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ABSTRACT. We find new "reasons" for a class of models for not having a universal model in a cardinal λ. This work, though has consequences in model theory, is really combinatorial. We concentrate on a prototypical class which is a simply defined class of models, of combinatorial character - models of $T_{\rm ceq}$ (essentially another representation of $T_{\rm feq}$ which was already considered but the proof with $T_{\rm ceq}$ is more transparent). Models of $T_{\rm ceq}$ consist essentially of an equivalence relation on one set and a family of choice functions for it. This class is not simple (in the model theoretic sense) but seems to be very low among the non-simple (first order complete countable) ones. We give sufficient conditions for the non-existence of a universal model for it in λ.

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Annotated Content

§ 0. INTRODUCTION

On a recent survey on the universality spectrum see [Shear], an earlier survey is [Mir05]; there have been several advances meanwhile (and this is one of the advances after [Shear]). The problem for general first order theories is a model theoretic one, but specific examples are combinatorial set theoretic ones (and serve as proto-types for suitable families of theories); so combinatorialists may ignore model theoretic notions like "T is simple, have the tree property, is TP_2", and consider only the concrete universal theories considered; so ignore 1.4(1),(2) and their proof. Here we concentrate on the theory T_{ceq}, which we considered as a proto-typical ''minimal" non-simple T, so are expecting it (under $\leq_{\mathrm{univ}}$) to be low, so is (like T_{feq}, see below, NSOP_1, see [She93], [DS04b], [SU08], [CR16], [KR17]). True, there were non-existence results near a strong limit singular cardinal (see on the T_{feq} in [She93], generalizing it the oak property [DS06], [She17, §3]) but there were weak consistency results on existence (see [She93], [DS04a]). We had considered T_{feq}, a prototypical example of such theories, now T_{ceq} is essentially equivalent to it for our aims, see 1.4(3),(4) but T_{ceq} seem more transparent; we intend to deal with "to what family of T's versions of our proof apply, in particular, NPT_2 and non-simple" elsewhere ([S$^+$a]), just lately we have learnt this is impossible, see [Shear, §11C].

We have hoped/expected that for the $\lambda > \mu = \mu^{<\mu}$ but $\lambda = \mu^+ < 2^\mu$ we shall have consistency results for theories like T_{feq} and the class of triangle free graphs, see [S$^+$b], [Shec].

We first give a case with stronger set theoretic assumptions, but more transparent proof in §1. In §2 we give such proof under reasonable set theoretic assumptions, (close to the so called club guessing) but then have to consider finer points in combinatorial set theory on guessing clubs. Elsewhere we hope to with relevant complimentary consistency (see [S$^+$c], [Shec]) and families of theories (so e.g. T_{feq} fit in, see [S$^+$d]).

A priory we think that T_{tfg}, the theory of triangle free graphs, is "more complicated" then $T_{\mathrm{feq}}, T_{\mathrm{ceq}}$, but now have doubts.

Question 0.1. 1) Does §1 apply to more theories than in §2?
2) Can we characterize the dividing line? Simple/non-simple in our context.
3) Does it help to have:

(∗) for some $\mu, \mu < \lambda < 2^\mu$ there is no $\mathscr{A} \subseteq [\lambda]^\lambda$ which is $\mu-\mathrm{AD}$ of cardinality $> \lambda$?

This would justify the use of $\mu - \mathrm{AD}$ family $\mathscr{A} \subseteq [\lambda]^\lambda$ in some consistency results, see [She90], [Shec], see below.

Discussion 0.2. Note that:

⊞ if $n \leq \omega, \theta \leq \mu \leq 2^\theta, \lambda \nrightarrow [\mu]^{<n}_\mu$ and we let T_n be the theory $\{P_k$ is a reflexive asymmetric k-place relation": $k < n\}$ and T_n has a universal model M_* in λ <u>then</u> there is a μ-disjoint $\mathscr{A} \subseteq [\lambda]^\lambda$ of cardinality 2^θ.

[Why? Let $\mathbf{c} : [\lambda]^{<n} \to \mu$ witness $\lambda \nrightarrow [\mu]^{<n}_\mu$ and for $u \subseteq \theta$ let $M_u = (\lambda, \ldots, P_k^{M_u}, \ldots)$ where $P_k^{M_u} = \{\eta \in {}^k\lambda : \eta$ is with no repetitions and $\mathbf{c}(\mathrm{Rang}(\eta)) \in u\}$. So there is an embedding f_u into M_*; now $\langle \mathrm{Rang}(f_u) : u \subseteq \theta \rangle$ is a family as promised.]

§ 0(A). **Preliminaries.**

Notation 0.3. 1) T is a theory with vocabulary $\tau_T = \tau(T)$ and is a first order, if not said otherwise.
2)

(a) $\mathrm{EC}_T = \{M : M \text{ a model of } T\}$
(b) $\mathrm{EC}_T(\lambda) = \{M \in \mathrm{EC}_T : M \text{ of cardinality } \lambda\}$
(c) $\mathrm{EC}_T(\lambda!) = \{M \in \mathrm{EC}_T : M \text{ has universe } \lambda\}$.
(d) for a set A of ordinals and ordinal α let $\mathrm{suc}_A(\alpha)$ be $\min\{\beta \in A : \beta > \alpha\}$

Convention 0.4. 1)

(A) If T is universal theory not complete (like $T^0_{\mathrm{ceq}}, T^0_{\mathrm{feq}}$), then embedding are the usual ones, (on EC_T) and $\subseteq_T$ (on EC_T) means $\subseteq$ and we assume EC_T has amalgamation and JEP.
(B) If T is complete, then embeddings are elementary (on EC_T) and $\subseteq_T$ means $\prec$ on EC_T.
(C) We say f is a T-embedding of M into N or $f : M \longrightarrow_T N$ when M, N are models of T, f embed M into N and $f(M) \subseteq_T N$.

2) If $\Delta \subseteq \mathbb{L}(\tau_T)$ then $\mathrm{univ}_{T,\Delta}(\lambda)$ is the minimal χ such that there is a sequence $\bar{M}$ which is a (λ, T, Δ)-universal sequence which means:

(a) $\bar{M} = \langle M_\alpha : \alpha < \chi \rangle$ is a sequence of models of T
(b) each M_α is of cardinality λ
(c) for every model M of T of cardinality λ there is a Δ-embedding of M into some M_α, see below.

3) For given T, Δ as above and models M, N of T, we say f is a Δ-embedding of M into N when :

(a) f is a function from M into N
(b) if $\varphi(x_0, \ldots, x_{n-1}) \in \Delta$ and $a_0, \ldots, a_{n-1} \in M$ and $M \models \varphi[a_0, \ldots, a_{n-1}]$ then $N \models \varphi[f(a_0), \ldots, f(a_{n-1})]$
(c) so f is one-to-one when $(x \neq y) \in \Delta$.

4) For T, Δ as above in part (2) we may omit Δ when :

(a) T is complete, $\Delta = \mathbb{L}(\tau_T)$, all first order formulas
(b) T not complete, Δ the set of quantifier formulas in $\mathbb{L}(\tau_T)$.

5) We may write at, ep for $\Delta_{\mathrm{at}(T)} = \{\varphi \in \mathbb{L}(\tau_T) : \varphi \text{ is atomic}\}, \Delta_{\mathrm{ep}(T)} = \{\varphi \in \mathbb{L}(\tau_T) : \varphi$ existential positive$\}$ respectively. We may write τ instead of T. We may write φ instead $\Delta = \{\varphi\}$ and $\pm\varphi$ instead $\Delta = \{\varphi, \neg\varphi\}$.

Notation 0.5. 1) Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, i, j$ denote ordinals.
2) Let $\kappa, \lambda, \mu, \chi, \partial, \theta, \Upsilon$ denote cardinals, infinite if not said otherwise.
3) Let k, ℓ, m, n denote natural numbers.
4) Let φ, ψ, ϑ denote formulas, f.o. if not said otherwise.

Definition 0.6. 1) $J^{\rm bd}_\theta = \{A \subseteq \theta : \sup(A) < \theta\}$, bd stands for bounding, for θ a regular cardinal or just a limit ordinal.
1A) For θ regular uncountable let:

- $D^{\rm club}_\theta = \{A \subseteq \theta$: there is a club (= closed unbounded subset) E of θ such that $E \subseteq A\}$.

2) For a regular θ let:

(a) $\mathfrak{d}_\theta = \mathrm{Min}\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\theta\theta$ is $<_{J^{\rm bd}_\theta}$-cofinal in $\mu_\mu\}$
(b) $\mathfrak{b}_\theta = \mathrm{Min}\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\theta\theta$ has no $<_{J^{\rm bd}_\theta}$-upper bound$\}$.

3) Let $\mathfrak{d}^{\rm club}_\theta$ be defined similarly using $<_{D^{\rm club}_\theta}$ when θ is regular uncountable.

§ 1. On $T_{\rm ceq}$ for Mahlo cardinals

As here we consider $T_{\rm ceq}$ the simplest, non-simple theory, we may consider how much does it behave like the class of graphs (equivalently random graph)? We prove that not by a non-existence result, but with quite specific set theoretic assumptions.

$T_{\rm feq}$ is a prime example for a theory with the tree order property (even TP_2 but neither the strict order property nor even just the SOP_2). For it we get here parallel and better results than [She93] where it is proved that there are limitations on the universality spectrum for $T_{\rm feq}$ and in [DS06] which generalize the results for any T with the so called oak property, see somewhat more in [She17, §3]. The results in those papers are meaningful when SCH fails, that is, consider a cardinal λ such that: for some strong limit singular $\mu, \mu^+ < \lambda < \mu$ if λ is regular in the interval then "usually" $T_{\rm feq}, T$ has no universal in λ.

Here we get further such non-existence results for Mahlo cardinals, but what about $\lambda \in [\mu, 2^\mu)$ when for transparency we assume $\mu = \mu^{<\mu}$?. In §2, we do better but the Mahlo case may cover more classes, comes first and the proofs are more transparent. The proof here can be axiomatized as in §2 using

⊞ $\mathrm{PGC}(\lambda, S), S$ a stationary set of regular cardinals (so $\mathscr{U} = S$).

First, recall (the reader can concentrate on the universal versions, $T^0_{\rm feq}, T^0_{\rm ceq}$, on $T_{\rm feq}$ see [She93, 2.1=Lb3,3.1=Lc3]):

Definition 1.1. $T_{\rm feq} = T^1_{\rm feq}$ is the model completion of the following theory, $T^0_{\rm feq}$ which is defined by:

(A) $\tau = \tau(T^0_{\rm feq})$ consists of:
  (a) predicates P, Q (unary)
  (b) E (three place predicate written as yE_xz)
(B) a τ-model M is a model of $T^0_{\rm feq}$ <u>iff</u>
  (a) the universe of M is the disjoint union of P^M and Q^M, each infinite
  (b) $yE^M_x z \to P(x) \wedge Q^M(y) \wedge Q^M(z)$
  (c) for any fixed $x \in P, E_x$ is an equivalence relation on Q.

Remark 1.2. 1) So if $M \models T_{\rm feq}$ <u>then</u>:

$(*)$ (a) in (B)(c), for each $x \in P, E_x$ is with infinitely many equivalence classes

(b) if $n < \omega, x_1, \dots, x_n \in P$ with no repetition and $y_1, \dots, y_n \in Q$ then for some $y \in Q$, $\bigwedge\limits_{\ell=1}^{n} y E_{x_\ell} y_\ell$

(c) if $n < \omega$ and $y_1, \dots, y_n \in Q$ and e is an equivalence relation on $\{1, \dots, n\}$ <u>then</u> for some $x \in P$ we have $y_\ell E_x y_k \Leftrightarrow \ell e k$.

2) Hence $T_{\rm feq}$ has elimination of quantifiers.

We present a close relative, the main one we consider here (and, as proved below, equivalent to $T_{\rm feq}$ for our purpose).

Definition 1.3. We define $T^0_{\rm ceq}, T_{\rm ceq} = T^1_{\rm ceq}$ as in Definition 1.1 replacing the subscript feq by ceq (or omitting it) and:

(A) $\tau = \tau(T^0_{\rm ceq}) = \tau(T_{\rm ceq})$ consists of: P, Q unary predicates, E a binary predicate and F a binary function symbol

(B) a τ-model M is a model of universal theory $T^0_{\rm ceq}$ <u>iff</u>

  (a) P^M, Q^M is a partition of M

  (b) E^M is an equivalence relation on Q^M

  (c) F^M is a function from $Q^M \times P^M$ into Q^M such that for every $c \in P^M, F^M(a,c)$ is choosing a representative for the a/E^M-equivalence class, that is, we have:

    (α) $a \in Q^M \Rightarrow F^M(a,c) \in a/E^M$

    (β) if $a, b \in Q^M$ are E^M-equivalence then $F^M(a,c) = F^M(b,c)$.

Concerning λ in the neighborhood of a strong limit singular we shall not give details as we can just quote.

Claim 1.4. *0) Concerning $T^{)}_{\rm ceq}$*

(a) For a model M of $T^{)}_{\rm ceq}$ and $A \subseteq M$ with n elements, the closure of A inside M has at most $n + n^2$ elements, (even at most $n + (n/2)^2$ elements)

(b) $T^{)}_{\rm ceq}$ has amalgamation and JEP

(c) $T^0_{\rm ceq}$ has a model completion, that is $T_{\rm ceq}$ is well defined.

1) $T_{\rm ceq}$ is not simple, is NSOP_2 and even NSOP_1 and has the oak property, in fact, by qf *(quantifier free) and even atomic formulas.*
2) We have $(A) \Rightarrow (B)$ <u>where</u>:

(A) (a) $\theta < \mu < \lambda < \chi$

  (b) $\mathrm{cf}(\lambda) = \lambda, \theta = \mathrm{cf}(\theta) = \mathrm{cf}(\mu), \mu^+ < \lambda$

  (c) $\chi := \mathrm{pp}_{\Gamma(\theta)}(\mu) > \lambda + |i^*|$

  (d) there is $\{(a_i, b_i) : i < i^*\}, a_i \in [\lambda]^{<\mu}, b_i \in [\lambda]^\theta$ and $|\{b_i : i < i^*\}| \le \lambda$ such that: for every $f : \lambda \to \lambda$ for some $i, f(b_i) \subseteq a_i$

(B) (a) $T_{\rm ceq}$ equivalently $T^0_{\rm ceq}$ has no universal model in λ

  (b) Moreover, $\mathrm{univ}(\lambda, T_{\rm ceq}) \ge \chi = \mathrm{pp}_{\Gamma(\theta)}(\mu)$.

3) $T_{\rm feq}$ can be interpreted in $T_{\rm ceq}$ hence $\mathrm{univ}_{T_{\rm feq}}(\lambda) \le \mathrm{univ}_{T_{\rm ceq}}(\lambda)$.
4) Also the inverse holds.

Proof. 1) Easy.
2) By (1) quoting [DS06] where the oak property was introduced.
3) For a model M of $T^0_{\rm ceq}$ we define a model $N = N[M]$ of $T_{\rm feq}$ as follows:

$(*)_{N,M}$ (a) $P^N = P^M, Q^N = Q^M/E^M$
(b) $E^N = \{(a,b,C) : C \in Q^N$ and $a,b \in P^M$ and $(\forall c \in C)[F^M(c,a) = F^M(c,b)]$ equivalently $(\exists c \in C)[F^M(c,a) = F^M(c,b)]\}$.

Now check.
4) For a model N of $T^0_{\rm feq}$ we define a model $M = M[N]$ of $T^0_{\rm ceq}$ as follows:

$(*)_{M,N}$ (a) $P^M = P^N$ and $Q^M = Q^N \times P^M$
(b) $E^M = \{((c_1,a_1),(a_2,c_2)) : c_1 = c_2 \in P^M, a_1,a_2 \in Q^N\}$
(c) $F^M : Q^M \times P^M \to Q^M$ is defined by: Iff $d \in Q^M, b \in P^M$ hence for some $c \in Q^N, a \in P^M$ we have $d = (c,a)$ <u>then</u> we let $F^M(d,b) = (c,b)$.

Now check. $\square_{1.4}$

We now point out a new reason involved "large $\mathfrak{d}_\theta$'s" for not having a universal model in λ, even for many non-simple T's. In this section we deal with a case where the proof is simpler using $T_{\rm ceq}$ and λ a Mahlo cardinal.

Claim 1.5. *1) Assume λ is a (weakly inaccessible) Mahlo cardinal and $S = \{\theta < \lambda : \theta$ regular (weakly inaccessible) and $\mathfrak{d}^{\rm club}_\theta > \lambda\}$ is stationary in λ.*
<u>Then</u>

(a) $\operatorname{univ}(\lambda, T_{\rm ceq})$ is $> \lambda$
(b) moreover, $\geq \sup\{\chi$: the set $\{\theta \in S : \mathfrak{d}_\theta > \chi\}$ is stationary$\}$.

2) If we add "S has club guessing", <u>then</u> we can replace $\mathfrak{d}^{\rm club}_\theta$ by $\mathfrak{d}_\theta$.

Remark 1.6. Recall that by $\mathfrak{d}^{\rm club}_\lambda = \mathfrak{d}_\lambda$ when $\lambda > \beth_\omega$, by Cummings-Shelah [CS95, Th.8].

Proof. 1) Note that we sometimes prepare for the proof of part (2) putting it in [].. Clearly

$(*)_0$ it suffices to find $\chi \geq \lambda$ such that $S_\chi = \{\theta \in S :$ $mathfrakd^{\rm club}_\theta > \chi[\mathfrak{d}_\theta > \chi]\}$ is stationary and prove $\operatorname{univ}_T(\lambda) > \chi$.

Let $T = T_{\rm ceq}$ and let:

$(*)_1$ $\langle C^*_\delta : \delta \in S\rangle$ witness "S has club-guessing"; exist and is relevant only for part (2)
$(*)_2$ if (A) below holds, <u>then</u> we define some objects in (B) where:
(A) (a) $M \in {\rm EC}_T(\lambda!)$
(b) $|P^M| = \lambda, \theta$ regular $\in (\aleph_0, \lambda)$
(c) E a club of θ
(B) we define:
(a) for $a \in Q^M$ hence $a < \lambda$ let $g_a = g_{M,E,a}$ be the following partial function from θ to θ:
• for $\alpha < \theta, g_a(\alpha)$ is as the minimal $\beta \in E$ such that: $\beta \in E \backslash (\alpha+1)$ and $(\beta_1 \in P^M \cap \beta) \wedge (F^M(\beta_1,a) < \theta) \Rightarrow F^M(\beta_1,a) < \beta$
(b) $\mathscr{G}^0_{M,E} = \{g_{M,E,a} : a \in Q^M\}$

(c) $\mathscr{G}^1_{M,\theta} = \{g_{M,C_\theta,a} : a \in Q^M\}$.

Now easily

$(*)_3$ for a, M, θ, E as above, $g_{M,E,a}$ is a non-decreasing function from θ into θ, in fact, into $E \subseteq \theta$

$(*)_4$ if $M, N \in \mathrm{EC}_T(\lambda!)$ and f embeds M into N then for some club E^* of λ: if $\theta \in S, \theta = \sup(E^* \cap \theta), E \subseteq E^* \cap \theta$ is a club of θ and $a \in Q^M$ then $g_{M,E,a} \leq g_{N,E,a}$.

So without loss of generality $\theta \in S \Rightarrow \mathfrak{d}^{\rm club}_\theta > \chi \geq \lambda$ [or $\theta \in S \Rightarrow \mathfrak{d}_\theta > \chi \geq \lambda$] and we shall prove that $\mathrm{univ}(\lambda, T) > \chi$; this suffices. So assume $\langle M_\alpha : \alpha < \chi \rangle$ is a sequence of members of $\mathrm{EC}_T(\lambda!)$. Now there is a sequence $\langle E_\alpha : \alpha < \chi \rangle$ such that E_α is a lcub of λ and $\theta \in S \cap E_\alpha \Rightarrow (M_\alpha \restriction \theta \prec M_\alpha) \wedge |P^{M_\alpha} \cap \theta| = \theta$.

So for each $\theta \in S$ the set $\mathscr{G}_\theta = \cup\{\mathscr{G}^1_{M_\alpha,\theta} : \alpha < \chi, \theta \in E_\alpha$ and $|P^{M_\alpha} \cap \theta| = \theta\}$ has cardinality $\leq \chi$ recalling $\lambda \leq \chi$.

As $[|\mathscr{G}^{\rm club}_\theta| < \mathfrak{d}_\theta]$ [or $|\mathscr{G}_\theta| < \mathfrak{d}_\theta$] necessarily there is $g_\theta \in {}^\theta\theta$ such that $g \in \mathscr{G}_\theta \Rightarrow g \nleq g_\theta$ mod $D^{\rm club}_\theta$.

Now we define a model $N \in \mathrm{EC}_T(\lambda!)$ with $\tau_N = \tau(T_{\rm ceq})$ as follows:

(A) universe is λ

(B) (a) P^N is the set of odd ordinals $< \lambda$

(b) if $\alpha = 4\beta + 1 < \lambda$ then α/E^N is disjoint to α

(c) E^M is an equivalence relation on P^N such that for every $\alpha < \beta < \lambda$ such that β is divisible by $|\alpha|, \alpha \in P^N$ we have $|\alpha/E^N \cap \beta| = |\beta|$

(d) if $\theta \in S$ then $F^M(\alpha, \theta) > g_\theta(\alpha)$.

This is easy to do and clearly N is not embeddable into M_α for every $\alpha < \chi$.

2) Similarly except in the choice of g_θ we use $|\mathscr{G}_\theta| < \mathfrak{d}_\theta$ and choose $g_\theta \in {}^\theta\theta$ such that $g \in \mathscr{G}_\theta \Rightarrow g \nleq g_\theta \mod J^{\rm bd}_\theta$. $\square_{1.5}$

§ 2. On successor Cardinals and club guessing

We first introduce the relevant notions (in 2.1); (we could add clause 2.1(3)(b) into the definition of $\mathbf{U}_{\lambda,\theta}$ in 2.1(1), but so far it does not matter[1]. We then investigate it and use it for sufficient conditions for no universal.

Definition 2.1. Assume $\lambda > \theta > \aleph_0$ are regular and $D \subseteq \mathscr{P}(\theta)$ satisfying $D \subseteq [\theta]^\theta$ is non-empty; omitting D means $D = \{\theta\}$, and $\mathfrak{B}$ is a model with universe λ and countable vocabulary.
1) Let $\mathbf{U}_{\lambda,\theta} = \{\bar{u} : \bar{u} = \langle u_i : i < \theta \rangle$ is $\subseteq$-increasing continuous, and $i < \theta \Rightarrow u_i \in [\lambda]^{<\theta}$ but $\cup\{u_i : i < \theta\} \in [\lambda]^\theta$ and (for transparency) $\bigwedge\limits_i u_i \cap \theta \in \theta$ and $\bigcup\limits_i u_i \supseteq \theta\}$.
1A) We shall say that $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ obeys $\mathfrak{B}$ when every $\bar{u} \in \mathbf{U}$ does, which means that for every $\varepsilon < \theta$ we have $\mathfrak{B} \restriction u_\varepsilon \subseteq \mathfrak{B}$, (if $\mathfrak{B}$ has Skolem functions this is equivalent to $\mathfrak{B} \restriction u_\varepsilon \prec \mathfrak{B}$) and $\varepsilon \subseteq u_\varepsilon$.
2) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ fully D-guess clubs <u>when</u> for every model M with universe λ and countable vocabulary there is $\bar{u} \in \mathbf{U}$ which D-guesses M meaning $(\exists \mathscr{X} \in D)(\forall \varepsilon)[\varepsilon \in \mathscr{X} \Rightarrow M \restriction u_\varepsilon \subseteq M]$, i.e. u_ε is closed under the functions of M, (in an equivalent definition $M_\varepsilon \restriction u_\varepsilon \prec M$ as we can expand M by Skolem functions).
3) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ almost D-guess clubs (but this is not implied by "fully D-guess clubs, because of clause (b)) <u>when</u>:

- (a) for every model M with universe λ and countable vocabulary and $A \in [\lambda]^\lambda$ for some $\bar{u} \in \mathbf{U}$ we have:
  - (α) $c\ell(u_\varepsilon, M) \subseteq \sup(u_\varepsilon)$ for every $\varepsilon < \theta$
  - (β) for some $\mathscr{X} \in D$ we have $\varepsilon \in \mathscr{X} \Rightarrow A \cap u_{\varepsilon+1} \nsubseteq \sup(u_\varepsilon)$
  - (γ) $c\ell(\bigcup\limits_\varepsilon u_\varepsilon, M) = \bigcup\limits_\varepsilon u_\varepsilon$, that is, $M \restriction (\bigcup\limits_\varepsilon u_\varepsilon) \subseteq M$
- (b) if $\bar{u} \in \mathbf{U}$ then $\mathrm{ord}(\bar{u}) = \langle \sup(u_\varepsilon) : \varepsilon < \theta \rangle$ is strictly increasing.

3A) We say $\mathbf{U}$ medium D-guess clubs <u>when</u> as in (3) omitting $(a)(\gamma)$.
3B) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ semi-guess clubs (we may write D-guess, but D is irrelevant) <u>when</u>:

- (a)′ as in part (3) but replacing (β) by:
  - $(\beta)'$ if $\varepsilon < \theta$ then for some $\zeta \in (\varepsilon, \theta)$ and $\alpha \in A$ we have $\alpha \in (u_{\zeta+1} \backslash u_\zeta) \cap (\sup(u_{\varepsilon+1}) \backslash \sup(u_\varepsilon))$
- (b) as in part (3)

3C) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ pseudo-guess clubs (we may write D-guess, but D is irrelevant) <u>when</u>:

- (a)″ if $A \in [\lambda]^\lambda$ then for some $\bar{u} \in \mathbf{U}$ we have $(\alpha), (\beta)'$ of (3B)(a)′
- (b) as above.

3D) We say $\mathbf{U}$ is (λ,θ)-reasonable (or just reasonable when (λ,θ) are clear from the context) <u>when</u> $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ satisfies clause (3)(b).
4) We say $\mathbf{U}$ does $X - D$-guess clubs (so may omit D for $X = S, X = P$) <u>when</u>:

- $\mathbf{U}$ does D-fully guess clubs and $X = F$

[1] but then "fully D-guess clubs" implies "almost guess clubs", see 2.2

- $\mathbf{U}$ almost D-guess clubs and $X = A$
- $\mathbf{U}$ semi-guess clubs and $X = S$
- $\mathbf{U}$ medium D-guess clubs and $X = M$
- $\mathbf{U}$ pseudo guess clubs and $X = P$.

5) Let $\text{XGC}_D(\lambda, \theta) = \min\{|\mathbf{U}| : \mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $\mathbf{U}$ does $X - D$-guess clubs$\}$ so if $X = S, P$ we may omit D.
5A) Similalry $\text{XGC}_D(\lambda, \theta, \mathfrak{B})$ when we restrict ourselves to $\mathbf{U}$ obeying $\mathfrak{B}$.
6) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ is bounded when there is an F witnessing it which means: F is a function from $\{\bar{u}\restriction\zeta : \bar{u} \in \mathbf{U}, \zeta < \theta\}$ into λ such that $F(\bar{u}_1\restriction\zeta_1) = F(\bar{u}_2\restriction\zeta_2) \Rightarrow \bar{u}_1\restriction\zeta_1 = \bar{u}_2\restriction\zeta_2$ and $F(\bar{u}\restriction(\zeta+1) < \sup(u_{\zeta+1})$.
7) We say "strongly bounded" when in addition $F(\bar{u}\restriction(\zeta+1)) \in u_{\zeta+1}$ for every $\zeta < \theta$.
8) We say $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ is weakly bounded, when there is a function F witnessing it which means:

(a) $\text{Dom}(F) = \{\text{ord}(\bar{u})\restriction\zeta : \bar{u} \in \mathbf{U}$ and $\zeta < \theta$ is a successor ordinal$\}$ where $\text{ord}(\bar{u}) = \langle \sup(u_\varepsilon) : \varepsilon < \theta \rangle$
(b) $\text{Rang}(F) \subseteq \lambda$ and $F(\text{ord}(\bar{u})\restriction(\zeta+1) < \sup(u_{\zeta+1})$ for $\bar{u} \in \mathbf{U}$ and $\zeta < \theta$
(c) if $\zeta_1, \zeta_2 < \theta$ are successor of successor ordinals and $\bar{u}_1, \bar{u}_2 \in \mathbf{U}$ and $F(\text{ord}(\bar{u}_1)\restriction\zeta_1) = F(\text{ord}(\bar{u}_2)\restriction\zeta_2)$ then $\text{ord}(\bar{u}_1)\restriction\zeta_1 = \text{ord}(\bar{u}_2)\restriction\zeta_2$.

9) Notation:

(a) if $\bar{u} \in \mathbf{U}_{\lambda,\theta}$ and $f : \theta \to \theta$ is $\leq$-increasing continuous with limit θ then $\bar{u}^{[f]} = \bar{u}[f] := \langle u_{f(\varepsilon)} : \varepsilon < \theta \rangle]$
(b) if $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $f : \theta \to \theta$ is $\leq$-increasing continuous with limit θ then $\mathbf{U}^{[f]} := \mathbf{U}[f] = \{\bar{u}^{[f]} : \bar{u} \in \mathbf{U}\}$
(c) if $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $\mathscr{F}$ is a set of $\leq$-increasing continuous function from θ into θ with limit θ then $\mathbf{U}[\mathscr{F}] = \{\bar{u}[f] : \bar{u} \in \mathscr{F}, f \in \mathscr{F}\}$
(d) if $w \in {}^\theta\theta$ then $f_w = f[w]$ is the $g : \theta \to \theta$ such that $g(\varepsilon) = \min\{\zeta : \text{otp}(w \cap \zeta) = \varepsilon\}$, so is $\leq$-increasing continuous with limit θ.

10) In (a),(b) of part (9) above we may write $\bar{u}[w], \mathbf{U}[w]$ for $w \in [\theta]^\theta$ means $\bar{u}[f], \mathbf{U}[f]$ where $f = f_w$, writing $\mathbf{U}[W], W \subseteq [\theta]^\theta$ mean $\cup\{\mathbf{U}[w] : w \in W\}$.
11) Now for $X \in \{F, A, S, M, P\}$ we let (naturally we may omit D if $X = S, P$, and we can add $\mathfrak{B}$ as in part (5A)):

(a) $\text{BXGC}_D(\lambda, \theta) = \text{Min}\{|\mathbf{U}| : \mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ does $X - D$-guess clubs and is strongly bounded$\}$
(b) $\text{CXGC}_D(\lambda, \theta)$ is defined as in (a) but $\mathbf{U}$ is just bounded
(c) $\text{WXGC}_D(\lambda, \theta)$ is defined as in clause (a) but $\mathbf{U}$ is weakly bounded.

Some of the obvious impliations are:

Observation 2.2. *1) If $\mathbf{U}$ is (λ, θ)-reasonable and fully D-guess clubs, then $\mathbf{U}$ almost D-guess clubs.*
2) If $\mathbf{U}$ almost D-guess clubs then $\mathbf{U}$ semi-guess-club and medium D-guess clubs.
3) If $\mathbf{U}$ semi-guess-clubs or medium D-guess clubs for D as above, then $\mathbf{U}$ does pseudo guess clubs.

4) If $D_1 \subseteq D_2 \subseteq [\theta]^\theta$ then "$\mathscr{U}$ does D_1-guess clubs" implies $\mathbf{U}$ does x D_2-guess clubs for $x \in \{\text{full}, \text{almost}, \text{medium}\}$.
6) Assume $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ and $\mathfrak{B}$ is as in 2.2. Then there is $\mathbf{U}'$ obeying $\mathfrak{B}$ such that

- *(a)* $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$
- *(b)* $\mathbf{U}, \mathbf{U}'$ *have the same cardinality*
- *(c) if $\mathbf{U}$ does x D-guess clubs, for x as in part (4). then so does $\mathbf{U}'$.*

Definition 2.3. 1) For the model theory: for a model $M \in \mathrm{EC}_T(\lambda!), \Delta \subseteq \mathbb{L}(\tau_T)$ and $u \subseteq \lambda, A \subseteq M$ let $M^{[A]}{\restriction}_\Delta u$ be the model $M{\restriction}u$ expanded by all the restriction to u of all relations definable by a Δ-formula with parameters from A.
1A) If $\Delta = \mathbb{L}_{\mathrm{qf}}(\tau_M)$ then we may omit Δ; writing $\bar{a}$ instead A means $\mathrm{Rang}(\bar{a})$.
2) For $M \in \mathrm{EC}_T(\lambda!), \bar{u} \in \mathbf{U}_{\lambda,\theta}$ and $\bar{a} \in {}^{\omega>}M$ let $g_{\bar{a},\bar{u},M}$ be the function from θ to θ such that for $\zeta < \theta, g_{\bar{a},\bar{u},M}(\zeta)$ is the minimal $\varepsilon \in (\zeta,\theta)$ such that $M^{[\bar{a}]}{\restriction}u_\varepsilon \prec M^{[\bar{a}]}{\restriction}\bigcup\limits_{\xi} u_\xi$.

Claim 2.4. *We assume that $\mathfrak{B}$ is a model with universe λ and countable vocabulary.*
1) We have

- *(A) If $\lambda = \mathrm{cf}(\lambda) = \theta^{++}$ and $\theta = \mathrm{cf}(\theta) > \aleph_0$, then $\mathrm{CSGC}(\lambda,\theta) = \lambda$, moreover $\lambda = \mathrm{CSGC}(\lambda,\theta,\mathfrak{B})$*
- *(B) in cluase (A) we have $\mathrm{BSGC}(\lambda,\theta) = \lambda$ provided[2] that*
  - *there is a stationary $S \subseteq S^{\theta^+}_\theta$ from $\check{I}_\theta[\theta^+]$*
- *(C) if $\lambda = \lambda^\theta$ and $\theta = \mathrm{cf}(\theta)$ then $\mathrm{BFGC}(\lambda,\theta)$ even with a reasonable witness.*

2) For regular $\lambda > \theta = \mathrm{cf}(\theta) > \aleph_0$ we have $\mathrm{PGC}(\lambda,\theta) = \lambda$ provided that:

$\boxplus^2_{\lambda,\theta}$ $\mathbf{U}_\theta(\lambda) = \lambda$*, that is, there is $\mathscr{S}$ such that:*

- *(a) $\mathscr{S} \subseteq [\lambda]^\theta$ has cardinality λ*
- *(b) if $A \in [\lambda]^\theta$ then for some $w \in \mathscr{S}$ we have $|A \cap w| = \theta$.*

3) For $\lambda > \theta = \mathrm{cf}(\theta) > \aleph_0$ we have $\mathrm{SGC}(\lambda,\theta) = \lambda$ provided that (e.g. $\lambda = \theta^{+n}$ for some n)

$\boxplus^3_{\lambda,\theta}$ $\mathrm{cf}([\lambda]^\theta, \subseteq) = \lambda$.

4) If $\mathbf{U}_1 \subseteq \mathbf{U}_{\lambda,\theta}$ pseudo guess clubs, then there is $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ which pseudo guess clubs of cardinality $\leq |\mathbf{U}_1|$ and

- *(c) if $u = \cup\{u_i : i < \theta\}$ and $\bar{u} \in \mathbf{U}_{\lambda,\theta}$ then $u \subseteq \delta = \sup(u)$ for some $\delta < \lambda$ of cofinality θ*
- *(d) if $\mathfrak{b}_\theta \leq \lambda$ then $u = \cup\{u_i : i < \theta\}$ and $\bar{u} \in \mathbf{U}_{\lambda,\theta}$ then $\mathrm{otp}(u) = \theta$*
- *(e) if $\mathfrak{d}_\theta \leq \lambda$ then $\mathrm{SGC}(\lambda,\theta) = \lambda$*
- *(f) if $\mathfrak{b}_\theta \leq \lambda$ and $D = \{S \subseteq \theta : S \text{ stationary}\}$ then $\mathrm{SGC}_D(\lambda,\theta) = \theta$.*

5) If $\lambda \geq \theta^+$ and $\theta = \mathrm{cf}(\theta) > \aleph_0$ and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is stationary and some $\bar{C} = \langle C_\delta : \delta \in S\rangle$ guess clubs, then $\mathrm{PGC}(\lambda,\theta) = \lambda$.

[2]We can waken the demand: if we weaken the demand in Definition 2.1(5) to "for stationary many $\varepsilon < \theta$" and $\theta \geq \aleph_2$.

Discussion 2.5. 1) In 2.4 we have ZFCresults, we shall get stronger results (on the full and almost versions) in some forcing extensions in 2.13.
2) We can look at the cases of Definition 2.1 for singular λ, replacing $(u_\zeta \backslash \sup(u_\varepsilon)) \cap A \neq P$ by $u_3 \backslash u_\varepsilon$. This works for $\mathrm{FGC}_D, \mathrm{AFC}_D, \mathrm{SGC}_D$.
3) When we have clause (a)(γ) of the Definition 2.1(3) there is no need of clause $(a)(\alpha)$. In this case we do not need "λ regular".

Proof. Without loss of generality$B$ has a pairing function $\mathrm{pr}^{\mathfrak{B}}$ 1) Clause (A): First, choose $S, S^*, \bar{C}$ such that (partial square guessing clubs):

$(*)_1$ (a) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta$ and $\delta > \theta^+\}$ is stationary
(b) $S \subseteq S^+ \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \leq \theta$ and $\delta > \theta^+\}$, moreover if $\delta \in S$ then $\delta = \sup(S^+ \cap \delta)$
(c) $\bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle$
(d) C_α is a closed subset of α of order type $\leq \theta$, (so $\mathrm{otp}(C_\alpha)$ is a limit ordinal iff $\alpha = \sup(C_\alpha))$
(e) for $\alpha \in S^+$ we have $\alpha \in S \Leftrightarrow \mathrm{otp}(C_\alpha) = \theta$
(f) if $\alpha \in C_\beta$ then $C_\alpha = C_\beta \cap \alpha$
(g) $\bar{C} \restriction S$ guess clubs, i.e.: if E is a club of λ then for stationarily many $\delta \in S$ we have $C_\delta \subseteq E$
(h) if $\alpha \in S^+$ then α is closed under $\mathfrak{B}$, that is $\mathfrak{B} \restriction \alpha \subseteq \mathfrak{B}$

[Why do they exist (provably in ZFC)? see [Shea, 1.3=L1.3(b)], but we elaborate; by [She91, 4.4,pg.47] there are $S, S^+, \bar{C}$ satisfying $(*)_1$ except possibly clauses clauses (g) and (h) which we can add as follows: choose $\langle C_\alpha : \alpha \in S^+ \rangle$ satisfying the rest. For any club E of λ define the sequence $\bar{C}[E] = \langle C_\alpha[E] : \alpha \in S^+_E \rangle$ as follows, first let $S^+_E = S^+ \cap E$ and $S_E = \{\delta \in S : \delta \in E, \delta = \sup(\delta \cap E)\}$ second for $\alpha \in S^+_E$ let $C_\alpha[[E] = C_\alpha \cap E$. Now $S_E, S^+_E, \bar{C}[E]$ satisfies clauses (a)-(f). next if $E_{\mathfrak{B}} = \{\delta < \lambda : \mathfrak{B} \restriction \delta \subseteq \mathfrak{B}\}$ is a club of λ and for any club $E \subseteq E_{\mathfrak{B}}$ clearly $\bar{C}[E]$ satisfies clause (h). Lastly for satisfying clause (g), just we shall try θ^+ times as in [She94, Ch.III,2.3(2)].

We choose $E_{<\varepsilon}, E^1_\varepsilon, E^2_\varepsilon$ by induction on $\varepsilon \leq \theta^+$ such that: $E_{<\varepsilon} = \{\delta siE_*$: if $\zeta < \varepsilon$ then $\delta \in E^2_\zeta\}$ and E^1_ε is a club of λ which if possible witness that $S_{E_{<\varepsilon}} \bar{C}[E_{<\varepsilon}] := \langle C_\delta \cap E_{<\varepsilon} : \delta \in S$ is equal to $\sup(C_\delta \cap E_\varepsilon)\rangle$ fails clause (g), that is, $\delta \in S \cap E_{<\varepsilon} \wedge \delta = \sup(C_\delta \cap E_{<\varepsilon}) \Rightarrow C_\delta \cap E_{<\varepsilon} \nsubseteq E^1_\varepsilon$. Lastly E^2_ε is a club of λ disjoint, if possible, to $\{\delta \in S \cap E_{<\varepsilon}, \delta = \sup(C_\delta \cap E_{<\varepsilon})$ but $C_\delta \cap S_{<\varepsilon} \subseteq E^1_\varepsilon$ If for some $\varepsilon < \theta^+, \bar{C}[E_{<\varepsilon}]$ satisfies clause (g) then we are done. Otherwise as S is stationary and $\theta = \mathrm{cf}(\theta) > \aleph_0$, clearly for some $\delta \in S$ we have $\delta = \sup(E_{<\theta^+} \cap \delta)$, hence $\langle C_\delta \cap E_{<\varepsilon} : \varepsilon < \theta^+ \rangle$ is a strictly decreasing sequence of sets but $|C_\delta| = \theta$, contradiction, so $(*)_1$ holds indeed].

$(*)_2$ For $\delta \in S$ let $\langle \gamma^\bullet_{\delta,\varepsilon} : \varepsilon < \theta \rangle$ list C_δ in increasing order.

Second, fix $\bar{f}, \bar{g}$ such that:

$(*)_3$ (a) $\bar{f} \equiv \langle f_\alpha : \alpha \in [\theta^+, \lambda) \rangle$
(b) f_α is a one-to-one function from θ^+ onto α
(c) $\bar{g} = \langle g_\xi : \xi \in [\theta, \theta^+) \rangle$
(d) g_ξ is a one-to-one function from θ onto ξ.

Third

$(*)_4$ (a) for $\delta \in S$ let $e_\delta = \{\xi < \theta^+$: if $\alpha \in C_\delta$ then $\mathrm{Rang}(f_\alpha \restriction \xi) = \alpha \cap \mathrm{Rang}(f_\delta \restriction \xi)$ and this set include $C_\delta \cap \alpha$ and has cardinality $\theta\}$

(b) e_δ is a club of θ^+.

[Why clause (b) holds? As $\mathrm{otp}(C_\delta) = \theta$ and $\alpha \in C_\alpha \cup \{\delta\} \Rightarrow |\alpha| = \theta^+$, this should be clear.]

$(*)_5$ for $\delta \in S$ and $\xi \in e_\delta$ let:

(a) $u_{\delta,\xi} = \mathrm{Rang}(f_\delta \restriction \xi) \in [\delta]^\theta$ and include C_δ

(b) we choose $\bar{u}_{\delta,\xi} = \langle u_{\delta,\xi,\varepsilon} : \varepsilon < \theta\rangle$ by $u_{\delta,\xi,\varepsilon} = c\ell_{\mathfrak{B}}(\{f_{\gamma^\bullet_{\delta,\zeta(1)}}(g_\xi(\zeta)) : \zeta(1) < \omega(1+\varepsilon)$ and $\zeta < \omega(1+\varepsilon)\} \cup \{\gamma^\bullet_{\delta,\zeta(1)} : \zeta(1) < \omega(1+\varepsilon)\})$

(c) for $w \in [\theta]^\theta$ let $\bar{u}^{[w]}_{\delta,\xi}$ be $\langle u^{[w]}_{\delta,\xi,\varepsilon} ; \varepsilon < \theta\rangle$ where $u^{[w]} = \cup\{u_{\delta,\xi,\iota} : \iota \in w$ and $\mathrm{otp}(w \cap \iota) < 1+\varepsilon\} : \iota < \theta\rangle$, this fit 2.1(9)(d).

Note that (recalling $(*)_2$)

$(*)_6$ For $\delta \in S, \xi \in e_\delta$ we have

(a) $\bar{u}_{\delta,\xi}$ is a $\subseteq$-increasing continuous sequence of subsets of $u_{\delta,\varepsilon}$

(b) each $u_{\delta,\xi,\varepsilon}$ include $C_{\gamma^\bullet_{\delta,\omega(1+\varepsilon)}}$ hence is an unbounded subset of $\gamma^\bullet_{\delta,\omega(1+\varepsilon)}$ and it is of cardinality $< \theta$

(c) $\cup\{u_{\delta,\xi,\varepsilon} : \varepsilon < \theta\}$ is equal to $u_{\delta,\varepsilon}$

(d) $u_{\delta,\xi,\varepsilon}$ is computable from $\mathrm{pr}^{\mathfrak{B}}(\gamma^\bullet_{\delta,\varepsilon}, \xi)$ recalling that $\mathrm{pr}^{\mathfrak{B}}$ is a pairing function.

[Why? should be clear]

Lastly,

$(*)_7$ let $\mathbf{U}_w = \{\bar{u}^{[w]}_{\delta,\xi} : \delta \in S$ and $\xi \in e_\delta\}$ for $w \in [\theta]^\theta$.

We shall prove that (why the w ? for the use in the proof of part (4) of the claim):

$(*)_8$ if $w \in [\theta]^\theta$ then $\mathbf{U}_w$ witness $\mathrm{WSGC}(\lambda, \theta) \leq \lambda$.

Fix w now and we shall deal with all the demands:

$(*)_{8.1}$ $\mathbf{U}_w$ has cardinality $\leq \lambda$; in fact is equal to λ.

[Why? As $|\mathbf{U}_w| \leq |\{(\delta,\xi) : \delta \in S, \xi \in e_\delta \subseteq \theta\}| \leq \lambda + \theta = \lambda$. The other inequality is also easy as $\cup\{u_{\delta,\xi} : \delta \in S, \xi \in e_\delta\} = \lambda$ and each $u_{\delta\xi}$ has cardinality $\theta < \lambda$.]

$(*)_{8.2}$ $\mathbf{U}_w \subseteq \mathbf{U}_{\lambda,\theta}$ is reasonable.

[Why? By the choices above.]

$(*)_{8.3}$ $\mathbf{U}_w$ semi-guess clubs.

[Why? Let M and A be as in Definition 2.1(3A)(a), let M^+ be the expansion of M by the relation $<^{M^+}$, the order of the ordinals $< \lambda$ and $P^{M+} = A$, and let $E := \{\delta < \lambda : M^+ \restriction \delta \prec M^+\}$, clearly E is a club of λ. By the choice of $\bar{C}$ there is $\delta \in S$ such that $C_\delta \subseteq E$ (hence $\delta \in S$). Note that if $\alpha \in C_\delta$ then $A \cap \alpha$ is unbounded in α.

Now recall $\langle u_{\delta,\xi} : \xi \in e_\delta\rangle$ is $\subseteq$-increasing continuous with union δ, each $u_{\delta,\xi}$ is of cardinality $\leq \theta$ and e_δ is a club of θ^+ hence $e = \{\xi \in e_\delta : M^+ \restriction u_{\delta,\xi} \prec M^+\}$ is a club of

θ^+ So if $\xi \in e_\delta$ then $A \cap u_{\delta,\xi}$ is unbounded in $u_{\delta,\xi}$. Now choose $\xi \in e$, so $\bar{u} = \bar{u}_{\delta,\xi}$ is as required.]

$(*)_{8.4}$ **U** is weakly bounded.

[Why? Just think, recalling $(*)_1$ and Definition 2.1(7), that is, note that $\langle C_\delta \cap \alpha : \delta \in S^+\rangle$ has cardinality $\leq \theta^+$ for each $\alpha < \lambda$, below we shall get more.]

$(*)_9$ **U** is bounded hence CSGC(λ,θ) holds, in fact:
- (a) if $u_1 = u_{\delta_1,\xi_1,\varepsilon_1}, u_2 = u_{\delta_2,\xi_2,\varepsilon_2}$ and $\text{pr}(\gamma^\bullet_{\delta_1,\varepsilon_1},\xi_1) = \text{pr}(\gamma^\bullet_{\delta_2,\varepsilon_2},\xi_2)$ then:
  - (α) $\langle \gamma^\bullet_{\delta_1,\varepsilon} : \varepsilon \leq \varepsilon_1\rangle = \langle \gamma^\bullet_{\delta_2,\varepsilon} : \varepsilon le \varepsilon_2\rangle$
  - (β) $u_1 = u_2$
- (b) $\text{pr}(\gamma^\bullet_{\delta,\varepsilon},\xi) < \gamma^\bullet_{\delta,\varepsilon+1}$.

[Why? Clause (a) holds by $(*)_6(d)$ and clause (b) by $(*)_1(h)$.]

We have finished proving $\lambda = \text{CSGC}(\lambda,\theta)$, and even CSGC$(\lambda,\theta,\mathfrak{B})$, that is clause (A) of part (1).

Clause (B): Choose $\zeta_{\xi,\varepsilon}$ for $\xi \in S^{\theta^+}_\theta$, i.e. $\xi < \theta^+$ of cofinality θ and $\varepsilon < \theta$ such that for any such $\xi \in S^{\theta^+}_\theta, \langle \zeta_{\xi,\varepsilon} : \varepsilon < \theta\rangle$ is increasing continuous with limit ξ. Now in the proof of part (1) we can restrict ourselves to $\xi \in S^{\theta^+}_\theta$ such that $u_{\delta,\xi}$ is closed under pr. Then we can restrict ourselves to (ω,δ,ξ) such that $\varepsilon_1 < \varepsilon_2 \in w \Rightarrow \text{pr}(\gamma^\bullet_{\delta,\varepsilon_1},\zeta_{\xi,\varepsilon_1}) \in u_{\delta,\xi,\varepsilon_2}$.

Clause (C): Easy.

2) By the assumption there is $\mathscr{S}$ such that:

$(*)_1$ (a) $\mathscr{S} \subseteq [\lambda]^\theta$ has cardinality λ
(b) if $A \in [\lambda]^\theta$ then for some $w \in \mathscr{S}$ we have $|A \cap w| = \theta$.

Without loss of generality

$(*)_2$ if $w \in \mathscr{S}$ then $w \subseteq \delta = \sup(w)$ for some δ of cofinality θ.

[Why? Let $\mathscr{S}' = \{w \cap \delta : \delta < \lambda$ has cofinality θ and $\delta = \sup(w \cap \delta)\}$, clearly it is as required.]

Now for every $w \in \mathscr{S}$, we choose $\bar{u} = \bar{u}_w$ such that:

- $\bar{u} = \langle u_\varepsilon : \varepsilon < \theta\rangle$
- $u_\varepsilon \in [w]^{<\theta}$ is increasing continuous with ε
- $w = \cup\{u_\varepsilon : \varepsilon < \theta\}$
- u_ε has no last element, $\langle \sup(u_\varepsilon) : \varepsilon < \theta\rangle$ is increasing continuous.

Let $\mathbf{U} = \{\bar{u}_w : w \in \mathscr{S}\}$, it is as required. Obviously $\mathbf{U} \subseteq [\lambda]^\theta$ has the right cardinality; so let M_* be as in $\boxplus^3(b)$.

3) By [She96] there is $\mathscr{S}$:

$(*)_1$ (a) $\mathscr{S} \subseteq [\lambda]^\theta$ has cardinality λ
(b) $\mathscr{S}$ is stationary, i.e. for every model M_* with universe λ and vocabulary $\leq \theta$ there is $w \in \mathscr{S}$ such that $M_* \restriction w \prec M_*$.

Now as we can increase $\mathscr{S}$, without loss of generality :

$(*)_2$ $\mathscr{S} \cap [\alpha]^\theta$ is a stationary subset of $[\alpha]^\theta$ for every $\alpha \leq \lambda$.

Also without loss of generality

$(*)_3$ if $u \in \mathscr{S}$ then $u \subseteq \delta = \sup(u)$ for some δ of cofinality θ.

[Why? Let $\mathscr{S}' = \{u \cap \delta : \delta < \lambda$ has cofinality θ and $\delta = \sup(u \cap \delta)\}$.

Why is $\mathscr{S}'$ as required? Clearly $\mathscr{S}^* \subseteq [\lambda]^\theta$ has cardinality λ. Given M_* as in $\boxplus^3(b)$, let $E = \{\delta < \lambda : M{\restriction}\delta \prec M\}$, a club of λ, now we can find $\delta \in E$ of cofinality θ, and applying $(*)_2$ we are done proving $(*)_3$.]

For $w \in \mathscr{S}$ choose $\bar{u} = \bar{u}_w = \langle u_\varepsilon : \varepsilon < \theta\rangle$ as in the proof of part (2). Now $\{\bar{u}_w : w \in \mathscr{S}\}$ is as required.

4) First without loss of generality $w \in \mathbf{U} \Rightarrow \mathrm{otp}(w) < \theta^\omega$ (ordinal exponentiation; by the Dashnik-Miller paradox). Second, without loss of generality $w \in \mathbf{U} \Rightarrow \mathrm{otp}(w) \in \{\theta^n : n < \omega\}$ by ordinal additivity theorem; this suffices for clause (a). Third, by "$\mathfrak{b}_\theta \le \lambda$" without loss of generality $\mathrm{otp}(w) = \theta$; this suffices for clause (b), clauses (c),(d) are easy, too (and not useful here).

5) As in part (1). $\square_{2.4}$

Discussion 2.6. Assume $\lambda > \theta \ge \sigma = \mathrm{cf}(\sigma), (2^\sigma > \lambda$ in the interesting case. Let $\mathscr{U}_{\lambda,\theta,\sigma} = \{\bar{u} : \bar{u} = \langle u_\varepsilon : \varepsilon < \sigma\rangle$ is $\subseteq$-increasing and $u_\varepsilon \in [\lambda]^\theta\}$ and repeat the definition. Of doubtful help, otherwise $(\theta^{++}, \theta^+, \theta)$ would have helped.

Theorem 2.7. *1) Assume* $\lambda = \mathrm{cf}(\lambda) \ge \theta = \mathrm{cf}(\theta) > \aleph_0, D = [\theta^\theta], \mathrm{AGC}_D(\lambda, \theta) = \lambda$ *and* $\mathfrak{b}_\theta > \lambda$. *Then* $\lambda \notin \mathrm{Univ}(T_{\mathrm{ceq}})$*; moreover,* $\mathrm{univ}_{T_{\mathrm{ceq}}}(\lambda) \ge \mathfrak{b}_\theta$.
2) If $\lambda = \mathrm{cf}(\lambda) > \theta = \mathrm{cf}(\theta) > \aleph_0$ *and* $\mathrm{AGC}(\lambda, \theta) = \lambda$ *and* $\mathfrak{d}_\theta > \lambda$, *then* $\mathrm{univ}_{T_{\mathrm{ceq}}}(\lambda) \ge \mathfrak{d}_\theta$.
3) If D is a uniform filter on $\theta, ({}^\theta\theta, <_D)$ *is* $(< \chi)$*-directed and* $\chi > \lambda, \mathrm{AGC}_D(\lambda, \theta) = \lambda$ *then* $\mathrm{univ}_{T_{\mathrm{ceq}}}(\lambda) \ge \chi$.
4) If $\lambda = \mathrm{cf}(\lambda) \ge \theta = \aleph_0$ *and* $\mathfrak{b} = \mathfrak{b}_{\aleph_0}$ *is* $> \lambda$. *then* $\mathrm{univ}(\lambda, T_{\mathrm{ceq}}) \ge \mathfrak{b}_\theta$

Remark 2.8. 2) The Claim 2.11 below shows that we cannot weaken the assumption on T too much.

Proof. So let ($T = T_{\mathrm{ceq}}$ and)

$(*)_1$ assume $\alpha_* < \mathfrak{b}_\theta$ and $M^*_\alpha \in \mathrm{EC}_T(\lambda!)$ for $\alpha < \alpha_*$; it suffices to find $N \in \mathrm{EC}_{T_{\mathrm{ceq}}}(\lambda!)$ not embeddable into M^*_α for every $\alpha < \alpha_*$

$(*)_2$ let $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ witness $\mathrm{PGC}(\lambda, \theta) = \lambda$

$(*)_3$ for $\bar{u} \in \mathbf{U}, \alpha < \alpha_*$ and $d \in P^{M^*_\alpha}$ the set $E_{\bar{u},d,\alpha}$ clearly is a club of θ where: $E_{\bar{u},d,\alpha} = \{\varepsilon < \theta : \varepsilon$ is a limit ordinal such that u_ε is closed inside $\cup\{u_\zeta : \zeta < \theta\}$ under the functions of M^*_α and the function $F^{M^*_\alpha}(-, d)$, i.e. if $a \in u_\varepsilon, b \in \bigcup_\zeta u_\zeta$ and $M^*_\alpha \models$ "$F(a, d) = b$" then $b \in u_\varepsilon\}$.

So $\mathscr{E} = \{E_{\bar{u},\alpha,a} : \bar{u} \in \mathbf{U}, \alpha < \alpha_*$ and $a \in P^{M^*_\alpha}\}$ is a set of clubs of θ of cardinality $\le |\mathbf{U}| + |\alpha_*| + |P^{M_*}| < \mathfrak{b}^\bullet_\theta$. Hence there is an increasing function $g : \theta \to \theta$ such that $(\forall E \in \mathscr{E})(\forall^\infty \varepsilon < \theta)(g(\varepsilon) > \mathrm{suc}_E(\varepsilon))$.

Now we can construct $N = N_g \in \mathrm{EC}_T(\lambda!)$ such that:

$(*)_4$ (a) $P^N = \{3\beta : \beta < \lambda\}$ hence $Q^N = \{3\beta + 1, 3\beta + 2 : \beta < \lambda\}$
(b) if $\alpha = 3\beta + 1 < \lambda$ (hence $\alpha \in Q^N$) then $\alpha = \min(\alpha/E^N)$
(c) if $\bar{u} \in \mathbf{U}$ then for some $\alpha(\bar{u}) = \alpha_{\bar{u},g} \in P^N$ we have: if $\beta \in (u_{\varepsilon+1} \backslash u_\varepsilon) \cap Q^N$ and $(\beta/E^N) \cap u_\varepsilon = \emptyset$ but $|(\beta/E^N) \cap \bigcup_\zeta u_\zeta)| = \theta$ then $F^N_{\alpha(\bar{u})}(\beta) \in \bigcup_\zeta u_\zeta \backslash u_{g(\varepsilon)}$.

Now toward contradiction assume that:

$(*)_5$ f embeds N_g into M^*_α and $\alpha < \alpha_*$.

Let M_* be a model with universe λ expanding M^*_α and (a renaming of) N_g and $f = G^N$ for some unary function symbol $G \in \tau(M^*_\alpha)$ and $<^{M_*} = \{(\alpha,\beta) : \alpha < \beta < \lambda\}$ and M_* has Skolem functions and $\tau(M_*)$ is countable.

By the choice of $\mathbf{U}$ there is $\bar{u}$ such that:

$(*)_6$ (a) $\bar{u} \in \mathbf{U}$
(b) $M_* \restriction \bigcup_\varepsilon u_\varepsilon \prec M_*$
(c) $c\ell(u_\varepsilon, M_*) \subseteq \sup(u_\varepsilon)$
(d) the set $\{\varepsilon < \theta : A \cap u_{\varepsilon+1} \setminus \sup(u_\varepsilon) \neq \emptyset\}$ has cardinality θ where $A = \{\alpha \in Q^M : \alpha \text{ minimal in } (\alpha/E^N) \cap \bigcup_\varepsilon u_\varepsilon\}$.

The rest should be clear.
2),3),4) Similarly. $\square_{2.7}$

Question 2.9. 1) Assume T (is countable complete first order) with the PT_2. If $\lambda > \theta > \aleph_0$ are regular, $\mathfrak{d}_\theta > \lambda$ and $\mathfrak{d}_\kappa > \mathrm{FGC}(\lambda,\theta)$ (maybe θ inaccessible), <u>then</u> $\mathrm{univ}_T(\lambda) \geq \mathfrak{d}_\kappa$.
2) Assume T (is countable complete first order) non-simple. If $\lambda > \theta > \aleph_0$ are regular, $\mathfrak{d}_\theta > \lambda$ and $\mathfrak{d}_\kappa > \mathrm{CFGC}(\lambda,\theta)$, <u>then</u> $\mathrm{univ}_T(\lambda) \geq \mathfrak{d}_\kappa$.

Remark 2.10. See [S$^+$e], [S$^+$d].

Claim 2.11. *Assume $\mu = \mu^{<\mu} \leq \theta = \mathrm{cf}(\theta) < \lambda = \mathrm{cf}(\lambda) < \chi = \chi^\lambda$ = (e.g. $\theta = \mu^+$ or $\theta = \mu$) and for transparency* GCH *holds in the interval $[\mu,\chi)$. For some $\mathbb{P}$:*

(a) $\mathbb{P}$ is a $(<\mu)$-complete forcing of cardinality χ neither collapsing any cardinal, in $\mathbf{V}^{\mathbb{P}}$ nor changing cofinalities
(b) $(2^\mu)^{\mathbf{V}[\mathbb{P}]} = \chi$
(c) $\mathfrak{b}_\theta = \chi$
(d) $\mathrm{FGC}(\lambda,\theta) = \lambda$ even with a reasonable witness hence $\mathrm{AGC}(\lambda,\theta) = \lambda$
(e) the results of [She90] *holds, i.e. there is a universal random graph in λ, and see* [Shear]

Proof. As in [She90], only in the preliminary forcing we also increase $\mathfrak{b}_\theta$, e.g. taking its product with $\mathbb{P}_{\chi^+}$ where $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \chi^+, \beta < \chi^+\rangle$ is a $(<\theta)$-support iteration, but for unboundedly many $\beta < \chi^+$ $\underset{\sim}{\mathbb{Q}}_\beta$ is the θ-dominating = θ-Hechler forcing and $\underset{\sim}{\eta}_\beta \in {}^\theta\theta$ dominating $({}^\theta\theta)^{\mathbf{V}(\mathbb{P}_\beta)}$ but see more in [S$^+$b]. Clause (c) holds by 2.13 below. $\square_{2.11}$

Question 2.12. 1) Can we for theories T satisfying $\mathrm{NSOP}_1 + \mathrm{PT}_2$ get similar results?
2) Is T_{ceq} in some sense minimal non-simple in a suitable family of theories?

Claim 2.13. *Assume $\lambda > \theta = \mathrm{cf}(\theta) > \aleph_0$ and $\lambda = \lambda^\theta$ and $\mathbb{P}$ is a θ-c.c. forcing notion*
1) In $\mathbf{V}^{\mathbb{P}}$ there is a reasonable strongly bounding $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ of cardinality λ witnessing $yl = \mathrm{FGC}(\lambda,\theta,\mathfrak{B})$
2) Assume $\mathbf{U} \subseteq \mathbf{U}_{\lambda,\theta}$ fully/almost guess clubs <u>then</u> in $\mathbf{V}^{\mathbb{P}}, \mathbf{U}$ still fully/almost (resp) guess clubs.
3) In part (2), if $\mathbf{U}$ is reasonable/bounding/strongly bounding/weakly bounding in $\mathbf{V}$, <u>then</u> so it is in $\mathbf{V}^{\mathbb{P}}$.

Remark 2.14. 1) This will help in consistency results, see [S$^+$b].
2) Similarly for the other versions of guessing clubs from 2.1, but take care of what is $\mathscr{D}$.

Proof. Part (1) follwos by parts (2),(3) because in **V** there is such **U** by 2.4(1)(C). The point is:

- $(*)$ $(A) \Rightarrow (B)$ where:
  - (A) if $\chi > \lambda$ and $\{\underset{\sim}{\mathfrak{B}}, \mathbb{P}, \lambda\} \cup \{\varepsilon : \varepsilon \leq \theta\} \subseteq N \prec (\mathscr{H}(\chi), \in)$ and $\|N\| < \theta$ where $\Vdash_{\mathbb{P}}$ "$\mathfrak{B}$ a model with universe λ and vocabulary of cardinality $< \theta$"
  - (B) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{N} \cap \lambda = c\ell\{|N|, \underset{\sim}{\mathfrak{B}}\}$ and $\underset{\sim}{\mathfrak{B}} \restriction |N|$ is an elementary submodel of $\underset{\sim}{\mathfrak{B}}$".

$\square_{2.13}$

REFERENCES

[CR16] Artem Chernikov and Nicholas Ramsey, *On model-theoretic tree properties*, J. Math. Log. **16** (2016), no. 2, 1650009, 41. MR 3580894

[CS95] James Cummings and Saharon Shelah, *Cardinal invariants above the continuum*, Ann. Pure Appl. Logic **75** (1995), no. 3, 251–268, arXiv: math/9509228. MR 1355135

[DS04a] Mirna Džamonja and Saharon Shelah, *On the existence of universal models*, Arch. Math. Logic **43** (2004), no. 7, 901–936, arXiv: math/9805149. MR 2096141

[DS04b] ______, *On $\triangleleft^*$-maximality*, Ann. Pure Appl. Logic **125** (2004), no. 1-3, 119–158, arXiv: math/0009087. MR 2033421

[DS06] ______, *On properties of theories which preclude the existence of universal models*, Ann. Pure Appl. Logic **139** (2006), no. 1-3, 280–302, arXiv: math/0009078. MR 2206258

[KR17] Alex Kruckman and Nicholas Ramsey, *Generic expansion and skolemization in $NSOP_1$ theories*.

[Mir05] Džamonja Mirna, *Club guessing and the universal models*, Notre Dame J. Formal Logic **46** (2005), 283–300.

[S+a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F1905.

[S+b] ______, *Tba*, In preparation. Preliminary number: Sh:F1840.

[S+c] ______, *Tba*, In preparation. Preliminary number: Sh:F1630.

[S+d] ______, *Tba*, In preparation. Preliminary number: Sh:F1960.

[S+e] ______, *Tba*, In preparation. Preliminary number: Sh:F1750.

[Shea] Saharon Shelah, *Analytical Guide and Corrections to [Sh:g]*, arXiv: math/9906022 Correction of [Sh:g].

[Sheb] ______, *Dependent dreams: recounting types*, arXiv: 1202.5795.

[Shec] ______, *Universals between strong limit singular and its power*.

[She90] ______, *Universal graphs without instances of CH: revisited*, Israel J. Math. **70** (1990), no. 1, 69–81. MR 1057268

[She91] ______, *Reflecting stationary sets and successors of singular cardinals*, Arch. Math. Logic **31** (1991), no. 1, 25–53. MR 1126352

[She93] ______, *The universality spectrum: consistency for more classes*, Combinatorics, Paul Erdős is eighty, Bolyai Soc. Math. Stud., vol. 1, János Bolyai Math. Soc., Budapest, 1993, arXiv: math/9412229, pp. 403–420. MR 1249724

[She94] ______, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912

[She96] ______, *Further cardinal arithmetic*, Israel J. Math. **95** (1996), 61–114, arXiv: math/9610226. MR 1418289

[She17] ______, *Universal structures*, Notre Dame J. Form. Log. **58** (2017), no. 2, 159–177, arXiv: math/0405159. MR 3634974

[Shear] ______, *Divide and Conquer: Dividing lines on universality*, Theoria (to appear).

[SU08] Saharon Shelah and Alexander Usvyatsov, *More on SOP_1 and SOP_2*, Ann. Pure Appl. Logic **155** (2008), no. 1, 16–31, arXiv: math/0404178. MR 2454629

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