Chapter 14

Torsionless Linearly Compact Modules

Rüdiger Göbel
FB 6, Mathematik, Universität Duisburg Essen, 45117 Essen, Germany
R.Goebel@Uni-Essen.De

Saharon Shelah
Institute of Mathematics, Hebrew University, Jerusalem, Israel and Department of Mathematics, Rutgers University, New Brunswick, NJ, USA shelah@math.huji.ac.il

Abstract The aim of this paper is to answer a problem raised in a recent monograph by Robert Colby and Kent Fuller [3, pp. 129, 130] concerning R-torsionless linearly compact R-modules; see the introduction for a precise definition of this class of modules. Over a ring R these modules are particular submodules of products R^κ. Are $\mathbb{Z}^{(\omega)}$ and $P = \mathbb{Z}^\omega$ $\mathbb{Z}$-torsionless linearly compact (for $R = \mathbb{Z}$)? Is this class closed under direct sums? Both questions can be answered to the negative. In fact we show much more and characterize $\mathbb{Z}$-torsionless linearly compact groups: They are the free groups of finite rank. The same result holds for all principal ideal domains which are neither fields nor complete discrete valuation rings.

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14.1 Introduction

Linearly compact modules are crucial objects for the structure theory of modules based on (extensions of) Morita duality; see Colby and Fuller [3, Section 4] for example. Linear compactness can easily be defined by inverse limits: A module M is linearly compact if with any related system $\sigma_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$ of epimorphisms as in Proposition 14.1.1 also the unique homomorphism $\sigma : M \longrightarrow \overline{M}$ is surjective; see [3, p.75]. It turned out that proofs using linearly compact modules often only require a weaker condition to obtain similarly strong results. This can be seen in recent publications [4, 5] by Colpi and Fuller. Thus Colby and Fuller suggested in their nice monograph [3, Section 5.7] to replace linear compactness by the weaker hypothesis torsionless linear compactness. Here the trivial cokernels C (of the surjective maps above) are replaced by cokernels C which may not be 0 but have trivial dual $C^* = 0$. This notion was inspired by the version that appeared in [8].

Colby and Fuller [3, Chapter 5.7, 5.8] succeeded to lay the ground for an extended theory and naturally posed related questions which we want to deal with. Thus we recall the central notions of a torsionless linearly compact R-module in detail from the new monograph [3]. If M is an R-

module, then traditionally and also in this paper $M^* = \mathrm{Hom}_R(M, R)$ denotes the dual module of M. Following Bass [1] an R-module M is torsionless if $M \subseteq R^\kappa$ for some cardinal κ. This is half of our central definition. The other half depends on the notion of inverse systems. Let us fix our notations.

Let $(I, \leq)$ be an *inverse directed set*, i.e., a partially ordered set so that for all $\beta, \gamma \in I$, there is $\alpha \in I$ with $\alpha \leq \beta, \gamma$. A set of R-modules and maps $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is an *inverse system of modules* if $\pi_\alpha^\beta : M_\beta \longrightarrow M_\alpha$ is an R-homomorphism, and whenever $\alpha < \beta < \gamma$, then $\pi_\alpha^\gamma = \pi_\beta^\gamma \pi_\alpha^\beta$ (maps are acting on the right). An R-module and R-homomorphisms $(\overline{M}, \pi_\alpha : \alpha \in I)$ is the *inverse limit* of this inverse system, if $\pi_\alpha : \overline{M} \longrightarrow M_\alpha$ is an R-homomorphism ($\alpha \in I$), and whenever $\alpha < \beta$, then $\pi_\alpha = \pi_\beta \pi_\alpha^\beta$. Recall the well-known proposition, which we apply several times just below.

Proposition 14.1.1 *Let $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ be an inverse system of modules with inverse limit $(\overline{M}, \pi_\alpha : \alpha \in I)$. For any* related inverse system *$\sigma_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$ with $\sigma_\alpha = \sigma_\beta \pi_\alpha^\beta$ for all $\alpha < \beta$ there is a unique homomorphism $\sigma : M \longrightarrow \overline{M}$ with $\sigma_a = \sigma \pi_\alpha$ $(\alpha \in I)$.*

Thus the system has a unique inverse limit $\overline{M} = \varprojlim_I M_\alpha$ with homomorphisms π_α. We can write

$$\overline{M} = \{m = \sum_{\alpha \in I} m_\alpha \in \prod_{\alpha \in I} M_\alpha \text{ such that } m_\beta \pi_\alpha^\beta = m_\alpha \ \forall\, \alpha < \beta \in I\} \subseteq \prod_{\alpha \in I} M_\alpha$$

as a submodule of the product and

$$\pi_\beta : \overline{M} \longrightarrow M_\beta \ (\sum_{\alpha \in I} m_\alpha \longrightarrow m_\beta).$$

It follows from the definition of an inverse limit that we may assume that the maps $\pi_\alpha^\beta : M_\beta \longrightarrow M_\alpha$ are epimorphisms (replacing M_α by $\mathrm{Im}\,\pi_\alpha^\beta$). Now we are ready to complete our central definition with the above notations.

Definition 14.1.2 An R-module M is *R-torsionless linearly compact* (we will say that M is an *R-TLC*–module and a *TLC-group* if $R = \mathbb{Z}$) if the following two conditions hold:

(i) M is a submodule of a cartesian product R^κ for a suitable cardinal κ.

(ii) If $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is an inverse system and if there is a related inverse system $\sigma_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$ of homomorphisms with cokernel having trivial dual $[(M_\alpha/M\sigma_\alpha)^* = 0]$, then also $\sigma : M \longrightarrow \overline{M}$ has cokernel with trivial dual $[(\overline{M}/M\sigma)^* = 0]$.

We want to prove the following theorem for abelian groups. By $P = \mathbb{Z}^\omega$ we denote the Baer-Specker group and $S = \mathbb{Z}^{(\omega)}$ is the free group of countable rank, hence $S \subseteq P$ canonically.

Theorem 14.1.3 *If $M \subseteq P$, then M is a* TLC-*group if and only if M is free of finite rank.*

The result has an immediate consequence.

Corollary 14.1.4 *A group is a* TLC-*group if and only if it is free of finite rank.*

Thus TLC-groups are well known and as a consequence the natural questions raised by Colby, Fuller [3, p. 129] are answered for $R = \mathbb{Z}$: For example, the groups $M = S$ or $M = P$ are not TLC-groups and the class is not closed under infinite direct sums; see [3, pp. 129, 130, questions (a),…,(d)]. But in this case the class obviously is closed under taking finite direct sums and extensions. The ring $\mathbb{Z}$ can be replaced by any principal ideal domain which is neither a field nor a complete discrete valuation ring. We would like to thank Kent Fuller for drawing our attention to these problems.

14.2 Proof of the Theorem

We also state the following two easy and well-known propositions used in this section; their proof can be found in [6, p. 330, 331, Proposition 1.2, 1.3], for instance. (The notion of a direct system is dual to the inverse system above. Also dually we can replace homomorphisms of the direct system by injective maps. When passing from one system to the other we will keep the same indexing set $(I, \leq)$, but the relevant maps act in the opposite direction.) Recall from the introduction that $M^* = \mathrm{Hom}(M, R)$. If $\rho : M \longrightarrow N$, then $\rho^* : N^* \longrightarrow M^*$ denotes the canonical map induced by ρ.

Proposition 14.2.1 *Suppose $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is an inverse system of modules. Then $(M_\alpha^*, (\pi_\alpha^\beta)^* : \alpha \leq \beta \in I)$ is a direct system of modules.*

Proposition 14.2.2 *Suppose $(M_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \in I)$ is a direct system of modules and let $(\overline{M}, \pi_\alpha : \alpha \in I)$ be its direct limit. Then $(M_\alpha^*, (\pi_\alpha^\beta)^* : \alpha \leq \beta \in I)$ is an inverse system of modules and $(\overline{M}^*, \pi_\alpha^* : \alpha \in I)$ is its inverse limit.*

We first consider the part of Theorem 14.1.3 showing that finitely generated free groups are TLC-groups. For this direction we must check the condition of our test lemma for TLC-groups, which is [3, Lemma 5.7.6] restricted to abelian groups.

Lemma 14.2.3 (Test Lemma) *Suppose that the abelian group M satisfies the following three conditions.*

(i) M is reflexive.

(ii) If $X \subseteq \mathbb{Z}^\kappa$ and $M \longrightarrow X \longrightarrow C \longrightarrow 0$ is an exact sequence with $C^* = 0$, then X is reflexive as well.

(iii) If $\eta : L \longrightarrow M^*$ is a monomorphism, then $(L^*/M^{**}\eta^*)^* = 0$.

Then M is a TLC*-group.*

For convenience we include the short proof which is more direct for abelian groups.

Proof We assume the notation from Proposition 14.1.1 and let $\overline{M} = \varprojlim_I M_\alpha$ be the inverse limit with homomorphisms $\pi_\alpha : M \longrightarrow M_\alpha$ $(\alpha \in I)$. Showing that M is a TLC-group we also assume $M_\alpha \subseteq \mathbb{Z}^\kappa$ for all $\alpha \in I$ and

$$M \xrightarrow{\ \sigma_\alpha\ } M_\alpha \longrightarrow C_\alpha \longrightarrow 0$$

with $C_\alpha^* = 0$ is the related system of maps. Thus $\sigma_\alpha^* : M_\alpha^* \longrightarrow M^*$ is injective and there is a unique monomorphism $\tau : \varinjlim_I M_\alpha \longrightarrow M^*$ by the dual result of Proposition 14.1.1. If $D = (\varinjlim_I M_\alpha^*)^*/M^{**}\tau^*$, then $D^* = 0$ by hypothesis (iii). By hypothesis (ii) for $X = M_\alpha$ follows that M_α is reflexive. Thus there is an isomorphism $\nu : \overline{M} \longrightarrow \varprojlim_I M_\alpha^{**} \longrightarrow (\varinjlim_I M_\alpha^*)^*$. Let $\delta : M \longrightarrow M^{**}$ be the evaluation map which is also an isomorphism by (i). We obtain the following diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{\ \sigma\ } & \overline{M} & \longrightarrow & C & \longrightarrow & 0 \\
\delta\big\downarrow & & \nu\big\downarrow & & \gamma\big\downarrow & & \\
M^{**} & \xrightarrow{(\varinjlim \pi_\alpha^*)^*} & (\varinjlim_I M_\alpha^*)^* & \longrightarrow & D & \longrightarrow & 0
\end{array}$$

with the induced isomorphism γ. Now we apply * to the last diagram and pass to its dual diagram. From $D^* = 0$ and γ^* follows $C^* = 0$. Hence M is a TLC-group. □

Finally we check the three conditions (i), (ii), (iii) of the Test Lemma 14.2.3 for finitely generated free abelian groups M. Clearly M is reflexive.

To show (ii) consider $X \subseteq \mathbb{Z}^\kappa$ and the sequence $M \stackrel{\varphi}{\longrightarrow} X \longrightarrow C \longrightarrow 0$ and note that $M\varphi \subseteq X \subseteq \mathbb{Z}^\kappa$ is also finitely generated. If $M' = (M\varphi)_*$ denotes the pure subgroup of $\mathbb{Z}^\kappa$ purely generated by $M\varphi$, then M' has finite rank. It follows that M' is free of finite rank because $\mathbb{Z}^\kappa$ is $\aleph_1$-free (see Fuchs [7, Vol. 1, p. 94, Theorem 19.2]), hence M' is finitely generated and must split because $\mathbb{Z}^\kappa$ is also separable; see [7, Vol. 2, Section 87]. Let $Z^\kappa = M' \oplus D$. If $x + M\varphi \in X/M\varphi \setminus M'/M\varphi$, then there are $y \in M'$ and $0 \neq z \in D$ with $x = y + z$ and there is a homomorphism $\psi : \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}$ with $M\varphi \subseteq M' \subseteq \operatorname{Ker}\psi$ and $z\psi \neq 0$. Hence ψ induces a non-trivial homomorphism $X/M\varphi \longrightarrow \mathbb{Z}$. This is a contradiction because $X/M\varphi \cong C$ and $C^* = 0$ by the above short exact sequence. Thus $M\varphi \subseteq X \subseteq M' \subseteq \mathbb{Z}^\kappa$ and X is also finitely generated and free, hence reflexive; (ii) follows.

If $\eta : L \longrightarrow M^*$ is a monomorphism, then $0 \longrightarrow L \longrightarrow M^* \longrightarrow D \longrightarrow 0$ is a short exact sequence, and D is a direct sum of a finite group E and a free group. It follows $0 \longrightarrow M \longrightarrow L^* \longrightarrow E \longrightarrow 0$ from $\operatorname{Ext}(D, \mathbb{Z}) \cong E$, $\operatorname{Ext}(M, \mathbb{Z}) = 0$ and $M \cong M^{**}$. In particular $E^* = 0$ and (iii) also holds. We derived the

Corollary 14.2.4 *All free groups of finite rank are* TLC-*groups.*

For the converse direction we recall that intersections of decreasing chains are inverse limits; see [7, Vol. 1, p. 62, Example 3]. This follows immediately from the preliminary remarks and Proposition 14.1.1.

Proposition 14.2.5 *Let $\{G_\alpha : \alpha \in \delta\}$ be a decreasing chain of subgroups of some group G with $G_\delta = \bigcap_{\alpha\in\delta} G_\alpha$. If $\alpha < \beta \in \delta$, then let $\pi_\alpha^\beta : G_\beta \longrightarrow G_\alpha$ be the injection map. Then*

$$\varprojlim_\delta G_\alpha \subseteq \prod_{\alpha\in\delta} G_\alpha$$

is the collection of all vectors with constant entry, thus with constant entry in $G_\delta = \bigcap_{\alpha\in\delta} G_\alpha$. Thus $\varprojlim_\delta G_\alpha = G_\delta$.

Next we will deal with subgroups M of the Baer-Specker group $P = \mathbb{Z}^\omega = \prod_{i\in\omega} \mathbb{Z}e_i$; recall that $S = \bigoplus_{i\in\omega} \mathbb{Z}e_i$ is its canonical free subgroup. The subgroups $P_n = \prod_{i\geq n} \mathbb{Z}e_i$ $(n \in \omega)$ of P generate the Hausdorff product topology on P. If $M \subseteq P$, then $\overline{M}$ denotes the closure of M in the product topology.

Lemma 14.2.6 *If $M \subseteq P$ is a subgroup and not finitely generated, then $\overline{M}$ is isomorphic to P and there is an isomorphism α of $\overline{M}$ onto P with $S \subseteq M\alpha \subseteq P$.*

Proof Subgroups of P of finite rank are finitely generated (and free), because P is $\aleph_1$-free, see [7]. If M is not finitely generated, then it must have infinite rank. An important observation by Nunke [10, p. 68, Lemma 2 (b)] applies; see also Chase [2]. There is an isomorphism of $\overline{M}$ with P, which carries M onto a subgroup of P containing S. □

Lemma 14.2.7 *If $M \subseteq P$ is not finitely generated then we can find $P' \cong P$ such that $P \subseteq P'$, and there is a descending chain $\{G_i : i \in \omega\}$ and $\bigcap_{i\in\omega} G_i = G_\omega$ of subgroups of P' such that*

(i) $M \subseteq G_\omega$ and $G_\omega/M \cong \mathbb{Z}$, hence $G_\omega \cong \mathbb{Z} \oplus M$ and $G_\omega^* \neq 0$.

(ii) G_i/M is divisible of rank 1, hence $(G_i/M)^* = 0$.

Proof We apply the previous lemma to M, which is not finitely generated, and get $S \subseteq M \subseteq \overline{M} \cong P$. If $P' = \prod_{i\in\omega} \mathbb{Z}e_i'$ is a copy of P and J_p is the ring of p-adic integers, then we consider the map

$$P' \longrightarrow J_p \ (e_n' \longrightarrow p^n).$$

This map extends linearly to $S' = \bigoplus_{i\in\omega} \mathbb{Z}e_i'$ and is continuous in the product topology on P' and the p-adic topology on J_p. Since S' is dense in P' it extends uniquely to an epimorphism from P' to J_p. Its kernel is a product $\prod_{i\in\omega} \mathbb{Z}(pe_i' - e_{i+1}')$. We put $e_i = pe_i' - e_{i+1}'$ and thus identify their product with $\overline{M}$. Hence

$$S \subseteq M \subseteq \overline{M} \subseteq P' \text{ and } P'/\overline{M} = J_p.$$

Moreover $0 \longrightarrow \overline{M}/S \longrightarrow P'/S \longrightarrow P'/\overline{M} \longrightarrow 0$ are canonical maps and $\overline{M}/S$ (by Hulanicki, see [7]) and $P'/\overline{M}$ are cotorsion. Thus also P'/S is cotorsion and in particular P'/M is cotorsion.

Now consider $1 \in J_p = P'/\overline{M}$ and its preimage $x \in P'$. Thus $0 \neq x + M \in P'/M$ is a torsion-free element which is not divisible (because its image 1 is torsion-free and p-reduced). By Harrison's characterization of cotorsion groups (see Fuchs [7, Vol. 1, p. 238]) we can write

$$P'/M = A \oplus C \oplus D$$

where D is divisible, A is torsion-free, algebraically compact and C is the adjusted part. Now let M_* be the pure closure of M in P'. As noted above, the element $x + \overline{M} = 1 \in P'/\overline{M}$ is not p-divisible, so $x + M$ does not belong to the maximal divisible subgroup D. The adjusted part C is the $\mathbb{Z}$-adic closure of the torsion subgroup $T = M_*/M$, hence C is divisible modulo T. Thus $x + M$ must have a non-trivial component in A and we may assume that $x + M \in A$ which is the completion of a product of J_p's for various primes p; so $x + M \in J_p$ (w.l.o.g.) which here is a direct summand of A. Now we are ready to use some simple structure theory.

Let $\mathbb{Q}_p \subseteq J_p$ be the p-localization of $\mathbb{Z}$, hence $\mathbb{Q}_p/\mathbb{Z} = \bigoplus_{j\in\omega} Z_{q_j^\infty}$, where $\{q_j : j \in \omega\}$ is the list of all primes different from p. Choose preimages $\mathbb{Z} \subseteq Q^i \subseteq \mathbb{Q}_p$ such that $Q^i/\mathbb{Z} = \bigoplus_{j\geq i} Z_{q_j^\infty}$. Moreover choose preimages $G_i \subseteq P'$ such that

$$G_i/M = Q^i \subseteq \mathbb{Q}_p \subseteq J_p \subseteq A \subseteq P'/M.$$

The family $\{G_i \subseteq P' : i \in \omega\}$ constitutes a descending chain of subgroups of P' satisfying the conditions of the lemma with $G_\omega = x\mathbb{Z} \oplus M$. □

Combining Lemma 14.2.7 and Proposition 14.2.5 we have the

Corollary 14.2.8 *Any* TLC*-subgroup of the Baer-Specker group is free of finite rank.*

Proof We rewrite the conditions for the infinitely generated group M in the last lemma using the notation of Definition 14.1.2: $\sigma_i = \mathrm{id} : M \longrightarrow G_i$ has cokernel $G_i/M = Q^i$ with trivial dual, $\sigma : M \longrightarrow G_\omega$ has cokernel $G_\omega/M = \mathbb{Z}$ with nontrivial dual. Thus M is not a TLC-subgroup. □

If $M \subseteq \mathbb{Z}^\kappa = \prod_{i\in\kappa} \mathbb{Z}e_i$ is a subgroup of a product for some infinite cardinal κ which is not finitely generated, then there is a countable infinite set of independent elements $x_k = \sum_{i\in\kappa} x_{ik}e_i \in M$ ($k \in \omega$). Inductively we can find a countable set $I \subset \kappa$ such that the elements $x_k \restriction I = \sum_{i\in I} x_{ik}e_i \in \mathbb{Z}^I$ ($k \in \omega$) are independent. Thus there is an epimorphism $\pi : \mathbb{Z}^\kappa \longrightarrow P$ such that $M\pi$ is not finitely generated. In the last proof we replace σ_i by $\pi\sigma_i$ and σ by $\pi\sigma$; hence M is not a TLC-group. This proves Corollary 14.1.4.

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