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Automorphism groups of countable stable structures

by

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Abstract. For every countable structure M we construct an $\aleph_0$-stable countable structure N such that $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are topologically isomorphic. This shows that it is impossible to detect any form of stability of a countable structure M from the topological properties of the Polish group $\mathrm{Aut}(M)$.

1. Introduction. The non-Archimedean Polish groups—those Polish groups admitting a basis at the identity of open subgroups—are known to be precisely the Polish groups that can be represented as the automorphism groups of countable structures. A common theme of the last decades has been the search for connections between model-theoretic properties of such structures and properties of their automorphism groups.

For example, the result of Engeler, Ryll-Nardzewski and Svenonius states that the theory of a countable structure is countably categorical if and only if its automorphism group is oligomorphic [2, 5, 6], and the theorem of Ahlbrandt and Ziegler states that two countable structures are bi-interpretable if and only if their automorphism groups are topologically isomorphic [1]. For more advanced results in this direction dealing with reconstruction up to bi-definability see [3, 4].

Perhaps the pre-eminent model-theoretic property is *stability*. In the present study we show that any attempt at a topological characterization of the group of automorphisms of a countable *stable* structure is doomed to fail. More strongly, we have

THEOREM 1.1. *For every countable* (1) *structure M there exists an $\aleph_0$-stable countable structure $N_M = N$ such that* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are*

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(1) In the present paper we consider only structures in a countable language.

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topologically isomorphic with respect to the naturally associated Polish group topologies.

In order to witness the continuity of the isomorphism constructed in the proof of Theorem 1.1 we use a new notion of interpretability, which we call $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretability. In fact, in our proof, given a countable structure M, we construct an $\aleph_0$-stable structure $N_M = N$ and show that not only is there an isomorphism of topological groups $\alpha : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$, but also this α can be chosen to be such that it is induced by the map witnessing that N is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretabile in M. Although the continuity of the isomorphism constructed in the proof of Theorem 1.1 is evident from the construction we believe that this new notion of interpretability is interesting per se, and that it gives more canonicity to our construction.

Finally, the theory $\mathrm{Th}(N_M)$ of Theorem 1.1 can be shown to be NDOP and NOTOP, but this will not be proved here, since it is outside of the scope of this study.

2. Proofs. To make the exposition complete we first introduce the classical notion of first-order interpretability (Definition 2.1), and then define the notion of $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretability (Definition 2.3). Next, we state two facts (Facts 2.4 and 2.5) which will be crucially used in the proof of Theorem 1.1, and then proceed to the proof.

DEFINITION 2.1. Let M and N be models. We say that N is *interpretable* in M if for some $n < \omega$ there are:

(1) a $\emptyset$-definable subset D of M^n;
(2) a $\emptyset$-definable equivalence relation E on D;
(3) a bijection $\alpha : N \to D/E$ such that for every $m < \omega$ and $\emptyset$-definable subset R of N^m the subset of M^{nm} given by

$$\hat{R} = \{(\bar{a}_1, \ldots, \bar{a}_m) \in (M^n)^m : (\alpha^{-1}(\bar{a}_1/E), \ldots, \alpha^{-1}(\bar{a}_m/E)) \in R\}$$

is $\emptyset$-definable in M.

NOTATION 2.2. Let τ be a language.

(1) For $R \in \tau$ a predicate, we denote by $k(R)$ the arity of R.
(2) Given a τ-structure M and a τ-formula $\varphi(\bar{x}) = \varphi(x_0, \ldots, x_{n-1})$, we let

$$\varphi(M) = \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}.$$

(3) Given a τ-structure M, we denote by $|M|$ the domain of M (although we will be sloppy in distinguishing between the two), and by $\|M\|$ the cardinality of M.
(4) Given a τ-structure M and $A \subseteq M$, we denote by $\mathrm{Aut}(M/A)$ the set of automorphisms of M which are the identity on A.

(5) We denote by $\mathfrak{L}_{\omega_1,\omega}(\tau)$ the logical language $\mathfrak{L}_{\omega_1,\omega}$ (admitting countable disjunctions and countable conjuctions) with respect to the vocabulary τ.

(6) Given a collection Δ of formulas in one free variable of the language $\mathfrak{L}_{\omega_1,\omega}(\tau)$, a τ-structure M, and $a \in M$, we let

$$\mathrm{tp}_{\Delta}(a, \emptyset, M) = \{\varphi \in \Delta : M \models \varphi(a)\}.$$

Definition 2.3. Let:

(i) τ_ℓ ($\ell = 1, 2$) be relational languages;
(ii) $\Delta_\ell \subseteq \mathfrak{L}_{\omega_1,\omega}(\tau_\ell)$ be sets of formulas, for $\ell = 1, 2$;
(iii) $\Delta_2 = \{\varphi \in \mathfrak{L}_{\omega_1,\omega}(\tau_2) : \varphi$ is an atomic τ_2-formula in one free variable$\}$;
(iv) M_ℓ be τ_ℓ-structures for $\ell = 1, 2$.

We say that M_2 is Δ_1-*interpretable* in M_1 by a scheme $\mathfrak{s}$ and a function $\bar{F}$ if:

(A) $\mathfrak{s} = \{\mathfrak{s}(p) : p \in \mathfrak{S}_{M_2}\} \cup \{\mathfrak{s}(R, \bar{p}) : R \in \tau_2,\ \bar{p} = (p_\ell : \ell < k(R)) \in \mathfrak{S}_{M_2}^{k(R)}\}$, where:

- (a) $\mathfrak{S}_{M_2} = \{\mathrm{tp}_{\Delta_2}(a, \emptyset, M_2) : a \in M_2\}$;
- (b) $\mathfrak{s}(p) = (r_p(\bar{x}_{m(p)}), E_p(\bar{y}_{m(p)}, \bar{z}_{m(p)})) \in \Delta_1 \times \Delta_1$, $m(p) < \omega$, and $E_p(M_1)$ is a non-empty equivalence relation on $r_p(M_1)$;
- (c) $\mathfrak{s}(R, \bar{p})$ is a τ_1-formula from Δ_1 of the $\varphi_{(R,\bar{p})}(\bar{x}^0_{m(p_0)}, \dots, \bar{x}^{k-1}_{m(p_{k-1})})$ with $\bar{x}^i_{m(p_i)} = (x^i_0, \dots, x^i_{m(p_i)-1})$ for every $i < k = k(R)$;

(B) $\bar{F} = (F_p : p \in \mathfrak{S}_{M_2})$, where:

- (a) F_p is a one-to-one function from
$$p(M_2) = \{a \in M_2 : p = \mathrm{tp}_{\Delta_2}(a, \emptyset, M_2)\}$$
onto $r_p(M_1)/E_p(M_1)$;
- (b) for every predicate R of τ_2, if $k = k(R)$, $\bar{a} \in M_2^k$, and, for every $\ell < k$, $p_\ell = \mathrm{tp}_{\Delta_2}(a_\ell, \emptyset, M_2)$, $\bar{b}_\ell \in r_{p_\ell}(M_1)$ and $F_{p_\ell}(a_\ell) = \bar{b}_\ell / E_{p_\ell}(M_1)$, then
$$M_2 \models R(a_0, \dots, a_{k-1}) \quad \text{iff} \quad M_1 \models \varphi_{(R,\bar{p})}(\bar{b}_0, \dots, \bar{b}_{k-1}).$$

Finally, we say that M_2 is $\mathfrak{L}_{\omega_1,\omega}$-*semi-interpretable* in M_1 when M_2 is Δ_1-interpretable in M_1 by some scheme $\mathfrak{s}$ and some function $\bar{F}$ for some Δ_1.

The proof of the following fact is essentially as in the case of first-order interpretability (cf. Definition 2.1).

Fact 2.4. *Let M and N be models, and suppose that N is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretable in M. Then every $\pi \in \mathrm{Aut}(M)$ induces a $\hat{\pi} \in \mathrm{Aut}(N)$, and the mapping $\pi \mapsto \hat{\pi}$ is a continuous homomorphism of* $\mathrm{Aut}(M)$ *into* $\mathrm{Aut}(N)$.

The following fact is well-known.

Fact 2.5. *Let G and H be Polish group and $\alpha : G \to H$ a group isomorphism. If α is continuous, then α is a topological isomorphism.*

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let M be a countable model. We construct a countable model $N_M = N$ such that:

(1) N is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretable in M (cf. Definition 2.3);
(2) for every $\pi \in \mathrm{Aut}(N)$ there is a unique $\pi_0 \in \mathrm{Aut}(M)$ such that $\pi = \hat{\pi}_0$ (cf. Fact 2.4);
(3) N is $\aleph_0$-stable.

Using Facts 2.4 and 2.5, and items (1)–(2) above, we find that $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are topologically isomorphic, and thus by (3) we are done.

We then proceed to the construction of a model $N_M = N$ as above. First of all notice that without loss of generality ([2]) we can assume that M is a relational structure in a language $\tau(M) = \{P_{(n,\ell)} : n < n_* \leq \omega,\ \ell < \ell_n \leq \omega\}$, where the $P_{(n,\ell)}$ are n-ary predicates, and, for transparency, we assume that if $M \models P_{(n,\ell)}(\bar{a})$, then $\bar{a}$ is without repetitions. We construct a structure N in the following language $\tau(N)$:

(i) $c \in \tau(N)$ is a constant;
(ii) $P \in \tau(N)$ is a unary predicate;
(iii) for $n < n_* \leq \omega$ and $\ell < \ell_n \leq \omega$, $Q_{(n,\ell)} \in \tau(N)$ is a unary predicate;
(iv) for $n < n_* \leq \omega$ and $\ell < \ell_n \leq \omega$, $E_{(n,\ell)} \in \tau(N)$ is a binary predicate;
(v) for $n < n_* \leq \omega$, $\ell < \ell_n \leq \omega$ and $\iota < n$, $F_{(n,\ell,\iota)} \in \tau(N)$ is a unary function;
(vi) for $n < n_* \leq \omega$, $\ell < \ell_n \leq \omega$ and $j < \omega$, $G_{(n,\ell,j)} \in \tau(N)$ is a unary function.

We define the structure N as follows:

(a) $|N|$ (the domain of N) is the disjoint union
$$P^N \cup \{c^N = e\} \cup \{Q^N_{(n,\ell)} : n < n_* \leq \omega \text{ and } \ell < \ell_n \leq \omega\};$$
(b) $P^N = |M|$ (the domain of M);
(c) $Q^N_{(n,\ell)} = \{(n,\ell,i,a_0,\ldots,a_{n-1}) : a_t \in M,\ i \leq \omega,\ (a_0,\ldots,a_{n-1}) \notin P^M_{(n,\ell)} \Rightarrow i < \omega\}$;
(d) $E^N_{(n,\ell)} =$
$$\{((n,\ell,i_1,\bar{a}),(n,\ell,i_2,\bar{a})) : i_1, i_2 \leq \omega,\ (n,\ell,i_t,\bar{a} = a_0,\ldots,a_{n-1}) \in Q^N_{(n,\ell)}\};$$
(e) for $\iota < n$, $F_{(n,\ell,\iota)}(x) = \begin{cases} a_\iota & \text{if } x = (n,\ell,i,a_0,\ldots,a_{n-1}), \\ e & \text{otherwise;} \end{cases}$
(f) for $j < \omega$, $G_{(n,\ell,j)}(x)$
$$= \begin{cases} (n,\ell,j,a_0,\ldots,a_{n-1}) & \text{if } x = (n,\ell,i,a_0,\ldots,a_{n-1}), \\ e & \text{otherwise.} \end{cases}$$

([2]) Recall that in this paper we only consider structures in a countable language.

We now prove items (1)–(3) from the list at the beginning of the proof. Item (3) is proved in Claim 2.6 below. We prove item (2). Let $\pi \in \mathrm{Aut}(N)$ and, for $a, b \in M$, let $\pi_0(a) = b$ iff $\pi(a) = b$. Clearly $\pi_0 \in \mathrm{Sym}(M)$. For the sake of contradiction, suppose that $\pi_0 \notin \mathrm{Aut}(M)$. Replacing π with π^{-1}, we can assume without loss of generality that there are $n < n_* \leq \omega$, $\ell < \ell_n \leq \omega$, $\bar{a} = (a_0, \ldots, a_{n-1}) \in M^n$ and $\bar{b} = (b_0, \ldots, b_{n-1}) \in M^n$ such that $\pi_0(\bar{a}) = \bar{b}$, $M \models P_{(n,\ell)}(\bar{a})$ and $M \not\models P_{(n,\ell)}(\bar{b})$. Then the element $(n, \ell, \omega, a_0, \ldots, a_{n-1}) \in N$ realizes the type

$$p = \{F_{(n,\ell,\iota)}(x) = a_\iota : \iota < n\} \cup \{G_{(n,\ell,j)}(x) \neq x : j < \omega\},$$

while the type

$$q = \{F_{(n,\ell,\iota)}(x) = b_\iota : \iota < n\} \cup \{G_{(n,\ell,j)}(x) \neq x : j < \omega\}$$

is not realized in N, a contradiction. Hence, $\pi_0 \in \mathrm{Aut}(M)$ and clearly $\pi = \hat{\pi}_0$ (cf. Fact 2.4), and for every $\pi_1 \in \mathrm{Aut}(M)$ such that $\pi = \hat{\pi}_1$ we have $\pi_0 = \pi_1$.

Finally, we prove item (1). Let $(k_{(n,\ell,i)} : n < n_* \leq \omega, \ell < \ell_n \leq \omega, i \leq \omega)$ be a sequence of natural numbers such that

$$(n_1, \ell_1, i_1) \neq (n_2, \ell_2, i_2) \quad \text{implies} \quad 1 < n_1 + k_{(n_1,\ell_1,i_1)} \neq n_2 + k_{(n_2,\ell_2,i_2)}.$$

Let also:

(i′) $n + k_{(n,\ell,i)} = m(n, \ell, i)$;
(ii′) $\bar{x}_{m(n,\ell,i)} = (x_0, \ldots, x_{m(n,\ell,i)-1})$;
(iii′) $\bar{y}_{m(n,\ell,i)} = (y_0, \ldots, y_{m(n,\ell,i)-1})$.

Consider now the following formulas:

(A) $\varphi_0(x_0) : x_0 = x_0$;
(B) $\theta_0(x_0, y_0) : x_0 = y_0$;
(C) for $n < n_* \leq \omega$, $\ell < \ell_n \leq \omega$ and $i < \omega$ let

$$\varphi_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}) : \bigwedge_{m < m(n,\ell,i)} x_m = x_m,$$

$$\theta_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}, \bar{y}_{m(n,\ell,i)}) : \bigwedge_{m<n} x_m = y_m;$$

(D) for $n < n_* \leq \omega$, $\ell < \ell_n \leq \omega$ and $i = \omega$ let

$$\varphi_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}) : \bigwedge_{m < m(n,\ell,i)} x_m = x_m \wedge P_{(n,\ell)}(x_0, \ldots, x_{n-1}),$$

$$\theta_{(n,\ell,i)}(\bar{x}_{m(n,\ell,i)}, \bar{y}_{m(n,\ell,i)}) : \bigwedge_{m<n} x_m = y_m.$$

Notice now that:

(I) $P^N = \varphi_0(M)/\theta_0(M)$;
(II) $Q^N_{n,\ell}$ is in bijection with $\bigcup\{\varphi_{(n,\ell,i)}(M)/\theta_{(n,\ell,i)}(M) : i \leq \omega\}$.

Using this observation it is easy to see how to choose Δ_M, $\mathfrak{s}$, and $\bar{F} = (F_p : p \in \mathfrak{S}_N)$ as in Definition 2.3 so as to witness that N is $\mathfrak{L}_{\omega_1,\omega}$-semi-interpretable in M. ■

Claim 2.6. *Let N be as in the proof of Theorem* 1.1. *Then* $\mathrm{Th}(N)$ *is* $\aleph_0$*-stable.*

Proof. Let N_1 be a countable model of $\mathrm{Th}(N)$. It is enough to show that there are only countably many 1-types over N_1. To this end, let N_2 be an $\aleph_1$-saturated model of $\mathrm{Th}(N)$ such that every countable non-algebraic type is realized by $\|N_2\|$-many elements, and define the following equivalence relation $E^* = E^*_{(N_1,N_2)}$ on N_2:

$$aE^*b \quad \text{iff} \quad \exists\pi \in \mathrm{Aut}(N_2/N_1) \text{ such that } \pi(a) = b.$$

We will show that E^* has $\aleph_0$ equivalence classes, which clearly suffices. To this end, notice that:

- $(\star_1)$ if π is a permutation of P^{N_2} which is the identity on P^{N_1}, then there is an automorphism $\check{\pi}$ of N_2 over N_1 extending it (recall that N_2 is $\aleph_1$-saturated);
- $(\star_2)_{(n,\ell)}$ if $b_1, b_2 \in E_{(n,\ell)}$, $(F_{n,\ell,\iota}(b_1) : \iota < n)$ and $(F_{n,\ell,\iota}(b_2) : \iota < n)$ realize the same $\{=\}$-type over P^{N_1}, and $b_t \notin \{G_{(n,\ell,j)}(b_t) : j < \omega\}$ for $t = 1, 2$, then there exists $\pi \in \mathrm{Aut}(N_2/N_1)$ such that $\pi(b_1) = b_2$;
- $(\star_3)_{(n,\ell,j)}$ if $b_1, b_2 \in E_{(n,\ell)}$, $(F_{n,\ell,\iota}(b_1) : \iota < n)$ and $(F_{n,\ell,\iota}(b_2) : \iota < n)$ realize the same $\{=\}$-type over P^{N_1}, and $G_{(n,\ell,j)}(b_t) = b_t$ for $t = 1, 2$, then there exists $\pi \in \mathrm{Aut}(N_2/N_1)$ such that $\pi(b_1) = b_2$.

Now, using $(\star_1)$, $(\star_2)_{(n,\ell)}$ and $(\star_3)_{(n,\ell,j)}$ and noticing that n, ℓ and j range over countable sets, it is easy to see that the relation E^* defined above has $\aleph_0$ equivalence classes. ■

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