for all $\delta \in E_\epsilon$, $\delta \in \bigcap_{\nu' \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}} C_{\nu'}$. For all $\eta \in \Lambda^2_\epsilon$, $S_{r_\eta} \subseteq S_{q_{\epsilon,\nu}}$ for some $\nu \in \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\alpha_\epsilon} \subseteq \Lambda^1_\epsilon \cap \boldsymbol{T}_{<\delta}$, so the set C_ν is disjoint to S_{r_η} and, in particular, $\delta \notin S_{r_\eta}$. Finally we have that $S_\epsilon \cap E_\epsilon = \emptyset$.

Define the condition.

The condition will be $p_\epsilon = p^*_{\varrho,\lambda,S_\epsilon}$ and so $p_\epsilon \in \mathbb{Q}_\lambda$. We would like $p_\epsilon \subseteq p_\zeta$ to hold, for the condition to be stronger than in the previous level; this is formed as we are using a larger tenuous set than the one of p_ζ.

Statement: *For all $\rho \in p_\zeta$, $\rho \in p_\epsilon$ if and only if ($\lg(\rho) < \alpha_\epsilon$) or ($\alpha_\epsilon \leq \lg(\rho)$ and $(\forall \eta \in \Lambda^2_\epsilon)(\rho \in r_\eta)$).*

Proof. (1) If $\rho \in p_\epsilon$, then either (a) $\lg(\rho) < \alpha_\epsilon$ or (b) $\alpha_\epsilon \leq \lg(\rho)$. In (b), let $\eta \in \Lambda^2_\epsilon$ and suppose δ_1 is minimal such that $\rho \restriction \delta_1 \notin r_\eta$ so $\delta_1 \in S_{r_\eta}$, in which case δ_1 is successful and

$$\rho \restriction \delta_1 \in \lim_{\delta_1}(r^*_{\delta_1}) \setminus \Big(\bigcup\{\lim_{\delta_1}(q^*_{\delta_1,\eta'}) : \eta' \in \Lambda^*_{\delta_1}\}\Big).$$

Thus $\rho \restriction \delta_1 \notin p_\epsilon \Rightarrow \rho \notin p_\epsilon$—a contradiction.

We then have that $\alpha_\epsilon \leq \lg(\rho) \to (\forall \eta \in \Lambda^2_\epsilon)(\rho \in r_\eta)$.

(2) For the other direction, if ρ is such that $\lg(\rho) < \alpha_\epsilon$, $\rho \in p_\zeta$ and if $\alpha_\epsilon \leq \lg(\rho)$, let $\rho \restriction \alpha_\epsilon =: \eta$; then $\eta \in \Lambda^2_\epsilon \wedge \rho \in r_\eta$. If $\rho \notin p_\epsilon$, for some $\lg(\varrho) < \delta_1 \in S_\epsilon$, $\rho \restriction \delta_1 \in \lim_{\delta_1}(r^*_{\delta_1}) \setminus (\bigcup\{\lim_{\delta_1}(q^*_{\delta_1,\eta'}) : \eta' \in \Lambda^*_{\delta_1}\})$.

(a) If $\delta_1 < \alpha_\epsilon$, then $\delta_1 \in S_\zeta$ and $\rho \restriction \delta_1 \notin p_\zeta$—a contradiction.

(b) If $\delta_1 > \alpha_\epsilon$, then $\delta_1 \in S^1_\epsilon$ so for some $\eta' \in \Lambda^2_\epsilon$, $\delta_1 \in S_{r_{\eta'}}$ and $\rho \restriction \delta_1 \notin r_{\eta'}$—a contradiction.

(c) If $\delta_1 = \alpha_\epsilon$, we have $\rho \in r_\eta = (q^*_{\alpha_\epsilon,\nu})^{[\eta]}$—a contradiction.

We are done with the statement.

Now observe:

- We can easily verify that $p_\zeta \leq_{\mathbb{Q}_\lambda} p_\epsilon$.
- The set $\{r_\eta : \eta \in \Lambda^2_\epsilon\}$ is predense above p_ϵ in $\mathbb{Q}_\lambda$: Let $p_\epsilon \leq q$; assume there are no forcing conditions in $\{q \cap r_\eta : \eta \in \Lambda^2_\epsilon\}$. Recall that

$$\rho \in p_\epsilon \Leftrightarrow \rho \in \bigcap\{r_\eta : \eta \in \Lambda^2_\epsilon\};$$

  then $q = q \cap p_\epsilon = \bigcap\{q \cap r_\eta : \eta \in \Lambda^2_\epsilon\}$—a contradiction, as the right side cannot be a condition.
- As in fact the pruning had been to get p_ϵ exactly by this set.
- Thus, as for all $\eta \in \Lambda^2_\epsilon$, $r_\eta \Vdash \underset{\sim}{\tau}(\zeta) = \gamma_{\epsilon,\nu_\eta}$ for some $\nu_\eta \trianglelefteq \eta$, we can write $u_\zeta = \{\gamma_{\epsilon,\nu_\eta} : \eta \in \Lambda^2_\epsilon\}$ and have $p_\epsilon \Vdash$ "$\underset{\sim}{\tau}(\zeta) \in u_\zeta$".

Clause (10) holds and so the construction is possible.

— Let $S' = \bigcup_{\epsilon<\lambda} S_\epsilon$; this is non-stationary because $\Delta_{\epsilon<\lambda} E_\epsilon \cap S' = \emptyset$ and a tenuous set as for all $\delta < \lambda$ there is $\epsilon < \lambda$ with $S' \cap \delta = S_\epsilon \cap \delta$ (by clause (9)).

— Lastly, let $q = p^*_{\varrho,\lambda,S'}$. Then indeed $p \leq q$ and we can define $g : \lambda \to \lambda$ by: for $\epsilon < \lambda$, let $g(\epsilon) = \sup\{u_\epsilon\}$ where u_ϵ is from clause (10)(b) in our induction, so as u_ζ is a subset of λ of cardinality $< \lambda$, clearly $g(\epsilon) < \lambda$ indeed. So g is a function from λ into λ which belongs to $\boldsymbol{V}$. Also by clause (10)(b) we have $p_{\epsilon+1} \Vdash$ "$\underset{\sim}{\tau}(\epsilon) \in u_\epsilon$", hence $p_{\epsilon+1} \Vdash$ "$\underset{\sim}{\tau}(\epsilon) \leq g(\epsilon)$". But q is above $p_{\epsilon+1}$ for every $\epsilon < \lambda$, hence $q \Vdash$ "$\underset{\sim}{\tau}(\epsilon) \leq g(\epsilon)$".

As p is stronger than our original p we are done proving the theorem. ∎

Corollary 41: *The forcing $\mathbb{Q}_\lambda$ resembles Random Real forcing for λ.*

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