

On possible restrictions of the null ideal

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We prove that the null ideal restricted to a non-null set of reals could be isomorphic to a variety of sigma ideals. Using this, we show that the following are consistent: (1) There is a non-null subset of plane each of whose non-null subsets contains three collinear points. (2) There is a partition of a non-null set of reals into null sets, each of size \aleph_1 , such that every transversal of this partition is null.

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1. Introduction

In [9], starting with a measurable cardinal, Shelah constructed a model in which the null ideal restricted to a non-null set of reals is \aleph_1 -saturated. We would like to prove the following generalization of this.

Theorem 1.1. *Suppose κ is a cardinal of uncountable cofinality and \mathcal{I} is a sigma ideal on κ that contains all bounded subsets of κ . Then there is a ccc forcing \mathbb{P} such that in $V^{\mathbb{P}}$, the null ideal restricted to some non-null set is isomorphic to the ideal generated by \mathcal{I} .*

An immediate corollary of Theorem 1.1 (see Lemma 5.1) is that the null ideal restricted to some non-null set of reals can be isomorphic to the non-stationary

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ideal on ω_1 . This answers a question of Fremlin [2, Problem 9; 3, Problem DX]. We also give the following applications.

Theorem 1.2. *The following is consistent: There is a non-null $X \subseteq \mathbb{R}^2$ such that for every $Y \subseteq X$ if Y is non-null, then Y contains three collinear points.*

Theorem 1.3. *The following is consistent: There exists a non-null $X \subseteq \mathbb{R}$ and a partition $\langle X_i : i < \omega_1 \rangle$ of X into null sets of size \aleph_1 such that for every $Y \subseteq X$, if $|Y \cap X_i| = 1$ for every $i < \omega_1$, then Y is null.*

Recently, we were able to combine the methods of this paper with that of Komjath's in [4] to construct a model where the null and the meager ideals are both somewhere \aleph_1 -saturated. This will appear in [7].

On notation: μ denotes the standard Lebesgue measure on 2^ω . A subset $W \subseteq 2^\omega$ is fat if for every clopen set C , either $W \cap C = \emptyset$ or $\mu(W \cap C) > 0$. A subtree $T \subseteq {}^{<\omega}2$ is fat if $[T] = \{x \in 2^\omega : (\forall n < \omega)(x \upharpoonright n \in T)\}$ is fat. For a clopen subset $C \subseteq 2^\omega$, define $\text{supp}(C)$ to be the smallest (finite) set F such that $(\forall x, y \in 2^\omega)((x \upharpoonright F = y \upharpoonright F) \Rightarrow (x \in C \Leftrightarrow y \in C))$. Random denotes the random real forcing. Note that $\{[T] : T \subseteq {}^{<\omega}2 \text{ is a fat tree}\}$ is dense in Random. Cohen_κ denotes the forcing for adding κ Cohen reals. In forcing we use the convention that a larger condition is the stronger one — $p \geq q$ means p extends q . If \mathbb{P}, \mathbb{Q} are forcing notions and $\mathbb{Q} \subseteq \mathbb{P}$, we write $\mathbb{Q} \triangleleft \mathbb{P}$ if every maximal antichain in \mathbb{Q} is also a maximal antichain in \mathbb{P} .

2. On Category

The category analogue of Theorem 1.1 follows from the argument in [4] where Komjath proved the following: Let κ be measurable. Then there is a ccc forcing \mathbb{P} such that in $V^\mathbb{P}$, the meager ideal restricted to some non-meager set of reals is \aleph_1 -saturated.

Let us sketch this argument. Suppose $\kappa \geq \omega_1$ and \mathcal{I} is a sigma ideal on κ that contains all singletons. Let $\mathbb{P} = \text{Cohen}_\kappa$ adding κ Cohen reals $\langle c_i : i < \kappa \rangle$ where each $c_i \in 2^\omega$. In $V^\mathbb{P}$, let \mathbb{Q} be the finite support product $\prod \{\mathbb{Q}_A : A \in \mathcal{I}\}$ where \mathbb{Q}_A is defined as follows: $p \in \mathbb{Q}_A$ if and only if $p = (F_p, N_p, \bar{n}_p, \bar{\sigma}_p) = (F, N, \bar{n}, \bar{\sigma})$ where:

- F is a finite subset of A .
- $N < \omega$.
- $\bar{n} = \langle n_k : k \leq N \rangle$ is a strictly increasing sequence of integers with $n_0 = 0$.
- $\bar{\sigma} = \langle \sigma_k : k < N \rangle$ where each $\sigma_k \in {}^{[n_k, n_{k+1})}2$.

For $p, q \in \mathbb{Q}_A$, define $p \leq q$ if and only if $F_p \subseteq F_q$, $N_p \leq N_q$, $\bar{n}_p \leq \bar{n}_q$, $\bar{\sigma}_p \leq \bar{\sigma}_q$ and for every $N_p \leq k < N_q$, for every $i \in F_p$, $\sigma_{q,k} \neq c_i \upharpoonright [n_{q,k}, n_{q,k+1})$. Note that \mathbb{Q}_A is a sigma centered forcing making $\{c_i : i \in A\}$ meager. The set of conditions $(p, q) \in \mathbb{P} \star \mathbb{Q}$ where $p \in \mathbb{P}$ and for each $A \in \text{dom}(q)$, p forces an actual value to $q(A)$ is dense in $\mathbb{P} \star \mathbb{Q}$. Put $\mathbb{S} = \mathbb{P} \star \mathbb{Q}$.

We claim that, in $V^\mathbb{S}$ the meager ideal restricted to $W = \{c_i : i < \kappa\}$ is isomorphic to $\mathcal{J} = \{X \subseteq \kappa : (\exists A \in \mathcal{I})(X \subseteq A)\}$ — the ideal generated by \mathcal{I} . It

is clear that the for each $X \in \mathcal{J}$, $\{c_i : i \in X\}$ is meager. Also, in $V^{\mathbb{S}}$, for every $X \subseteq \kappa$, if $X \notin \mathcal{J}$, then $\{c_i : i \in X\}$ is non-meager. To see this, let $B \subseteq 2^\omega$ be a meager F_σ -set coded in $V^{\mathbb{S}}$. Since \mathbb{S} is ccc, we can find a countable $\mathcal{F} \subseteq \mathcal{I}$ such that B is coded in $V[\langle c_i : i \in \bigcup \mathcal{F} \rangle][\prod\{G_{\mathbb{Q}_A} : A \in \mathcal{F}\}]$. Choose $i_* \in X \setminus (\bigcup \mathcal{F})$. Note that for each $A \in \mathcal{F}$, $\mathbb{Q}_A \in V[\langle c_i : i \in A \rangle]$. It follows that c_{i_*} is Cohen over $V[\langle c_i : i \in \bigcup \mathcal{F} \rangle][\prod\{G_{\mathbb{Q}_A} : A \in \mathcal{F}\}]$. Hence $c_{i_*} \notin B$ and therefore $\{c_i : i \in X\}$ is non-meager.

This proof does not have an obvious analogue in the case of the null ideal since e.g. if we start by adding a set $X = \{r_i : i < \kappa\}$ of κ random reals, and do a finite support iteration (for ccc) to make certain subsets of X null, then we would inevitably add Cohen reals at stages of countable cofinality making X null. To get around this difficulty, Shelah came up with the following idea in [8, 9].

Let $\langle X_\alpha : \alpha < \lambda \rangle$ be a list of members of \mathcal{I} , each occurring $\lambda = 2^\kappa$ times. Perform a finite support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \lambda + \kappa \rangle$ with limit \mathbb{P} where for $\alpha < \lambda$, \mathbb{Q}_α is Cohen forcing with generic real τ_α and for $\xi < \kappa$, $\mathbb{Q}_{\lambda+\xi} = (\text{Random})^{V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi} \rangle]}$, where $A_{\lambda+\xi} = \{\alpha < \lambda : \xi \notin X_\alpha\} \cup [\lambda, \lambda + \xi)$, with generic partial random real $\tau_{\lambda+\xi}$. Using the fact that \mathbb{P} is ccc, it is easy to show that if $A \subseteq \kappa$ is not in the ideal generated by \mathcal{I} , then $\{\tau_{\lambda+\xi} : \xi \in A\}$ is not null in $V^{\mathbb{P}}$. The other direction is supposed to follow from the fact that for $\alpha < \lambda$, τ_α codes a null G_δ -set that should cover $\{\tau_{\lambda+\xi} : \xi \in X_\alpha\}$ because for $\xi \in X_\alpha$, the memory $A_{\lambda+\xi}$ of the partial random $\tau_{\lambda+\xi}$ does not contain α . But $A_{\lambda+\xi}$ contains $[\lambda, \lambda + \xi)$ and the partial random added at these stages can use τ_α so this is not a simple product forcing argument as in the category case.

Some remarks on the structure of the proof of Theorem 1.1 follow. Section 2 introduces the model witnessing the theorem and proves some preliminary facts about the iteration used in defining it. The main difficulty in the verification appears in Claim 3.8 whose proof is reduced to constructing a condition p_* satisfying the hypothesis of Lemma 3.9. The concluding remarks in Sec. 3 explain the difficulty in an inductive construction of p_* due to the use of partial memories for the random reals. Section 4 introduces the main tools to get around this difficulty via Definition 4.6 and Claim 4.7 where the existence of p_* is reduced to the existence of certain finitely additive measures on $\mathcal{P}(\omega) \cap V^{\mathbb{P}}$. Lemma 4.8 completes the proof by showing that such measures exist. The use of blueprints is to allow a sufficiently general statement in this lemma so that an inductive proof using automorphisms of higher iterations can be given.

3. Forcing

Let κ, \mathcal{I} be as in the hypothesis of Theorem 1.1. Put $\lambda_0 = 2^\kappa$. For $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, define the following.

- (1) $\langle X_\alpha : \alpha < \lambda_0^{+\omega} \rangle$ is a sequence of members of \mathcal{I} .
- (2) For every $n < \omega$ and $X \in \mathcal{I}$, $|\{\alpha < \lambda_0^{+n} : X_\alpha = X\}| = \lambda_0^{+n}$.
- (3) For $\xi < \kappa$, $A_{\lambda+\xi} = A_{\lambda+\xi}^\lambda = \{\alpha < \lambda : \xi \notin X_\alpha\} \cup [\lambda, \lambda + \xi)$.

(4) $\bar{\mathbb{P}}_\lambda = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \lambda + \kappa \rangle$ is a finite support iteration with limit $\mathbb{P}_{\lambda+\kappa}$ such that

- (a) For $\alpha < \lambda$, \mathbb{Q}_α is Cohen forcing with generic real $\tau_\alpha \in \omega^\omega$.
- (b) For $\xi < \kappa$, $\mathbb{Q}_{\lambda+\xi} = (\text{Random})^{V[\langle \tau_i : i \in A_{\lambda+\xi} \rangle]}$ with generic partial random $\tau_{\lambda+\xi} \in 2^\omega$.

The reason for considering iterations $\bar{\mathbb{P}}_\lambda$ for $\lambda > \lambda_0$ will become clear during the proof of Lemma 4.8 where we use automorphisms of $\mathbb{P}_{\lambda+\kappa}$ for $\lambda > \lambda_0$ to construct certain finitely additive measures on $\mathcal{P}(\omega) \cap V^{\mathbb{P}_{\lambda_0+\kappa}}$.

The following is easily proved by induction on $\xi \leq \kappa$ using the standard properties of Cohen and random forcings.

Claim 3.1. *For every $\xi \leq \kappa$, $\dot{x} \in 2^\omega \cap V^{\mathbb{P}_{\lambda+\xi}}$, there are a Borel function $B : \omega^\omega \rightarrow 2^\omega$ and $\langle (n_k, \gamma_k) : k < \omega \rangle$ such that every $\gamma_k < \lambda + \xi$ and $n_k < \omega$, and $\Vdash_{\mathbb{P}} \dot{x} = B(\langle \tau_{\gamma_k}(n_k) : k < \omega \rangle)$.*

Definition 3.2. (1) Let $\mathbb{P}'_{\lambda+\kappa}$ be the set of conditions $p \in \mathbb{P}_{\lambda+\kappa}$ satisfying the following requirements.

- (a) For each $\alpha \in \lambda \cap \text{dom}(p)$, $p(\alpha) \in {}^{<\omega}\omega$. In this case, define $\text{supp}(p(\alpha)) = \phi$.
- (b) Let $\text{dom}(p) \cap [\lambda, \lambda + \kappa) = \{\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}\}$. For each $j < n$, there are a Borel function B , $\rho \in {}^{<\omega}2$ and $\langle (n_k, \gamma_k) : k < \omega \rangle$ such that $n_k < \omega$, $\gamma_k \in A_{\alpha_j}$, the range of B consists of fat trees in ${}^{<\omega}2$ and $\Vdash_{\mathbb{P}_\alpha} p(\alpha_j) = [B(\langle \tau_{\gamma_k}(n_k) : k < \omega \rangle)]$ is a subset of $[\rho]$ of relative measure more than $1 - 2^{-(n-j+10)}$. In this case, define $\text{supp}(p(\alpha_j)) = \{\gamma_k : k < \omega\}$.

- (2) For $\xi \leq \kappa$, define $\mathbb{P}'_{\lambda+\xi} = \{p \in \mathbb{P}'_{\lambda+\kappa} : \text{dom}(p) \subseteq \lambda + \xi\}$.
- (3) For $A \subseteq \lambda + \kappa$, define $\mathbb{P}'_{\lambda,A} = \mathbb{P}'_A = \{p \in \mathbb{P}'_{\lambda+\kappa} : \text{dom}(p) \subseteq A, (\forall \alpha \in \text{dom}(p))(\text{supp}(p(\alpha)) \subseteq A)\}$. Note that if $A = \lambda + \xi$ for some $\xi \leq \kappa$, then this agrees with the notation in (2).

Using Claim 3.1 and the Lebesgue density theorem, it is easy to see that $\mathbb{P}'_{\lambda+\xi}$ is dense in $\mathbb{P}_{\lambda+\xi}$.

Suppose $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, $\xi_* \leq \kappa$. Let $h : \lambda + \xi_* \rightarrow \lambda + \xi_*$ be a bijection satisfying the following:

- (1) $h \upharpoonright [\lambda, \lambda + \xi_*)$ is the identity.
- (2) For each $\xi < \xi_*$ and $\alpha < \lambda$, $\alpha \in A_{\lambda+\xi} \Leftrightarrow h(\alpha) \in A_{\lambda+\xi}$.

Define $\hat{h} : \mathbb{P}'_{\lambda+\xi_*} \rightarrow \mathbb{P}'_{\lambda+\xi_*}$ as follows. For $p \in \mathbb{P}'_{\lambda+\xi_*}$, put $\hat{h}(p) = p'$ where:

- (a) $\text{dom}(p') = \{h(\alpha) : \alpha \in \text{dom}(p)\}$.
- (b) For $\alpha \in \text{dom}(p) \cap \lambda$, $p'(h(\alpha)) = p(\alpha)$.
- (c) For $\alpha \in \text{dom}(p) \cap [\lambda, \lambda + \xi_*)$, $\Vdash_{\mathbb{P}_\alpha} p'(\alpha) = B(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega \rangle)$ where B , $\langle (n_k, \gamma_k) : k < \omega \rangle$ are as in Definition 3.2 for coordinate α .

Claim 3.3. *\hat{h} is an automorphism of $\mathbb{P}'_{\lambda+\xi_*}$.*

Proof. By induction on ξ_* . □

Lemma 3.4. *Let $\xi_* \leq \kappa$, $A \subseteq \lambda + \xi_*$ and $[\lambda, \lambda + \xi_*) \subseteq A$. Suppose for every countable $B \subseteq \lambda$, there is a bijection $h : \lambda + \xi_* \rightarrow \lambda + \xi_*$ such that $h \upharpoonright ((B \cap A) \cup [\lambda, \lambda + \xi_*))$ is the identity, $(\forall \xi < \xi_*)(\forall \alpha < \lambda)(\alpha \in A_{\lambda+\xi} \Leftrightarrow h(\alpha) \in A_{\lambda+\xi})$, $h[B] \subseteq A$ and $h[B \cap A_{\lambda+\xi_*}] \subseteq A \cap A_{\lambda+\xi_*}$. Then $\mathbb{P}'_A \leq \mathbb{P}'_{\lambda+\xi_*}$.*

Proof. By induction on ξ_* . If $\xi_* = 0$ or limit this is clear. So assume $\xi_* = \xi + 1$ and put $\alpha = \lambda + \xi$. By inductive hypothesis, $\mathbb{P}'_{A \cap \alpha} \leq \mathbb{P}'_\alpha$ so it suffices to check the following: If $\{p_n : n < \omega\} \subseteq \mathbb{P}'_A$, $p \in \mathbb{P}'_{A \cap \alpha}$ and $p \Vdash_{\mathbb{P}'_{A \cap \alpha}} \{p_n(\alpha) : n < \omega, p_n \upharpoonright \alpha \in G_{\mathbb{P}'_{A \cap \alpha}}\}$ is predense in $(\text{Random})^{V[\langle \tau_\beta : \beta \in A \cap A_\alpha \rangle]}$, then $p \Vdash_{\mathbb{P}'_\alpha} \{p_n(\alpha) : n < \omega, p_n \upharpoonright \alpha \in G_{\mathbb{P}'_\alpha}\}$ is predense in $(\text{Random})^{V[\langle \tau_\beta : \beta \in A_\alpha \rangle]}$. Suppose this fails and choose $q \in \mathbb{P}'_\alpha$, D , $\langle (n_k, \gamma_k) : k < \omega \rangle$ such that $q \geq p$, D is a Borel function whose range consists of fat trees, $\gamma_k \in A_\alpha$ and $q \Vdash_{\mathbb{P}'_\alpha} r = [D(\langle \tau_{\gamma_k}(n_k) : k < \omega \rangle)]$ is incompatible with every member of $\{p_n(\alpha) : n < \omega, p_n \upharpoonright \alpha \in G_{\mathbb{P}'_\alpha}\}$. Let $W = \text{dom}(q) \cup \bigcup \{\text{supp}(q(\beta)) : \beta \in \text{dom}(q)\} \cup \bigcup \{\text{supp}(p(\beta)) : \beta \in \text{dom}(p)\} \cup \bigcup \{\text{dom}(p_n) : n < \omega\} \cup \bigcup \{\text{supp}(p_n(\beta)) : n < \omega, \beta \in \text{dom}(p_n)\} \cup \{\gamma_k : k < \omega\}$. Put $B = W \cap \lambda$. Choose $h : \alpha \rightarrow \alpha$ satisfying the hypothesis for this B . So \hat{h} is an automorphism of \mathbb{P}'_α . As $h[B] \subseteq A$, $\hat{h}(q) \in \mathbb{P}'_A$. Since $h \upharpoonright (B \cap A)$ is identity, it follows that $\hat{h}(p) = p$ and for every $n < \omega$, $\hat{h}(p_n) = p_n$. Since $\{\gamma_k : k < \omega\} \subseteq W$ and $h[B \cap A_\alpha] \subseteq A \cap A_\alpha$, we have that $\Vdash_{\mathbb{P}'_\alpha} r' = \hat{h}(r) = [D(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega \rangle)] \in (\text{Random})^{V[\langle \tau_\beta : \beta \in A \cap A_\alpha \rangle]}$. It follows that $\hat{h}(q) \Vdash_{\mathbb{P}'_\alpha} r'$ is incompatible with every condition in $\{p_n(\alpha) : n < \omega, p_n \upharpoonright \alpha \in G_{\mathbb{P}'_\alpha}\}$. Hence also $p = \hat{h}(p) \leq \hat{h}(q) \Vdash_{\mathbb{P}'_{A \cap \alpha}} r'$ is incompatible with every condition in $\{p_n(\alpha) : n < \omega, p_n \upharpoonright \alpha \in G_{\mathbb{P}'_{A \cap \alpha}}\}$ — contradiction. □

Corollary 3.5. *For every $\xi_* < \kappa$, $\mathbb{P}'_{A_{\lambda+\xi_*}} \leq \mathbb{P}'_{\lambda+\xi_*}$.*

Proof. Let $B \subseteq \lambda$ be countable. By Lemma 3.4, it suffices to construct a bijection $h : \lambda + \xi_* \rightarrow \lambda + \xi_*$ such that $h \upharpoonright ((B \cap A_{\lambda+\xi_*}) \cup [\lambda, \lambda + \xi_*))$ is identity, $(\forall \xi < \xi_*)(\forall \alpha < \lambda)(\alpha \in A_{\lambda+\xi} \Leftrightarrow h(\alpha) \in A_{\lambda+\xi})$ and $h[B] \subseteq A_{\lambda+\xi_*}$. For each $x \subseteq \xi_*$, let $W_x = \{\alpha < \lambda : X_\alpha \cap \xi_* = x\}$. Let $W_{x,0} = \{\alpha \in W_x : \xi_* \in X_\alpha\}$ and $W_{x,1} = \{\alpha \in W_x : \xi_* \notin X_\alpha\}$ so $W_x = W_{x,0} \sqcup W_{x,1}$. Since every bounded subset of κ is in \mathcal{I} , it follows that for every $x \subseteq \xi_*$, $|W_{x,0}| = |W_{x,1}| = \lambda$. So for each $x \subseteq \xi_*$, we can choose a bijection $h_x : W_x \rightarrow W_x$ such that $h_x[W_{x,0} \cap B] \subseteq W_{x,1}$ and $h_x \upharpoonright (W_{x,1} \cap B)$ is identity. Put $h \upharpoonright \lambda = \bigcup \{h_x : x \subseteq \xi_*\}$. □

Put $\mathbb{P} = \mathbb{P}_{\lambda+\kappa}$. In $V^\mathbb{P}$, let $\mathcal{J} = \{X \subseteq \kappa : (\exists Y \in \mathcal{I})(X \subseteq Y)\}$ be the ideal generated by \mathcal{I} . Since \mathbb{P} is ccc, \mathcal{J} is a sigma ideal.

Claim 3.6. *In $V^\mathbb{P}$, for every $A \in \mathcal{J}^+$, $\{\tau_{\lambda+\xi} : \xi \in A\}$ is not null.*

Proof. Let N be a null Borel set coded in $V^\mathbb{P}$. Since \mathbb{P} is ccc we can find a countable family $\{p_k : k < \omega\}$ such that $\{k < \omega : p_k \in G_\mathbb{P}\}$ determines N . Let $W = \bigcup \{\text{dom}(p_k) : k < \omega\} \cup \{\text{supp}(p_k(\alpha)) : k < \omega, \alpha \in \text{dom}(p_k)\}$. Note that W is

countable. Since $cf(\kappa) \geq \aleph_1$ and \mathcal{I} contains all bounded subsets of κ , we can find $\xi \in A$ such that $\lambda + \xi > \sup(W)$ and $\xi \notin \bigcup \{X_i : i \in W \cap \lambda\}$. It follows that N is coded in $V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi} \rangle]$ and hence $\tau_{\lambda+\xi} \notin N$. \square

Definition 3.7. For each $n < \omega$, let $\langle C_k^m : k < \omega \rangle$ be a one-to-one listing of all clopen subsets of 2^ω of measure 2^{-n} . For $\alpha < \lambda$, define $\dot{N}_\alpha = \bigcap_k \bigcup_{n>k} C_{\tau_\alpha(n)}^m$. So \dot{N}_α is a null G_δ -set coded by τ_α .

To complete the proof of Theorem 1.1, we need to show that in $V^{\mathbb{P}}$, for every $A \in \mathcal{I}$, $\{\tau_{\lambda+\xi} : \xi \in A\}$ is null. For this it suffices to show the following. Its proof will be completed at the end of Sec. 4 with the proof of Lemma 4.8.

Claim 3.8. *Suppose $\xi < \kappa$ and $\alpha \in \lambda \setminus A_{\lambda+\xi}$. Then $\Vdash_{\mathbb{P}} \tau_{\lambda+\xi} \in \dot{N}_\alpha$.*

Proof. Suppose not. Choose $p \in \mathbb{P}'_{\lambda+\xi+1}$, $k_* < \omega$ such that $p \Vdash (\forall k \geq k_*)(\tau_{\lambda+\xi} \notin C_{\tau_\alpha(k)}^k)$. We can assume that $\alpha \in \text{dom}(p)$ and $p(\alpha) = \sigma_* \in {}^{l_*}\omega$ for some $l_* > k_*$. Choose a Borel function B and $\langle (n_j, \gamma_j) : j < \omega \rangle$ such that $\gamma_j \in A_{\lambda+\xi}$ (see Definition 3.2(1)(b)), range of B consists of fat trees and $\Vdash_{\mathbb{P}_{\lambda+\xi}} B(\langle \tau_{\gamma_j}(n_j) : j < \omega \rangle) = \dot{T}$ and $[\dot{T}] = p(\lambda+\xi)$. It follows that $p \upharpoonright (\lambda+\xi) \Vdash_{\mathbb{P}_{\lambda+\xi}} (\forall k \geq k_*)([\dot{T}] \cap C_{\tau_\alpha(k)}^k = \phi)$. Choose $\langle \alpha_i : i < \lambda \rangle$ such that the following hold.

- For all $i < j < \lambda$, $\alpha_i < \alpha_j < \lambda$.
- $X_{\alpha_i} = X_\alpha$.
- $\alpha_i \notin \text{dom}(p) \cup \{\text{supp}(p(\beta)) : \beta \in \text{dom}(p)\}$.

For $i < \lambda$, define $q_i \in \mathbb{P}'_{\lambda+\xi}$ as follows.

- $\text{dom}(q_i) = (\text{dom}(p \upharpoonright (\lambda + \xi)) \setminus \{\alpha\}) \cup \{\alpha_i\}$.
- $q_i(\alpha_i) = p(\alpha) = \sigma_*$.
- If $\beta \in \text{dom}(p) \cap (\lambda \setminus \{\alpha\})$, then $q_i(\beta) = p(\beta)$.
- If $\beta \in \text{dom}(p) \cap [\lambda, \lambda + \xi)$, and $B, \langle (n_j, \gamma_j) : j < \omega \rangle$ are such that $\Vdash_{\mathbb{P}_\beta} [B(\langle \tau_{\gamma_j}(n_j) : j < \omega \rangle)] = p(\beta)$, then $q_i(\beta) = [B(\langle \tau_{\gamma'_j}(n_j) : j < \omega \rangle)]$ where $\gamma'_j = \gamma_j$ if $\gamma_j \neq \alpha$ and α_i otherwise.

Since interchanging α, α_i coordinates gives rise to an automorphism of $\mathbb{P}'_{\lambda+\xi}$ that fixes the name \dot{T} (as $\alpha, \alpha_i \notin A_{\lambda+\xi}$), it follows that, for each $i < \lambda$, $q_i \Vdash (\forall k \geq k_*)([\dot{T}] \cap C_{\tau_{\alpha_i(k)}}^k = \phi)$.

Let $\text{dom}(p) \cap \lambda = \{\alpha\} \sqcup \{\beta_0 < \beta_1 < \dots < \beta_{r_*-1}\}$ and $\text{dom}(p) \cap [\lambda, \lambda + \xi) = \{\lambda + \xi_0 < \lambda + \xi_1 < \dots < \lambda + \xi_{n_*-1}\}$. For $k < n_*$, let ρ_k be as in clause (1)(b) of Definition 3.2 for the $(\lambda + \xi_k)$ th coordinate of p . It follows that $\{(q_j, \alpha_j) : j < \omega\}$ satisfies the following.

- (a) $\text{dom}(q_j) = \{\alpha_j\} \sqcup \{\beta_k : k < r_*\} \sqcup \{\lambda + \xi_k : k < n_*\}$.
- (b) For every $j < \omega$, $k < r_*$, $q_j(\beta_k) = p(\beta_k) = \sigma_k$.
- (c) For every $j < \omega$, $q_j(\alpha_j) = p(\alpha) = \sigma_*$.

- (d) For every $j < \omega$, $k < n_*$, $\Vdash_{\mathbb{P}_{\lambda+\xi_k}} q_j(\lambda + \xi_k)$ is a fat subset of $[\rho_k]$ of fractional measure more than $1 - 2^{-(n_*-k+10)}$.
- (e) For every $j < \omega$, $q_j \Vdash_{\mathbb{P}_{\lambda+\xi}} (\forall k \geq k_*)([\dot{T}] \cap C_{\tau_{\alpha_j}(k)}^k = \phi)$.

Recall that $|\sigma_*| = l_* > k_*$. We will extend each q_j on the α_j th coordinate to get p_j as follows. For each $n < \omega$, let $K_n = \{k < \omega : \text{supp}(C_k^{l_*}) \subseteq n\}$. Note that for all $n \geq l_*$, $|K_n| = \binom{2^n}{2^{n-l_*}}$. Define $\langle k_n : n < \omega \rangle$ by $k_0 = 0$, $k_{n+1} - k_n = \binom{2^n}{2^{n-l_*}}$. Let $f : \omega \rightarrow \omega$ be such that $f[[k_n, k_{n+1}]] = K_n$. For each $j < \omega$, $\gamma \in \text{dom}(q_j)$ define

$$p_j(\gamma) = \begin{cases} q_j(\gamma) & \text{if } \gamma \neq \alpha_j, \\ \sigma_* \cup \{(l_*, f(j))\} & \text{if } \gamma = \alpha_j. \end{cases} \quad \square$$

Our plan to construct a condition p_* which will force a “large number” of p_j 's in the generic filter which will imply that $[\dot{T}]$ is finite giving us the desired contradiction. The following lemma makes this precise.

Lemma 3.9. *Suppose for some $p_* \in \mathbb{P}$ and $\varepsilon > 0$,*

$$p_* \Vdash (\exists^\infty n) \frac{|\{j \in [k_n, k_{n+1}) : p_j \in G_{\mathbb{P}}\}|}{k_{n+1} - k_n} \geq \varepsilon.$$

Then, $p_ \Vdash [\dot{T}]$ is finite.*

Proof. For $n < \omega$, let $\dot{W}_n = \{j \in [k_n, k_{n+1}) : p_j \in G_{\mathbb{P}}\}$ and $\dot{a}_n = |\dot{T} \cap n \cdot 2|$. Note that $p_* \Vdash (\forall j \in \dot{W}_n)(\forall \sigma \in \dot{T} \cap n \cdot 2)(C_{f(j)}^{l_*} \cap \sigma = \phi)$ because $l_* > k_*$, $p_j(\alpha_j)(l_*) = f(j)$ and $p_j \Vdash (\forall k \geq k_*)(C_{\tau_{\alpha_j}(k)}^k \cap [\dot{T}] = \phi)$. It follows that $|\dot{W}_n| \leq \binom{2^n - \dot{a}_n}{2^{n-l_*}}$. Hence

$$\frac{|\dot{W}_n|}{k_{n+1} - k_n} \leq \frac{\binom{2^n - \dot{a}_n}{2^{n-l_*}}}{\binom{2^n}{2^{n-l_*}}} = \prod_{i=1}^{\dot{a}_n} \left(1 - \frac{2^{n-l_*}}{2^n - \dot{a}_n + i}\right) \leq \prod_{i=1}^{\dot{a}_n} \left(1 - \frac{2^{n-l_*}}{2^n}\right).$$

Hence

$$\frac{|\dot{W}_n|}{k_{n+1} - k_n} \leq (1 - 2^{-l_*})^{\dot{a}_n}.$$

As \dot{a}_n is increasing with n , it follows that p_* forces that $\lim_n \dot{a}_n < \infty$ and hence $[\dot{T}]$ is finite. \square

Note that $\dot{a}_n \approx 2^n \mu([\dot{T}])$ which is much better asymptotic behavior than just \dot{T} being a perfect tree (which was what was used in [8]) but this does not seem to help simplify the construction of p_* . Toward constructing p_* , we can further refine $\langle p_j : j < \omega \rangle$ as follows.

Lemma 3.10. *Suppose $K < \omega$, $F \subseteq [\lambda, \lambda + \kappa]$ is finite, $\langle \rho_\alpha : \alpha \in F \rangle$ is a sequence in ${}^{<\omega}2$, $\langle a_\alpha : \alpha \in F \rangle$ is a sequence in $(0.5, 1)$ and $\langle q_j : j < K \rangle$ is a sequence of conditions in \mathbb{P} such that for every $j < K$, $\text{dom}(q_j) = F$ and for each $\alpha \in F$, $\Vdash_{\mathbb{P}_\alpha} q_j(\alpha)$ is a subset of $[\rho_\alpha]$ of relative measure $\geq a_\alpha$. Then there exists $q^* \in \mathbb{P}'$*

with $\text{dom}(q^*) = F$ such that for every $\alpha \in F$, $\Vdash_{\mathbb{P}_\alpha} q^*(\alpha)$ is a subset of $[\rho_\alpha]$ of relative measure $\geq 2a_\alpha - 1$ and

$$q^* \Vdash_{\mathbb{P}} |\{j < K : q_j \in G_{\mathbb{P}}\}| \geq K2^{-|F|} \prod_{\alpha \in F} a_\alpha.$$

Proof. By induction on $|F|$. Suppose $F = \{\alpha\}$. Working in $V^{\mathbb{P}^\alpha}$, define $\phi = \sum_{j < K} 1_{q_j(\alpha)}$ where $1_{q_j(\alpha)}$ is the characteristic function of $q_j(\alpha)$. Put $A = \{x \in [\rho_\alpha] : \phi(x) \geq \frac{Ka_\alpha}{2}\}$. It suffices to show that $\mu(A) > \mu([\rho_\alpha])(2a_\alpha - 1)$. We have

$$Ka_\alpha \mu([\rho_\alpha]) \leq \int \phi d\mu = \int_A \phi d\mu + \int_{2^\omega \setminus A} \phi d\mu \leq K\mu(A) + (\mu([\rho_\alpha]) - \mu(A)) \frac{Ka_\alpha}{2}.$$

$$\text{Solving gives } \frac{\mu(A)}{\mu([\rho_\alpha])} \geq \frac{a_\alpha}{2 - a_\alpha} > 2a_\alpha - 1.$$

Now suppose $|F| \geq 2$ and β is the largest member of F . Let $F' = F \setminus \{\beta\}$, $q'_j = q_j \upharpoonright F'$. Choose $q' \in \mathbb{P}'$ with domain F' such that for every $\alpha \in F'$, $\Vdash_{\mathbb{P}_\alpha} q'(\alpha)$ is a subset of $[\rho_\alpha]$ of relative measure $2a_\alpha - 1$ and $q' \Vdash_{\mathbb{P}} |\{j < K : q'_j \in G_{\mathbb{P}}\}| \geq K2^{-|F'|} \prod_{\alpha \in F'} a_\alpha$. Let $\dot{W} = \{j < K : q'_j \in G_{\mathbb{P}}\}$. Let $\{W_i : i < N\}$ list all subsets of K of size $\geq K2^{-|F'|} \prod_{\alpha \in F'} a_\alpha$. Choose a maximal antichain $\{r_i : i < N\}$ in \mathbb{P}_β above q' such that each $r_i \Vdash_{\mathbb{P}} \dot{W} = W_i$. Working in $V^{\mathbb{P}^\beta}$, for each $i < N$, arguing as above, we can get a condition $s_i \in \mathbb{Q}_\beta$ such that $r_i \Vdash_{\mathbb{P}_\beta} \mu(s_i) \geq 2a_\beta - 1$ and $s_i \Vdash_{\mathbb{Q}_\beta} |\{j \in W_i : q_j(\beta) \in G_{\mathbb{Q}_\beta}\}| \geq \frac{|W_i|a_\beta}{2}$. Choose q^* such that $q^*(\alpha) = q'(\alpha)$ if $\alpha \in F'$ and for each $i < N$, $r_i \Vdash_{\mathbb{P}_\beta} q^*(\beta) = s_i$. \square

Using Lemma 3.10, construct $\langle q_n^* : n < \omega \rangle$ such that:

- For every $n < \omega$, $q_n^* \in \mathbb{P}'_{\lambda+\xi}$ and $\text{dom}(q_n^*) = \{\lambda + \xi_k : k < n_\star\}$.
- For every $n < \omega$ and $k < n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_k}} q_n^*(\lambda + \xi_k)$ is a subset of $[\rho_k]$ of relative measure $\geq 2(1 - 2^{-(n_\star - k + 10)}) - 1 = 1 - 2^{-(n_\star - k + 9)}$.
- For every $n < \omega$, $q_n^* \Vdash_{\mathbb{P}_{\lambda+\xi}} |\{j \in [k_n, k_{n+1}) : p_j \upharpoonright [\lambda, \lambda + \xi) \in \mathbb{G}_{\mathbb{P}_{\lambda+\xi}}\}| \geq (k_{n+1} - k_n)2^{-n_\star} \prod_{k < n_\star} (1 - 2^{-(n_\star - k + 10)}) > (k_{n+1} - k_n)4^{-n_\star}$.

Definition 3.11. Define $\bar{p}^* = \langle p_j^* : j < \omega \rangle$ as follows:

- For every $j < \omega$, $p_j^* \in \mathbb{P}'_{\lambda+\xi}$ and $\text{dom}(p_j^*) = \text{dom}(p_j)$.
- For every $k < r_\star$, $p_j^*(\beta_k) = p_j(\beta_k) = \sigma_k$.
- For every $j < \omega$, $p_j^*(\alpha_j) = p_j(\alpha_j) = \sigma_\star \cup \{(l_\star, f(j))\}$.
- For every $n < \omega$, $k < n_\star$ and $j \in [k_n, k_{n+1})$, $p_j^*(\lambda + \xi_k) = p_j(\lambda + \xi_k) \cap q_n^*(\lambda + \xi_k)$.

$$\text{Hence } \Vdash_{\mathbb{P}_{\lambda+\xi_k}} \mu(p_j^*(\lambda + \xi_k)) \geq 1 - 2^{-(n_\star - k + 8)}.$$

Let $\dot{A} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p_k^* \upharpoonright \lambda \in \mathbb{G}_{\mathbb{P}})\}$. Note that $q = \{(\beta_k, \sigma_k) : k < r_\star\} \Vdash_{\mathbb{P}} \dot{A}$ is infinite. By Lemma 3.9, it suffices to construct a condition $p_\star \geq q$ that forces that $\dot{A} \cap \{j < \omega : p_j^* \in G_{\mathbb{P}}\}$ is infinite. What prevents us from arguing as we did in Lemma 3.10 to construct p_\star ? If $n_\star = 1$, there is no problem. But the inductive step fails. Having constructed $p^* \upharpoonright (\lambda + \xi_k)$, we must

somehow pass the information about the set $\dot{A} \cap \{j < \omega : p_j^* \upharpoonright (\lambda + \xi_k) \in G_{\mathbb{P}_{\lambda+\xi_k}}\}$ to $V[(\tau_\alpha : \alpha \in A_{\lambda+\xi_k})]$. This will be materialized by introducing a coherent family of finitely additive measures on $\mathcal{P}(\omega)$ that interact with the partial random reals appearing at stages $\{\lambda + \xi_k : k < n_\star\}$ of the iteration in a sense made precise in the next section (Definition 4.6).

4. Measures and Blueprints

An algebra \mathcal{A} is a family of subsets of ω that contains all finite subsets of ω and is closed under complementation and finite union. A finitely additive measure on an algebra \mathcal{A} is a function $m : \mathcal{A} \rightarrow [0, 1]$ that satisfies the following.

- For every finite $F \subseteq \omega$, $m(F) = 0$.
- $m(\omega) = 1$.
- If $A_1, A_2 \in \mathcal{A}$ and $A_1 \cap A_2 = \emptyset$, then $m(A_1 \cup A_2) = m(A_1) + m(A_2)$.

Suppose $m : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure and $f : \omega \rightarrow [0, 1]$. Define

$$\int f dm = \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n} \frac{ka_k}{2^n},$$

where $a_k = m(\{n < \omega : k/2^n \leq f(n) < (k+1)/2^n\})$.

The following is a standard application of the Hahn–Banach theorem.

Lemma 4.1. *Suppose $m : \mathcal{A} \rightarrow [0, 1]$ is a finitely additive measure on an algebra \mathcal{A} and $X \subseteq \omega$. Let $a \in [0, 1]$ be such that for every $A, B \in \mathcal{A}$, if $A \subseteq X \subseteq B$, then $m(A) \leq a \leq m(B)$. Then, there exists a finitely additive measure $m' : \mathcal{P}(\omega) \rightarrow [0, 1]$ that extends m and $m'(X) = a$.*

The proofs of the next two lemmas can be found in [1].

Lemma 4.2. *Suppose $m : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure. For $i = 1, 2$, let \mathbb{R}_i be a forcing notion and $\dot{m}_i \in V^{\mathbb{R}_i}$ be such that $\Vdash_{\mathbb{R}_i} \dot{m}_i : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure extending m . Then, there exists $\dot{m}_3 \in V^{\mathbb{R}_1 \times \mathbb{R}_2}$ such that $\Vdash_{\mathbb{R}_1 \times \mathbb{R}_2} \dot{m}_3 : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure extending both \dot{m}_1 and \dot{m}_2 .*

Lemma 4.3. *Suppose that $m : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure. Let $\mathbb{B} = \text{Random}$, $r \in \mathbb{B}$. Define $\dot{m}_r \in V^{\mathbb{B}}$ as follows. For $\dot{X} \in \mathcal{P}(\omega) \cap V^{\mathbb{B}}$, define*

$$\dot{m}_r(\dot{X}) = \sup \left\{ \inf \left\{ \int \frac{\mu(q \cap \llbracket n \in \dot{X} \rrbracket_{\mathbb{B}})}{\mu(q)} dm : q \geq p \right\} : p \geq r, p \in G_{\mathbb{B}} \right\}.$$

Then, the following hold.

- (1) $r \Vdash \dot{m}_r : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure extending m .
- (2) If $\dot{X} \in \mathcal{P}(\omega) \cap V^{\mathbb{B}}$ and $a > 0$ satisfy for every $n < \omega$, $\frac{\mu(r \cap \llbracket n \in \dot{X} \rrbracket_{\mathbb{B}})}{\mu(r)} \geq a$, then there exists $s \geq r$ such that $s \Vdash \dot{m}_r(\dot{X}) \geq a$.

Definition 4.4. For $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, let \mathcal{T}_λ be the set of tuples $t = (\bar{\alpha}, \bar{\eta}, \bar{\beta}, r, \bar{\xi}, n, \bar{\sigma}, \bar{\rho}, \bar{\varepsilon}) = (\bar{\alpha}^t, \bar{\eta}^t, \bar{\beta}^t, r^t, \bar{\xi}^t, n^t, \bar{\sigma}^t, \bar{\rho}^t, \bar{\varepsilon}^t)$ where

- (i) $n, r < \omega$.
- (ii) $\bar{\alpha} = \langle \alpha_j : j < \omega \rangle$ where each $\alpha_j < \lambda$ and they are pairwise distinct.
- (iii) $\bar{\eta} = \langle \eta_j : j < \omega \rangle$ where each $\eta_j \in {}^{<\omega}\omega$.
- (iv) $\bar{\beta} = \langle \beta_k : k < r \rangle$ is a sequence of pairwise distinct ordinals in $\lambda \setminus \{\alpha_j : j < \omega\}$.
- (v) $\bar{\sigma} = \langle \sigma_k : k < r \rangle$ where each $\sigma_k \in {}^{<\omega}\omega$.
- (vi) $\bar{\xi} = \langle \xi_k : k < n \rangle$ is an increasing sequence in κ .
- (vii) $\bar{\rho} = \langle \rho_k : k < n \rangle$ where each $\rho_k \in {}^{<\omega}2$.
- (viii) $\bar{\varepsilon} = \langle \varepsilon_k : k < n \rangle$, where $\varepsilon_{n-1} \in (0, 2^{-8})$ and $2\varepsilon_k \leq \varepsilon_{k+1}$ for every $k < n - 1$.

We call members of \mathcal{T}_λ blueprints. They are intended to code information about certain sequences of conditions in \mathbb{P}'_λ that look like \bar{p}^* from Definition 3.11 in the following sense.

Definition 4.5. Suppose $\bar{p} = \langle p_j : j < \omega \rangle$ is a sequence in \mathbb{P}'_λ and $t = (\bar{\alpha}, \bar{\eta}, \bar{\beta}, r, \bar{\xi}, n, \bar{\sigma}, \bar{\rho}, \bar{\varepsilon}) \in \mathcal{T}_\lambda$. We say that \bar{p} is of type t if the following hold.

- (a) For every $j < \omega$, $\text{dom}(p_j) = \{\alpha_j\} \sqcup \{\beta_k : k < r\} \sqcup \{\lambda + \xi_k : k < n\}$.
- (b) For every $j < \omega$ and $k < r$, $p_j(\beta_k) = \sigma_k$ and $p_j(\alpha_j) = \eta_j$.
- (c) For every $j < \omega$ and $k < r$, $\Vdash_{\mathbb{P}_{\lambda+\xi_k}} p_j(\lambda + \xi_k)$ is a subset of $[\rho_k]$ of relative measure more than $1 - \varepsilon_k$.

For $t \in \mathcal{T}_\lambda$, $\xi < \kappa$, we write $t \upharpoonright \xi$ for the blueprint which is obtained by restricting the sequence $\bar{\xi}^t$ to ordinals below ξ and modifying $\bar{\rho}^t, \bar{\varepsilon}^t$ and n^t accordingly.

Definition 4.6. Suppose $t = (\bar{\alpha}, \bar{\eta}, \bar{\beta}, r, \bar{\xi}, n, \bar{\sigma}, \bar{\rho}, \bar{\varepsilon}) \in \mathcal{T}_\lambda$, $\xi_{n-1} < \xi \leq \kappa$ and $\dot{m} \in V^{\mathbb{P}_{\lambda+\xi}}$. We say that \dot{m} satisfies t if the following hold.

- (1) $\Vdash_{\mathbb{P}_{\lambda+\xi}} \dot{m} : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure.
- (2) For every $k < n$, letting $V_k = V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi_k} \rangle]$, we have $\Vdash_{\mathbb{P}_{\lambda+\xi}} \dot{m} \upharpoonright (\mathcal{P}(\omega) \cap V_k) \in V_k$.
- (3) For every $\bar{p} = \langle p_j : j < \omega \rangle$ of type t , there exists $p_{\bar{p}} \in \mathbb{P}'_{\lambda+\xi}$ such that:
 - (a) $\text{dom}(p_{\bar{p}}) = \{\beta_k : k < r\} \cup \{\lambda + \xi_k : k < n\}$.
 - (b) For every $k < r$, $p_{\bar{p}}(\beta_k) = \sigma_k$.
 - (c) For every $k < n$, $\Vdash_{\mathbb{P}_{\lambda+\xi_k}} p_{\bar{p}}(\lambda + \xi_k) \subseteq [\rho_k]$.
 - (d) $p_{\bar{p}} \upharpoonright \lambda \Vdash_{\mathbb{P}_{\lambda+\xi}} \dot{m}(\dot{A}_{\bar{p}}) = 1$ where

$$\dot{A}_{\bar{p}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall j \in [k_n, k_{n+1})) (p_j \upharpoonright \lambda \in \mathbb{G}_{\mathbb{P}})\}.$$

- (e) For every $k < n$, $p_{\bar{p}} \Vdash_{\mathbb{P}_{\lambda+\xi}} \dot{m}(\dot{X}_{\bar{p},k}) \geq 1 - 2\varepsilon_k > 0$ where

$$\dot{X}_{\bar{p},k} = \{j < \omega : p_j \upharpoonright [\lambda, \lambda + \xi_k + 1) \in \mathbb{G}_{\mathbb{P}}\}.$$

Let $\bar{p}^* = \langle p_j^* : j < \omega \rangle$ be as in Definition 3.11. The next claim says that it is enough to construct a measure $\dot{m} \in V^{\mathbb{P}^{\lambda+\xi}}$ that satisfies the blueprint associated with \bar{p}^* .

Claim 4.7. *Suppose for every $t \in \mathcal{T}_\lambda$, for some $\xi_{n-1} < \xi < \kappa$, there exists $\dot{m} \in V^{\mathbb{P}^{\lambda+\xi}}$ such that \dot{m} satisfies t . Then there exists $p_\star \in \mathbb{P}$ such that p_\star satisfies the hypothesis of Lemma 3.9.*

Proof. Let $t^* \in \mathcal{T}_\lambda$ be such that \bar{p}^* is of type t^* . Choose $\dot{m} \in V^{\mathbb{P}^{\lambda+\xi}}$ where $\xi_{n-1} < \xi < \kappa$ such that \dot{m} satisfies t^* . Let $p_\star = p_{\bar{p}^*}$, $\dot{X}_{\bar{p}^*} = \{j < \omega : p_j^* \upharpoonright [\lambda, \lambda + \xi] \in \mathbb{G}_{\mathbb{P}}\}$ and

$$\dot{A}_{\bar{p}^*} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall j \in [k_n, k_{n+1})) (p_j^* \upharpoonright \lambda \in \mathbb{G}_{\mathbb{P}})\}.$$

Then p_\star forces that $\dot{m}(\dot{A}_{\bar{p}^*}) = 1$ and $\dot{m}(\dot{X}_{\bar{p}^*}) > 0$ and hence that $\dot{A}_{\bar{p}^*} \cap \dot{X}_{\bar{p}^*}$ is infinite. It follows that (see Definition 3.11(3))

$$p_\star \Vdash (\exists^\infty n) \frac{|\{j \in [k_n, k_{n+1}) : p_j \in G_{\mathbb{P}}\}|}{k_{n+1} - k_n} \geq 4^{-n} \quad \square$$

The next lemma completes the proof of Claim 3.8 and therefore of Theorem 1.1.

Lemma 4.8. *Suppose $\lambda_0 \leq \lambda < \lambda_\omega$, $t = (\bar{\alpha}, \bar{\eta}, \bar{\beta}, r, \bar{\xi}, n, \bar{\sigma}, \bar{\rho}, \bar{\varepsilon}) \in \mathcal{T}_\lambda$ and $\xi_{n-1} < \xi < \kappa$. Then there exists $\dot{m} \in V^{\mathbb{P}^{\lambda+\xi}}$ such that \dot{m} satisfies t .*

Proof. By induction on $n = |\bar{\xi}|$.

Suppose $n = 0$. Fix $\xi < \kappa$. Put $p_{\bar{p}} = \{(\beta_k, \sigma_k) : k < r\}$. Note that there is a unique \bar{p} of type t . Let $\dot{A}_{\bar{p}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall j \in [k_n, k_{n+1})) (p_j \upharpoonright \lambda \in \mathbb{G}_{\mathbb{P}})\}$. Then $p_{\bar{p}} \Vdash_{\mathbb{P}_\lambda} \dot{A}_{\bar{p}}$ is infinite. Hence we can choose $\dot{m} \in V^{\mathbb{P}^{\lambda+\xi}}$ such that $\Vdash_{\mathbb{P}_\lambda+\xi} \dot{m} : \mathcal{P}(\omega) \rightarrow [0, 1]$ is a finitely additive measure and $q \Vdash_{\mathbb{P}_\lambda+\xi} \dot{m}(\dot{A}_{\bar{p}}) = 1$. It follows that \dot{m} satisfies t .

Next fix $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, $t = (\bar{\alpha}, \bar{\eta}, \bar{\beta}, r, \bar{\xi}, n+1, \bar{\sigma}, \bar{\rho}, \bar{\varepsilon}) \in \mathcal{T}_\lambda$. We will construct $\dot{m}_1 \in V^{\mathbb{P}^{\lambda+\xi_n+1}}$ such that \dot{m}_1 satisfies t . By the inductive hypothesis, there is a measure \dot{m} satisfying $t \upharpoonright \xi_n$. The next claim says that we can do slightly better.

Claim 4.9. *There exists $\dot{m} \in V^{\mathbb{P}^{\lambda, \lambda+\xi_n}}$ such that \dot{m} satisfies $t \upharpoonright \xi_n = (\bar{\alpha}, \bar{\eta}, \bar{\beta}, r, \bar{\xi} \upharpoonright n, n, \bar{\sigma}, \bar{\rho} \upharpoonright n, \bar{\varepsilon} \upharpoonright n)$ and $\Vdash_{\mathbb{P}^{\lambda, \lambda+\xi_n}} \dot{m} \upharpoonright (\mathcal{P}(\omega) \cap V^{\mathbb{P}^{\lambda, \lambda+\xi_n}}) \in V^{\mathbb{P}^{\lambda, \lambda+\xi_n}}$.*

Proof. Let $\mathcal{T}'_{\lambda^+} = \{t' \in \mathcal{T}_{\lambda^+} : t' = (\bar{\alpha}^{t'}, \bar{\eta}, \bar{\beta}^{t'}, r, \bar{\xi} \upharpoonright n, n, \bar{\sigma}, \bar{\rho} \upharpoonright n, \bar{\varepsilon} \upharpoonright n)\}$. By inductive assumption, for every $t' \in \mathcal{T}'_{\lambda^+}$, there exists $\dot{m}^{t'} \in V^{\mathbb{P}^{\lambda^+, \lambda^++\xi_n}}$ such that $\dot{m}^{t'}$ satisfies t' . Fix such a map $t' \mapsto \dot{m}^{t'}$.

Let χ be sufficiently large. Choose M_0, M_1 elementary submodels of $(\mathcal{H}_\chi, \in, <_\chi)$ such that $M_0 \in M_1$, $|M_0| = |M_1| = \lambda$, and for $l \in \{0, 1\}$, \mathbb{P}_{λ^+} , \mathcal{T}'_{λ^+} and the map $t' \mapsto \dot{m}^{t'}$ are in M_l , $\lambda+1 \subseteq M_l$ and ${}^{<\kappa}M_l \subseteq M_l$. Note that if $B_j \in \{\lambda \cap A_{\lambda+\xi_j}^\lambda, \lambda \setminus A_{\lambda+\xi_j}^\lambda\}$ for $j < n+1$, then $|\bigcap_{j < n+1} B_j| = \lambda$. Also, if $D_j \in \{\lambda^+ \cap A_{\lambda^++\xi_j}^{\lambda^+}, \lambda^+ \setminus A_{\lambda^++\xi_j}^{\lambda^+}\}$ for

$j < n + 1$, then $|M_0 \cap \bigcap_{j < n+1} D_j| = \lambda$ and $|(M_1 \setminus M_0) \cap \bigcap_{j < n+1} D_j| = \lambda$. So we can choose a bijection $h : \lambda + \xi_n \rightarrow M_1 \cap (\lambda^+ + \xi_n)$ such that:

- (i) For every $\xi < \xi_n$, $h(\lambda + \xi) = \lambda^+ + \xi$.
- (ii) For every $k < n$ and $\alpha < \lambda$, $\alpha \in A_{\lambda+\xi_k}^\lambda$ if and only if $h(\alpha) \in A_{\lambda^++\xi_k}^{\lambda^+}$.
- (iii) For every $\alpha < \lambda$, $\alpha \in A_{\lambda+\xi_n}^\lambda$ if and only if $h(\alpha) \in M_0$.

Let $t' = (\langle h(\alpha_j) : j < \omega \rangle, \bar{\eta}, \langle h(\beta_k) : k < r \rangle, r, \bar{\xi} \upharpoonright n, n, \bar{\sigma}, \bar{\rho} \upharpoonright n, \bar{\varepsilon} \upharpoonright n)$. As M_l is countably closed, t' and therefore $\dot{m}^{t'}$ are in M_1 .

Define $\hat{h} : \mathbb{P}'_{\lambda, \lambda+\xi_n} \rightarrow \mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_1}$ as follows: $\hat{h}(p) = p'$ where $\text{dom}(p') = \{h(\alpha) : \alpha \in \text{dom}(p)\}$. If $\alpha \in \text{dom}(p) \cap \lambda$, then $p'(h(\alpha)) = p(\alpha)$. If $\alpha \in \text{dom}(p) \cap [\lambda, \lambda + \xi_n)$, then $p'(\alpha) = B(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega \rangle)$ where $B, \langle (n_k, \gamma_k) : k < \omega \rangle$ are as in Definition 3.2(1)(b) for coordinate α .

Claim 4.10. *The following hold.*

- (1) $\hat{h} : \mathbb{P}'_{\lambda, \lambda+\xi_n} \rightarrow \mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_1}$ is an isomorphism.
- (2) $\mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_0} \leq \mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_1} \leq \mathbb{P}'_{\lambda^+, \lambda^++\xi_n}$.
- (3) For $k < n$, put $A_k = A_{\lambda^++\xi_k}^{\lambda^+} \cap M_1$. Then $\Vdash_{\mathbb{P}'_{\lambda^+, \lambda^++\xi_n}} \dot{m}^{t'} \upharpoonright (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+, A_k}}) \in V^{\mathbb{P}'_{\lambda^+, A_k}}$.
- (4) For $l \in \{0, 1\}$, $\Vdash_{\mathbb{P}'_{\lambda^+, \lambda^++\xi_n}} \dot{m}^{t'} \upharpoonright (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_l}}) \in V^{\mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_l}}$.

Proof. (1) and (4) should be clear. For (2), use Lemma 3.4. For (3), recall that $\dot{m}^{t'}$ satisfies t' . □

Choose $\dot{m}' \in V^{\mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_1}}$ such that $\Vdash_{\mathbb{P}'_{\lambda^+, \lambda^++\xi_n}} \dot{m}' = \dot{m}^{t'} \upharpoonright (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+, (\lambda^++\xi_n) \cap M_1}})$ and define $\dot{m} \in V^{\mathbb{P}'_{\lambda, \lambda+\xi_n}}$ by $\hat{h}(\dot{m}) = \dot{m}'$. By Claim 4.10, \dot{m} satisfies $t \upharpoonright \xi_n = (\bar{\alpha}, \bar{\eta}, \bar{\beta}, r, \bar{\xi} \upharpoonright n, n, \bar{\sigma}, \bar{\rho} \upharpoonright n, \bar{\varepsilon} \upharpoonright n)$ and $\Vdash_{\mathbb{P}'_{\lambda, \lambda+\xi_n}} \dot{m} \upharpoonright (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda, A_{\lambda+\xi_n}}}) \in V^{\mathbb{P}'_{\lambda, A_{\lambda+\xi_n}}}$. This completes the proof of Claim 4.9.

Put $V_n = V^{\mathbb{P}'_{\lambda, A_{\lambda+\xi_n}}}$ and $\mathbb{B} = (\text{Random})^{V_n}$. Working in V_n , apply Lemma 4.3 to $\dot{m} \upharpoonright (\mathcal{P}(\omega) \cap V_n)$, with $r = [\rho_n]$ to obtain the extension $\dot{m}_r \in (V_n)^\mathbb{B}$ as defined there. Since $\mathbb{P}'_{\lambda, A_{\lambda+\xi_n}} \leq \mathbb{P}'_{\lambda+\xi_n}$, we can write $V^{\mathbb{P}'_{\lambda+\xi_n}} = (V_n)^\mathbb{Q}$ for some $\mathbb{Q} \in V_n$. By Lemma 4.2, it follows that $\dot{m}_r \in (V_n)^\mathbb{B}$ and $\dot{m} \in (V_n)^\mathbb{Q}$ have a common extension $\dot{m}_1 \in (V_n)^\mathbb{Q} \times \mathbb{B} = V^{\mathbb{P}'_{\lambda+\xi_n+1}}$. It suffices to check that \dot{m}_1 satisfies t . So fix $\bar{p} = \langle p_j : j < \omega \rangle$ of type t and construct $p_{\bar{p}}$ as follows. Put $\bar{q} = \langle p_j \upharpoonright (\lambda + \xi_n) : j < \omega \rangle$. Since \dot{m} satisfies $t \upharpoonright \xi_n$, we can find $p_{\bar{q}} \in \mathbb{P}'_{\lambda+\xi_n}$ satisfying clauses (3)(a)–(d) in Definition 4.6. Working in $V_n = V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi_n} \rangle]$, let $\mathbb{B} = (\text{Random})^{V_n}$, $r = [\rho_n]$ and $\dot{X} \in \mathcal{P}(\omega) \cap (V_n)^\mathbb{B}$ be such that $[[j \in \dot{X}]]_{\mathbb{B}} = p_j(n)$. By Lemma 4.3, we can choose $s \geq r$, such that $s \Vdash_{\mathbb{B}} \dot{m}_r(\dot{X}) \geq 1 - \varepsilon_n$. Define $p_{\bar{p}} = p_{\bar{q}} \cup \{(\lambda + \xi_n, s)\}$ and note that it satisfies clauses (3)(a)–(e) in Definition 4.6.

5. Avoiding Collinear Points

Note that under the continuum hypothesis (or just $\text{add}(\text{Null}) = \mathfrak{c}$), every non-null subset X of the plane has a subset Y of the same two-dimensional Lebesgue outer measure which omits collinear points. In [5], Komjáth showed the following. It is consistent that there is a non-meager $A \subseteq \mathbb{R}$ such that every injective $f : A \rightarrow A$ is meager in the plane. It follows that A^2 is a non-meager subset of the plane each of whose non-meager subsets contains three collinear points. To see this, note that if $Y \subseteq A^2$ is such that every vertical and horizontal section of Y has ≤ 2 points, then Y can be covered by four injective functions from A to A . Let us prove the measure analogue of this.

Lemma 5.1. *It is consistent that there is a non-null $A \subseteq \mathbb{R}$ such that the null ideal restricted to A is isomorphic to the non-stationary ideal on ω_1 .*

Proof. Let $\kappa = \omega_1$ and \mathcal{I} be the non-stationary ideal on κ . Apply Theorem 1.1 and use the fact that the forcing is ccc. \square

Proof of Theorem 1.2. Let $A \subseteq \mathbb{R}$ be as in Lemma 5.1. List $A = \langle x_\alpha : \alpha < \omega_1 \rangle$ such that for every $W \subseteq \omega_1$, W is non-stationary if and only if $\{x_\alpha : \alpha \in W\}$ is null. Put $X = A^2$. Then X is non-null. As noted above, it suffices to show that if $f : A \rightarrow A$ is injective, then f is null in the plane. Let $A_1 = \{x_\alpha : (\exists \beta < \alpha)(f(x_\alpha) = x_\beta)\}$, $A_2 = \{x_\alpha : (\exists \beta > \alpha)(f(x_\alpha) = x_\beta)\}$ and $A_3 = \{x_\alpha : f(x_\alpha) = x_\alpha\}$. Note that $f \upharpoonright A_3$ is null. Toward a contradiction, suppose $f \upharpoonright A_1$ is non-null in the plane. Then A_1 is non-null in \mathbb{R} . Hence $\{\alpha : x_\alpha \in A_1\}$ is stationary. But f is injective so this contradicts Fodor's lemma. Similarly the inverse of $f \upharpoonright A_2$ is null in the plane. Hence f is null in the plane. \square

6. On Transversals

In [6], the following was shown: For $X \subseteq [0, 1]$ and for every partition $\{X_i : i \in S\}$ of X into countable sets, there exists $Y \subseteq X$ such that for every $i \in S$, $|Y \cap X_i| = 1$ and $\mu^*(Y) = \mu^*(X)$. Bill Weiss asked what happens if we consider partitions into null sets of size \aleph_1 . Note that under the continuum hypothesis the result continues to hold. Theorem 1.3 says that it can consistently fail too.

Proof of Theorem 1.3. Define $\mathcal{I} = \{A \subseteq \omega_1 \times \omega_1 : (\exists i_0 < \omega_1)(\forall i > i_0)(|A_i| \leq \aleph_0)\}$ where $A_i = \{j : (i, j) \in A\}$ and apply Theorem 1.1. \square

7. A Question

The diligent reader might have noticed that there are sigma ideals for which our method does not work. This is because of the requirement in Theorem 1.1 that the sigma ideal contain all bounded subsets of κ . So we can ask the following.

Question 7.1. Can the null ideal restricted to some non-null set of reals be isomorphic to $(\omega_3, [\omega_3]^{\leq \aleph_1})$?

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