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CANONICAL UNIVERSAL LOCALLY FINITE GROUPS SH1175

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Abstract. We prove that for $\lambda = \beth_\omega$ or just λ strong limit singular of cofinality $\aleph_0$, if there is a universal member in the class $\mathbf{K}^{\rm lf}_\lambda$ of locally finite groups of cardinality λ, then there is a canonical one (parallel to special models for elementary classes, which is the replacement of universal homogeneous ones and saturated ones in cardinals $\lambda = \lambda^{<\lambda}$).

For this we rely on the existence of enough indecomposable such groups. We also more generally deal with the existence of universal members of in general classes for such cardinals.

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The author thanks Alice Leonhardt for the beautiful typing. First typed February 18, 2016 as part of [Shec]. In References [She17a, 0.22=Lz19] means [She17a, 0.22] has label z19 there, L stands for label; so will help if [She17a] will change. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.

Annotated Content

§ 0. Introduction

§ 0(A). **Review.**

Our motivation is the class of locally finite groups so the reader may consider only this case ignoring the general case; or consider universal classes (see 0.3). We continue [She17a], see history there.
We wonder:

Question 0.1. 1) Is there a universal $G \in \mathbf{K}^{\rm lf}_\lambda$, e.g. for $\lambda = \beth_\omega$? Or just λ strong limit of cofinality $\aleph_0$ (which is not above a compact cardinal)?
2) May there be a universal $G \in \mathbf{K}^{\rm lf}_\lambda$, when $\lambda < \lambda^{\aleph_0}$, e.g. for $\lambda = \aleph_1 < 2^{\aleph_0}$, i.e. consistently?

Concerning 0.1(1) recall that by Grossberg-Shelah [GS83], if λ is strong limit of cofinality $\aleph_0$ is above a compact cardinal κ, then there is $G \in K^{\rm lf}_\lambda$ which is universal.

Here we return to the universality problem for $\mu = \beth_\omega$ or just strong limit of cofinality $\aleph_0$. We prove for $\mathbf{K}_{\rm lf}$ and similar classes that if there is a universal model of cardinality μ, then there is something like a special model of cardinality μ, in particular, universal and unique up to isomorphism. This relies on [Sheb], which proves the existence and even density of so-called θ-indecomposable (i.e. θ is not a possible cofinality) models in $\mathbf{K}_{\rm lf}$ of various cardinalities continuing Carson-Shelah [CS] which deal with the class of groups.

Returning to Question 0.1(1), a possible avenue is to try to prove the existence of universal members in μ when $\mu = \Sigma_{n<\omega}\mu_n$ each μ_n measurable $< \mu$, i.e. maybe for some reasonable classes this holds.

Context 0.2. $\mathbf{K}$ will be one of the following cases

Case 1: $\mathbf{K} = \mathbf{K}_{\rm lf}$, the class of locally finite groups, so the submodel relation is just being a subgroup,

Case 2: $\mathbf{K}$ is a universal class, see below, the submodel is just a submodel,

Case 3 $\mathbf{K}$ is $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ an a.e.c. with $|LST_{\mathfrak{k}} < \mu$, see [Shea]; we shall only comment on it. In particular, in this context, in the definitions $M \subseteq N$ should be replaced by $M \le_{\mathfrak{k}} N$..

Definition 0.3. 1) We shall say that $\mathbf{K}$ is a universal class when for some vocabulary $\tau = \tau_{\mathbf{k}}$:

(a) $\mathbf{K}$ is a class of τ-models
(b) a τ-model belongs to $\mathbf{K}$ iff every finitely generated sub-model belongs to it

3) Let $\mathbf{K}_\mu$ be the class of $M \in \mathbf{K}$ of cardinality μ. We define $\mathbf{K}_{<\mu}, \mathbf{K}_{\le\mu}$
4) For cardinals $\lambda \le \mu$ let $\mathbf{K}_{\mu,\lambda}$ be the class of pairs (N, M) such that $N \in \mathbf{K}_\mu, M \in \mathbf{K}_\lambda$ and $M \subseteq N$
5) Let $(N_1, M_1) \le_{\mu,\lambda} (N_2, M_2)$ mean that $(N_\ell, M_\ell) \in \mathbf{K}_{\mu,\lambda}$ and for $\ell = 1, 2$ and $M_1 \subseteq M_2, N_1 \subseteq N_2$.
6) For $\mu \le \lambda$ we define $\mathbf{K}_{\mu,<\lambda}$ and $\le_{\mu,<\lambda}$ similarly.
7)A universal class $\mathbf{K}$ can be considered as the a.e.c. $\mathfrak{k} = (\mathbf{K}, \subseteq)$

Notation 0.4. 1) Let M, N and also G, H, L denote members of $\mathbf{K}$.
2) Let $|M|$ be the universe = set of elements of M and $\|M\|$ its cardinality.
2) Let a, b, c, d denote members of such M-s

Definition 0.5. 1) We say the pair (N, M) is an (χ, μ, κ)-amalgamation base (or amalgamation pair, but may omit χ when $\chi = \mu$) when:

- (a) $(N, M) \in \mathbf{K}_{\mu,\kappa}$
- (b) if $N \subseteq N_\ell \in \mathbf{K}_{\leq \chi}$ for $\ell = 1, 2$, then for some N_3, f_1, f_2 we have $M \subseteq N_3 \in \mathbf{K}$ and f_ℓ-embeds N_ℓ into N_3 over M.

2) We say that the (N, M) is a universal (μ, λ)-amalgamation base when:

- (a) $(N, M) \in \mathbf{K}\mu, \lambda$
- (b) if $N \subseteq N' \in \mathbf{K}_\mu$ then N' can be embedded into N over M.

3) We define when (N, M) is a universal $(\mu, < \lambda)$-amalgamation base similarly.
4) We may in part (1) omit χ when $\chi = \mu$, we then may in parts (1),(2) omit μ, κ when $(\mu, \lambda) = (\|N\|, \|M\|)$.

§ 1. Indecomposability

In this section we deal with indecomposability, equivalently $\mathrm{CF}(M)$, see e.g. [ST97]; we have $\mathbf{K}_{\mathrm{lf}}$ in mind but still is meaningful and of interest also for other classes.

Definition 1.1. 1) We say M is θ-decomposable <u>when</u>: θ is regular and if $\langle M_i : i < \theta\rangle$ is $\subseteq$-increasing with union M, <u>then</u> $M = M_i$ for some i.
2) We say M is Θ-indecomposable <u>when</u> it is θ-indecomposable for every $\theta \in \Theta$. We say M is Θ^{orth}-indecomposable <u>when</u> it is θ-indecomposable for every regular $\theta \notin \Theta$.
3) We say G is θ-indecomposable inside G^+ <u>when</u>:

(a) $\theta = \mathrm{cf}(\theta)$;
(b) $G \subseteq G^+$;
(c) if $\langle G_i : i \le \theta\rangle$ is $\subseteq$-increasing continuous and $G \subseteq G_\theta = G^+$ then for some $i < \theta$ we have $G \subseteq G_i$.

4) For $\theta = \mathrm{cf}(\theta) \le \lambda \le \mu$ we say $\mathbf{K}$ is $(\mu, < \lambda, \theta)$-indecomposable when for every pair $(N, M) \in \mathbf{K}_{\mu,\lambda}$ there is $(N_1, M_1) \in \mathbf{K}_{\mu,\lambda}$ which is $\le_{\mu,\lambda}$-above it and M_1 is θ-indecomposable inside N_1, For *theta* $= \mathrm{cf}(\theta) < \lambda \le \mu$ we define $\mathbf{K}$ being $(\mu, < \lambda, \theta$-indecomposable similarly,
5) We say $\mathbf{c} : [\lambda]^2 \to S$ is θ-indecomposable <u>when</u>: if $\langle u_i : i < \theta\rangle$ is $\subseteq$-increasing with union λ then $S = \{\mathbf{c}\{\alpha, \beta\} : \alpha \ne \beta \in u_i\}$ for some $i < \theta$;
6) We may replace above the cardinal θ by a set or class Θ of regular cardinals.

A group G may be indecomposable as a group or as a semi-group; our default choice is semi-group; but note that for locally finite groups the two are the same.

Theorem 1.2. *1) If $\lambda \ge \aleph_1$ and we let $\Theta_\lambda = \{\mathrm{cf}(\lambda)\}$ except that $\Theta_\lambda = \{\mathrm{cf}(\lambda), \partial\}$ when $(c)_{\lambda,\partial}$ below holds, <u>then</u> (a),(b) holds, where:*

(a) *some $\mathbf{c} : [\lambda]^2 \to \lambda$ is θ-indecomposable for every $\theta = \mathrm{cf}(\theta) \notin \Theta_\lambda$*
(b) *for every $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\le\lambda}$ there is an extension $G_2 \in \mathbf{K}^{\mathrm{lf}}_\lambda$ which is $\Theta^{\mathrm{orth}}_\lambda$-indecomposable*
$(c)_{\lambda,\partial}$ *for some $\mu, \lambda = \mu^+, \mu > \partial = \mathrm{cf}(\mu)$ and $\mu = \sup\{\theta < \mu : \theta$ is a regular Jonsson cardinal$\}$.*

2) If $\mu \ge \lambda, \theta = \mathrm{cf}(\theta) < \lambda, \theta \notin \Theta_\lambda \cup \Theta_\mu$ and $\mathbf{K} = \mathbf{K}$ <u>then</u> $\mathbf{K}$ is (μ, λ, θ)-indecomposable

Proof. 1) By [Sheb, Th.1.6=Lb24] .

2) Let $(N, M) \in \mathbf{K}_{\mu,\lambda}$ be given. By induction of $\alpha \le \theta$ we choose H_α, L_α, such that:

(a) $(H_\alpha, L_\alpha) \in \mathbf{K}_{\mu,\lambda}$ is increasing continuous with α
(b) $(H_0, L_0) = (N, M)$
(c) if $\alpha = \beta + 1 < \theta$ then and L_β is θ-indecomposable inside L_α

Why can we carry the induction? For $\alpha = 0$ this is trivial; similarly for α a limit ordinal. Lastly by clause (b) of part (1), for $\alpha = \beta + 1 \le \alpha_*$, recalling [Sheb, 1.4=Lb15]. . $\square_{1.2}$

Now comes the central definition

Definition 1.3. We say that $\mathbf{K}$ is μ-nice <u>when</u>:

(a) $\tau_{\mathbf{k}}$ has cardinality $< \mu$,
(b) for every $M \in \mathbf{K}_{<\mu}$ there is $N \in \mathbf{K}_\mu$ extending M.
(c) $\mathbf{K}$ has the JEP (joint embedding property)
(d) $\mathbf{K}$ is $(\mu, < \lambda, \mathrm{cf}(\mu))$-indecomposable
(e) $\mathbf{K}$ is $(\mu, \mathrm{cf}(\mu))$-indecomposable.

Claim 1.4. $\mathbf{K}_{\mathrm{lf}}$ *is μ -nice when $\mu > \aleph_1$*

Proof. In Def 1.3 clauses (a),(b),(c) are clear and clause (d) holds by 1.2(2) and 1.5(2) below $\square_{1.4}$

We give below more then what is strictly needed.

Claim 1.5. *Assume $\mathbf{K} = \mathbf{K}_{\mathrm{lf}}$.*
1) We have (A) $\Rightarrow$ (B) where:

(A) (a) $\lambda \geq \aleph_1$
 (b) $\alpha_* \leq \lambda$ *and* $\lambda_\alpha \in [\aleph_1, \lambda]$ *for* $\alpha < \alpha_*$
 (c) $\lambda_\alpha \in \Theta^\bullet_\mu[\mathbf{K}]$ $\Theta_\lambda, \Theta_{\lambda_\alpha}$ *for* $\alpha < \alpha_*$ *are as above*
 (d) $G_1 \in \mathbf{K}_{\leq\lambda}$
 (e) $G_{1,\alpha} \in \mathbf{K}_{\leq\lambda_\alpha}$ *for* $\alpha < \alpha_*$
(B) *There are* $G_2, \bar{G}_2$ *such that:*
 (a) $G_2 \in \mathbf{K}_\lambda$ *extends* G_1
 (b) $\bar{G}_2 = \langle G_{2,\alpha} : \alpha < \alpha_* \rangle$
 (c) $G_{2,\alpha} \in \mathbf{K}_{\lambda_\alpha}$ *extend* $G_{1,\alpha}$
 (d) G_2 *is* $\Theta^{\mathrm{orth}}_\lambda$*-indecomposable*
 (e) $G_{2,\alpha}$ *is* $\Theta^{\mathrm{orth}}_{\lambda_\alpha}$*-indecomposable for every* $\alpha < \alpha_*$

2) If $\mu \geq \lambda \geq \aleph_1$ <u>then</u> $\aleph_0 \in \lambda \cap \Theta^{\mathrm{orth}}_\mu \cap \Theta^{\mathrm{orth}}_\lambda$ except possibly when $\lambda = \aleph_1, \mu = \chi^+, \mathrm{cf}(\chi) = \aleph_0$ for some χ.

Proof. 1) For $\alpha < \alpha_*$ let $\mathbf{c} : [\lambda_\alpha]^2 \to \lambda_\alpha$ be $\Theta^{\mathrm{orth}}_{\lambda_\alpha}$-indecomposable.

Now by induction of $\alpha \leq \alpha_*$ we choose H_α, L_α, but L_α is chosen together with $H_{\alpha+1}$ and not chosen for $\alpha = \alpha_*$, such that

(a) $H_\alpha \in \mathbf{K}_\lambda$ is increasing continuous with α
(b) $H_0 = G_1$
(c) $(H_\alpha, L_\beta) \in \mathbf{K}_{\lambda,\lambda_\beta}$ when $\alpha = \beta + 1 \leq \alpha_*$
(d) $G_{1,\beta} \subseteq L_\beta$ for $\beta < \alpha_*$ and L_β is $\Theta^{\mathrm{orth}}_{\lambda_\beta}$-indecomposable

Why can we carry the induction? For $\alpha = 0$ this is trivial (note that L_α is not chosen), similarly for α a limit ordinal. Lastly by clause (b) of part (1), for $\alpha = \beta + 1 \leq \alpha_*$, Def 1.1(4) gives the desired conclusion. Lastly let $G_2 \in \mathbf{K}_\lambda$ extend H_{α_*} and satisfies the indecomposablity demand, and letting $G_{2,\alpha} = L_\alpha$ we are done.
2) Easy. $\square_{1.5}$

Claim 1.6. *1 If μ is strong limit singular and $N \in \mathbf{K}_\mu$ <u>then</u> the set $\mathrm{IDC}_{\mathrm{cf}(\mu)}(N)$ has cardinality $\leq \mu$, if fact even equal; where, for $N \in \mathbf{K}_\mu$,*

(*) $\mathrm{IDC}_{\mathrm{cf}(\mu)}(M) = \{M : M \subseteq N$ *has cardinality* $< \mu$ *and is* $\mathrm{cf}(\mu)-$*indecomposable* $\}$.

Proof. Easy. $\square$

§ 2. Universality

For quite many classes, there are universal members in any (large enough) μ which is strong limit of cofinality $\aleph_0$, see [She17b] which include history. Below we investigate "is there a universal member of $\mathbf{K}^{\rm lf}_\mu$ for such μ". We prove that if there is a universal member, e.g. in $\mathbf{K}^{\rm lf}_\mu$, then there is a canonical one.

Theorem 2.1. *Assume μ be strong limit of cofinality $\aleph_0$ and $\mathbf{K}$ is μ-nice.*
1) The following conditions are equivalent:

- (A) *there is a universal $G \in \mathbf{K}_\mu$*
- (B) *if $H \in \mathbf{K}_\lambda$ is $\aleph_0$-indecomposable for some $\lambda < \mu$, <u>then</u> there is a sequence $\bar{G} = \langle G_\alpha : \alpha < \alpha_* \leq \mu \rangle$ such that:*
  - (a) *$H \subseteq G_\alpha \in \mathbf{K}_\mu$*
  - (b) *if $G \in \mathbf{K}_\mu$ extend H, <u>then</u> for some α, G is embeddable into G_α over H.*
- $(B)^+$ *We can add in (B)*
  - (c) *if $\alpha_1 < \alpha_2 < \alpha_*$, <u>then</u> there are no G, f_1, f_2 such that $H \subseteq G \in \mathbf{K}$ and f_ℓ embeds G_{α_ℓ} into G over H for $\ell = 1, 2$.*
  - (d) *(H, G_α) is an amalgamation pair (see Definition 0.5, moreover a universal amalgamation base (see 0.5(2))*

2) We can add in part (1):

- (C) *there is G_* such that:*
  - (a) *$G_* \in \mathbf{K}_\mu$ is universal in $\mathbf{K}_{<\mu}$;*
  - (b) *$\mathscr{E}^{\aleph_0}_{G_*,<\mu}$, see see 2.2 below, is an equivalence relation with $\leq \mu$ equivalence classes;*
  - (c) *G_* is μ-special see below.*
  - (d) *If $G \in \mathbf{K}_\mu$ is μ-special then G, G_* are isomorphic, (that is uniqueness).*

Before we prove 2.1,

Definition 2.2. For $\theta = \mathrm{cf}(\theta) < \mu$ we define for $M_* \in \mathbf{K}_\mu$:

- (A) $\mathrm{IND}^\theta_{M_*,<\mu} = \{N : N \leq_{\mathfrak{k}} M_*$ has cardinality $< \mu$ and is θ-indecomposable$\}$.
- (B) $\mathscr{F}^\theta_{M_*,<\mu} = \{f$: for some θ-indecomposable $N = N_f \in K_{<\mu}$ with universe an ordinal, f is a $\leq$-embedding of N into $M_*\}$.
- (C) $\mathscr{E}^\theta_{M_*,<\mu} = \{(f_1, f_2) : f_1, f_2 \in \mathscr{F}^\theta_{M_*,<\mu}, N_{f_1} = N_{f_2}$ and there are embeddings g_1, g_2 of M_* into some extension M of M_* such that $g_1 \circ f_1 = g_2 \circ f_2\}$.
- (D) M_* is $\theta - \mathscr{E}^\theta_{M_*,<\mu}$-indecomposably homogeneous (or just M_* is θ-indecomposably homogeneous) <u>when</u>: if $f_1, f_2 \in \mathscr{F}^\theta_{M_*,<\mu}$ and $(f_1, f_2) \in \mathscr{E}^\theta_{M_*,<\mu}$ and $A \subseteq M_*$ has cardinality $< \mu$ <u>then</u> there is $(g_1, g_2) \in \mathscr{E}^\theta_{M_*,<\mu}$ such that $f_1 \subseteq g_1 \wedge f_2 \subseteq g_2$ and $A \subseteq \mathrm{Rang}(g_1) \cap \mathrm{Rang}(g_2)$; it follows that if $\mathrm{cf}(\mu) = \aleph_0$ then for some $g \in \mathrm{aut}(M_*)$ we have $f_2 = g \circ f_1$.
- (E) We say that $M_* \in \mathbf{K}_\mu$ is μ-special when it is θ-indecomposably homogeneous and every $M \in \mathbf{K}_{<\mu}$ is embeddable into it.

Remark 2.3. We may consider in 2.1 also $(A)_0 \Rightarrow (A)$ where

$(A)_0$ if $\lambda < \mu, H \subseteq G_1 \in \mathbf{K}_{<\mu}$ and $|H| \le \lambda$, <u>then</u> for some G_2 we have $G_1 \subseteq G_2 \in \mathbf{K}_{<\mu}$ and (H, G_2) is a (μ,μ,λ)-amalgamation base.

Proof. It suffices to prove the following implications:

<u>$(A) \Rightarrow (B)$</u>:

Let $G_* \in \mathbf{K}_\mu$ be universal and choose a sequence $\langle G^*_n : n < \omega\rangle$ such that $G_* = \bigcup\limits_n G^*_n, G^*_n \subseteq G^*_{n+1}, |G^*_n| < \mu$.

Let H be as in 2.1(B) and let $\mathscr{G} = \{g : g \text{ embed } H \text{ into } G^*_n \text{ for some } n\}$. So clearly $|\mathscr{G}| \le \sum\limits_n |G^*_n|^{|H|} \le \sum\limits_{\lambda<\mu} 2^\lambda = \mu$, (an over-kill).

Let $\langle g^*_\alpha : \alpha < \alpha_* \le \mu\rangle$ list $\mathscr{G}$ and let (G_α, g_α) be such that (exist by renaming):

$(*)_1$ (a) $H \subseteq G_\alpha \in \mathbf{K}_\mu$;

(b) g_α is an isomorphism from G_α onto G_* extending g^*_α.

It suffices to prove that $\bar G = \langle G_\alpha : \alpha < \alpha_*\rangle$ is as required in (B). Now clause (B)(a) holds by $(*)_1(a)$ above. As for clause (B)(b), let G satisfy $H \subseteq G \in \mathbf{K}_{\le\mu}$, hence there is an embedding g of G into G_*. We know that $g(H) \subseteq G = \bigcup\limits_n G_n$ hence $\langle g(H) \cap G_n : n < \omega\rangle$ is $\subseteq$-increasing with union $g(H)$; but $g(H)$ by the assumption on H is $\aleph_0$-indecomposable, hence $g(H) = g(H) \cap G^*_n \subseteq G^*_n$ for some n, so $g{\restriction}H \in \mathscr{G}$ and so for some α we have $g = g^*_\alpha$. Hence $g^{-1}_\alpha g$ is an embedding of G into G_* extending $(g_\alpha{\restriction}H)^{-1})(g{\restriction}H) = (g^*_\alpha)(g^*_\alpha) = \mathrm{id}_H$ as promised.

<u>$(B) \Rightarrow (B)^+$</u>:

What about $(B)^+(c)$? while $\bar G$ does not necessarily satisfy it, we can "correct it", e.g. we choose u_α, v_α and if $\alpha \notin \cup\{v_\beta : \beta < \alpha\}$ also G'_α by induction on $\alpha < \alpha_*$ such that (the idea is that if $\beta \in v_\alpha, G_\beta$ is discarded being embeddable into some G'_α and G'_α is the "corrected" member):

$(*)^2_\alpha$ (a) $G_\alpha \subseteq G'_\alpha \in \mathbf{K}_\mu$ if $\alpha \notin \cup\{v_\beta : \beta < \alpha\}$;

(b) $u_\alpha \subseteq \alpha$ and $v_\alpha \subseteq \alpha_* \backslash \alpha$;

(c) if $\beta < \alpha$ then $u_\beta = u_\alpha \cap \beta$ and $u_\alpha \cap v_\beta = \emptyset$;

(d) if $\alpha = \beta + 1$ then $\beta \in u_\alpha$ <u>iff</u> $\beta \notin \cup\{v_\gamma : \gamma < \beta\}$;

(e) if $\alpha \notin \cup\{v_\gamma : \gamma < \alpha\}$, <u>then</u>:

- $\bullet_1$ $\gamma \in v_\alpha$ <u>iff</u> ($\gamma > \alpha$ and) G_γ is embeddable into G'_α over H;
- $\bullet_2$ if $\gamma \in \alpha_* \backslash (\alpha + 1) \backslash \cup \{v_\beta : \beta \le \alpha\}$ then G_γ is not embeddable over H into any G' satisfying $G'_\alpha \subseteq G' \in \mathbf{K}$;

(f) if $\alpha = \beta + 1$ and $\beta \notin u_\alpha$ then $v_\beta = \emptyset$.

This suffices because if we let $u_{\alpha_*} = \alpha_* \backslash \cup \{v_\gamma : \gamma < \alpha_*\}$, then $\langle G'_\alpha : \alpha \in u_{\alpha_*}\rangle$ is as required. Why can we carry the induction?

For $\alpha = 0, \alpha$ limit we have nothing to do because u_α is determined by $(*)^2_\alpha(c)$ and $(*)^2_\alpha(d)$. For $\alpha = \beta + 1$, if $\beta \in \bigcup\limits_{\gamma<\beta} v_\gamma$ we have nothing to do, in the remaining case we choose $G'_{\beta,i} \in \mathbf{K}_\mu$ by induction on $i \in [\alpha, \alpha_*]$, increasing continuous with $i, G'_{\beta,\alpha} = G_\alpha, G'_{\beta,i+1}$ make clause (e) true, i.e. if $G'_{\beta,i}$ has an extension into which G_i is embeddable over H, then there is such an extension of cardinality μ and choose $G'_{\beta,i+1}$ as such an extension.

Lastly, let $G'_\alpha = G'_{\alpha,\alpha_*}$ and $u_\alpha = u_\beta \cup \{\alpha\}$ and $v_\alpha = \{i : i \in \alpha_*, i \notin \cup\{v_\gamma : \gamma < \beta\}$ and G_i is embeddable into G_β over $H\}$.

So clause $(B)^+(c)$ holds; and clause $(B)^+(d)$ follows from $(B)(b) + (B)^+(c)$, so we are done.

$\underline{(B)^+ \Rightarrow (A)}$:

We prove below more: there is something like "special model", i.e. part (2), now $\underline{(C) \Rightarrow (A)}$

is trivial so we are left with the following.

$\underline{(B)^+ \Rightarrow (C)}$:

Choose $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle, \lambda_n \in \{\aleph_{\alpha+2} : \alpha \in \mathrm{Ord}\} \cap \mu \setminus |\tau|^+$ such that $2^{\lambda_n} < \lambda_{n+1}$ and $\mu = \sum\limits_n \lambda_n$ and $\lambda_n \in \Theta^\bullet_\mu$, see 1.3.(d).

Let $\mathbf{K}^{\mathrm{spc}}_\mu$ be the class of G such that:

$(*)^3_{\bar{G}}$ (a) $G \in \mathbf{K}_\mu$

(b) if $H \subseteq G, |H| < \lambda$, $\underline{\text{then}}$ there is an $\aleph_0$-indecomposable $H' \in \mathbf{K}^{<\mu}$, such that $H \subseteq H' \subseteq G$

(c) if $H \subseteq G$ is $ha0$-indecomposable of cardinality $< \mu$ $\underline{\text{then}}$ the pair (G, H) is an universal amalgamation base (see Definition 0.5(2));

(d) if $H \subseteq G$ is $\aleph_0$-indecomposable of cardinality $< \mu, H \subseteq H' \in \mathbf{K}_{<\mu}, H'$ is $\aleph_0$-indecomposable[1], and G, H' are compatible over H (in $\mathbf{K}_{\le\mu}$), $\underline{\text{then}}$ H' is embeddable into G over H.

Now we can finish by proving $(*)_4 + (*)_5$ below.

$(*)_4$ if $G \in \mathbf{K}_{\le\mu}$ $\underline{\text{then}}$ for some $\bar{G} \in \mathbf{K}^{\mathrm{spc}}_{\bar{\lambda}}, G$ is embeddable into $\bigcup\limits_n G_n$;

We break the proof to some stages, $(*)_{4,3}$. gives the desired conclusion of $((*)_4$

$(*)_{4.1}$ if $N_1 \in \mathbf{K}_\mu$ $\underline{\text{then}}$ there is N_2 such that

(a) $N_2 \in \mathbf{K}_\mu$

(b) $N_1 \subseteq N_2$

(c) if $H \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(N_1)$ $\underline{\text{then}}$ (N_2, H) is a universal amalgamation base.

Why? by 1 it is enough to deal with one such H, which is O.K. by clause (d) of Def 1.3]

$(*)_{4.2}$ like $(*)_{4.1}$ but in clause (c) is replaced by

(c)' if $H_1 \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(N_1)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and, we may add, H_2 is $\aleph_0$-indecomposable) then $\underline{\text{either}}$ N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\le\mu}$ $\underline{\text{or}}$ H_2 is embeddable into N_2 over H_1

[Why? Again it is enough to deal with one pair (H_1, H_2)] which is done by hand.]

$(*)_{4.3}$ If $N_1 \in \mathbf{K}_{\le\mu}$ then there is N_2 such that

(a) $N_2 \in \mathbf{K}_\mu$

(b) $N_1 \subseteq N_2$

(c) if $H \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(N_2)$ $\underline{\text{then}}$ (N_2, H) is a universal amalgamation base

(d) if $H_1 \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(N_2)$ and $H_1 \subseteq H_2 \in \mathbf{K}_{<\mu}$ (and, we may add, H_2 is $\aleph_0$-indecomposable) then $\underline{\text{either}}$ N_2, H_1 are incompatible over H_1 in $\mathbf{K}_{\le\mu}$ $\underline{\text{or}}$ H_2 is embeddable into N_2 over H_1

[1]The $\aleph_0$-indecomposability is not always necessary, but we need it sometimes.

[Why? We choose $L_\varepsilon \in \mathbf{K}_\mu$ by induction on $\varepsilon < \mathrm{cf}((\mu)$, such that

(a) $L_\alpha \in \mathbf{K}_\mu$
(n) $\langle L_\beta : \beta \le \alpha\rangle$ is increasing continuous
(c) $G_1 \subseteq L_0$
(d) if $\alpha = 2\beta+1$ then L_α relate to $L_{2\beta}$ as N_2 relate to N_1 is $(*)_{4.1}$
(e) if $\alpha = 2\beta+2$ then L_α relate to $L_{2\beta+1}$ as N_2 relate to N_1 is $(*)_{4.2}$

There is no problem to carry the induction and then $N_2 = L_{\mathrm{cf}(\mu)}$ is as required in $(*)_{4.3}$ hence in $(*)_4$.

$(*)_5$ (a) if $G_1, G_2 \in \mathbf{K}^{\mathrm{spc}}_\mu$ then G_1, G_2 are isomorphic;
(b) if $G_1, G_2 \in \mathbf{K}^{\mathrm{spc}}_\mu, H \in \mathbf{K}$ is $\aleph_0$-indecomposable and f_ℓ embeds H into G_ℓ, for $\ell = 1,2$, and this diagram can be completed, (i.e. there are $G \in \mathbf{K}_\mu$ and embedding $g_\ell : G_\ell \to G_*$ such that $g_1 \circ f_1 = g_2 \circ f_2$) <u>then</u> there is h such that:
(α) h is an isomorphism from G_1 onto G_2;
(β) $h \circ f_1 = f_2$;

Why? Let $\mathscr{F} = \mathscr{F}[G_1, G_2]$ be the set of f such that:

(a) f is an isomorphism from $G_{1,f} \in \mathrm{IDC}_{\mathrm{cf}(\mu)}$ onto $G_{2.f} \in \mathrm{IDC}_{\mathrm{cf}(\mu)}(G_2)$
(b) G_1, G_2 are f-compatible in $\mathbf{K}_\mu$ which means that there is $G \in \mathbf{K}_\mu$ and embeddings g_ℓ of G_ℓ into G for $\ell = 1,2$ such that $g_2 \circ f = g_1 {\restriction} G_{1,f}$.

First $\mathscr{F}$ is non-empty (the function f with domain $\{e_{G_1}\}$ and range $\{e_{G_2}\}$ will do.) Second use the hence and forth argument]

$\square_{2.1}$

Remark 2.4. 1) Can we prove for strong limit singular μ of uncountable cofinality κ a parallel result? Well we have to consider the following game:

(x) a play last θ moved
(x) in the ε move, first Player I choose M_ε and then player II choose N_ε
(x) $M_\varepsilon \in \mathbf{K}_{<\mu}$
(x) $\langle M_\zeta : \zeta \le \varepsilon\rangle$ is increasing continuous
(x) $M_\varepsilon \subseteq N_\varepsilon \subseteq M_{\varepsilon+1}$
(x) in the end of the play, th eplayer II wins <u>iff</u>for every limite ordinal $\varepsilon < \theta$ is an amalgamtion base inside $\mathbf{K}_{<\mu}$

Now if player II does not lose then we can imitate the proof above; but does not seem exciting.

2) The proof works for any a.e.c. $\mathfrak{k}$ with $\mathrm{LST}_{\mathfrak{k}} < \mu$. But We may wonder can we weaken the demand on $\mathfrak{k}$. Actually we can: there is no need of smoothness (that is: if $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing then $\cup\{M_\alpha : \alpha < \delta\} \le M_\delta$. Moreover while we need the existence of an upper bound for any $\le_{\mathfrak{k}}$-increasing sequence, its being the union can be demanded only for the cofinality $\mathrm{cf}(\mu)$. Again, do not look exciting.

§ 3. Universal in $\beth_\omega$

In §(3B) we have characterized when there are special models in $\mathbf{K}$ of cardinality, e.g. $\beth_\omega$. We try to analyze a related combinatorial problem. Our intention is to first investigate $\mathfrak{k}_{\rm fnq}$ (the class structures consisting of a set and a directed family of equivalence relations on it, each with a finite bound on the size of equivalence classes). So $\mathfrak{k}_{\rm fnq}$ is similar to $\mathbf{K}$ but seems easier to analyze. We consider some partial orders on $\mathfrak{k} = \mathfrak{k}_{\rm fnq}$.

First, under the substructure order, $\le_1 = \subseteq$, this class fails amalgamation. Second, another order, $\le_2$ demanding TV for countably many points, finitely many equivalence relations, we have amalgamation. Third, we add: if $M \le_3 N$ then $M \le_1 N$ and the union of $(P^n, E_d)_{d \in Q(M)}$ is the disjoint union of models isomorphic to $(P^M, E_d)_{d \in Q(M)}$, the equivalence relation is $E_{M,N}$. This is intended to connect to locally finite groups. So we may instead look at $\{f \in {\rm Sym}(N)$: if $a \in N \backslash M$ and $a/E_{M,N} \ncong M$ then $f{\restriction}(a/E_{M,N}) = {\rm id}(a/E_{M,N})$; no need of representations.

The model in $\beth_\omega$ will be $\bigcup\limits_n M_n, \|M_n\| = \beth_{n+1}$, gotten by smooth directed unions of members of cardinality $\beth_\omega$ by $\mathbf{I}_n \subseteq P^{M_{n+1}}$ is a set of representatives for $E_{M_n, M_{n+1}}$.

Definition 3.1. Let $\mathbf{K} = \mathbf{K}_{\rm fnq}$ be the class of structures M such that (the vocabulary is defined implicitly and is $\tau_{\mathbf{K}}$, i.e. depends just on $\mathbf{K}$):

- (a) P^M, Q^M is a partition of M, P^M non-empty;
- (b) $E^M \subseteq P^M \times P^M \times Q^M$ (is a three-place relation) and we write $aE^M_c b$ for $(a,b,c) \in E^M$;
- (c) for $c \in Q^M, E^M_c$ is an equivalence relation on P^M with $\sup\{|a/E^M_c| : a \in P^M\}$ finite (see more later);
- (d) $Q^M_{n,k} \subseteq (Q^M)^n$ for $n, k \ge 1$
- (e) if $\bar{c} = \langle c_\ell : \ell < n \rangle \in {}^n(Q^M)$ we let $E^M_{\bar{c}}$ be the closure of $\bigcup\limits_\ell E_\ell$ to an equivalence relation;
- (f) ${}^n(Q^M) = \bigcup\limits_{k \ge 1} Q^M_{n,k}$;
- (g) if $\bar{c} \in Q^M_{n,k}$ then $k \ge |a/E^M_{\bar{c}}|$ for every $a \in P^M$.

Definition 3.2. We define some partial order on $\mathbf{K}$.
1) $\le_1 = \le^1_{\mathbf{K}} = \le^1_{\rm fnq}$ is being a sub-model.
2) $\le_3 = \le^3_{\mathbf{K}} = \le^3_{\rm fnq}$ is the following: $M \le_3 N$ iff:

- (a) $M, N \in \mathbf{K}$
- (b) $M \subseteq N$
- (c) if $A \subseteq N$ is countable and $A \cap Q^N$ is finite, <u>then</u> there is an embedding of $N{\restriction}A$ into M over $A \cap M$ or just a one-to-one homomorphism.

3) $\le_2 = \le^2_{\mathbf{K}} = \le^2_{\rm fnq}$ is defined like $\le_3$ but in clause (c), A is finite.

Claim 3.3. *1)* $\mathbf{K}$ *is a universal class, so* $(\mathbf{K}, \subseteq)$ *is an a.e.c.*
2) $\le^3_{\mathbf{K}}, \le^2_{\mathbf{K}}, \le^1_{\mathbf{K}}$ *are partial orders on* $\mathbf{K}$.
3) $(\mathbf{K}, \le^2_{\mathbf{K}})$ *is an a.e.c.*
4) $(\mathbf{K}, \le^2_{\mathbf{K}})$ *has disjoint amalgamation.*

Proof. 1),2),3) Easy.
4) By 3.4 below. $\square_{3.3}$

Claim 3.4. *If* $M_0 \le^1_{\mathbf{K}} M_1, M_0 \le^3_{\mathbf{K}} M_2$ *and* $M_1 \cap M_2 = M_0$, <u>*then*</u> $M = M_1 + M_2$, *the disjoint sum of* M_1, M_2 *belongs to* $\mathbf{K}$ *and extends* M_ℓ *for* $\ell = 0, 1, 2$ *and even* $M_1 \le^3_{\rm fnq} M$ *and* $M_0 \le^2_{\mathbf{K}} M_1 \Rightarrow M_2 \le^2_{\mathbf{K}} M$ <u>*when*</u>*:*

- $(*)$ $M = M_1 + M_2$ *means* M *is defined by:*
  - (a) $|M| = |M_1| \cup |M_2|$;
  - (b) $P^M = P^{M_1} \cup P^{M_2}$;
  - (c) $Q = Q^{M_1} \cup Q^{M_2}$;
  - (d) *we define* E^M *by defining* E^M_c *for* $c \in Q^M$ *by cases:*
    - (α) *if* $c \in Q^{M_0}$ *then* E^M_c *is the closure of* $E^{M_1}_\ell \cup E^{M_2}_\ell$ *to an equivalence relation;*
    - (β) *if* $c \in Q^{M_\ell} \backslash Q^{M_0}$ *and* $\ell \in \{1, 2\}$ *then* E^M_c *is defined by*
      - $aE^M_c b$ <u>*iff*</u> $a = b \in P^{M_{3-\ell}} \backslash M_0$ *or* $aE^{M_\ell}_c b$ *so* $a, b \in P^{M_\ell}$;
  - (e) $Q^M_{n,k} = Q^{M_1}_{n,k} \cup Q^{M_2}_{n,k} \cup \{\bar c : \bar c \in {}^n(Q^M) \backslash ({}^n(Q^{M_1})) \cup {}^n(Q^{M_2})\}$.

Proof. Clearly M is a well defined structure, extends M_0, M_1, M_2 and satisfies clauses (a),(b),(c) of Definition 3.1. There are two points to be checked: $a \in P^M, \bar c \in Q^M_{n,k} \Rightarrow |a/E^M_{\bar c}| \le k$ and ${}^n(Q^M) = \bigcup\limits_{k \ge 1} Q^M_{n,k}$

$(*)_1$ if $a \in P^M$ and $\bar c \in Q^M_{n,k}$ <u>then</u> $|a/E^M_{\bar c}| \le k$.

Why? If $\bar c \in Q^M_{n,k} \backslash (Q^{M_1}_{n,k} \cup Q^{M_2}_{n,k})$ this holds by the definition, so assume $\bar c \in Q^{M_\iota}_{n,k}$. If this fails, then there is a finite set $A \subseteq M$ such that $\bar c \subseteq A, a \in A$ and letting $N = M{\restriction}A$ we have $|a/E^N_{\bar c}| > k$. By $M_0 \le^1_{\mathbf{K}} M_1, M_0 \le^3_{\mathbf{K}} M_2$ (really $M_0 \le^2_{\mathbf{K}} M_2$ suffice) there is a one-to-one homomorphism f from $A \cap M_2$ into M_0. Let $B' = (A \cup M_1) \cup f(A \cap M_2)$ and $N' = M{\restriction}B$ and let $g = f \cup \mathrm{id}_{A \cap M_1}$. So g is a homomorphism from N onto N' and $g(a)/E^{N'}_{g(\bar c)}$ has $> k$ members, which implies $g'(a)/E^{M_1}_{g'(\bar c)}$ has $> k$ members. Also $g(\bar c) \in Q^{M_1}_{n,k}$. (Why? If $\iota = 1$ trivially, if $\iota = 2$ by the choice of f, contradiction to $M \in \mathbf{K}$.)]

$(*)_2$ if $\bar c \in {}^n(Q^M)$ then $\bar c \in \bigcup\limits_k Q^M_{n,k}$.

Why? If $\bar c \in M_1$ or $\bar c \subseteq M_2$, this is obvious by the definition of M, so assume that they fail. By the definition of the $Q^M_{n,k}$'s we have to prove that $\sup\{a/E^M_{\bar c} : a \in P^M\}$ is infinite. Toward contradiction assume this fails for each $k \ge 1$ there is $a_k \in P^M$ such that $a_k/E^M_{\bar c}$ has $\ge k$ elements hence there is a finite $A_k \subseteq M$ such that $a_k/E^{M \restriction A_k}_{\bar c}$ has $\ge k$ elements. Let $A = \bigcup\limits_{k \ge 1} A_k$, so A_k is a countable subset of M and we continue as in the proof of $(*)_1$.

Additional points (not really used) are proved like $(*)_2$:

$(*)_3$ $M_1 \leq^3_{\mathbf{K}} M$;
$(*)_4$ $M_0 \leq^2_{\mathbf{K}} M_1 \Rightarrow M_2 \leq^2_{\mathbf{K}} M$;
$(*)_5$ $M_1 +_{M_0} M_2$ is equal to $M_2 +_{M_0} M_1$.

$\square_{3.4}$

Claim 3.5. *If $\lambda = \lambda^{<\mu}$ and $M \in \mathbf{K}$ has cardinality $\leq \lambda$ then there is N such that:*

(a) *$N \in \mathbf{K}_\lambda$ extend M;*
(b) *if $N_0 \leq^3_{\mathbf{K}} N_1$ and N_0 has cardinality $< \mu$ and f_0 embeds N_0 into N, then there is an embedding f_1 of N_1 into N extending f_0.*

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