Foreword

It is natural for me and for this book to consider what the right frameworks are for developing model theory. Being asked to write an introductory note, it seems reasonable to decide to do what nobody else can do better: try to somewhat describe the evolution of my thoughts on the subject.

In the fall of 1968, having started my Ph.D. studies at the Hebrew University, I started to read systematically what was available on model theory. Why model theory? I liked mathematical logic not for an interest in philosophical questions, but for its generality, which appeared to me as the epitome of mathematics; and so model theory seemed right; like algebra, but replacing fields, rings, etc. by the class of models of a first order theory. There was also some randomness — my M.Sc. thesis was on infinitary logics.

The central place of first order theory, equivalently, elementary classes, is not in doubt. The reasons are inherent and also rooted in history — in answering how to axiomatize mathematical theories, in particular set theory, and later Tarski and Malcev starting model theory. Though in the late sixties some thought that first order model theory was done, the news of its demise was premature, as it continued to be the central case in pure model theory, as well as in many applications.

But from quite early on, there were other frames. Mostowski had suggested various infinitary logics and generalized quantifiers.

A syntax-free way to treat model theory is to deal with Abstract Elementary Classes (aec). Why were they introduced ([Sh:88])? Looking at sentences in the logics $\mathbb{L}_{\aleph_1,\aleph_0}$ or $\mathbb{L}(Q)$ or their combinations, with exactly one or just few models in $\aleph_1$ (up to isomorphism, of course, [Sh:48]), there was no real reason to choose exactly those logics. We can expand the logic by various quantifiers stronger than Q (there are unaccountably many), and still have similar results. The answer was to try to axiomatize "an elementary class" using only the most basic properties. This, of course, does not cover some interesting cases (like the $\aleph_1$-saturated models of a first order theory, or classes of complete metric spaces), but still, it covers lots of ground on one hand, and has an interesting model theory on the other.

Later, attending a conference in California in 1986 led me to start to develop a more restricted frame — universal classes (encouraged by [Sh:155]).

Editor's note: The complete list of Shelah's publications can be found online at http://shelah.logic.at.

This Foreword is listed as E:84 in Shelah's list.

This led to introducing orbital types. The point was that an important step in the sixties was moving to saturation, i.e., using elementary mappings adding one element at a time, rather than using elementary submodels; this led to moving from "model homogeneous" to "sequence homogeneous". This move to types of singletons was essential for developing the theory of superstable classes. A major point of [Sh:300] (mainly for aec with amalgamation) was having the best of all possible worlds — dealing with types of elements, but saturation is expressed by embedding of models (where embedding is of the relevant kind, as axiomatized in aec). This is justified, as it is proved that the two definitions of saturated are equivalent (for aec with amalgamation).

Now aec seem close to classes defined as "the class of models of a sentence from $\mathbb{L}_{\lambda^+,\aleph_0}$" when the class has LST-number $\leq \lambda$. However they have closure properties which are better for some aspects; my belief in this direction being worthwhile is witnessed by having written a book. Why not logics like $\mathbb{L}_{\aleph_1,\aleph_1}$ and stronger? For a long time, everybody, including me, took for granted that for such logics we can generalize "elementary model theory" but not classification theory, though we can for more special cases (as mentioned above). Relatively recently, reexamining this has changed my view to some extent ([Sh:1019]).

However, all this does not imply for me that using and looking for logics has passed its day. Moreover, maybe, maybe there is a logic hidden from our view, by which we can get aec's or even a better family of classes. This is given some support by the following (see [Sh:797]).

Assuming for notational simplicity that λ is an inaccessible cardinal, we know that $\mathbb{L}_{\lambda,\aleph_0}$ fails interpolation, and also $\mathbb{L}_{\lambda,\lambda}$ fails it; but the pair satisfies it, so it was asked: is there a "nice" logic which lies between those two logics? Now $\mathbb{L}^1_\lambda$ from [Sh:797] seems a reasonable solution: it lies between them (i.e., $\mathbb{L}_{\lambda,\aleph_0} \leq \mathbb{L}^1_\lambda \leq \mathbb{L}_{\lambda,\lambda}$), has interpolation, has a characterization (as Lindström's characterization of first order logic), well ordering is not definable in it, and it has addition theorems; see more in [Sh:1101]. But this logic, $\mathbb{L}^1_\lambda$, strongly fails the upward Löwenheim-Skolem theorem (quite naturally being maximal under conditions as above). Can we find a similar logic for which this holds, for example, that has EM models? The search for such a logic has failed so far. Still, every aec can be characterized by a suitable sentence in $\mathbb{L}_{\lambda^+,\kappa^+}$ where κ is the (downward) LST number of the class and λ not much larger than κ plus the cardinality of the vocabulary.

Another exciting direction is dealing with complete metric spaces. Though we have learned much, there is much to be desired. For me, the glaring omission is that we do not have a theory generalizing the one for superstable theories.

In conclusion, our horizons have widened, but much remains a mystery. Surely, the interplay of syntax and semantics will continue to puzzle us on one hand and provide us with illuminations on the other.

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