# PARTITION THEOREMS FOR EXPANDED TREES 1176

SAHARON SHELAH

Abstract. We look for partition theorems for large subtrees for suitable uncountable trees and colourings. We then apply it to model theoretic problems. The simple cases are to prove the consistency of: if $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ has a model $M$ of cardinality $\beth_n$ with $\text{otp}(P^M, <^u) = \omega_1$, <u>then</u> $\psi$ has a model of cardinality $\beth_n$ such that $(P^M, <^u)$ is not well ordered Contrary to a priory expectation no large cardinal is used.

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Anotated Content

## § 0. Introduction

### §(0A) Background and Results

We continue two lines of research One is set theoretic: pure partition relations on trees and the other is model theoretic-Hanf numbers and non-definabliity of well ordeiring, in partiucualr related to $\omega_1$ . This relate to the existence of GEM (generalized Eherenfuecht-Mostowski) templates (see [Sheb]), and applications to descriptive set theory.

Halpern-Levy [HL71] had proved a milestone theorem on independence of versions of the axiom of choice: in $\mathrm{ZF}, \mathrm{AC}$ is strictly stronger than the maximal prime ideal theorem (i.e. every Boolean algebra has a maximal ideal).

This work isolated a partition theorem[1] on the tree ${}^{\omega>}2$, sufficient for the proof. This partition theorem was then proved by Halpern-Lauchli [HL66] and was a major and early theorem in Ramsey theory.

See more Laver [Lav71], [Lav73] and [She90, AP,§2] and Milliken [Mil79], [Mil81].

The [HL66] proof uses induction, later Harrington found a different proof using forcing-adding many Cohens and a name of a (non-principal) ultrafilter on $\mathbb{N}$. Earlier, see Silver proof on $\pi^1_1$-equivalence relations, see [Sil80]

Now [She92, §4] turn to uncountable trees, i.e. for some $\kappa > \aleph_0$, we consider trees $\mathscr{T}$ which are are sub-trees of $({}^{\kappa>}2, \triangleleft)$, again for every level $\varepsilon < \kappa$, either $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\eta\hat{}\langle 0\rangle, \eta\hat{}\langle 1\rangle \in \mathscr{T})$ or $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\exists!)[\eta\hat{}\langle \iota\rangle \in \mathscr{T}]$. But a new point is that we have to use a well ordering of $\mathscr{T} \cap {}^{\varepsilon}2$ for $\varepsilon < \kappa$ and is closed enough (that is under unions of increasing sequences of length $< \kappa$. Also colouring with infinite number of colours, the proof uses a "measurable $\kappa$" which remains so when we add $\lambda$ many $\kappa$-Cohens for appropriate $\lambda$; it generalizes Harrington's proof. This was continued, see Dobrinen-Hathaway [DH17] and see references there.

We are here mainly interested in a weaker version which are enough for model theoretic applications we have in mind, see more in §(1B). In this case the embedding does not preserve the equality of levels, also we may start with a large tree and get one of smaller cardinality, in a sense this is solving ?/[She92]/ [HL66] = (Erdos-Rado theorem)/(weakly compact cardinal)/(Ramsey theorem).

Note that by [She89], [Sheb], consistently we have GEM (generalized Eherenfuecht-Mostowski) model for ordered graphs as index models.

We intend to prove for $n < \omega$:

$(*)_n$ consistently

- (a) if $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ has a model $M$ of cardinality $\beth_{n+1}$ with $(P^M, <^M)$ having order type $\omega_1$ <u>then</u> $\psi$ has a model $N$ of cardinality $\beth_{n+1}$ and $(P^N, <^N)$ is not well ordered
- (b) moreover, it is enough that $M$ will have cardinality $\aleph_\delta.\delta < \beth^{++}_n$
- (c) preferably not using large cardinals

This require consistency of many cases of partition relations on trees and more complicated structures, analysing GEM models.

we can hope for more:

$(*)$ (a) $\alpha_\bullet < \omega_1, \beth_{\alpha+1} = (\beth_\alpha)^{+\omega_1+1}$ for $\alpha < \alpha_\bullet$ and well ordering of $\omega_1$ is not definable in $\{\mathrm{EC}_\psi(\beth_{\alpha_*}) : \psi \in \mathbb{L}_{\aleph_1,\aleph_0}\}$ or at least

[1] Using not splitting to 2 but other finite splitting make a minor difference; similarly here

(b) as above but for $\beth_{\alpha+1} = \aleph_{\beth_\alpha^{++}}$.
(c) parallel results replacing $\aleph_0$ by $\mu$

This is supposed to have consequences in descriptive set theory, see [SU19]

## § 0(A). Preliminaries.

**Definition 0.1.** If $\mu = \mu^{<\kappa}$ then "for a $(\mu,\kappa)$-club of $u \subseteq X$ we have $\varphi(u)$" means that: for some $x \in \mathscr{H}(\chi)$ if $x \in \mathscr{B} \prec (\mathscr{H}(\chi), \in), \|\mathscr{B}\| = \mu, [\mathscr{B}]^{<\kappa} \subseteq \mathscr{B}$ and $\mu, \kappa \in \mathscr{B}$, then the set $u = \mathscr{B} \cap X$ satisfies $\varphi(u)$; there are other variants.

**Definition 0.2.** For $\kappa$ regular (usually $\kappa = \kappa^{<\kappa}$) and ordinal $\gamma$ the forcing $\mathbb{P} = \mathrm{Cohen}(\kappa, \gamma)$ of adding $\alpha$ many $\kappa$-Cohen is defined as follows:

(A) $p \in \mathbb{P}$ iff:
  (a) $p$ is a function with domain from $[\gamma]^{<\kappa}$
  (b) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \in {}^{\kappa>}2$

(B) $\mathbb{P} \models p \leq q$ iff:
  (a) $p, q \in \mathbb{P}$
  (b) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$
  (c) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \trianglelefteq q(\alpha)$

(C) for $\alpha < \gamma$ let $\underset{\sim}{\eta}_\alpha = \cup\{p(\alpha) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ satisfies $\alpha \in \mathrm{dom}(p)\}$, so $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta}_\alpha \in {}^\kappa 2$"

(D) for $u \subseteq \lambda$ let $\mathbb{P}_u = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq u\}$ so $\mathbb{P}_u \lessdot \mathbb{P}$ and $\bar{\eta}_u = \langle \underset{\sim}{\eta}_\alpha : \alpha \in u \rangle$ is generic for $\mathbb{P}_u$.

# § 1. Partition theorems

*Question* 1.1. Use $\mathbf{M}_1, \mathbf{M}_2$ instead?

## § 1(A). The First Frame for Partition Theorems.

**Definition 1.2.** 1) Let $\mathbf{M}_{1,1}$ be the class of objects $\mathbf{m}$ consisting of (so $\kappa = \kappa_{\mathbf{m}}$, etc.):

- (a) $\kappa$, a regular cardinal
- (b) $\lambda \geq \kappa$
- (c) $\mathbb{Q}$ a quasi-order, $(< \kappa)$-complete
- (d) val, a function from $\mathbb{Q}$ into $[\lambda]^\lambda$
- (e) (monotonicity) if $p \leq_{\mathbb{Q}} q$ then $\mathrm{val}(p) \supseteq \mathrm{val}(q)$
- (f) (decidability) if $\mathrm{val}(p) = A \subseteq \lambda$ and $A \subseteq A_0 \cup A_1$, then for some $\ell \in \{0,1\}$ and $q \in \mathbb{Q}$ above $p$ we have $\mathrm{val}(q) \subseteq A_\ell$.

2) Let $\mathbf{M}_{1,2}$ be the class of $\mathbf{m} \in \mathbf{M}_{1,1}$ such that:

- (g) (non-atomicity) if $p \in \mathbb{Q}$ then for some $q_1, q_2 \in \mathbb{Q}$ above $p$, the sets $\mathrm{val}(q_1), \mathrm{val}(q_2)$ are disjoint.

**Claim 1.3.** *Assume* $\mathbf{m} \in \mathbf{M}_{1,1}$.
*1) If there is no $(< \kappa)$-complete uniform ultrafilter on $\lambda$ then* $\mathbf{m} \in \mathbf{M}_{1,2}$.
*2) The cofinality of $\lambda_{\mathbf{m}}$ is $\geq \kappa$.*
*3) In Definition 1.2(1) we can weaken clause (f) demanding $A_0, A_1 \in [\lambda]^\lambda$ and $A = A_0 \cup A_1$.*

*Proof.* 1) Toward contradiction assume that $\mathbf{m} \notin \mathbf{M}_{1,2}$, that is, clause (g) of 1.2(2) fails, let $p_*$ witness this, so:

$(*)_1$ if $q_1, q_2 \in \mathbb{Q}$ are above $p_*$, then $\mathrm{val}(q_1) \cap \mathrm{val}(q_2) \neq \emptyset$.

Define

$(*)_2$ let $D = D_{p_*} = \{A \subseteq \lambda$: for some $q \in \mathbb{Q}$ above $p_*$ we have $\mathrm{val}(q) \subseteq A\}$.

We shall prove that $D$ is a $\kappa$-complete uniform ultrafilter on $\lambda$, thus arriving at the promised contradiction.

$(*)_3$ $D \subseteq [\lambda]^\lambda$ and $\lambda \in D$ and $D$ is upward closed.

[Why? Check the definition of $D$ in $(*)_2$ and 1.2(1)(d)].

$(*)_4$ $D$ is a $(< \kappa)$-complete filter on $\lambda$.

[Why? Let $\zeta < \kappa$ and $A_\varepsilon \in D$ for $\varepsilon < \zeta$ and we shall prove that $\bigcap\limits_{\varepsilon<\zeta} A_\varepsilon \in D$. As $A_\varepsilon \in D$ we can choose $q_\varepsilon \in \mathbb{Q}$ above $p_*$ such that $\mathrm{val}(q_\varepsilon) \subseteq A_\varepsilon$. Now choose $p_\varepsilon$ by induction on $\varepsilon \leq \zeta$ such that:

$(*)_{4.1}$
- (a) $p_\varepsilon \in \mathbb{Q}$ is above $p_*$
- (b) if $\varepsilon(1) < \varepsilon$ then $p_{\varepsilon(1)} \leq_{\mathbb{Q}} p_\varepsilon$
- (c) if $\varepsilon = \varepsilon(1) + 1$ then $\mathrm{val}(p_\varepsilon) \subseteq A_{\varepsilon(1)}$.

(d) if $\xi < \varepsilon$ then $\mathrm{val}(p_\varepsilon) \subseteq \mathrm{val}(p_{\xi+1}) \subseteq A_\xi$; this actually follows.

If we succeed, then $\mathrm{val}(p_\zeta) \subseteq \bigcap\limits_{\varepsilon<\zeta} \mathrm{val}(p_{\varepsilon+1}) \subseteq \bigcap\limits_{\varepsilon<\zeta} A_\varepsilon$ hence $p_\zeta$ witnesses $\bigcap\limits_{\varepsilon<\zeta} A_\varepsilon \in D$ and we are done, that is, contradict $(*)_1$.

Why can we carry the induction? For $\varepsilon = 0$ let $p_\varepsilon = p_*$, for limit $\varepsilon$ recall $\mathbb{Q}$ is $(< \kappa)$-complete and $\zeta < \kappa$ is assumed.

Lastly, for $\varepsilon$ a successor ordinal, let $\varepsilon = \varepsilon(1) + 1$, now by clause (f) of 1.2(1) applied to the pair $(A_{\varepsilon(1)}, \lambda\backslash A_{\varepsilon(1)})$ there is $p_\varepsilon \in \mathbb{Q}$ above $p_{\varepsilon(1)}$ such that $\mathrm{val}(p_\varepsilon) \subseteq A_{\varepsilon(1)}$ or $\mathrm{val}(p_\varepsilon) \subseteq \lambda\backslash A_{\varepsilon(1)}$. But if $\mathrm{val}(p_\varepsilon) \subseteq \lambda\backslash A_{\varepsilon(1)}$ then, recalling $q_\varepsilon$ was chosen before $(*)_{4.1}$, the pair $p_\varepsilon, q_\varepsilon$ contradict the choice of $p_*$.

Hence we can carry the induction in $(*)_{4.1}$. Together we are done proving $(*)_4$.]

$(*)_5$ $D$ contains all co-bounded subsets of $\lambda$

[Why? Let $B \in [\lambda]^\lambda$ and choose $A_0 = B, A_1 = \lambda \setminus b$, so by clause (f) of Def 1.2(1), there is $p \in \mathbb{Q}$ above $p_*$ such that $\mathrm{val}(p) \in [\lambda]^\lambda$ and $\mathrm{val}(p)$ is a subset of $A_0$ or of $A_1$. But in the former case $\mathrm{val}(p)$ is a subset of $B$ hence has cardinality $\leq |B| < \lambda$ contradiction to clause ((d) of 1.2(1), so $\mathrm{val}(p) \subseteq A_1 = \lambda \setminus B$ and it belongs to $D$, so we are done proving $(*)_5$].

$(*)_6$ if $A \subseteq \lambda$ then $A \in D$ or $\lambda\backslash A \in D$.

[Why? By clause (f) of Definition 1.2(1).]

By $(*)_2, (*)_3, (*)_4, (*)_5$ clearly $D$ is a uniform $\kappa$-complete ultrafilter on $\lambda$, contradicting an assumption, so we are done proving 1.3(1).

2)Toward contradiction assume $\theta = \mathrm{cf}(\lambda) < \kappa$. Choose $\langle \alpha_i : i < \theta\rangle$ an increasing sequence of ordinals $< \lambda$ with limit $\lambda$. Now choose $p_i$ by induction on $i \leq \theta$ such that:

(*) $p_i \in \mathbb{Q}$
(*) $j < i \Rightarrow p_j \leq_{\mathbb{Q}} p_i$
(*) if $i = j + 1$ then $\mathrm{val}(p_i) \cap \alpha_i = \emptyset$.

For $i = 0$ choose any $p_i \in \mathbb{Q}$ and for $i$ a limit ordinal let $p_i$ be any upper bound of $\langle p_j : j < i\rangle$ recalling $\mathbb{Q}$ is $(< \kappa)$-complete. Lastly for $i = j + 1$ use the proofs of $(*)_5$ above.

3) Easy.

$\square_{1.3}$

**Claim 1.4.** *If (A) then (B) where:*

(A) (a) $\mathbf{m} \in \mathbf{M}_{1,1}$ *and let* $(\lambda, \kappa) = (\lambda_{\mathbf{m}}, \kappa_{\mathbf{m}})$
(b) $\theta < \kappa$
(c) $\mathbf{c} : [\lambda]^2 \to \theta$
(d) $\mathscr{T}$ *satisfies:*
($\alpha$) $\mathscr{T} \subseteq {}^{\kappa>}2$ *is a sub-tree and is closed under initial segments*
($\beta$) $\mathscr{T}$ *is of cardinality* $\kappa$, *and* $\alpha < \kappa \Rightarrow |\mathscr{T} \cap {}^\alpha 2| < \kappa$
($\gamma$) *notation:* $\mathrm{suc}(\mathscr{T}) = \{\eta \in \mathscr{T} : \ell g(\eta)$ *is a successor ordinal*$\}$
($\delta$) $\eta \in \mathscr{T} \Rightarrow \eta\hat{\ }\langle 0\rangle, \eta\hat{\ }\langle 1\rangle \in \mathscr{T}$

($\varepsilon$) *for every $\eta \in \mathscr{T}$ and $\zeta \in (\ell g(\eta), \kappa)$ there is $\nu \in \mathscr{T} \cap {}^{\zeta}2$ such that $\eta \triangleleft \nu$ (needed for $(*)_1(b)(c)$; it follows that: for every limit $\delta < \kappa, \mathscr{T} \cap {}^{\delta}2$ is infinite and even of cardinality $\geq |\delta|$)*

(B) *there are $u$ and $g$ such that:*
- (a) *$u \subseteq \lambda$ has order-type $\kappa$*
- (b) *$\{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u\}$ has at most two members*
- (c) *$g$ is a one-to-one function from $u$ onto $\mathscr{T}$*
- (d) *if $\alpha \leq \beta$ are from $u$, then $\ell g(g(\alpha)) \leq \ell g(g(\beta))$*
- (e) *for $\alpha < \beta$ from $u, \mathbf{c}\{\alpha, \beta\}$ depends only on:*
  - ($\alpha$) *the truth value of $g(\alpha) \triangleleft g(\beta)$*
  - ($\beta$) *if the answer is no and we let $\rho = g(\alpha) \cap g(\beta) \notin \{g(\alpha)\}$ then $\mathbf{c}\{\alpha, \beta\}$ depends also on $(g(\beta))(\ell g(\rho))$.*

*Remark* 1.5. 0) We can deduce clause (e) of 1.4(B) by making $\mathbf{c}\{\alpha, \beta\}$ code such information when we start with a one-to-one function $g^* : \lambda \to {}^{\lambda>}\kappa$ or $g^* : \lambda \to {}^{\kappa}2$, see 1.7 below. There we try to comment on how to adapt the present proof for proving 1.7.
1) Note that if $\kappa$ is strongly inaccessible then $\mathscr{T} = {}^{\kappa>}2$ is O.K. and even ${}^{\kappa>}\alpha$ for some $\alpha < \kappa$.
2) Is it worthwhile to allow $|\mathscr{T} \cap {}^{\alpha}2| \leq \kappa$ for $\alpha < \kappa$? It seems we shall not have a real gain.

*Proof.* Here $\mathbb{Q} = \mathbb{Q}_{\mathbf{m}}$. As a warm-up:

$(*)_1$ for $p \in \mathbb{Q}$ let:
- (a) $\mathrm{col}_{\mathbf{c}}(p) = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in \mathrm{val}_{\mathbf{m}}(p)\}$
- (b) $p \in \mathbb{Q}$ is $\mathbf{c}$-minimal when: for every $q \in \mathbb{Q}$ above $p$ we have $\mathrm{col}_{\mathbf{c}}(q) = \mathrm{col}_{\mathbf{c}}(p)$.

Now

$(*)_2$ (a) if $p \leq_{\mathbb{Q}} q$, then $\mathrm{col}_{\mathbf{c}}(p) \supseteq \mathrm{col}_{\mathbf{c}}(q) \neq \emptyset$
- (b) a dense open set of $p \in \mathbb{Q}$ is $\mathbf{c}$-minimal.

[Why? Clause (a) holds as $\leq_{\mathbb{Q}}$ is monotonic. Clause (b) holds as $|\mathrm{Rang}(\mathbf{c})| \leq \theta < \kappa$ and $\mathbb{Q}$ is $(< \kappa)$-complete.]
Next

$(*)_3$ For $p_1, p_2 \in \mathbb{Q}$ let:
- (a) $\mathrm{col}_{\mathbf{c}}(p_1, p_2) = \{\xi < \theta$: for some $\alpha \in \mathrm{val}_{\mathbf{m}}(p_1)$ the set $\{\beta \in \mathrm{val}_{\mathbf{m}}(p_2) : \mathbf{c}\{\alpha, \beta\} = \xi\}$ includes $\mathrm{val}_{\mathbf{m}}(q_1)$ for some $q_1$ above $p_2\}$
- (b) the pair $(p_1, p_2) \in \mathbb{Q} \times \mathbb{Q}$ is $\mathbf{c}$-minimal when: for every $q_1, q_2$ above $p_1, p_2$ respectively we have $\mathrm{col}_{\mathbf{c}}(q_1, q_2) = \mathrm{col}_{\mathbf{c}}(p_1, p_2)$ and $\mathrm{col}_{\mathbf{c}}(q_2, q_1) = \mathrm{col}_{\mathbf{c}}(p_2, p_1)$
- (c) let $h\mathrm{col}_{\mathbf{c}}(p_1, p_2)$ be $\cap\{\mathrm{col}_{\mathbf{c}}(q_1, q_2) : q_1, q_2 \in \mathbb{Q}$ are above $p_1, p_2$ respectively$\}$.

So as above

$(*)_4$ (a) if $p_1 \leq_{\mathbb{Q}} q_1$ and $p_2 \leq_{\mathbb{Q}} q_2$ then $\mathrm{col}_{\mathbf{c}}(p_1, p_2) \supseteq \mathrm{col}_{\mathbf{c}}(q_1, q_2)$

(b) if $p_1, p_2 \in \mathbb{Q}$ <u>then</u> for some $q_1, q_2 \in \mathbb{Q}$ we have $p_1 \le_{\mathbb{Q}} q_1, p_2 \le_{\mathbb{Q}} q_2$ and $(q_1, q_2)$ is $\mathbf{c}$-minimal

(c) $(p_1, p_2)$ is $\mathbf{c}$-minimal <u>iff</u> $\mathrm{col}(p_1, p_2) = \hbar\mathrm{col}(p_1, p_2)$ and $\mathrm{col}(p_2, p_1) = \hbar\mathrm{col}(p_2, p_1)$

(d) if $p_1, p_2 \in \mathbb{Q}$ then $\mathrm{col}_{\mathbf{c}}(p_1, p_2) \neq \emptyset$.

Now

$(*)_5$ we say $p \in \mathbb{Q}$ is $(\xi_1, \xi_2)$-minimal <u>when</u> $\xi_1, \xi_2 < \theta$ and for every $q \in \mathbb{Q}$ above $p$ there are $q_1, q_2$ above $q$ such that $\xi_1 \in \hbar\mathrm{col}_{\mathbf{c}}(q_1, q_2)$ and $\xi_2 \in \hbar\mathrm{col}_{\mathbf{c}}(q_2, q_1)$

$(*)_6$ the set $\{p \in \mathbb{Q} : p$ is $(\xi_1, \xi_2)$-minimal for some $\xi_1, \xi_2 < \theta\}$ is a dense (and open) subset of $\mathbb{Q}$.

[Why? As above by $\mathbb{Q}$ being $(< \kappa)$-complete and $\theta < \kappa$.]

$(*)_7$ (a) fix $\xi_1, \xi_2 < \theta$ and $p_* \in \mathbb{Q}$ such that $p_*$ is $(\xi_1, \xi_2)$-minimal

(b) fix $\bar\eta$ such that:

($\alpha$) $\bar\eta = \langle \eta_i : i < \kappa \rangle$ lists the elements of $\mathscr{T}$

($\beta$) $\ell g(\eta_i) < \ell g(\eta_j) \Rightarrow i < j$

($\gamma$) $\eta_i = \rho\char`^\langle 0 \rangle \Rightarrow \eta_{i+1} = \rho\char`^\langle 1 \rangle$ and then $(\exists \zeta)(i = 1 + 2\zeta)$

($\delta$) for $i < j < \kappa$ we have $\eta_i = \eta_j$ <u>iff</u> $(\exists \zeta)(i = 1 + 2\zeta \wedge j = i + 1)$ and $\ell g(\eta_i)$ is a limit ordinal

($\varepsilon$) if $j < \kappa$ <u>then</u> $\{i : \ell g(\eta_i) = j\}$ is an interval, (follows by clause ($\beta$)) and if $j$ is a limit ordinal then it is $[\zeta_1, \zeta_2)$ for some limit ordinals $\zeta_1, \zeta_2$.

[Why such $(\xi_1, \xi_2)$ as in clause (a) exists? By $(*)_6$. Why $\bar\eta$ as in clause (b) exists? Note that by clause (A)(d) of the claim's assumption, for every limit ordinal $\varepsilon < \kappa$, $\mathscr{T} \cap {}^\varepsilon 2$ has cardinality $\ge |\varepsilon|$.]

Toward our inductive construction:

$(*)_{7.1}$ For $\zeta < \kappa$ let:

(a) $I^{\mathrm{is}}_\zeta = \{\eta_\varepsilon : \varepsilon < 1 + (2\zeta)\}$, where "is" stands for "initial segment"

(b) $I^{\mathrm{fr}}_\zeta = \{\eta \in \mathscr{T} : \eta \in I^{\mathrm{is}}_\zeta \wedge [\eta\char`^\langle 0 \rangle \notin I^{\mathrm{is}}_\zeta]$ or $\ell g(\eta)$ is a limit ordinal and $\eta \notin I^{\mathrm{is}}_\zeta \wedge (\forall i < \ell g(\eta))(\eta{\restriction}i \in I^{\mathrm{is}}_\zeta)\}$; where fr stands for "front"

(c) hence $I^{\mathrm{fr}}_\zeta$ is a maximal $\triangleleft$-antichain of $\mathscr{T}$

(d) let $I^{\mathrm{ac}}_\zeta = \{\eta_\varepsilon : \varepsilon < 1 + 2\zeta$ and $\ell g(\eta_\varepsilon)$ is a successor ordinal$\}$

(e) $I^{\mathrm{fs}}_\zeta = I^{\mathrm{is}}_\zeta \cap I^{\mathrm{fr}}_\zeta$.

Clearly

$(*)_{7.2}$ (a) if $\varepsilon, \zeta < \kappa$ then $\eta_\varepsilon\char`^\langle 0 \rangle \in I^{\mathrm{is}}_\zeta \Leftrightarrow \eta_\varepsilon\char`^\langle 1 \rangle \in I^{\mathrm{is}}_\zeta$

(b) $I = \bigcup\limits_{\zeta < \kappa} I^{\mathrm{is}}_\zeta$ and $\mathscr{T}^{\mathrm{is}}_\varepsilon$ is $\subseteq$-increasing continuous with $\varepsilon$ and if $[\zeta_1, \zeta_2] = \{\varepsilon < \kappa : \ell g(\eta_\varepsilon) = \delta\}, \delta < \kappa$ a limit ordinal then $\langle I^{\mathrm{fr}}_\zeta : 1 + 2\zeta \in [\zeta_1, \zeta_2]\rangle$ is constantly $\mathscr{T} \cap {}^\delta 2$

(c) if $\varepsilon < \zeta$ and $\eta \in I^{\mathrm{fr}}_\zeta$ then $(\exists! \nu \in I^{\mathrm{fr}}_\varepsilon)[\nu \trianglelefteq \eta]$.

Now we choose $(\bar{p}_\zeta, \bar{\alpha}_\zeta)$ by induction on $\zeta < \kappa$ such that:

$(*)_{7.3}$ (a) $\bar{p}_\zeta = \langle p_{\zeta,\eta} : \eta \in I^{\rm is}_\zeta \rangle$

(b) $p_{\zeta,\eta} \in \mathbb{Q}$

(c) $p_* \le_{\mathbb{Q}} p_{\zeta,\eta}$ recallin $(*)_7(a)$

(d) if $\varepsilon < \zeta$ and $\nu \in I^{\rm is}_\varepsilon, \nu \trianglelefteq \eta \in I^{\rm is}_\zeta$ <u>then</u>

($\alpha$) $p_{\varepsilon,\nu} \le_{\mathbb{Q}} p_{\zeta,\eta}$

($\beta$) if $\nu \notin I^{\rm fr}_\varepsilon$ then $p_{\varepsilon,\nu} = p_{\zeta,\eta}$

(e) $\bar{\alpha}_\zeta = \langle \alpha_\eta = \alpha(\eta) : \eta \in I^{\rm is}_\zeta \rangle$ where $\alpha_\eta < \lambda$ so $\varepsilon < \zeta \Rightarrow \bar{\alpha}_\varepsilon = \bar{\alpha}_\zeta {\restriction} I^{\rm ac}_\varepsilon$ but if $\eta \triangleleft \nu$ are from $I^{\rm is}_\zeta$ and $\ell g(\eta) \le i < \ell g(\nu) \Rightarrow \nu(i) = 1$, <u>then</u> $\alpha_\eta = \alpha_\nu$

(f) if $\varepsilon_1 < \varepsilon_2 < 1 + 2\zeta$ and $\eta_{\varepsilon_1}, \eta_{\varepsilon_2} \in I^{\rm ac}_\zeta$, <u>then</u>

- $\bullet_1$ $\alpha(\eta_{\varepsilon_1}) \le \alpha(\eta_{\varepsilon_2})$ and equality holds iff $(\forall i([\lg(\eta_{\varepsilon_1} i < \lg(\eta_{\varepsilon_2}) \Rightarrow \eta_{\varepsilon_2}(i) = 1$
- $\bullet_2$ $\mathbf{c}\{\alpha(\eta_{\varepsilon_1}), \alpha(\eta_{\varepsilon_2})\} \in \{\xi_1, \xi_2\}$
- $\bullet_3$ in fact, $\mathbf{c}\{\alpha(\eta_{\varepsilon_1}), \alpha(\eta_{\varepsilon_2})\} = \xi_1$ <u>iff</u> letting $\rho = \eta_{\varepsilon_1} \cap \eta_{\varepsilon_2}$ we have $\rho\hat{\ }\langle 0 \rangle \trianglelefteq \eta_{\varepsilon_1}$

(g) if $\rho\hat{\ }\langle 0 \rangle, \rho\hat{\ }\langle 1 \rangle \in I^{\rm fr}_\zeta$, <u>then</u> $\xi_1 \in {\rm hcol}(p_{\rho\hat{\ }\langle 0 \rangle}, p_{\rho\hat{\ }\langle 1 \rangle})$ and $\xi_2 \in {\rm hcol}(p_{\rho\hat{\ }\langle 1 \rangle}, p_{\rho\hat{\ }\langle 0 \rangle})$.

Before carrying the induction:

$(*)_{7.4}$ it suffice to carry the induction

[Why? Let

- $u = \{\alpha_{\nu\hat{\ }\langle 0 \rangle} : \nu \in \mathscr{T}\}$
- $g(\alpha) = \nu$ iff $\alpha_{\nu\hat{\ }\langle 0 \rangle}$

Now clearly $u$ is well defined as $\nu \in \mathscr{T} \Rightarrow \nu\hat{\ }\langle 0 \rangle \in \mathscr{T} \Rightarrow (\alpha_{\nu\hat{\ }\langle 0 \rangle}$ is well defined). Also $g$ is a function from $u$ onto $\mathscr{T}$, it is onto again because for every $\nu \in \mathscr{T}$ also $\nu\hat{\ }\langle 0 \rangle \in \mathscr{T}$. Lastly, if $\eta_\varepsilon \ne \eta_\zeta \in \mathscr{T}, \varepsilon < \zeta$ then $\mathbf{c}\{\alpha_{\eta_\varepsilon\hat{\ }\langle 0 \rangle}\}, \alpha_{\eta_\zeta\hat{\ }\langle 0 \rangle}\}$ belongs to $\xi_1, \xi_2$ by $(*)_{7.3}(f)\bullet_3$.]

Now we turn to carrying the induction in $(*)_{7.3}$.

<u>Case 1</u>: $\zeta = 0$

We can find $p', p'' \in \mathbb{Q}_{\mathbf{m}}$ above $p_*$ such that $\xi_1 \in hcol(p', p''), \xi_2 \in hcol(p'', p')$. Let $\alpha_{\langle \rangle} \in {\rm val}(p')$ and $p_{\langle \rangle}$ above $p''$ be such that $\beta \in {\rm val}(p_{\langle \rangle}) \Rightarrow \mathbf{c}\{\alpha_{\langle \rangle}, \beta\} = \xi_2$.

<u>Case 2</u>: $\zeta = \varepsilon + 1, \eta_{1+2\varepsilon} = \rho\hat{\ }\langle 0 \rangle$

So $\rho \in I^{\rm fr}_\varepsilon$ and let $\langle \nu_{\varepsilon,i} : i < i_\varepsilon = i(\varepsilon) \rangle$ list with no repetitions the set $I^{\rm fs}_\varepsilon \backslash \{\rho\}$. Now we choose $q_{\varepsilon,i}$ by induction on $i \le i_\varepsilon$ and if $i = j+1$ also $r_{\varepsilon,j}$ such that:

(a) $q_{\varepsilon,i} \in \mathbb{Q}$ is above $q_{\varepsilon,j}$ for $j < i$

(b) if $i = 0$ then $q_{\varepsilon,i} = p_{\varepsilon,\rho}$

(c) if $i = j+1$ then

($\alpha$) $p_{\varepsilon,\nu_{\varepsilon,j}} \le r_{\varepsilon,j}$ and

($\beta$) if $\alpha \in {\rm val}(q_{\varepsilon,i})$ <u>then</u> for some $q \ge r_{\varepsilon,j}$ we have: $\beta \in {\rm val}(q) \Rightarrow \mathbf{c}\{\alpha, \beta\} \in \{\xi_1, \xi_2\}$; moreover, the one which is as promised in $(*)_{7.3}(f)\bullet_3$

There is no problem to carry the induction. Then choose $q', q''$ above $q_{\varepsilon,i(\varepsilon)}$ such that $\xi_1 \in \mathrm{hcol}(q', q''), \xi_2 \in \mathrm{hcol}(q'', q')$ and $\alpha \in \mathrm{val}(q') \Rightarrow (\exists q)(q'' \le q \wedge (\forall \beta \in \mathrm{val}(q))(\mathbf{c}\{\alpha, \beta\} = \xi_2)$.

Lastly, we choose our objects for $\zeta$:

(∗) (a) let $\alpha_{\rho\hat{}\langle 0\rangle}$ be any member of $\mathrm{val}(q')$, this is the only new case of an $\alpha_\rho, \rho \in I^{\mathrm{ac}}_\zeta)$

(b) let $p_{\zeta,\nu_{\varepsilon,i}}$ be above $r_{\varepsilon,i}$ such that $\beta \in \mathrm{val}(p_{\zeta,\nu_{\varepsilon,i}}) \Rightarrow \mathbf{c}\{\alpha_{\rho\hat{}\langle 0\rangle}, \beta\} \in \{\xi_1, \xi_2\}$; moreover, one which is as promised

(c) let $(p_{\zeta,\rho\hat{}\langle 0\rangle}, p_{\zeta,\rho\hat{}\langle 1\rangle})$ be a pair of members of $\mathbb{Q}$ above $q''$ such that $\xi_1 \in h\mathrm{col}(p_{\zeta,\rho\hat{}\langle 0\rangle}, p_{\zeta,\rho\hat{}\langle 1\rangle})$ and $\xi_2 \in h\mathrm{col}(p_{\zeta,\rho\hat{}\langle 0\rangle}, p_{\zeta,\rho\hat{}\langle 1\rangle})$.

<u>Case 3</u>: $\zeta = 2\varepsilon + 1$ and $\ell g(\eta_{1+2\varepsilon})$ is a limit ordinal.

As $\eta_{1+2\varepsilon} = \eta_{1+2\varepsilon+1}$, we deal only with $\eta_{1+2\varepsilon}$. Let us choose $p'_{\eta_{1+2\varepsilon}} \in \mathbb{Q}_{\mathbf{m}}$ above $p_{\xi,\eta_{1+2\varepsilon}\restriction i}$ for every $i < \ell g(\eta_{1+2\varepsilon}), \xi \le \varepsilon$ such that $\eta_{1+2\varepsilon}\restriction i \in I^{\mathrm{is}}_\xi$.

Then we choose a condition $p''_{\eta_{1+2\varepsilon}} \in \mathbb{Q}_{\mathbf{m}}$ above $p'_{\eta_{1+2\varepsilon}}$, and ordinal $\alpha_{\eta_{1+2\varepsilon}} < \lambda$ and a condition $p_{\zeta,\nu} \in \mathbb{Q}_{\mathbf{m}}$ above $p_{\eta_{\varepsilon,\nu}}$ for every $\nu \in I^{\mathrm{fr}}_\varepsilon$ as in Case 2 we have chosen $p_{\zeta,\rho\hat{}\langle 0\rangle}, \langle p_{\zeta,\nu} : \nu \in I^{\mathrm{fr}}_\varepsilon \backslash \{\rho\}\rangle$ there.

Lastly, we let $p_{\zeta,\eta_{1+2\varepsilon}} = p_{\zeta,\eta_{1+2\varepsilon+1}}, p''_{\eta_1+2_\varepsilon}$ and $\alpha_{\zeta,\eta_{1+2_\varepsilon}} = \alpha_{\zeta,\eta_{1+2\varepsilon+1}} = \alpha_{\eta_{1+2\varepsilon}}$.

<u>Case 4</u>: all othe rcases

Just use "$\mathbb{Q}$ is $(< \kappa)$-complete". $\square_{1.4}$

**Discussion 1.6.** We shall later turn to "$k$-place colourings and "end extension $k$-uniformity" as in [She92, §4].

**Claim 1.7.** *In 1.4 if we add (A)(e),(f) to (A)* <u>*then*</u> *we can add (B)(f) to (B) where:*

(A) (e) *$g'$ is a one-to-one function from $\lambda$ into $\lim_\kappa(\mathscr{T})$ so necessarily $\lambda \le 2^\kappa$*

(f) *$\bar{<} = \langle <_\varepsilon : \varepsilon < \kappa\rangle$ where $<_\varepsilon$ is a well ordering of $\mathscr{T} \cap {}^\alpha 2$*

(B) (f) *there is an increasing function $h$ from $\kappa$ to $\kappa$ such that: assuming $\alpha \ne \beta \in u$ and $g(\alpha) = \eta, g(\beta) = \nu$ we have:*

(∗) *if $\rho, \eta, \nu \in \mathscr{T}$ and $(\rho\hat{}\langle 0\rangle \trianglelefteq \eta)$ and $(\rho = \nu) \vee \rho\hat{}\langle 1\rangle \trianglelefteq \nu$* <u>*then*</u>*:*

- *if $\ell g(\eta) < \ell g(\nu)$ then $\mathbf{c}\{\alpha, \beta\} = \xi_2$*
- *if $\ell g(\eta) > \ell g(\nu)$ then $\mathbf{c}\{\alpha, \beta\} = \xi_1$*
- *if $\ell g(\eta) = \ell g(\nu) \le \varepsilon$ and $\eta <_\varepsilon \nu$ then $\mathbf{c}\{\alpha, \beta\} = \xi_2$ and*
- *if $\ell g(\eta) = \ell g(\nu) = \varepsilon$ and $\nu <_\varepsilon \eta$ and $\mathbf{c}\{\alpha, \beta\} = \xi_1$.*

*Proof.* Similarly to 1.4. $\square_{1.7}$

## § 1(B). **Expanded Trees and Second Frame for partition Theorem.**

*Question* 1.8. Replacing ${}^{\kappa>}2$ by ${}^{\kappa>}I$, is of interest ? use creature tree forcing?

Here we consider partition on tree. Now in [HL66] (and [She92]) the embedding of the trees preserves level (which is a plus), for uncountable trees we find the need to consider a well ordering of each level, still preserving equality of level. But for the model theoretic applications we have in mind it is enough to consider embeddings where levels are not preserved, see Dzamonja-Shelah [DS04] in the web.

We have two versions- for $\iota = 1, 2$, according to whther the embedding preserve the level, which implies more differences and $\iota = 0$ is a variant of $\iota = 1$.

**Convention 1.9.** Here $\iota$ will be 0, or 1 or 2

**Definition 1.10.** 1) For $\iota = 0, 1, 2$ let $\mathbf{T}_\iota$ be the class of structures $\mathscr{T}$ such that (let $\mathbf{T} = \mathbf{T}_1$):

(a) $\mathscr{T} = (u, <_*, E, <, \cap, S, R_0, R_1) = (u_{\mathscr{T}}, <^*_{\mathscr{T}}, E_{\mathscr{T}}, <_{\mathscr{T}}, \cap_{\mathscr{T}}, S_{\mathscr{T}}, R^0_{\mathscr{T}}, R^1_{\mathscr{T}})$ but we may write $s \in \mathscr{T}$ instead of $s \in u$

(b) $(u, <_*)$ as a well ordering, linear, $u$ non-empty

(c) $<_{\mathscr{T}}$ is a partial order included in $<_*$

(d) $(u, <_{\mathscr{T}})$ is a tree, i.e. if $t \in \mathscr{T}$ <u>then</u> $\{s : s <_{\mathscr{T}} t\}$ is linearly ordered by $<_{\mathscr{T}}$; the tree is with $\mathrm{ht}(\mathscr{T})$ levels

(e) $E$ is an equivalence of $u$, convex under $<_*$

(f) ($\alpha$) each $E$-equivalence class is the set of $t \in \mathscr{T}$ of level $\varepsilon$ for some $\varepsilon$
 ($\beta$) we denote the $\varepsilon$-the equivalence class by $\mathscr{T}_{[\varepsilon]}$
 ($\gamma$) $E$ has no last $E$-equivalence class if not said otherwise
 ($\delta$) let $\mathrm{lev}_{\mathscr{T}}(s) = \mathrm{lev}(s, \mathscr{T})$ be $\varepsilon$ when $s \in \mathscr{T}_{[\varepsilon]}$, equivalently $\{t : t <_{\mathscr{Y}} s\}$ has order type $\varepsilon$ under the order $\leq_{\mathscr{T}}$
 ($\varepsilon$) let $\mathrm{ht}(\mathscr{T})$ be $\cup\{\mathrm{lev}(s) + 1 : s \in \mathscr{T}\}$

(g) for each $s \in u$ there is $t \in \mathrm{spt}(\mathscr{T})$ above (i.e. $\leq_{\mathscr{T}}$-above) $s$ <u>where</u>:
 - $\mathrm{spt}(\mathscr{T}) = \{t \in \mathscr{T}$: there are at least two $<_{\mathscr{T}}$-immediate successors to $t\}$

(h) each $s \in \mathscr{T}$ has at most two immediate successors by $<_{\mathscr{T}}$

(i) for $s \in \mathscr{T}$
 $\bullet_1$ $\mathscr{T}_{\geq s} = \{t \in \mathscr{T} : s \leq_{\mathscr{T}} t\}$
 $\bullet_2$ $\mathrm{suc}_{\mathscr{T}}(s) = \{t : t \in \mathscr{T}_{[\mathrm{lev}(s)+1]}$ satisfies $s <_{\mathscr{T}} t\}$
 $\bullet_3$ let $t{\restriction}\varepsilon$ means that $\mathrm{lev}_{\mathscr{T}}(s) = \varepsilon \leq \mathrm{lev}_{\mathscr{T}}(t)$

(j) $\bullet_1$ if $\iota = 2$ then for every $\varepsilon < \mathrm{lev}(\mathscr{T})$ for every $t \in \mathscr{T}_{[\varepsilon]}$ we have $|\mathrm{suc}_{\mathscr{T}}(t)| = 2$
 $\bullet_2)$ if $\iota = 0$ then for every $\varepsilon < \mathrm{lev}(\mathscr{T})$ for exactly one $t \in \mathscr{T}_{[\varepsilon]}$ we have $|\mathrm{suc}_{\mathscr{T}}(t)| = 2$

(k) for $t_1, t_2 \in \mathscr{T}, t_1 \cap_{\mathscr{T}} t_2$ is the maximal common lower bound of $t_1, t_2$ so we demand it always exists, i.e. $(\mathscr{T}, <)$ is normal

(l) for $\ell = 0, 1$ we have $R_\ell \subseteq \{(s, t) : s \in \mathrm{spt}(\mathscr{T})$ and $s <_{\mathscr{T}} t\}$ and if $s \in \mathrm{spt}(\mathscr{T})$ then for some $t_0 \neq t_1$ we have $\mathrm{suc}_{\mathscr{T}}(s) = \{t_0, t_1\}$ and $(\forall t)(sR_\ell t$ <u>iff</u> $t_\ell \leq_{\mathscr{T}} t)$; so $sR_\ell t$ is the parallel to $\eta\hat{\ }\langle \ell \rangle \trianglelefteq \nu$

(m) $S_{\mathscr{T}} = \mathrm{spt}(\mathscr{T})$
(n) if $s \in u, \mathrm{lev}_{\mathscr{T}}(s) < \zeta < \mathrm{ht}(\mathscr{T})$ then there is $t \in \mathscr{T}_{[\zeta]}$ which is $<_{\mathscr{T}}$-above $s$.

2) Let $\mathbf{T}^{\iota}_{\theta,\kappa} = \{\mathscr{T} \in \mathbf{T}_{\iota}$: the tree $\mathscr{T}$ has $\delta$ levels, for some ordinal $\delta$ of cofinality $\kappa$ and for every $\varepsilon < \delta, \theta > |\{s \in \mathscr{T} : s$ of level $\leq \varepsilon\}|\}$.
3) Let $\mathscr{T}_1 \subseteq_{\iota} \mathscr{T}_2$ mean

(a) $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$
(b) $<_{\mathscr{T}_1} = <_{\mathscr{T}_2} \restriction u_{\mathscr{T}_1}$
(c) if $\mathscr{T}_1 \models$ "$\eta \cap \nu = \rho$ then $\mathscr{T}_2 \models lqlq\eta \cap \nu = \rho$" $R_{\mathscr{T}_1,\ell} = R_{\mathscr{T}_2,\ell} \restriction u_{\mathscr{T}_1}$ for $\ell = 0, 1$

(d) $\mathrm{spt}(\mathscr{T}_1) \subseteq \mathrm{spt}(\mathscr{T}_2) \cap u_{\mathscr{T}_1}$
(e) $<^*_{\mathscr{T}_1} = <^*_{\mathscr{T}_2} \restriction u_{\mathscr{T}_1}$.
(f) if $\iota = 2$ then $E_{\mathscr{T}_1} = E_{\mathscr{T}_1} \restriction u_{\mathscr{T}_1}$ and $\mathrm{spt}(\mathscr{T}_1) = \mathrm{spt}(\mathscr{T}_2) \cap u_{\mathscr{T}_1}$

4) For $\iota = 0$ we define $\subseteq_{\iota}$ by: $\mathscr{T}_1 \subseteq_{\iota} \mathscr{T}_2$ iff clauses (a), (b), (e) above holds and

(c)' if $\rho R_{\mathscr{T}_1,\ell}\eta_\ell$ and $\mathscr{Y}_2 \models qlql\varrho = \eta_1 \cap \eta_2$" then $\varrho R_{\mathscr{T}_2,\ell}\eta_\ell$

5) For $s \in \mathrm{spt}(\mathscr{T})$ and $\ell \in \{0, 1\}$, let $\mathrm{suc}_{\mathscr{T},\ell}(s)$ be the unique immediate successor of $s$ in $\mathscr{T}$ such that $(s,t) \in R^{\mathscr{T}}_{\ell}$.
6) We say $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ are neighbors <u>when</u> they are equal except that on each equivalence class we can change the order.

**Definition 1.11.** 1) We say $f$ is a $\subseteq_{\iota}$-embedding of $\mathscr{T}_1 \in \mathbf{T}$ into $\mathscr{T}_2 \in \mathbf{T}$ <u>when</u>: when $f$ is an isomorphism from $\mathscr{T}_1$ onto $\mathscr{T}'_1$ where $\mathscr{T}'_1 \subseteq_{\iota} \mathscr{T}_2$
2) For any ordinal $\alpha$ and sequence $\bar{<} = \langle <_{\beta} : \beta < \alpha \rangle, <_{\beta}$ a well ordering of ${}^{\beta}2$ we define $\mathscr{T} = \mathscr{T}^{\bullet}_{1,\alpha,\bar{<}}$ as follows:

(a) universe ${}^{\alpha>}2$
(b) $<_{\mathscr{T}}$ is $\triangleleft \restriction {}^{\alpha>}2$
(c) $E_{\mathscr{T}} = \{(\eta,\nu) : \eta, \nu \in {}^{\beta}2$ for some $\beta < \alpha\}$
(d) $<^*_{\mathscr{T}} = \{(\eta,\nu) : \eta,\nu \in {}^{\alpha>}2$ and $\ell g(\eta) < \ell g(\nu)$ or $(\exists \beta < \alpha)(\ell g(\eta) = \beta = \ell g(\nu) \wedge \eta <_{\beta} \nu)\}$
(e) $S_{\mathscr{T}} = {}^{\alpha>}2$.

**Claim 1.12.** *If $\theta = \sup\{(2^{|\alpha|})^+ : \alpha < \kappa\}$ and $\bar{<} = \langle <_{\beta} : \beta < \kappa \rangle$ as above, <u>then</u> $\mathscr{T}^{\bullet}_{1,\kappa,\bar{<}}$ is well defined and belongs to $\mathbf{T}_{\theta,\kappa}$.*

*Proof.* Should be clear. $\square_{1.12}$

**Definition 1.13.** 1) For $\mathscr{T} \in \mathbf{T}$ let $\mathrm{eseq}^{\iota}_n(\mathscr{T})$ be the sequence of $\bar{a}$ such that:

(a) $\{\bar{a} : \bar{a}$ is an $<^*_{\mathscr{T}}$-increasing sequence of length $n$ of members of $^{(}\mathscr{T})$
(b) if $\iota = 1, 2$ then $k < \ell < n \Rightarrow a_k \cap a_\ell \in \{a_m : m < n\}$
(c) if $\iota = 2$ then $k, \ell < n \wedge \mathrm{lev}(a_k) \leq \mathrm{lev}(a_\ell) \Rightarrow a_\ell \restriction \mathrm{lev}(a_k) \in \{a_m : m < n\}$
(d) if $\iota = 0$ and $k < \ell < n$ then for some $m(*) < k$ we have $a_{m(*)} \leq_{\mathscr{T}} a_k \wedge a_{m(*)} \leq_{\mathscr{T}} a_\ell$ and $m < n \wedge a_a M \leq_{\mathscr{T}} a_k \wedge a_m \leq_{\mathscr{T}} a_\ell \Rightarrow a_m \leq_{\mathscr{Y}} a_{m(*)}$

1A) $\mathrm{eseq}^{\iota}_n(\mathscr{U}, \mathscr{T})$ is $\mathrm{eseq}^{\iota}_n(\mathscr{T}) \cap ({}^n\mathscr{U})$, similarly for part (2).
2) Let $\mathrm{eseq}_{\iota} = (\mathscr{T}) = \mathrm{eseq}^{\iota}_{<\omega} = \cup\{\mathrm{eseq}_n(\mathscr{T}) : n < \omega\}$.
3) We say $\bar{a}, \bar{b} \in \mathrm{eseq}_{\iota}(\mathscr{T})$ are $\mathscr{T} - \iota$-similar or $\bar{a} \sim_{\mathscr{T},\iota} \bar{b}$ <u>when</u> for some $n$ we have:

(a) $\bar{a}, \bar{b} \in \mathrm{eseq}_n(\mathscr{T})$

(b) for any $k, \ell, m < n$ we have:
- $a_k \le_{\mathscr{T}} a_\ell$ <u>iff</u> $b_k \le_{\mathscr{T}} b_\ell$
- if $\iota = 1, 2$ then $(a_k, a_i) \in R^{\mathscr{T}}_\ell$ <u>iff</u> $(b_k, b_i) \in R^{\mathscr{T}}_\ell$ for $\ell = 0, 1$
- assuming $\iota = 0, \ell = 0, 1$ for $k, m, i < n$ we have $(a_k \cap a_m) R_{\mathscr{Y},\ell} a_m$ iff $(b_k \cap b_m) R_{\mathscr{Y},\ell} b_m$
- $\mathrm{lev}_{\mathscr{T}}(a_k) \le \mathrm{lev}_{\mathscr{T}}(a_\ell)$ <u>iff</u> $\mathrm{lev}_{\mathscr{T}}(b_k) \le_{\mathscr{T}} (b_\ell)$
- $a_k \cap_{\mathscr{T}} a_\ell = a_m$ <u>iff</u> $b_k \cap_{\mathscr{T}} b_\ell = b_m$ if $\iota = 1, 2$
- $a_\ell \in S_{\mathscr{T}}$ <u>iff</u> $b_\ell \in S_{\mathscr{T}}$ if $\iota = 1, 2$
- if $\iota = 2$ and $k < \ell < n$ we have $a_k = a_\ell \restriction \mathrm{lev}(a_k)$ iff $b_k = b_\ell \restriction \mathrm{lev}(b_k)$
- note that $a_k <^*_{\mathscr{T}} a_\ell \Leftrightarrow b_k <^*_{\mathscr{T}} a_\ell$ follows by part (1).

4) For $\bar{a} \in {}^n\mathscr{T}$ let $\mathrm{lev}(\bar{a}) = \{\mathrm{lev}_{\mathscr{T}}(a_\ell) : \ell < \lg(\bar{a})\}$
5) We say that $\mathscr{Y} \in \mathbf{T}_\iota$ is weakly $(\aleph_0, \iota)$-saturated when

(a) if $\iota = 0, 2$ and for every $\varepsilon < \zeta < \mathrm{ht}(\mathscr{T})$ and $s_0, \dots, s_{n-1}$ from $\mathscr{T}_{[\varepsilon]}$ there are $t_o \le^*_{\mathscr{T}} \cdots \le^*_{\mathscr{T}} t_{n-1}$ from $\mathscr{T}_{[\zeta]}$ satisfying $k < n \Rightarrow s_k <_{\mathscr{T}} t_k$
(b) if $\iota = 1$[2] then as above but we add $m < n$ and demand $t_\in \mathrm{spt}(\mathscr{T})$, other wise we have to replace $m$ by a set $w \subseteq n$ and demand

Now comes the main property

**Definition 1.14.** 1) $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ and $n < \omega$ and a cardinal $\sigma$ let $\mathscr{T}_1 \rightarrow_\iota (\mathscr{T}_2)^n_\sigma$ mean:

$(*)$ if $\mathbf{c} : \mathrm{eseq}_n(\mathscr{T}_1) \to \sigma$, <u>then</u> there is a $\iota$-embedding $g$ of $\mathscr{T}_2^\bullet$ into $\mathscr{T}_1$ such that the colouring $\mathbf{c} \circ g$ is homogeneous for $\mathscr{T}_2$ which means:
- if $\bar{a}, \bar{b} \in \mathrm{seq}^\iota_n(\mathscr{T}_2)$ are $\mathscr{T}_2 - \iota$ - similar then $\mathbf{c}(g(\bar{a})) = \mathbf{c}(g(\bar{b}))$.

2) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ and $n$ and $\sigma$ let $\mathscr{T}_1 \rightarrow_\iota (T_2)^{\mathrm{end}(k)}_\sigma$ mean that:

$(*)$ $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ and: if $\mathbf{c} : \mathrm{eseq}_\iota(\mathscr{T}) \to \sigma$ then there is an $\subseteq_\iota$-embedding $g$ of $\mathscr{T}_2$ into $\mathscr{T}_1$ such that the colouring $\mathbf{c}' = \mathbf{c} \circ g$ satisfies $\mathbf{c}'(\bar{\eta})$ does not depend on the last $k$ levels, that is, if $\bar{a}, \bar{b} \in \mathrm{eseq}^\iota_n(\mathscr{T}_2$ and $\ell < n \wedge (k \le |\mathrm{lev}(\bar{a}) \setminus \mathrm{lev}(a_\ell)| \Rightarrow b_\ell = a_\ell$ then $\mathbf{c}(\bar{a}) = \mathbf{c}(\bar{b})$

**Claim 1.15.** *Let $\mathscr{T} \in \mathbf{T}_\iota$.*
*1) If $A \subseteq \mathscr{T}$ is finite non-empty with $m$ elements <u>then</u>:*

(a) *if $\iota = 0, 1$ then for some $n \le (2m - 1)$ and $\bar{a} \in \mathrm{ccseq}_n(\mathscr{T})$ we have $A \subseteq \mathrm{Rang}(\bar{a})$; moreover for every $\eta \in \mathrm{Rang}(\bar{a}), \eta$ is $<_{\mathscr{T}}$-maximal in $\mathrm{Rang}(\bar{a})$ <u>iff</u> $\eta \in A$.*
(b) *if $\iota = 2$ then for some $n \le (2m - 1)m$ and $\bar{a} \in \mathrm{eseq}^\iota_n(\mathscr{T})$ we have $A \subseteq \mathrm{Rang}(\bar{a})$; moreover $\max\{\mathrm{lev}_{\mathscr{T}}(a) : a \in A\} = \max\{\mathrm{lev}_{\mathscr{T}}(a_\ell) : \ell < n\}$*

*2) The number of quantifier free complete $n$-types realized by some $\mathscr{T} \in \mathbf{T}_\iota$ and $\bar{a} \in \mathrm{eseq}^\iota_n(\mathscr{T})$ is, e.g. $\le 2^{2n^2+n}$ but $\ge n$.*
*3) Assume $\iota = 0, 2$ or $\iota = 1$ and we restrict ourselves to trees with at most one splitting point in each level. If $\mathscr{T} \in \mathbf{T}_\iota$ is weakly $\aleph_0$-saturated <u>then</u> $\mathscr{T}$ realizes all possible such types, i.e. one realized in some $\mathscr{T}' \in \mathbf{T}_\iota$.*

[2]it is very natural when we deal with trees with unique splitting nod in each level, but to deal with the three cases we did not succeed to formulate it better; anyhow a minor point as it is used only in 1.15(3).

*Proof.* 1) Clause (a): Let $B_1 = \{\eta \cap_{\mathscr{T}} \nu : \eta, \nu \in A\}$ and note that $\eta \in A \Rightarrow \eta = \eta \cap \eta \in B_1$. Now by induction on $|A|$ easily $|B_1| \leq 2m - 1$.

Now the second statement is easy too.

Clause (b) Let $w = \{\mathrm{lev}(_{\mathscr{T}}(a)\} : a \in A\}$; and let $\zeta = \max(w)$. Now we shall choose $\bar{b}$ such that:

(a) $\bar{b} = \langle b_a : a \in A \rangle$
(b) $b_a \in \mathscr{Y}_{[\zeta]}$
(c) $a \leq_{\mathscr{Y}} b_a$
(d) if $a_1 \leq_{\mathscr{T}} a_2$ then $b_{a_1} = b_{a_2}$

Clealry possible. Lastly let

$B_2 = \{b_a {\restriction} \varepsilon : a \in A, \varepsilon \in w\}$.

Now clearly $|w| \leq |B_1|$ and $|B_2| \leq |B_1| \times |A| \leq (2m-1)m$ as promised. (We may improve th eboudn but this deos not matter here)

2) Considering the class of such pairs $(\bar{a}, \mathscr{T})$ (fixing $n$) the number of $E_{\bar{a}} = \{(k,\ell) : a_k E_{\mathscr{T}} a_\ell\}$ is $\leq 2^{n^2}$ and the number of $<_{\bar{a}} = \{(k,\ell) : a_k <_{\mathscr{T}} a_\ell\}$ is $\leq 2^{n^2}$ and the number of $\{(a_k,a_\ell) : (a_k,a_\ell) \in R_1^{\mathscr{T}}$ and for no $i, a_k <_{\mathscr{T}} a_i <_{\mathscr{T}} a_\ell\}$ is $\leq 2^n$.

Lastly, from those we can compute $\{(a_k,a_\ell) : (a_k,a_\ell) \in R_0^{\mathscr{T}}\}$ as $\{(a_k,a_\ell,a_m) : a_k \cap_{\mathscr{T}} a_\ell = a_m\}$, so together the number is $\leq 2^{2n^2+n}$.

Clearly we can get a better bound, e.g. letting $m_n^\bullet(\mathscr{T}) = |\{\mathrm{tp}_{\mathrm{qf}}(\bar{a}{\restriction}n, \emptyset, \mathscr{T}) : \bar{a} \in \mathrm{eseq}(\mathscr{T})$ has length $\geq n\}|$ then:

- $m_n^\bullet(\mathscr{T}) = 1$ for $n = 0, 1$
- $m_{n+1}^\bullet(\mathscr{T}) \leq 2n(m_n^\bullet(\mathscr{T}))$
- hence $m_n^\bullet(\mathscr{T}) \leq 2^{n-1}(n-1)!$

3) Should be clear. $\square_{1.15}$

**Claim 1.16.** *1) If $\theta = \Sigma\{(2^{|\alpha|})^+ : \alpha < \kappa\}$ there is $\mathscr{T} \in \mathbf{T}_{\theta,\kappa}$ with $\kappa$ levels and in the $\alpha$-th level at $2^{|\alpha|}$ elements, expanding $({}^{\kappa>}2, \triangleleft)$ which means that $\mathscr{T}$ expand s* $(\mathrm{spt}(\mathscr{T})$.
*2) For any $\kappa$ letting $\theta = \Sigma\{(2^{|\alpha|})^+ : \alpha < \kappa\}$, there is $\mathscr{T}$ such that* $\mathrm{ht}(\mathscr{T}) = \Sigma\{2^{|\alpha|} : \alpha < \kappa\}$, *ordinal sum.*

*Proof.* 1),2) Should be clear. $\square_{1.16}$

Next we define $\mathbf{N}_\iota$ like $\mathbf{M}_{1,1}, \mathbf{M}_{1,2}$ in §(1A) using $\mathscr{T}$ from Definition 1.10, that is:

**Definition 1.17.** 1) For $\iota = 0, 1$ let $\mathbf{N}_\iota$ be the class of objects $\mathbf{m}$ consisting of (so $\kappa = \kappa_{\mathbf{m}}$, etc.):

(a) $\kappa$, a regular cardinal
(b) $\mathscr{T} \in \mathbf{T}_\iota$ and $\mathbb{B}$ a Boolean algebra of subsets of $\mathrm{dom}(\mathbb{B})$ which is $\mathrm{spt}(\mathscr{T})$ when $\iota = 1, 2$ and is $u_{\mathscr{Y}}$ when $\iota = 0, 2$
(c) $\mathbb{Q}$ is a quasi-order, $(< \kappa)$-complete
(d) ($\alpha$) val, a function from $\mathbb{Q}$ into $\mathbb{B}$
 ($\beta$) if $p \in \mathbb{Q}$ then $\{\mathrm{lev}_{\mathscr{T}}(\eta) : \eta \in \mathrm{val}(p)\}$ is unbounded in $\mathrm{ht}(\mathscr{T})$
 ($\gamma$) if $\varepsilon < \mathrm{ht}(\mathscr{T})$ and $p \in \mathbb{Q}$ then for some $s, t, p_0, p_1$ we have: $s \in \mathrm{val}_{\mathscr{T}}(p), \mathrm{lev}_{\mathscr{T}}(s) \geq \varepsilon, s \leq_{\mathscr{T}} t$ and for $\ell = 0, 1$ we have $p \leq_{\mathbb{Q}} p_\ell$ and $(\forall r \in \mathrm{val}(p_\ell))(t R_{\mathscr{T},\ell} r)$

($\delta$) above, if $\iota = 1$ then we can add $s \in \mathrm{val}(p)$

(e) (monotonicity) if $p \le q$ then $\mathrm{val}(p) \supseteq \mathrm{val}(q)$

(f) (decidability) if $\mathrm{val}(p) = A \subseteq \lambda$ and $A = A_0 \cup A_1$ and $A_0, A_1 \in \mathbb{B}$ then for some $\ell \in \{0,1\}$ and $q \in \mathbb{Q}$ above $p$ we have $\mathrm{val}(q) \subseteq A_\ell$.

2) For $\iota = 2$ Let $\mathbf{N}_\iota$ be the class of $\mathbf{m} \in \mathbf{N}_2$ such that clauses (a)-(f) as above and:

(g) if $\iota = 2$ and $\gamma < \kappa$ and $p_\alpha \in \mathbb{Q}$ for $\alpha < \gamma$ then $\bigcap_{\alpha<\gamma} \{\ell g(\eta) : \eta \in \mathrm{val}(p_\alpha)\}$ is unbounded in $\mathrm{ht}(\mathscr{T})$ BUT WE need new decidability xob xob XOB

**Definition 1.18.** If $\mathbf{m} \in \mathbf{N}_\iota$ then we shall say $\mathbf{c}$ is an $\mathbf{m}$-colouring when

(*) $\mathbf{c}$ is a function from $\mathrm{eseq}_\iota(\mathscr{T}_\mathbf{m})$

(*) $\mathbf{c}$ is a function into some $\sigma < \kappa_\mathbf{m}$

(*) if $\bar a \in \mathrm{eseq}_\iota$ and $j < \sigma$ then the set $\{b \in \mathscr{T}_\mathbf{m} : \bar a \hat{\ } \langle b\rangle \in \mathrm{eseq}_\iota(\mathscr{T}, \mathbf{c}(\bar a \hat{\ } \langle b\rangle) = j\}$ belongs to $\mathbb{B}_\mathbf{m}$

**Crucial Claim 1.19.** *Assume $\iota = 0, 1, \mathbf{m} \in \mathbf{N}_\iota$ so $\kappa = \kappa_\mathbf{m}$, etc. and $\sigma < \kappa, \mathscr{S} \in \mathbf{T}^\iota_{\theta,\partial}$ and $\partial \le \kappa, \theta \le \kappa$ and $\alpha < \theta \Rightarrow \sigma^{|\alpha|} < \kappa$. Then $\mathscr{T} \rightarrow_{\mathbf{m},\iota} (\mathscr{S})^{\mathrm{end}(1)}_\sigma$ which means that we restrict ourselves to $\mathbf{m}$-colourings, that is: for every $\mathbf{m}$-colouring $\mathbf{c} : \mathrm{eseq}^\iota_{<\omega}(\mathscr{T}) \rightarrow \sigma$ there is an $\iota$-embedding $g$ of $\mathscr{S}$ into $\mathscr{T}_\mathbf{m}$ such that $\mathbf{c}' = \mathbf{c} \circ g$ which has domain $\mathrm{eseq}^\iota_{<\omega}(\mathscr{S})$ is 1-end homogeneous, recalling 1.14(2).*

*Remark* 1.20. 1) Note that our embedding (in 1.19) preserves $<^*$ but not necessarily equality of levels, i.e. $E$.
2) A delicate point to check: consider sequence $\bar\eta$ in ${}^{n+1}\mathrm{spt}(\mathscr{T}), <^*_\mathscr{T}$-increasing, consider a colouring $\mathbf{c}_1$ such that $\mathbf{c}_1(\langle \eta_\ell : \ell \le n\rangle) = \mathbf{c}_2(\langle \eta_\ell : \ell < n\rangle \hat{\ } \langle \eta_n {\restriction} \mathrm{lev}_\mathscr{T}(\eta_{n-1})\rangle)$. See colouring below.
3) Recall $\kappa$ regular but $\mathrm{cf}(\mathrm{ht}(\mathscr{T})) \ge \kappa$ is enough.

*Proof.* So we Fix $\mathbf{m} \in \mathbf{T}_\iota, \mathscr{T} = \mathscr{T}_\mathbf{m}$ etc, $\sigma < \kappa$ and $\mathbf{m}$-colouring $\mathbf{c} : \mathrm{eseq}^\iota_{<\omega}(\mathscr{T}) \rightarrow \sigma$. Now:

$(*)_1$ if $\mathscr{U} \subseteq \mathscr{T}$ has cardinality $< \theta$ and $p \in \mathbb{Q}$ then we let:

(a) $\mathrm{Col}_\mathscr{U}(p)$ is the set of functions $\mathbf{d}$ from $\mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})$ into $\sigma$ such that for some $s \in \mathrm{val}(p) \backslash \mathscr{U}$ we have:
  - if $\bar a \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})$ then $\bar a \hat{\ } \langle s\rangle \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{T}) \Leftrightarrow \bar a \in \mathrm{Dom}(\mathbf{d})$ and if this holds then $\mathbf{c}(\bar a \hat{\ } \langle s\rangle) = \mathbf{d}(\bar a)$

(b) $\mathrm{col}_\mathscr{U}(p) = \mathrm{col}(p, \mathscr{U})$ is the function $\mathbf{d}$ from $\mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})$ into $\sigma + 1$ such that (if there is no such function, then $\mathrm{col}_\mathscr{U}(p)$ is is not defined) for every $s \in \mathrm{val}(p)$ and $\bar a \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})$ we have $\sup\{\mathrm{lev}_\mathscr{T}(t) + 1 : t \in \mathscr{U}\} \le \mathrm{lev}_\mathscr{T}(s)$ and if $\bar a \hat{\ } \langle s\rangle \in \mathrm{eseq}^\iota(\mathscr{U}, \mathscr{T})$ and $i < \sigma$ then $\mathbf{d}(\bar a = i$ iff $\bar a \hat{\ } \langle s\rangle \in \mathrm{eseq}_\iota(\mathscr{T})$ is well defined and $\mathbf{c}(\bar a \hat{\ } \langle s\rangle) = i$ .

Next

$(*)_3$ if $\mathscr{U} \subseteq \mathscr{T}, p \in \mathbb{Q}$ and $|\mathscr{U}| < \theta$, then for some $q$ we have:

- $p \le_\mathbb{Q} q$
- $\mathrm{col}_\mathscr{U}(q)$ is well defined
- if $r \in \mathbb{Q}$ is above $q$ then $\mathrm{col}_\mathscr{U}(r) = \mathrm{col}_\mathscr{U}(q)$ so both are well defined.

[Why? Let $\langle(\bar a_i, c_i) : i < \mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})\times\sigma\rangle$ list the pairs $(\bar a, c) \in \mathrm{eseq}^\iota_{<\omega}(\mathscr{U}, \mathscr{T})\times\sigma$. Now choose $p_i$ by induction on $i \le |\mathrm{eseq}_{<\omega}(\mathscr{U}, \mathscr{T}) \times \sigma|$ such that:

- $\bullet_1$ $p_0 = p$
- $\bullet_2$ $j < i \Rightarrow p_j \le p_i$
- $\bullet_3$ if $i = j + 1$ then $\mathrm{val}(p_i)$ is included in $\{s \in \mathrm{spt}(\mathscr{T}) : \mathbf{c}(\bar a_i{}^\frown\langle s\rangle) = c_j\}$ or is disjoint to it.

Recall that by the properties of val, we can ensure $s \in \mathrm{val}(p_i) \Rightarrow (\forall t \in \mathscr{U})(t <_{\mathscr{T}} s)$, see 1.17(1)(e),(f).

This suffices (note that: $(\alpha < \theta \Rightarrow \sigma^{|\alpha|} < \kappa)$).]

$(*)_4$ fix $\bar\eta$ such that:
- (a) $\bar\eta = \langle\eta_i : i < i(*)\rangle$ lists the elements of $\mathscr{S}$ where $i(*) < \theta$
- (b) $i < j \Rightarrow \eta_i <^*_{\mathscr{S}} \eta_j$
- (c) if $\eta_i = \mathrm{suc}_{\mathscr{S},0}(\rho)$ then $\eta_{i+1} = \mathrm{suc}_{\mathscr{S},1}(\rho)$ and $i \in \{1+2j : 1+2j < i(*)\}$

[Why exists? Just think.]

Toward our inductive construction:

$(*)_5$ for $\zeta < \kappa$ let:
- (a) $I^{\mathrm{is}}_\zeta = \{\eta_\varepsilon : \varepsilon < 1 + (2\zeta)\}$, where "is" stands for "initial segment"
- (b) $I^{\mathrm{fr}}_\zeta$, where "fr" stands for front, is the set of $\eta$ satisfying one of the following
  - (*) $\eta \in \mathscr{T}, \eta \in I^{\mathrm{is}}_\zeta$ but $\neg(\exists\nu)(\eta <_{\mathscr{T}} \nu \in I^{\mathrm{is}}_\zeta)$;
  - (*) $\eta \in \mathscr{T}, \eta \notin I^{is}$ and $\lg(\eta)$ is a limit ordinal but $(\forall\nu)[\nu <_{\mathscr{T}} \eta \Rightarrow \nu \in I^{\mathrm{is}}_\zeta]$
- (c) hence $I^{\mathrm{fr}}_\zeta$ is a maximal $\triangleleft$-antichain of $\mathscr{T}$.

Now we shall choose $(\bar p_\zeta, \bar s_\zeta, \bar t_\zeta, \bar{\mathbf{d}}_\zeta)$ by induction on $\zeta < \theta$ such that:

$(*)_6$
- (a) $\bar p_\zeta = \langle p_{\zeta,\eta} : \eta \in I^{\mathrm{is}}_\zeta \cup I^{\mathrm{fr}}_\zeta\rangle$
- (b) $p_{\zeta,\eta} \in \mathbb{Q}$
- (c) $p \le_{\mathbb{Q}} p_{\zeta,\eta}$ for $\eta \in I^{\mathrm{is}}_\zeta \cup I^{\mathrm{fr}}_\zeta$
- (d) if $\varepsilon < \zeta$ and $\nu \in I^{\mathrm{is}}_\varepsilon, \nu \trianglelefteq \eta$ then
  - $p_{\varepsilon,\nu} \le_{\mathbb{Q}} p_{\zeta,\eta}$
  - if $\beta \notin I^{\mathrm{fr}}_\zeta$ then equality holds
- (e) ($\alpha$) $\bar s_\zeta = \langle s_\eta = s(\eta) : \eta \in I^{\mathrm{is}}_\zeta\rangle$ so $\varepsilon < \zeta \Rightarrow \bar s_\varepsilon = \bar s_\zeta \restriction I^{\mathrm{is}}_\varepsilon$
  - ($\beta$) if $\eta_\varepsilon, \eta_\xi \in I^{\mathrm{fr}}_\zeta$ then $s_{\eta_\varepsilon} <^*_{\mathscr{T}} s_{\eta_\xi}$
  - ($\gamma$) if $\nu <_{\mathscr{T}} \eta$ then $s_\eta \in \mathrm{val}(p_{\zeta,\eta}$
- (f) ($\alpha$) $\bar t_\zeta = \langle t_\eta : \eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta\rangle$
  - ($\beta$) $s_\eta \le_{\mathscr{T}} t_\eta \in \mathrm{spt}(\mathscr{T})$ for $\eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta$
  - ($\gamma$) $\nu \triangleleft \eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta \Rightarrow t_\nu \le_{\mathscr{T}} s_\eta$
  - ($\delta$) if $\iota = 1$ then $s_\eta = t_\eta$
- (g) if $\eta \in I^{\mathrm{is}}_\zeta \backslash I^{\mathrm{fr}}_\zeta, \ell < 2$ and $\nu = \mathrm{Suc}_{\mathscr{S},\ell}(\eta)$ then (necessarily $\nu \in I^{\mathrm{is}}_\zeta$ and) $(\forall t)[t \in \mathrm{val}(p_{\zeta,\nu}) \rightarrow t_\eta R_{\mathscr{T},\ell} t]$
- (h) $\bar{\mathbf{d}}_\zeta = \langle \mathbf{d}_{\zeta,\eta} : \eta \in I^{\mathrm{is}}_\zeta\rangle$ but $\mathbf{d}_{\zeta,\eta} = \mathbf{d}_\eta$ does not depend on $\zeta$

(i) $\mathbf{d}_\eta = \mathrm{col}_{I^{\mathrm{is}} \setminus I^{\mathrm{fr}}}(p_{\zeta,\eta})$ is well defined.

Clearly it is enough to carry the induction, because the mapping $\eta \mapsto s_\eta$ is a $\subseteq_\iota$-embedding as required.

Case 1: $\zeta = 0$

Let $p_{\langle\rangle} \in \mathbb{Q}$ be above $p$ and such that $\mathbf{d}_{0,\emptyset} = \mathrm{col}_\emptyset(p_{\langle\rangle})$ is well defined. Let $\alpha_{\langle\rangle} \in \mathrm{val}(p_{\langle\rangle})$.

Case 2: $\zeta = \varepsilon + 1, \eta_{1+2\varepsilon} = \mathrm{suc}_{\mathscr{S},0}(\rho)$

Choose $\xi < \kappa$ such that:

$\eta \in I^{\mathrm{is}}_\varepsilon \setminus I^{\mathrm{fr}}_\varepsilon \Rightarrow \mathrm{lev}(\eta) < \xi$ .

Now we choose a quadruple $(s_\rho, t_\rho, p^0_\rho, p^1_\rho)$ such that

(a) $s_\rho \in \mathrm{val}(p_{\varepsilon,\rho})$
(b) $s_\rho \leq_{\mathscr{T}} t_\rho$
(c) $p^0_\rho, p^1_\rho$ are above $p_{\varepsilon,\rho}$ in $Q$
(d) for every $r \in \mathrm{val}(p^\ell_\rho$ we have $t_\rho R \mathscr{Y}, \ell r$
(e) if $\iota = 1$ then $s_\rho = t_\rho$

[Why such a quadruple exists? by 1.17(1)(d)($\gamma$)]

Nexst we choose $p_{\zeta,\eta}$ for $\eta \in I^{\mathrm{is}}_\varepsilon \cup I^{\mathrm{fr}}_\varepsilon$ as follows

(*) if $\eta \notin I^{\mathrm{is}}_\zeta$ then $p_{\zeta,\eta} = p_{\varepsilon,\eta}$
(*) if $\eta \in I^{\mathrm{is}}_\varepsilon \wedge \{\rho\}$ then we choose $p_{\zeta,\eta} \in \mathbb{Q}$ above $p_{\varepsilon,\eta}$ such that $\mathrm{val}(p_{\zeta,\eta}, \{s_\nu : \nu \in (I^{\mathrm{is}}_\varepsilon \setminus I^{\mathrm{fr}}_\varepsilon)\}) \cup \{\rho\}$
(*) if $\ell = 0,1$ and $\eta = \rho\char`^\langle \ell \rangle$ then we choose $p_{\zeta,\eta} \in \mathbb{Q}$ above $p^\ell_{\rho,\eta}$ such that $\mathrm{val}(p_{\zeta,\eta}, \{s_\nu : \nu \in (I^{\mathrm{is}}_\varepsilon \setminus I^{\mathrm{fr}}_\varepsilon)\}) \cup \{\rho\}$

Now check.

Case 3: $\zeta = 1 + 2\zeta$ and $\eta_{1+2\varepsilon}\rho = \eta_{1+2\varepsilon+1}$

Note that $\{\rho\} \cup (I^{\mathrm{is}}_\varepsilon \setminus I^{\mathrm{fr}}_\varepsilon)$ is equal to $I^{\mathrm{is}}_\zeta \cup I^{\mathrm{fr}}_\zeta$. Now we choose $p_{\zeta,\eta} = p_{\varepsilon,\eta}$ for $\eta \in I^{\mathrm{is}}_\varepsilon \setminus I^{\mathrm{fr}}_\varepsilon$ and choose $p_{\zeta,\rho}$ as an upper bound of $\{p_{\varepsilon,\beta} : \nu <_{\mathscr{T}} \rho\}$ which satisfies the demand in 1.17(1)(d)($\gamma$) BDOQ

Case 4: $\zeta$ is a limit ordinal

This is easy, recalling $\mathbb{Q}$ is $(< \kappa)$-complete. the demand in and 1.17(1)(d)($\gamma$

So $\rho \in I^{\mathrm{is}}_\varepsilon$. Hence $s_\rho \in \mathrm{val}(p_{\varepsilon,\rho})$ is well defined and is $<_{\mathscr{T}}$-above $\{s_\nu : \nu \triangleleft \rho\}$ and choose $t_\rho \in \mathscr{T}$ $<_{\mathscr{T}}$-above $s_\rho$ and $p^0_{\zeta,\rho}, p^1_{\zeta,\rho}$ above $p_{\zeta,\rho}$ such that $\mathrm{val}(p^\ell_{\zeta,\rho}) \subseteq \{s \in \mathscr{T} : t_\rho R^{\mathscr{T}}_\ell s\}$ for $\ell = 0,1$ and note that $I^{\mathrm{fr}}_\zeta = (I^{\mathrm{fr}}_\varepsilon \setminus \{\rho\}) \cup \{\rho_0, \rho_1\}$ where $\mathrm{suc}_{\mathscr{S},\iota}(\rho) = \rho_\iota$ for $\iota = 0,1$. This is possible by 1.17(1)(d)($\delta$).

We now shall choose $p_{\zeta,\nu}$ for $\nu \in I^{\mathrm{fr}}_\zeta$. Choose $s_{\rho_0} \in \mathrm{val}(p^0_{\varepsilon,\rho})$ and then choose $p^{1,*}_{\varepsilon,\rho}$ which is above $p^1_{\varepsilon,\rho}$ and $\mathrm{col}_{\{s_\xi:\xi<1+2\varepsilon\}\cup\{s_{\rho_0}\}}(p^{1,*}_{\varepsilon,\rho})$ is well defined and choose $s_{\rho_1} \in \mathrm{val}(p^{1,*}_{\varepsilon,\rho_1})$ and $p^{0,*}_{\varepsilon,\rho} = p^0_{\varepsilon,\rho}$.

If $\nu \in I^{\mathrm{fr}}_\zeta \setminus \{\rho_0, \rho_1\} \subseteq I^{\mathrm{fr}}_\varepsilon \setminus \{\rho\}$ then $p_{\zeta,\nu}$ is a member of $\mathbb{Q}$ above $p_{\varepsilon,\nu}$ such that $\mathrm{col}_{\{s_\xi:\xi<1+2\zeta\}}(p_{\zeta,\nu})$ is well defined.

Lastly, for $\ell = 0,1$ let $p_{\zeta,\rho_\ell}$ be a member of $\mathbb{Q}$ above $p^{\ell,*}_{\zeta,\rho}$ such that $\mathrm{col}_{\{s_\xi:\xi<1+2\zeta\}}(p_{\zeta,\rho_\ell})$ is well defined. So we are done with Case 2.

$\square_{1.19}$

Now we derive the obvious conclusion

**Claim 1.21.** *If* $\iota < 3$.
*1) If* $\mathscr{T} \in \mathbf{T}$ *and* $\mathscr{T} \to (\mathscr{T})^{\mathrm{end}(1)}_\sigma$ *then* $\mathscr{T} \to_\iota (\mathscr{T})^{\mathrm{end}(k)}_\sigma$ *for every* $k$.
*2) If* $\mathscr{T} \in \mathbf{T}$ *and* $\mathscr{T} \to (\mathscr{T})^{\mathrm{end}(1)}_\sigma$ *then* $\mathscr{T} \to (\mathscr{T})^n_\sigma$ *for every* $n$.
*3) If* $k \geq 1$ *and* $\mathscr{T}_\ell \in \mathbf{T}$ *for* $\ell = 0, \ldots, k$ *and* $\mathscr{T}_{\ell+1} \to_\iota (T_\ell)^{\mathrm{end}(1)}_\sigma$ *for* $\ell < n$, *then* $\mathscr{T}_n \to (\mathscr{T}_0)^{\mathrm{end}(k)}_\sigma$ *hence* $\mathscr{T}_n \to (\mathscr{T}_0)^k_\sigma$.

*Proof.* Should be clear. $\square_{1.21}$

## § 2. Examples

### § 2(A). Consistency with no Large Cardinal.

**Claim 2.1.** *Assume $\iota = 0, 1$ and (A) then (B) where:*

(A) (a) $\kappa = \kappa^{<\kappa} < \mu = \kappa^+$ *(can get it by a preliminary forcing; let* $\mu_0 = \mu(0) = \mu$ *and force by* $\text{Levy}(\kappa, 2^{<\kappa})$ *for* $\kappa$ *regular)*

(b) $\lambda > \mu_1 = \mu(1) = 2^{<\mu} = 2^\kappa$

(c) $\mathbb{P}$ *is the forcing* $\text{Cohen}(\kappa, \lambda)$*, adding* $\lambda$ *many* $\kappa$*-Cohens*

(d) *the* $\mathbb{P}$*-name* $\underset{\sim}{\mathscr{T}}_1 \in \mathbf{T}_\iota$ *expand* $({}^{\mu>}2, \triangleleft)$ *as in 1.16*

(e) $\mathscr{T}_2 \in \mathbf{T}\iota$ *expand* $({}^{\kappa>}2, \triangleleft)$ *as in 1.16, so can be chosen in* $\mathbf{V}$

(B) *in* $\mathbf{V}^{\mathbb{P}}$*, for every* $\sigma < \kappa$ *we have* $\mathscr{T}_1 \rightarrow_\iota (\mathscr{T}_2)^{\text{end}(1)}_\sigma$

*Proof.* First

$\boxplus_1$ so $\mathbb{P}$ is $\text{Cohen}(\kappa, \lambda)$ and $\underset{\sim}{\eta}_\alpha$ for $\alpha < \lambda$ are as in 0.2

For the rest of the proof we assume:

$\boxplus_2$ $\Vdash_{\mathbb{P}}$: $\underset{\sim}{\mathbf{c}} : \text{eseq}_\iota(\mathscr{T}_1) \rightarrow \sigma$ and $\sigma$ is a cardinal $< \kappa$

Next, in $\mathbf{V}$, we choose:

$(*)_1$ (a) let $\chi > \lambda$ and $<^*_\chi$ a well ordering of $\mathscr{H}(\chi)$

(b) let $\mathfrak{B} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be of cardinality $\kappa$ such that $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\lambda, \kappa, \mu, \sigma, \mathscr{T}_1, \mathscr{T}_2, \underset{\sim}{\mathbf{c}} \in \mathfrak{B}$

(c) let $u_* = \mathfrak{B} \cap \lambda \in [\lambda]^\kappa$ and $\delta_* = \min(\lambda \backslash u_*)$

(d) let $\mathbf{G}_{u_*} \subseteq \mathbb{P}_{u_*}$ be generic over $\mathbf{V}_0 = \mathbf{V}$

(e) let $\bar{\eta}_{u_*} = \langle \underset{\sim}{\eta}_\alpha[\mathbf{G}_{u_*}] : \alpha \in u_* \rangle$

(f) let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_{u_*}] = \mathbf{V}_0[\bar{\eta}_{u_*}]$

$(*)_2$ (a) let $\underset{\sim}{\mathscr{T}}_0$ be the sub-structure of $\underset{\sim}{\mathscr{T}}_1$ with set of elements $\mathscr{S} = \{\underset{\sim}{\eta} : \underset{\sim}{\eta}$ is a canonical $\mathbb{P}$-name of a member of $\underset{\sim}{\mathscr{T}}_1$ and this name belongs to $\mathfrak{B}\}$; ("canonical" mean it is defined by $\kappa$ maximal anti-chains of $\mathbb{P}$ each of cardinality $\kappa$ etc)

(b) let $\delta_* = \delta(*)$ be $\min(\lambda \backslash u_*) = \min(\mu \setminus u_*) = \mu \cap u_*$

Clearly

$(*)_3$ (a) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathscr{T}_0}$ is a sub-tree of $\underset{\sim}{\mathscr{T}_1}$ which expand $({}^{\mu}2, \triangleleft)^{\mathbf{V}[\mathbb{P}]}$ , $\mathscr{S}$ is closed under initial segments, is of cardinality $\kappa$ and closed under unions of increasing chains of length $< \kappa$" and $\nu \in \underset{\sim}{\mathscr{S}} \Rightarrow \nu\hat{\ }\langle 0 \rangle, \nu\hat{\ }\langle 1 \rangle \in \underset{\sim}{\mathscr{S}}$

(b) $\underset{\sim}{\mathscr{T}}_0, \underset{\sim}{\mathscr{S}} S$ are actually $\mathbb{P}_{u_*}$-names and we can use $\mathfrak{B} \cap \mu$ as its set of levels,

$(*)_4$ (a) let $\mathscr{T}_0 = \underset{\sim}{\mathscr{T}}_0[\mathbf{G}_{u_*}], \mathscr{S} = \underset{\sim}{\mathscr{S}}[\mathbf{G}_{u_*}]$, so they are from $\mathbf{V}_1$

(b) let $\mathbb{P}_2 = \mathbb{P}/\mathbf{G}_{u_*} = \mathbb{P}_{\lambda \backslash u_*}$

$(*)_5$ in $\mathbf{V}_1$ there are

(a) a $\mathbb{P}_2$-name $\underset{\sim}{\eta}$ of a branch of $\mathscr{T}_0$ generic over $\mathbf{V}_1$, i.e. for the forcing notion $(\mathscr{S}, \triangleleft)$.

(b) hence $\underset{\sim}{\eta} \in {}^{\delta(*)}2 \subseteq \mathscr{T}_1$ and $\varepsilon < \delta_* \Rightarrow \Vdash_{\mathbb{P}_{u_*}}$ "$\underset{\sim}{\eta} \restriction \varepsilon \in \underset{\sim}{\mathscr{T}}_1$"

[Why? By the character of $\mathscr{S}$ see $(*)_3$ being of cardinality $\kappa$.]

$(*)_6$ (a) let $\mathscr{A}$ be the set of object from $\mathfrak{B}$ which are $\mathbb{P}$-names of a subset of $\underset{\sim}{\mathscr{T}}_1$.

(b) if the object $\underset{\sim}{A}$ belongs to $\mathscr{A}$ then we let $\underset{\sim}{\check{A}}$ be the $\mathbb{P}_{u_*}$-name $\underset{\sim}{A} \cap \underset{\sim}{\mathscr{S}}$

(c) $\underset{\sim}{\check{A}}$ is actually a $\mathbb{P}_{u_*}$-name

(d) in $\mathbf{V}_1$ let $\mathbb{B} = \underset{\sim}{\check{A}}[\mathbf{G}_{u_*}] : \underset{\sim}{A} \in \mathscr{A}\}$ and so $\underset{\sim}{\mathbb{B}} = \{\underset{\sim}{\check{A}} : \underset{\sim}{A} \in \mathscr{A}\}$ is a $\mathbb{P}_{u_*}$-name such that $\underset{\sim}{\mathbb{B}}[\mathbf{G}_{u_8}] = \mathbb{B}$

(e) for $p \in \mathbb{P}_2$ we let (in $\mathbf{V}_1$) $D_p = \{\underset{\sim}{\check{A}}[\mathbf{G}_{u_*}] : p \Vdash_{\mathbb{P}_2}$ "$\eta \in \underset{\sim}{A}\}$

Now comes the main point - find appropriate $\mathbf{m}$ in the universe $\mathbf{V}_1$

$(*)_7$ we define $\mathbf{m}$ as follows:

(a) $\mathscr{T}_{\mathbf{m}} = \underset{\sim}{\mathscr{T}}_0[\mathbf{G}_{u_*}]$

(b) the forcing notion $\mathbb{Q}_{\mathbf{m}} = \mathbb{P}_2$ is defined by
- the set of elements is $\{(p, \underset{\sim}{A}) : p \in \mathbb{P}_2, \underset{\sim}{A} \in \mathscr{A}$ such that $\Vdash$ "$\eta \in \underset{\sim}{A}\}$
- the order is $(p_1, \underset{\sim}{A}_1) \leq_{\mathbb{Q}_{\mathbf{m}}} (p_2, \underset{\sim}{A}_2)$ iff (both are from $\mathbb{Q}$ and) $p_1 \leq_{\mathbb{Q}} p_2, p_2 \Vdash$ "$\underset{\sim}{A}_2 \subseteq \underset{\sim}{A}_1$"

(c) $\mathbb{B}_{\mathbf{m}}$ is the family $\mathbb{B}$ defined above except that we intersect it with $\mathscr{P}(\mathrm{spt}(\mathscr{T}_0)$ when $\iota = 1$

(d) $\mathrm{val}_{\mathbf{m}}((p, \underset{\sim}{A})) = \underset{\sim}{\check{A}}[\mathbf{G}_{u_*}]$

$(*)_8$ $\mathbf{m} \in \mathbf{N}_\iota$

gggggggggggggggg Why? We should check all the clauses in Definition 1.17 in order to prove that $\mathbf{m} \in \mathbf{N}_\iota$ indeed.

Clause (a): $\kappa$ is a regular cardinal

Recall clause (A)(a) of 2.1(1).

Clause (b): $\mathscr{T} \in \mathbf{T}_\iota$

On $\mathscr{T}_{\mathbf{m}}$ recall clause (A)(d) of 2.1(1) and $(*)_2 + (*)_3$. On $\mathbb{B}_{\mathbf{m}}$ see its definition.

Clause (c): $\mathbb{Q}_{\mathbf{m}}$ is a quasi order, $(< \kappa)$-complete

For being a quasi-order, see the choice of $\mathbb{Q}_{\mathbf{m}}$ in $(*)_7(b)$; and $\mathbb{P}_2$ being a quasi-order and $(\underset{\sim}{D}_p, \supseteq)$ forced to be. As for being $(< \kappa)$-complete recall $\mathbb{P}_2$ is $(< \kappa)$-complete by clause (A)(b) of 2.1(1) and $\underset{\sim}{D}_p$ is forced to be $(< \kappa)$-complete.

Clause (d)($\alpha$): $\mathrm{val} = \mathrm{val}_{\mathbf{m}}$ being a function from $\mathbb{Q}$ into $\mathscr{P}(\mathrm{spt}_\iota(\mathscr{T}))$ HOSEP HAGDARAH yovil gam qodem!!

See the choice of val in $(*)_7(d)$ above. In particular, why the value belongs to $\mathbb{B}_{\mathbf{m}}$? see Def 1.19(xyz).

Clause (d)(($\beta$): if $p \in \mathbb{Q}$ then $\{\mathrm{lev}_{\mathscr{T}}(\eta) : \eta \in \mathrm{val}(p)\}$ is unbounded in $\mathrm{ht}(\mathscr{T})$

Why? as it is forced that $\underset{\sim}{\eta}$ is a branch of $\underset{\sim}{\mathscr{T}}_0$

Clause (d)($\gamma$) if $\varepsilon < \mathrm{ht}(\mathscr{T})$ and $p \in \mathbb{Q}$ then for some $s, p_0, p_1$ we have: $\mathrm{lev}_{\mathscr{T}}(s) \geq \varepsilon$ and for $\ell = 0, 1$ we have $p \leq_{\mathbb{Q}} p_\ell$ and $(\forall t \in \mathrm{val}(p_\ell))(s R_{\mathscr{T},\ell} t)$

The reason is that it is forced (for $\mathbb{P}_2$)) that $\underset{\sim}{\eta}$ is not from $\mathbf{V}_1$

Clause (d)($\delta$): in (d)($\gamma$) above, if $\iota = 1$ then we can add $s \in \mathrm{val}(p)$

This is easy.

Clause (e): monotonicity

Just check the definition.

Clause (f): decidability

Why? This is the point wher $\mathbb{B}_{\mathbf{m}}$ help us.

So we are given $(p, \underset{\sim}{A} \in \mathbb{Q}_{\mathbf{m}}$ and $A_0, A_1 \in \mathbb{B}$ such that letting $A = \underset{\sim}{A}[\mathbf{G}_{u_*}]$. Recalling $(p, \underset{\sim}{A}) \in \mathbb{Q}_{\mathbf{m}}$ clearly $p \cup r_1 \Vdash \underset{\sim}{\eta} \in \underset{\sim}{A}$ for some $r_1 \in \mathbf{G}_{u_*}$. Also $A \subseteq A_0 \cup A_1$ hence some $r_2 \in \mathbf{G}_{u_*}$ $forces(for\ \mathbb{P}_{u_*})$ that $\underset{\sim}{A} \subseteq \underset{\sim}{A}_0 \cup \underset{\sim}{A}_1$, and let $r \in \mathbf{G}_{u_*}$ be a common upper bound of $r_1, r_2$. Hence $p \cup r$ forces (for $\mathbb{P}$) that $\eta \in \underset{\sim}{A}_0$ or $\underset{\sim}{A}_1$. Hence for some $q, r', \ell$ we have $\ell \in \{0, 1\}$ and $\mathbb{P}_2 \models$ "$p \le q$" and $r' \in \underset{\sim}{\mathbf{G}} - [u_*]$ is above $r$ and $q \cup r' \Vdash$ "$\underset{\sim}{\eta} \in \underset{\sim}{A}_\ell$, So $q, \ell$ are as required.

So we are done proving $(*)_8$. hhhhhhhhhhhhhhh ]

$(*)_9$ let $\mathbf{c} =$ be the function from $\mathrm{eseq}_\iota(\mathscr{T}_0)$ to $\sigma$ such that $\mathbf{c}(\bar{a}) = j$ iff for some $r \in \mathbf{G}_{u_*}$ we have $r \Vdash$ "$\underset{\sim}{\mathbf{c}}(\bar{a} = j$"

$\mathbf{c}$ is indeed a function from $\mathrm{eseq}_\iota(\mathscr{T}_0$ to $\sigma$ (and belong to $\mathbf{V}_1$)

$(*)_{10}$ there is an $\subseteq_\iota$-embedding $g$ of $\mathscr{T}_2$ into $\mathscr{S}$ and $\mathbf{c} \circ g$ is an end(1)-extension homogeneous colouring of $\mathrm{eseq}(\mathscr{T}_2')$.

[Why? By §(1B), that is 1.19].

Now $g$ witness our desired conclusion. $\square_{2.1}$

**Claim 2.2.** *XOB BDOQ XOB 2020-05-13 04:05 1) Assume $\mu = (2^{<\kappa})^+$ and $\mathscr{T} \in \mathbf{T}_{\mu,\kappa}$ is as in §(1B) and $\mathscr{S} \in \mathbf{T}_{\theta,\partial}$ and $\theta + \partial \le \kappa, \sigma < \kappa$. A sufficient condition for $\mathscr{T} \to_{\mathrm{reg}} (\mathscr{S})^{\mathrm{end}(1)}_\sigma$ is:*

$(*)$ *for every family $\mathscr{P} \subseteq \mathscr{P}(\mathrm{spt}(\mathscr{T}))$ of cardinality $\le 2^{<\kappa}$ there is $\mathbf{m} \in \mathbf{M}_1[\mathscr{P}]$ which means $\mathbf{m}$ is "almost from $\mathbf{N}_2$"; we weaken:*
- $\mathrm{val}_{\mathbf{m}}(p) \in \mathscr{P}$ *for every* $p \in \mathbb{Q}$
- *in the decidability clause demanding $A_1, A_2 \in \mathscr{P}$.*

*2) If $\kappa$ is weakly compact, $\mathscr{T} \in \mathbf{T}$ is as in §(1B) then $\mathscr{T} \to (\mathscr{T})^{\mathrm{end}(1)}_\sigma$ for $\sigma < \kappa$ hence $\mathscr{T} \Rightarrow (\mathscr{T})^{\mathrm{end}(k)}_\sigma$ for $k < \omega$.*

*Proof.* 1) Similarly to 1.19, first we fix $\mathbf{c} : [\mathscr{T}_2]^{<\aleph_0} \to \sigma$, let $\chi$ be large enough, and then choose $\mathscr{B} \prec (\mathscr{H}(\chi), <^*_\chi)$ be of cardinality $2^{<\kappa}$ such that $2^{<\kappa} + 1 \subseteq \mathscr{B}$ and $[\mathscr{B}]^{<\kappa} \subseteq \mathscr{B}$ and $\{\mathscr{T}_2, \mathscr{S}, \kappa, \theta_2, \mu, \partial\} \subseteq \mathscr{B}$. Now choose $\mathbf{m}$ as in $(*)$ of 1.19(2) for $\mathscr{P} = \mathscr{P}(\mathrm{spt}(\mathscr{T})) \cap \mathscr{B}$. Now proceed as in the proof of 1.19.

2) By §(1B) (and more there). $\square_{2.2}$

QUESTION: 1) where do we prove that we can force with no large cardinals that between $\mu$ and $2^\mu$ there are many $\lambda$ and trees $\mathscr{T}_\lambda \in \mathbf{T}_\iota$ such that $\theta < \lambda \Rightarrow \mathscr{T}_\lambda \rightarrow_\iota (\mathscr{T}_\theta)^{\mathrm{end}(1)}_\sigma$ for $\sigma < \mu$ .

Needed for the model theory.

2) Should we go back to $\iota = -1$ that is having embedding $f_0, f_1,\ f_) isa \subseteq_1$-embedding, $f_)(\eta) \leq f)1(\eta)$ and the homogeneity is for $f_1$ 2020-05-13 05:21
3) 3).

2. Have we staed the obvious conclusion of 2.1? wher e?

eeeeeeeeeeeeeee cccccccccccccccccccccccccccccccc

§ 2(B). .

[Here?] 1) Is it reasonable to resurrect the difference $\iota = 3, 4$ here? I.e. have splitting/non-splitting nodes.
2) Deal with non-full trees?
3) Definitely we use $\mathscr{T}_{[\varepsilon]}$ correctly.
4) Better $\rightarrow$ than $\rightarrow_{\rm reg}$?

- recheck.

We start with a weaker version of [She92, §4], i.e. getting $\rightarrow_{\rm reg}$ in 2.3. The gain compare to [She92, §4] and subsequent works [19.10.05 - check] we need a weaker assumption, add $\lambda = \kappa^+$ many $\kappa$-reals, i.e. forcing with Cohen$(\kappa, \kappa^+)$.

Rather than $\lambda = \beth_{(2^\kappa)^+}$, this has serious consequences on consistency strength.

**Claim 2.3.** *1) If (A) then (B) <u>where</u>:*

(A) (a) *$\kappa$ is (strongly) inaccessible and $\lambda \geq \kappa^+$*
  (b) *$\mathbb{P}$ is the forcing* Cohen$(\kappa, \lambda)$*, adding $\lambda$ many $\kappa$-Cohens*
  (c) *in $\mathbf{V}^{\mathbb{P}}, \kappa$ is measurable; in fact*
    - *there is a normal ultrafilter on $\kappa$ extending one from $\mathbf{V}$*
  (d) *$\mathscr{T} \in \mathbf{T}$ expand $({}^{\kappa>}2, \triangleleft)$ so is full*

(B) (a) *in $\mathbf{V}$ we have $\mathscr{T} \rightarrow_{\rm reg} (\mathscr{T})^{\rm end(1)}_\sigma$ for every $\sigma < \kappa$*
  (b) *for[3] some (really many) $u \in [\lambda]^\kappa$, see below, in $\mathbf{V}^{\mathbb{P} \restriction u}$ there is $\mathbf{m} \in \mathbf{N}_1$ with $\mathscr{T}_{\mathbf{m}} = \mathscr{T}$ so $\kappa_{\mathbf{m}} = \kappa$*
  (b)$^+$ *moreover, we can have $\mathbf{m} \in \mathbf{N}_2$.*

*2) We can replace clause (A)(b) by:*

($\alpha$) *$\mathbb{P}$ is a forcing notion*
($\beta$) *if $\underset{\sim}{\eta}$ is a $\mathbb{P}$-name of a $\kappa$-real <u>then</u> for a dense set of $p \in \mathbb{P}, \mathbb{P}_{\geq p}$ is equivalent to some forcing notion $\mathbb{P}_p * \underset{\sim}{\mathbb{Q}}_p$ satisfying $\underset{\sim}{\eta}$ is a $\mathbb{P}_p$-name and $\mathbb{Q}_p$ is a $(< \kappa)$-complete forcing adding a new $\kappa$-real (i.e. $\notin \mathbf{V}^{\mathbb{P}_\beta}$).*

*Proof.* 1) Clause (B)(a) follows from clause (B)(b) by 1.19, so we shall deal with clause (B)(b).

So

$\boxplus_1$ recall that $\mathbb{P}$ is defined by:
  (A) $p \in \mathbb{P}$ <u>iff</u>:
    (a) $p$ is a function with domain from $[\lambda]^{<\kappa}$
    (b) if $a \in {\rm dom}(p)$ then $p(\alpha) \in {}^{\kappa>}2$
  (B) $\mathbb{P} \models$ "$p \leq q$" <u>iff</u>:
    (a) $p, q \in \mathbb{P}$
    (b) ${\rm dom}(p) \subseteq {\rm dom}(q)$

[3]That is, if $\chi > 2^\lambda, \mathscr{B} \prec (\mathscr{H}(\chi), \in)$ has cardinality $\kappa$ and $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\{\kappa, T, \lambda\} \in \mathfrak{B}$ then $u = \mathfrak{B} \cap \lambda$ satisfies this.

(c) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \trianglelefteq q(\alpha)$

$\boxplus_2$ (C) for $\alpha < \lambda$ let $\underset{\sim}{\eta}_\alpha = \cup\{p(\alpha) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$ satisfies $\alpha \in \mathrm{dom}(p)\}$, so $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\eta}_\alpha \in {}^\kappa 2$"

(D) for $u \subseteq \lambda$ we let $\mathbb{P}_u = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq u\}$, so $\mathbb{P}_u \lessdot \mathbb{P}$ and $\bar{\eta}_u = \langle \underset{\sim}{\eta}_\alpha : \alpha \in u\rangle$ is a generic for $\mathbb{P}_u$

$\boxplus_3$ (a) Let $\mathscr{T} \in \mathbf{T}, \mathbf{T}_{\mathrm{full}}$ is as in 1.11(2),(3) expanding ${}^{\kappa>}2$

(b) fix $\sigma < \kappa$ and $\underset{\sim}{\mathbf{c}}$ a $\mathbb{P}$-name of a function from $\mathrm{eseq}(\mathscr{T})$ into $\sigma$

$\boxplus_4$ (a) to prove clause (B)(a) it suffices to prove it for a $\sigma$-colouring $\mathbf{c}$, i.e. is in the ground model.

We shall prove that there is $\mathbf{m}$ as promised.

$(*)_1$ (a) let $\chi > \lambda$ and $<^*_\chi$ a well ordering of $\mathscr{H}(\chi)$

(b) let $\mathfrak{B} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be of cardinality $\kappa$ such that $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\lambda, \mathscr{T}, \underset{\sim}{\mathbf{c}} \in \mathfrak{B}$

(c) let $u_* = \mathfrak{B} \cap \lambda \in [\lambda]^\kappa$

(d) let $\mathbf{G} \subseteq \mathbb{P}$ be generic over $\mathbf{V}_0 = \mathbf{V}$ and let $\mathbf{G}_{u_*} \subseteq \mathbb{P}_{u_*}$ be $\mathbf{G} \cap \mathbb{P}_{u_*}$

(e) let $\bar{\eta}_{u_*} = \langle \underset{\sim}{\eta}_\alpha[\mathbf{G}_{u_*}] : \alpha \in u_*\rangle$

(f) let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_{u_*}]$ and $\mathbf{V}_2 = \mathbf{V}_0[\mathbf{G}]$.

Clearly it suffices to prove (B)(b) of Claim 2.3 for $u = u_*$.

Now

$(*)_3$ in $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P}}, \mathfrak{B}_2 = \mathfrak{B}[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}] \prec (\mathscr{H}(\chi)[\underset{\sim}{\mathbf{G}}_{\mathbb{P}}], \in, <^*_\chi)$ and let $\delta_* = \mathfrak{B} \cap \kappa^+ = \mathfrak{B}_2 \cap \kappa^+ = \mathfrak{B} \cap \kappa^+$.

Note that

$(*)_4$ we can choose $\underset{\sim}{D} \in \mathfrak{B}$, a $\mathbb{P}$-name of a normal ultrafilter on $\kappa$.

Lastly, we define $\mathbf{m}$:

$(*)_5$ $\mathbf{m} \in \mathbf{V}_1$ is defined by:

(a) $\mathscr{T}_{\mathbf{m}}$ is $\mathscr{T}$ which is as in 1.11, so, e.g. $E_{\mathbf{m}} = \cup\{{}^\alpha 2 \times {}^\alpha 2 : \alpha < \kappa\}$

(b) $\mathbb{Q}_{\mathbf{m}}$ is defined by:

($\alpha$) $q \in \mathbb{Q}_{\mathbf{m}}$ <u>iff</u> $q = (p_q, \underset{\sim}{A}_q), p_q \in \mathbb{P}_2 := \mathbb{P}/\mathbf{G}_{u_*}$ and $p_q \Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{A}_q \in \underset{\sim}{D}$"

($\beta$) $\mathbb{Q}_{\mathbf{m}} \models$ "$q \le r$" <u>iff</u> $q, r \in \mathbb{Q}_{\mathbf{m}}$ and $\mathbb{P}_2 \models$ "$p_q \le p_r$" hence $p_r \Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{A}_q \supseteq \underset{\sim}{A}_r$"

(c) $\mathrm{val}(q) = \{\nu \in {}^{\kappa>}2$: there is $p \in \mathbb{P}$ above $p_q$ which forces (i.e. $p \Vdash_{\mathbb{P}_2}$ "...") that $\ell g(\nu) \in \underset{\sim}{A}_q$ and $\nu \triangleleft \underset{\sim}{\eta}_{\delta_*}\}$

$(*)_6$ $\mathbf{m}$ is as required in the claim.

Why? We should check all the clauses in Definition 1.17 in order to prove that $\mathbf{m} \in \mathbf{M}_{2,4}$ indeed.

<u>Clause (a)</u>: $\kappa$ is a regular cardinals

Recall clause (A)(a) of 2.3.

<u>Clause (b)</u>: $\mathscr{T} \in \mathbf{T}$

Recall clause (A)(d) of 2.3 and 1.12.

Clause (c): $\mathbb{Q}_{\mathbf{m}}$ is a quasi order, $(<\kappa)$-complete

For being a quasi-order, see the choice of $\mathbb{Q}$ in $(*)_5(b)$; and $\mathbb{P}$ being a quasi-order and $(\underset{\sim}{D}, \supseteq)$ forced to be. As for being $(<\kappa)$-complete recall $\mathbb{P}_2$ is $(<\kappa)$-complete by clause (A)(b) of 2.3 and $\underset{\sim}{D}$ is forced to be $(<\kappa)$-complete.

Clause (d)($\alpha$): val $=$ val$_{\mathbf{m}}$ being a function from $\mathbb{Q}$ into $\mathscr{P}(\mathrm{spt}(\mathscr{T}))$

See the choice of val in $(*)_5(c)$ above. Debt: Check the Wednesday version.

Clause (e): monotonicity

Just check the definition.

Clause (f): decidability

As $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{D}$ is an ultrafilter on $\kappa$".

Clause (g):

Let $q \in \mathbb{Q}$. If $\delta_* \in \mathrm{dom}(p_q)$ then let $q' = q$ and if not, then let $q' = (p_q \cup \{\{(\delta_*, \langle\rangle)\}\}$. So without loss of generality $\delta_* \in \mathrm{dom}(p_q)$ and then for $\ell = 0, 1$ let $q_\ell = (p_\ell, \underset{\sim}{A}_\ell) \in \mathbb{Q}$ be defined by:

- $\underset{\sim}{A}_\ell = \underset{\sim}{A}_q$
- $p_\ell$ is a funtion with domain $\mathrm{dom}(p_q)$
- $p_\ell(\alpha) = p_q(\alpha)$ if $\alpha \in \mathrm{dom}(p_q)\backslash\{\delta_*\}$
- $p_\ell(\delta_*) = p_q(\delta_*)\hat{\ }\langle\ell\rangle$
- let $t_\ell = p_\ell(\delta_*)$ for $\ell = 0, 1$.

Clearly $q_0, q_1$ are as required.

2) By a similar proof. $\square_{2.3}$

We shall consider weakening in clause (A)(c) of the assumption measurable to weakly 2.4, 2.5 below.

**Definition 2.4.** Consider any $\sigma, \theta \le \kappa = \mathrm{cf}(\kappa)$ as $D$ a filter on $\kappa$ (the default value is the co-bounded filter) and a set $\mathscr{F} \subseteq \{f : f$ a function from ${}^{\kappa>}2$ into some $\alpha < \sigma\}$. We define the game $\Game_{\kappa,\mathscr{F},\theta,\sigma,D}$ (the case $\kappa$ inaccessible is more transparent; $\theta$ may be an ordinal)

$(*)$ (a) a play last $\theta$ moves, it is between the player COM and INC

(b) ($\alpha$) before the $\varepsilon$-th move $\mathscr{X}_\varepsilon \subseteq {}^{\kappa>}2$ is chosen

($\beta$) $\varepsilon = 0 \Rightarrow \mathscr{X}_\varepsilon = {}^{\kappa>}2$ and $\zeta < \varepsilon \Rightarrow \mathscr{X}_\zeta \supseteq \mathscr{X}_\varepsilon$ and for limit $\varepsilon, \mathscr{X}_\varepsilon = \bigcap\limits_{\zeta<\varepsilon} \mathscr{X}_\zeta$

($\gamma$) the set $\{\ell g(\eta) : \eta \in \mathscr{X}_\varepsilon\} \in D^+$

(c) in the $\varepsilon$-th move

- INC chooses $f_\varepsilon \in \mathscr{F}$
- COM chooses $i \in \mathrm{Rang}(f_\varepsilon)$ and let $\mathscr{Y}_{\varepsilon+1} = \{\alpha \in \mathscr{X}_\varepsilon : f_\varepsilon(\alpha) = i\}$

- COM chooses $\rho \in {}^{\kappa>}2$
- INC chooses $\ell < 2$, we let $\mathscr{X}_{\varepsilon+1} = \{\eta \in \mathscr{Y}_\eta : \rho\hat{\ }\langle\ell\rangle \triangleleft \eta\}$

(d) COM wins when the play does not stop (as $\mathrm{cf}(\mathrm{ht}(\mathscr{T})) \ge \kappa$ this can occur only for some $\varepsilon$, $\mathscr{X}_\varepsilon$ fails Clause $(b)(\gamma)$

2) Omitting $\mathscr{F}$ means INC chooses such $\mathscr{F}$ of cardinality $\le \kappa$ in the first move. Omitting $\sigma, \theta$ means $\theta = \kappa = \sigma$.

**Claim 2.5.** *1) If (A) then (B) where:*

(A) (a) $\kappa$ *is regular,* $\mathscr{T}_1 \in \mathbf{T}$ *expand* $({}^{\kappa >}2, \triangleleft)$
   (b) $\mathscr{T}_2 \in \mathbf{T}_{\theta,\partial}$ *so* $\mathrm{ht}(\mathscr{T}) = \partial$
   (c) $\delta_0 = \Sigma\{|\mathscr{T}_{[\varepsilon]}| : \varepsilon < \partial\}$, *ordinal sum,* $\sigma_\bullet = \Sigma\{(\sigma^{|\mathscr{T}_{[<\varepsilon]}|})^+ : \varepsilon < \partial\}$
   (d) *the* COM *player has a winning strategy in the game* $\eth_{\kappa,\delta_\bullet,\sigma_\bullet}$

(B) (a) $\mathscr{T}_1 \to (\mathscr{T}_2)^{\mathrm{end}(1)}_\sigma$
   (b) *there is* $\mathbf{m} \in \mathbf{N}_2$.

*Proof.* Similar to the proof of 2.3. $\square_{2.5}$

*Question* 2.6. 1) If $\kappa$ is weakly compact and $\mathscr{F} \subseteq \{f : f$ is a function from ${}^{\kappa>}2$ into some $\alpha < \kappa\}$ has cardinality $\le \kappa$ <u>then</u> COM wins the game $\eth_{\kappa,\mathscr{F}}$.
2) [Here?] If COM wins in $\eth_{\kappa_1,\theta_2}$, $\mathscr{T}_1 \in \mathbf{T}_{a,\kappa_1,\theta_1}$, $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}_4$ and $(\kappa(\mathscr{T}_1), \kappa(\mathscr{T}_2)) = (\kappa_1, \kappa_2)$, $\mathscr{T}_2 \in \mathbf{T}_{4,\kappa_2,\theta_2}$ and $\partial < \theta_2 \Rightarrow \partial < \theta_1$ <u>then</u> $\mathscr{T}_1 \to_4 (T_2)^{\mathrm{end}(n)}_\sigma$ for every $n < \omega, \sigma <?$.

*Proof.* Debt $\square_{2.5}$

We naturally may wonder when does the assumptions of 2.3(1) hold?.

**Fact 2.7.** If (A) then (B) <u>where</u>:

(A) (a) $\kappa$ is a measurable cardinal
   (b) $\mathbf{C}$ is a set of inaccessibles $< \kappa$ which belong to some normal ultrafilter on $\kappa$
   (c) let $\mathbb{P} = \mathbb{P}_{\kappa+1}$ where $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \kappa + 1, \beta \le \kappa \rangle$ be an Easton support iteration
   (d) in $\mathbf{V}^{\mathbb{P}_\beta}, \mathbb{Q}_\beta$ is the forcing Cohen$(|\beta|, |\beta|^+)$, see 0.2, when $\beta \in \mathbf{C}$ and trivial otherwise

(B) in $\mathbf{V}^{\mathbb{P}}, \kappa$ is measurable.

*Proof.* See Gitik [Git10] if known, anyhow for completeness we give a proof.
The point to check is as follows.
We can assume:

$(*)_1$ (a) $\mathbf{j} : \mathbf{V} \to \mathbf{M}$
   (b) $\mathbf{M}^\kappa \subseteq M$; moreover, $\mathbf{M}$ is the Mostowski Collapse of $\mathbf{M}^\kappa/E$, for some normal ultrafilter $E$ on $\kappa$
   (c) critical$(\mathbf{j}) = \kappa$
   (d) without loss of generality $\mathbf{V} \models 2^\kappa = \kappa^+$ (as Levy$(\kappa^+, 2^\kappa)$ preserve "$\kappa$ is measurable").

$(*)_2$ (a) let $\mathbf{G}_{<\kappa} \subseteq \mathbb{P}_\kappa$ be generic over $\mathbf{V}_0 = \mathbf{V}$
   (b) let $\mathbf{G}_{=\kappa} \subseteq \mathbb{Q}_\kappa[\mathbf{G}_{<\kappa}]$ be generic over $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_{<\kappa}]$ so $\mathbf{G}_{<\kappa} * \mathbf{G}_{=\kappa}$ is a subset of $\underset{\sim}{\mathbb{P}}_{\kappa+1}$ generic over $\mathbf{V}_b$
   (c) let $\mathbf{G}_{<\mathbf{j}(\kappa)}$ be generic over $\mathbf{M}$ extending $\mathbf{G}_{<\beta<\kappa}]$
   (d) without loss of generality $\mathbf{G}_{<\mathbf{j}(\kappa)}$ belongs to $\mathbf{V}$

$(*)_3$ (a) $\mathbb{P}_{\kappa+1} \in \mathbf{M}$ so $\mathbf{G}_{<\kappa+1}$ is a subset of $\mathbb{P}_\kappa$ generic over $\mathbf{M}$

(b) $\mathbf{j}(\mathbf{q})$ has the form $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \mathbf{j}(\kappa)+1, \beta < \mathbf{j}(\kappa)\rangle$, i.e. $\mathbb{P}_\alpha, \mathbb{Q}_\beta$ are the same for $\mathbf{q}, \mathbf{j}(\mathbf{q})$ when $\alpha \leq \kappa+1, \beta < \kappa+1$

$(*)_4$ the set $A = \{\mathbf{j}(\alpha) : \alpha < \kappa^+\}$ is a subset of the ordinal $\mathbf{j}(\kappa^+)$ which is unbounded in it, and every (proper) initial segment of it belongs to $M$ though it does not.

[Why unbounded? By $(*)_1(b)$.

Why initial segment of $A$ belongs to $\mathbf{M}$? Because $\mathbf{M}^\kappa \subseteq \mathbf{M}$ and $A$ has orer type $\kappa^+$.

Why $A$ not in $M$? Because $\mathbf{M} \models$ "$j(\kappa^+)$ is a successor cardinal $> \kappa^+$".]

$(*)_5$ For $\gamma < \kappa^+$ we define $q_\gamma$:

(a) $q$ is a function with domain $A \cap \mathbf{j}(\gamma)$

(b) if $\beta < \gamma$ then $q(\mathbf{j}(\beta))$ is $\cup\{p(\beta) : p \in \underset{\sim}{\mathbf{G}}_{=\kappa}\}$

$(*)_6$

(a) $q_\gamma \in \mathbf{M}$ and $\mathbf{M}[\mathbf{G}_{<j(\kappa)}] \models$ "$q_\gamma \in \mathbb{Q}_{\mathbf{j}(\kappa)}$"

(b) if $\gamma(1) < \gamma$ then $q_{\gamma(j)} = \mathbf{q}_\gamma \restriction \gamma(1)$.

Lastly, in $\mathbf{V}[\mathbf{G}_{<\kappa}]$ let $\langle \underset{\sim}{B}_\zeta : \zeta < \kappa^+\rangle$ list the $\mathbb{P}_{\kappa+1}$-name of subsets of $\kappa$; without loss of generality the name $\underset{\sim}{B}_\zeta$ depends just on the first $\zeta, \kappa$-Cohens (in the generic of $\mathbb{Q}_\kappa$).

Now we choose $\mathbf{q}_\zeta$ by induction on $zeta < \kappa^+$ such that:

- $\bullet_1$ $q_\zeta \in \mathbf{M}[\mathbf{G}_{<\mathbf{j}(\kappa)}]$
- $\bullet_2$ $q_\zeta \in \underset{\sim}{\mathbb{Q}}_{j(\kappa)}]$
- $\bullet_3$ $\mathrm{dom}(q_\zeta) \subseteq \mathbf{j}(\zeta)$
- $\bullet_4$ if $\xi < \zeta$ then $\underset{\sim}{\mathbb{Q}}_{j(\kappa)}] \models$ "$q_\xi \leq q_\zeta$"
- $\bullet_5$ if $\zeta = \xi+1$ then $q_\zeta$ forces a truth value to "$\kappa \in \mathbf{j}(\underset{\sim}{B}_\zeta)$"
- $\bullet_6$ fixing a well order $<_*$ of $bb\underset{\sim}{Q}_{\mathbf{j}(\kappa)}[\mathbf{G}_{<\mathbf{j}(\kappa)}]$ from $\mathbf{M}[\mathbf{G}_{<\mathbf{j}(\kappa)}], q_\zeta$ is $<_*$-minimal satisfying the demands above.

Now the induction cannot be carried out in $\mathbf{M}[\mathbf{G}_{<\mathbf{j}(\kappa)}]$ because the set $A$ does not belong to it but all initial segments can be out $\gamma < \kappa^+ \Rightarrow \langle \mathbf{j}(\alpha) : \alpha < \gamma\rangle \in \mathbf{M}$. As we leave no freedom $\langle q_\zeta : \zeta < \kappa^+\rangle$ is well defined.

Now lastly $E_2 = \{\underset{\sim}{B}_\zeta : \zeta < \kappa^+$ and $q_{\zeta+1}$ forces $\kappa \in \mathbf{j}(\underset{\sim}{B}_\zeta)\}$ is a normal ultrafilter on $\kappa$. $\square_{2.7}$

**Fact 2.8.** 1) Like 2.7 replacing "measurable" by "weakly compact".
2) In the proof of 2.7, 2.8(1), we can change $\mathbb{Q}_\alpha, \alpha \in \mathbf{C}$ to $\mathrm{Cohen}(\alpha, \kappa)$.

*Proof.* We do not elaborate because it is similar to that of 2.7. $\square_{2.8}$

**Claim 2.9.** *In 2.8(1) we get that the player* COM *wins the game* $\Game_\kappa$.

*Proof.* Similar. $\square_{2.9}$

*Question* 2.10. For weakly compact, can we get the full partition as in [She92, §4], as in 2.8(2).

We now improve the theorem in some ways.

*Question* 2.11. 1) Do we have: if (A) then (B) <u>where</u>:

(A) (a) $\kappa$ is strongly inaccessible and $\lambda > \kappa$
 (b) $\mathbf{C}$ and $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \kappa + 1, \beta \le \kappa \rangle$ are as in 2.7 but $\underset{\sim}{\mathbb{Q}}_\theta$ is adding one $\theta$-Cohen
 (c) $\mathbb{P} = \mathbb{P}_{\kappa+1}$
 (d) $\kappa$ is measurable

(B) in $\mathbf{V}^{\mathbb{P}_\kappa}$ if $\underset{\sim}{\mathbb{Q}} = (\mathbb{P}_\kappa)_{[u]}, u \in [\lambda]^{<\kappa}$ then in $\mathbf{V}^{\mathbb{P}_\kappa * \underset{\sim}{\mathbb{Q}}}, \kappa$ is measurable.

2) Similarly replacing measurable by weakly compact.

*Remark* 2.12. If we like in the claim on $\mathbf{V}^{\text{Levy}(\kappa,<\mu)}, \mu$ measurable, to have $\mathscr{S} \in \mathbf{T}_{\kappa,\kappa}$ which have $\mu = \kappa^+, \kappa$-branches we should force also the tree and $\mu, \kappa$-branches.

## § 2(C). Collapsing a Large Cardinal.

**Claim 2.13.** *1) Assume $\mu$ is a measurable cardinal and $\kappa < \mu$ is regular and $\mathbb{Q} = \text{Levy}(\kappa, < \mu)$, e.g. $\kappa = \aleph_1, \mu$ first measurable. Let $\iota = 3, 4$.*
 *<u>Then</u>*

(A) *$\mathbb{Q}$ satisfies the $\mu$-c.c. is $(< \kappa)$-complete and in $\mathbf{V}^{\mathbb{Q}}, \mu = \kappa^+$ and cardinals, cofinality, power are not changed otherwise*

(B) *in $\mathbf{V}^{\mathbb{Q}}$*
 (a) *for some $\mathbf{m} \in \mathbf{N}_{\iota,2}$ we have $\lambda_{\mathbf{m}} = \mu, \kappa_{\mathbf{m}} = \kappa, |\mathbb{Q}_{\mathbf{m}}| \le 2^\mu, \mathscr{T}_{\mathbf{m}} = [{}^{\kappa>}2]^{\mathbf{V}}$*
 (b) *if $\theta \le \kappa$ and $\mathscr{S} \in \mathbf{T}^1_{\kappa,\theta}$ the pair $(\mathbf{m}, \mathscr{S})$ is as in 1.19.*

*2) To deduce (A) + (B), it is enough to assume that $\mathbb{Q}$ satisfies the $\kappa$-c.c. and is strategically $(< \kappa)$-complete (i.e. a play last $\kappa$-moves,* COM *wins if it has a legal choice for every move).*

*Proof.* Like 2.3(3) (follows by 2.12 below), using Jech-Magidor-Mitchel-Prikry [JMMP80]; see below or xyz.

But we have to use the new ${}^{\mu>}2$ and $\underset{\sim}{\eta} \in {}^{\mu}2$ such that $\underset{\sim}{\eta} \restriction (\theta, \theta^+)$ is computed from the generic of Levy$(\kappa, \theta^+)$ - FILL. $\square_{2.13}$

Now we quote [She96] in 2.8, 2.12.

**Definition 2.14.** 1) Let $\kappa$ be a cardinal and $D$ a filter on $\kappa$ and $\theta$ be an ordinal $\le \kappa$ and $\mu < \chi$ but $\mu \ge 2$ and $\chi \le \kappa$. Let $\text{GM}_{\kappa,\chi,\theta,\mu}(D)$ be the following game: a play last $\theta$ moves, in the $\zeta$'s move INC, the first player chooses a function $h_\zeta$ from $\kappa$ to some ordinal $\gamma_\zeta < \chi$ and COM the second player chooses a subset $B_\zeta$ of $\gamma_\zeta$ of cardinality $< \mu$.

The second player wins a play if for every $\zeta < \theta$ the set $\cap\{\{\beta < \kappa : h_\varepsilon(\beta) \in B_\varepsilon\} : \varepsilon \le \zeta\}$ is $\ne \emptyset \mod D$.
2) If $\mu = 2$ we may omit it, if $\mu = 2$ and $\chi = \kappa$ we omit $\chi$ and $\mu$.

*Remark* 2.15. We wonder: assume that $\kappa, D, \theta, \mu, \chi$ are as in Definition 2.8 and the COM player has a winning strategy in the game $\text{GM}_{\kappa,\chi,\theta,\mu}(D)$. <u>Then</u> there is $\mathbf{m}$ such that:

(a) $\mathbf{m} \in \mathbf{M}_4$
(b) FILL.

Probably the proof is similar to [She96].

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§ 2(D). **Consistency with no Large Cardinal.**

**Claim 2.16.** *1) If (A) then (B) where:*

(A) (a) *$\kappa < \mu = \kappa^+$ are regular, moreover, $\kappa = \kappa^{<\kappa}$ (can get it by a preliminary forcing; let $\mu_0 = \mu(0) = \mu$ and force by* $\mathrm{Levy}(\kappa, 2^{<\kappa})$ *for $\kappa$ regular)*
(b) $\lambda > \mu_1 = \mu(1) = 2^{<\mu}$
(c) *$\mathbb{P}$ is the forcing* $\mathrm{Cohen}(\kappa, \lambda)$*, adding $\lambda$ many $\kappa$-Cohens*
(d) *the $\mathbb{P}$-name $\underset{\sim}{\mathscr{T}}_1 \in \mathbf{T}$ expand $({}^{\mu>}2, \triangleleft)$ or just FILL*

(B) *in $\mathbf{V}^{\mathbb{P}}$; $\mathscr{T}_1 \rightarrow_{\mathrm{reg}} (\mathscr{T}_2)^{\mathrm{end}(1)}_\sigma$ when $\mathscr{T}_2 \in \mathbf{T}$ expand $({}^{\kappa>}2, \triangleleft)$,* without loss of generality $\mathscr{T}_2 \in \mathbf{V}$.

*Proof.* First

$\boxplus_1$ so $\mathbb{P}$ is defined by:
for the rest of the proof we assume:
$\boxplus_2$ $\Vdash_{\mathbb{P}}$: $\underset{\sim}{\mathbf{c}} : [\mathrm{spt}(\underset{\sim}{\mathscr{T}}_1)]^{<\aleph_0} \rightarrow \sigma$.

Next:

$(*)_1$ (a) let $\chi > \lambda$ and $<^*_\chi$ a well ordering of $\mathscr{H}(\chi)$
(b) let $\mathfrak{B} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be of cardinality $\kappa$ such that $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\lambda, \kappa, \mu, \mathscr{T}_1, \mathscr{T}_2, \underset{\sim}{\mathbf{c}} \in \mathfrak{B}$
(c) let $u_* = \mathfrak{B} \cap \lambda \in [\lambda]^\kappa$ and $\delta_* = \min(\lambda \backslash u_*)$
(d) let $\mathbf{G}_{u_*} \subseteq \mathbb{P}_{u_*}$ be generic over $\mathbf{V}_0 = \mathbf{V}$
(e) let $\bar{\eta}_{u_*} = \langle \underset{\sim}{\eta}_\alpha[\mathbf{G}_{u_*}] : \alpha \in u_* \rangle$
(f) $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_{u_*}] = \mathbf{V}_0[\bar{\eta}_{u_*}]$

$(*)_2$ (a) let $\underset{\sim}{\mathscr{T}}_0 = (\underset{\sim}{\mathscr{T}}_1 \restriction \mathfrak{B})$ that is $\underset{\sim}{\mathscr{T}}_2 \restriction \mathscr{S}$, that is, the sub-structure of $\underset{\sim}{\mathscr{T}}_1$ with set of elements $\mathbf{S} = \{\underset{\sim}{\eta} : \underset{\sim}{\eta}$ is a $\mathbb{P}$-name of a member of $\underset{\sim}{\mathscr{T}}_1$ and this name belongs to $\mathfrak{B}\}$
(b) let $\delta_* = \delta(*)$ be $\min(\lambda \backslash u_*)$.

Clearly

$(*)_3$ (a) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathscr{L}}$ is a subtree of $({}^\mu 2, \triangleleft)^{\mathbf{V}[\mathbb{P}]}$ closed under initial segments of cardinality $\kappa$ and closed under unions of increasing chains of length $< \kappa$" and $\nu \in \underset{\sim}{\mathscr{L}} \Rightarrow \nu{}^\frown\langle 0 \rangle, \nu{}^\frown\langle 1 \rangle \in \underset{\sim}{\mathscr{L}}$
(b) $\mathbf{S}$ is actually a $\mathbb{P}_{u_*}$-name

$(*)_4$ (a) let $\mathscr{S} = \underset{\sim}{\mathscr{L}}[\mathbf{G}_{u_*}]$
(b) let $\mathbb{P}_2 = \mathbb{P}/\mathbf{G}_{u_*} = \mathbb{P}_{\lambda \backslash u_*}$

$(*)_5$ there is a $\mathbb{P}_2$-name $\underset{\sim}{\eta}$ of a branch of $\underset{\sim}{\mathscr{S}}$ generic over $\mathbf{V}_1$, i.e. for the forcing notion $(\mathscr{S}, \triangleleft)$.

[Why? By the character of $\mathbf{S}$ see $(*)_3$ being of cardinality $\kappa$.]

Now comes the main point - find appropriate $\mathbf{m}$ in $\mathbf{V}_1$

$(*)_6$ we define $\mathbf{m}$ as follows:
- (a) $\mathscr{T}_{\mathbf{m}} = \underset{\sim}{\mathscr{T}}_0[\mathbf{G}_{u_*}]$
- (b) $\mathbb{Q}_{\mathbf{m}} = \mathbb{P}_2$
- (c) $\mathrm{val}_{\mathbf{m}}(p) = \{\nu \in \mathbf{S} : q \Vdash_{\mathbb{P}_2}$ "$\nu \triangleleft \underset{\sim}{\eta}$" for some $q \in \mathbb{P}_2$ above $p\}$

$(*)_7$ $\mathbf{m} \in \mathbf{N}_2$

[Why? As in the proof of 2.3.]

$(*)_8$ there is an $\subseteq_{\mathrm{reg}}$-embedding $g$ of $\mathscr{T}_2$ into $\mathscr{S}$ such that a neighbor $\mathscr{T}_2'$ of $\mathscr{T}_2$ and $\mathbf{c} \circ g$ is an end(1)-extension homogeneous colouring of $\mathrm{eseq}(\mathscr{T}_2')$.

[Why? By §(1B). Now $g$ witness our desired conclusion.] $\square_{2.16}$

**Claim 2.17.** *1) Assume $\mu = (2^{<\kappa})^+$ and $\mathscr{T} \in \mathbf{T}_{\mu,\kappa}$ is as in §(1B) and $\mathscr{S} \in \mathbf{T}_{\theta,\partial}$ and $\theta + \partial \leq \kappa, \sigma < \kappa$. A sufficient condition for $\mathscr{T} \rightarrow_{\mathrm{reg}} (\mathscr{S})^{\mathrm{end}(1)}_{\sigma}$ is:*

*$(*)$ for every family $\mathscr{P} \subseteq \mathscr{P}(\mathrm{spt}(\mathscr{T}))$ of cardinality $\leq 2^{<\kappa}$ there is $\mathbf{m} \in \mathbf{M}_1[\mathscr{P}]$ which means $\mathbf{m}$ is "almost from $\mathbf{N}_2$"; we weaken:*
- *$\mathrm{val}_{\mathbf{m}}(p) \in \mathscr{P}$ for every $p \in \mathbb{Q}$*
- *in the decidability clause demanding $A_1, A_2 \in \mathscr{P}$.*

*2) If $\kappa$ is weakly compact, $\mathscr{T} \in \mathbf{T}$ is as in §(1B) then $\mathscr{T} \rightarrow (\mathscr{T})^{\mathrm{end}(1)}_{\sigma}$ for $\sigma < \kappa$ hence $\mathscr{T} \Rightarrow (\mathscr{T})^{\mathrm{end}(k)}_{\sigma}$ for $k < \omega$.*

*Proof.* 1) Similarly to 1.19, first we fix $\mathbf{c} : [\mathscr{T}_2]^{<\aleph_0} \rightarrow \sigma$, let $\chi$ be large enough, and then choose $\mathscr{B} \prec (\mathscr{H}(\chi), <^*_\chi)$ be of cardinality $2^{<\kappa}$ such that $2^{<\kappa} + 1 \subseteq \mathscr{B}$ and $[\mathscr{B}]^{<\kappa} \subseteq \mathscr{B}$ and $\{\mathscr{T}_2, \mathscr{S}, \kappa, \theta_2, \mu, \partial\} \subseteq \mathscr{B}$. Now choose $\mathbf{m}$ as in $(*)$ of 1.19(2) for $\mathscr{P} = \mathscr{P}(\mathrm{spt}(\mathscr{T})) \cap \mathscr{B}$. Now proceed as in the proof of 1.19.
2) By §(1B) (and more there). $\square_{2.17}$

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## § 3. Toward the model theoretic application

### § 3(A). The Successor Case.

Here we try to deal with $\beth_n(\mu), \psi \in \mathbb{L}_{\mu^+,\aleph_0}$ so we can use an induction. We hope to deal with $\beth_\alpha(\mu), \alpha < \mu^+$.

Note our tree indiscernibility is suitable even for $\lambda < \mathrm{trp}^+(\mu)$ not just $\lambda < \mathrm{trp}_\theta(\mu)$ for some specific $\mu$, not a great matter, still a gain.

**Definition 3.1.** Let $K$ be an indiscernible model class $\mu \geq |\tau_K|$. We say $\mathbf{b}$ is a $(K,\mu)$-flowing parameter (if $\mu = \aleph_0$ we may omit it) <u>when</u> $\mathbf{p}$ consists of:

- (A) (a) $\psi_1, \psi_2 \in \mathbb{L}_{\mu^+,\aleph_0}\psi_2$ universal
- (B) for any model $M$ of $\psi_1$
  - (a) $(|M|, \in^M, <^M_\bullet)$ is a moel of enough set theory $<^M_\bullet$ a well ordering in respect to $(|M|, \in^M)$
  - (b) $\mathscr{T}_M = (P^M, <^M, R^M_0, R^M_1, \cap^M, <^M_*)$ is $\mathbf{T}_3$ (so necessarily traces of $E$, the order not necessarily well founded), $x \cap y$ is lub, every node has 2 successors, left(0) and right(1), ADC ?
  - (c) for $r = r(*) \in \mathbf{S}^n(K)$, see Definition xyz. $F^M_r$ is a function with domain $\mathrm{fseq}(\mathscr{T}_M)$ (only maximal nodes) such that for $\bar{t} \in \mathrm{fseq}_n(T_M), F^M_r(\bar{t}) = N_{\mu,\bar{t},s}$ is a model of $\psi_2$ generated by $\langle G_{(\bar{a},\ell)}(\bar{t}) = \ell < \ell g(t)\rangle$
  - (d) if $\bar{s}_\ell \in \mathrm{fseq}_{n(\ell)}(\mathscr{T}_\mu), r_\ell \in \mathbf{S}^{n(\ell)}_{\mathrm{qf}}(K)$ for $\ell = 1,2$ and $h : n(1) \to n(2)$ is one-to-one and commutes with the types above (see xyz) then $H_{r_1,r_2,h}(\bar{s},\bar{t})$ is an embedding of $N_{M,\bar{s}_1,r_1}$ into $N_{M,\bar{s}_2,r_2}$ mapping $G_{r_1,\ell}(\bar{s}_1)$ to $G_{r_2,h(\ell)}(\bar{s}_2)$.

**Claim 3.2.** *1) If $\mathbf{b}$ is a $(\mu,K)$-blowing parameter $M$ a model of $\psi_{\mathbf{b}}, I \in K$ has cardinality at most that of $\lim(\mathscr{T}_M) = \{B : B$ a branch of $\mathscr{T}_M\}$ <u>then</u>:*

- (a) *$\psi_{\mathbf{b},2}$ has a model $M_2$ of cardinality $\geq |\mathscr{T}|$*
- (b) *moreover letting $\bar{\eta} = \langle \eta_t : t \in I\rangle$ be a sequence of pairwise distinct members of $\lim(\mathscr{T}_M)$ then for some $\mathbf{a}$*
  - ($\alpha$) *$\mathbf{a} = \langle \bar{a}_t : t \in I\rangle$ generate $\psi_{\mathbf{b},1}$*
  - ($\beta$) *if $\bar{s} \in \mathrm{fseq}_n(\mathscr{T}_M)$ realizes $r \in \mathscr{S}^n_{\mathrm{qf}}(K), \ell < n \Rightarrow s_\ell < \eta_{T_\ell}$ then there is an embedding $g$ of $N_{M,\bar{s},r}$ into $M_2$ mapping $a_{M,\bar{s},r,\ell}$ to $a_{t_\ell}$.*

*2) We can replace $\eta_\alpha \in \mathrm{Lim}(\mathscr{T}_M)$ are pairwise and can be replaced by "$\eta_\alpha$ is an initial segment of a branch and $\alpha \neq \beta \Rightarrow \eta_\alpha \not\trianglelefteq \eta_\beta$" (this simplifies 3.3).*

**Definition 3.3.** We say $\mathbf{b}$ is a tree $\mu$-blowing parameter when:

- (a) $\mathbf{b}$ is a $(\mu, K_{\mathrm{tr}(3)})$-blowing parameter but:
  - ($\alpha$) $\psi_2 \vdash$ "$(P_2, <_2) \in \mathbf{T}_3$"
- (b) if $\mathscr{T} \in K_{\mathrm{tr}(3)}$ expand ${}^{\lambda >}2, M$ a model of $\psi, \bar{\eta} = \langle \eta_t : t \in I\rangle$ is a sequence of pairwise distinct branches of $\mathscr{T}_\mu, (N, \mathbf{a})$ as above <u>then</u>:
  - ($\alpha$) $a_s \in P^M_2$
  - ($\beta$) $s_1 <_{\mathscr{T}} N_2 \Leftrightarrow a_{s_\eta} <_{\mathscr{T}[M_2]} a_{s_2}$

**Definition 3.4.** 1) A model $M$ of $\psi_{\mathbf{b},1}$ is $\kappa$-standard <u>when</u> $({}^{\kappa >}2, \triangleleft)$ can be embedded into $\mathscr{T}_{\underset{\sim}{\mu}}$, i.e. $(\mathscr{T}_m u, <^{\mathscr{T}[M]})$.
2) We say $\bar{\mathbf{b}}^+$ is a $\kappa$-standard tree $\theta$-blowing parameter <u>when</u> it is a tree parameter $\mathbf{b}$ expanded by a $\kappa$-model $M^{\mathbf{b}(+)}$ of $\psi_{\mathbf{b},1}$.

**Theorem 3.5.** *If (A0 then (B) where:*

(A) (a) $\theta \le \mu = \mu^{<\mu}, \delta = \delta(\mu)$, *see* [She90, Ch.VII,§5] *and* [GS] *and references there*
(b) $\langle \lambda_i = \lambda(i) : i \le \delta \rangle$ *is increasing*
(c) $\mathscr{T}_i \in \mathscr{T}$ *expands* ${}^{\lambda(i)}2$ *for* $i < \delta$
(d) $\mathscr{T}_{i+1} \rightarrow_{\text{reg}} (\mathscr{T}_i)^{\text{end}(1)}$ *or just the conclusion*
(d)′ *if* $i + n \le j < \delta_\bullet$ <u>*then*</u> $\mathscr{T}_j \rightarrow_{\text{reg}} (\mathscr{T}_i)^n_{\lambda_i}$
(e) $\mathbb{P} = \text{Cohen}(\mu, \lambda_{\delta_\bullet})$ *hence* $\Vdash_{\mathbb{P}}$ " $<^\mu = ((\lambda_{\delta_\bullet})^{M^{\mathbf{V}!}})$"
(f) *in* $\mathbf{V}^{\mathbb{P}}, M_i$ *is a model of* $\psi_2 \in \mathbb{L}_{\mu^+,\aleph_0}$ *such that* $(P^{M_i}, <^{M_i}) \in \mathbf{T}_2$ *and* ${}^{\lambda(i)}2$ *is embeddable into it*

(B) (a) *there is a* $\mu$*-standard* $Q$*-blowing parameter* $\mathbf{b}^+$ *with* $\psi_{\mathbf{b},(T),2} = \psi_2$ *and*
(b) *if* $(Q^{M_i}, <^{M_1}_\bullet) \cong \delta_\bullet$ *for* $i < \delta_\bullet$ *then in* $M_{\mathbf{b}(+)}, (Q, <_\bullet)$ *is not well ordered.*

**Conclusion 3.6.** *Assum* $\mu = \mu^{<\mu} = \lambda_0 < \lambda_n \ldots < \lambda_n$*, all regular and for transparency GCH holds in* $[\lambda_0, \lambda_n)$*.*

*Then for some* $\mathbb{P}$*:*

(a) $\mathbb{P}$ *is a* $(< \mu)$*-complete forcing not of cardinality* $\lambda_n$*, not collapsing cardinals nor changing cofinalities*
(b) *in* $\mathbf{V}^{\mathbb{P}}, 2^{\lambda_\ell} = 2^{<\lambda_\ell} = \lambda_{\ell+1}$ *for* $\ell < n$
(c) *each pair* $(\lambda_\ell, \lambda_{\ell+1})$ *satisfies the conclusion on* $(\mu, \lambda_{\delta(\lambda_\ell)})$
(d) *if* $\ell < k \le n, \psi \in \mathbb{L}_{\mu^+_\ell,\aleph_0}$ *has a model* $M$ *expanding* $(\aleph_{\lambda^{++}_{k-1}}, <), P^M = \delta(\lambda_\ell)$ *then* $\psi$ *has a model* $N$ *of cardinality* $\lambda_k$ *such taht* $(P^N, <)$ *is not well ordered.*

*Proof.* Debt. $\square_{3.6}$

## § 3(B). .

We try to deal with $\beth_\alpha(\mu), \alpha < \mu^+$ (now for $\mu = \aleph_0$ or $\mu = \beth_\delta$), $\text{cf}(\delta) = \aleph_0$, this is enough but for other $\mu$'s the $\delta \in [\mu, \mu^+)$ remain in limbo. One way is to start with a measurable $\lambda > \mu$ of suitable rank $\alpha$. First, have enough cases of §(1B) (FILL) then add an increasing continuous sequence $\bar{\lambda} = \langle \lambda_i : i \subseteq \alpha \rangle$ such that $\lambda_0 = \mu, \lambda_\alpha = \lambda$ and $2^{\lambda_i} = \lambda_{i+1}$.

When we think of ${}^\beth\alpha(\mu), \alpha < \mu^+$ a limit ordinal. Have not yet tried this.

A second way is to allow, e.g. $\lambda_{i+2} = 2^{(\lambda_i)^+}, 2^{\lambda_i} = \lambda_i^+$ (or more), O.K. because $2\alpha = \alpha$ when $\alpha$ is a limit ordinal. Note that if $\alpha = \omega\alpha_\bullet + n$ then $\langle \lambda_{\omega\alpha_i+\ell} : \ell \le n \rangle$ behave differently we cannot make.

Have to sort out that this... ?

Implicit in this is the expectation that:

(∗) if we have indiscernibles nice enough to deal with $\alpha$-exponential gotten from case $\lambda_i = \lambda_1^{<\lambda_\alpha}$, we can get a $(\theta_0, \theta_\alpha)$ when $\bar{\theta} = \langle \theta_i : i \le \alpha \rangle$ is increasing continuous, $\theta_{i+1} < \mathrm{trp}^+(\theta_i)$ - well after transforcing from $\mu$ to $\theta_i$ (easily by DLS, i.e. $\theta_0 \le \mu$, with more definitely if $\theta_0 >$ ...???

Particularly for $\alpha$ limit, given $\theta_0$ as bove, $Q_\alpha = \beth_\alpha(\theta_0)$ is O.K. by [CS16] tell us we can "make vacation along the way", i.e. use $\langle \lambda_\beta : \beta \le \alpha \rangle$ such that for limit $\beta \le \alpha, \lambda_\beta = (\sum\limits_{\gamma<\beta} \lambda_\gamma)^+$. But have we the right indiscernibility proved? In §(3A) this is hidden - Debt.

**Theorem 3.7.** *1) In 3.6(d) for $\ell = 0$ we can replace $\aleph_{\delta(0)}$ where $\delta(\ell) = \delta_i = \delta(\lambda_i)$ by $\lambda_i^{+\delta(0)}$.*

*Proof.* Debt. $\square_{3.7}$

**Discussion 3.8.** Think this $\bar{\lambda} = \langle \lambda_i : i \le \delta(\mu) \rangle$ is increasing $\mu = \lambda_0 = \mu^{<\mu}, \lambda_{i+1} \to_{\mathrm{reg}} (\lambda_i)^{\mathrm{end}(1)}_{\lambda_i}$ (or other variants) and $\mathbb{P}$ is ading $\lambda_{\delta(\mu)}$-many $\mu$-Cohens $\langle \underset{\sim}{\eta}_\alpha : \alpha < \lambda_{\delta(\mu)} \rangle$ generic of $\mathbb{P}, \mathscr{T}_i \in \mathbf{T}$ expand $({}^{\lambda+1}2)$ all in $\mathfrak{B} = (\mathscr{H}(\chi), \in, <^*_\chi, \mathbb{P}, \underset{\sim}{M})$ as usual $\underset{\sim}{\bar{T}}, \mathbb{P}, \underset{\sim}{M} \in \mathfrak{B}_0 \prec \mathfrak{B}, \|\mathfrak{B}_0\| = \mu, \mathfrak{B}_0 \prec \mathfrak{B}_1, \mathbb{B}_1 \restriction (\mu+1) = \mathfrak{B}_0 \restriction (\mu+1), \delta(\mu)^{\mathfrak{B}_1}$ not well ordered $\langle a_n : n < \omega \rangle$ decreasing, $\mathscr{T}_\gamma$ expands.
0) We for each $i \in \delta(\mu)^{\mathfrak{B}_1}$ interprets $\eta_\alpha$ as $\underset{\sim}{\eta}'_\alpha : {}^{\mu>}2 \to \{0,1\}$ giving a perfect set of finite-wise generics along the branches.
1) We have tree indescernibles as in §(1B), $\mathscr{S}_n \le \mathscr{T}_{a_1}, \mathscr{S}_n \le {}^{\lambda(a(n))>}2, \mathscr{S}_{n+1} \le \mathscr{S}_1$. Then use $\underset{\sim}{\eta}_{a_0}$ to describe a model. This fits finite iteration prove 3.7.

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Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 9190401, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

*Email address*: shel ah@math.huji.ac.il

*URL*: http://shelah.logic.at