SUPERSTABLE THEORIES AND REPRESENTATION
1043

SAHARON SHELAH

Abstract. In this paper we give an additional characterizations of the first order complete superstable theories, in terms of an external property called representation. In the sense of the representation property, the mentioned class of first-order theories can be regarded as "not very complicated". This was done for "stable" and for "$\aleph_0$-stable." Here we give a complete answer for "superstable".

Date: 2020-03-15.

2010 *Mathematics Subject Classification.* Primary:03C45; Secondary:03C55.

Key words and phrases. model theory, classification theory, stability, representation, superstable.

The author thanks Alice Leonhardt for the beautiful typing. First typed June 3, 2013.

§ 0. Introduction

Our motivation to investigate the properties under consideration in this paper comes from the following

Thesis: It is very interesting to find dividing lines and it is a fruitful approach in investigating quite general classes of models. A "natural" dividing property "should" have equivalent internal, syntactical, and external properties. (see [Shea] and lately [Shear], [Bal88] for more)

Of course, we expect the natural dividing lines will have many equivalent definitions by internal and external properties.

The class of stable (complete first order theories) T is well known (see [She90]), it has many equivalent definitions by "internal, syntactical" properties, such as the order property. As for external properties, one may say "for every $\lambda \geq |T|$ for some model M of T we have $\mathbf{S}(M)$ has cardinality $> \lambda$" is such a property (characterizing instability). Anyhow, the property "not having many κ-resplendent models (or equivalently, having at most one in each cardinality)" is certainly such an external property (see [Sheb]).

Here we deal with another external property, *representability.* This notion was a try to formalize the intuition that "the class of models of a stable first order theory is not much more complicated than the class of models $M = (A, \dots, E_t, \dots)_{s \in I}$ where E_t^M is an equivalence relation on A refining E_s^M for $s < t$; and I is a linear order of cardinality $\leq |T|$. It was first defined in Cohen-Shelah [CS16], where it was shown that one may characterize stability and $\aleph_0$-stability by means of representability. In this paper we give a complete answer also for the superstable case. If T is uncountable we may consider other values of $\kappa(T)$. That is, recall that for a stable (complete first order) theory T, $\kappa(T)$ can be any cardinal in the interval $[\aleph_0, |T|^+)$. So if T is countable there are two possible values- $\aleph_0, \aleph_1$, the second is dealt with in [CS16] and the first in Theorem 2.1. But if T is uncountable, the result above gives a representation in a class which depends just on $|T|$, so it is natural to suspect that if $\kappa(T) < |T|^+$ we can restrict this class further. We hope to consider this later.

The results are phrased below, and the full definition appears in Definition 1.2, but first consider a simplified version. We say that a a model M is $\mathfrak{k}$-representable for a class $\mathfrak{k}$ when there exists a structure $\mathbf{I} \in \mathfrak{k}$ with the universe extending M such that for any n and two sequences of length n from M, if they realize the same quantifier free type in $\mathbf{I}$ then they realize the same (first order) type in M. Of course, T is $\mathfrak{k}$-representable if every model of T is $\mathfrak{k}$-representable. We prove, e.g. that T is superstable iff for some κ, it is representable in the class of locally finite structures with exactly κ unary functions (and nothing else), see Definition 1.7.

This raises various further questions

Problem:

(1) Can we characterize, by representability "T is strongly dependent ", similarly for the various relatives (see [Sh:863])
(2) For a natural number n , what is the class of T representable by $\mathfrak{k}_\kappa^n$ of structures with just κ n-place functions (or relations)
(3) What about strong representability (meaning we demand in addition that
(∗) if $a, b \in \mathbf{I}$ realise the same qf-type in $\mathbf{I}$ then $a \in M \leftrightarrow b \in M$.

Concerning the last demand, even for general stable T this fail for $\mathbf{I} = \mathscr{M}_{\mu,\kappa}(\mathbf{I}')$ for $\mathbf{I}' \in \mathfrak{k}^{\rm eq}$, but a relative called 'medium" we can but this is delayed

The main result presented in this paper is:

Characterization of superstable theories (Theorem 2.1):

In the attempt to extend the framework of representation it seemed natural, initially, to conjecture that if we consider representation over linear orders rather than over sets, we could find an analogous characterizations for dependent theories. However, such characterizations would imply strong theorems on existence of indiscernible sequences. In [KS14], some dependent theories were discovered for which it is provably "quite hard to find indiscernible subsequences", implying that this conjecture would fail in its original formulation. However in [She14] it was proved that such results hold for strongly dependent T.

The reader that would like to avoid the reference [CS16] can restrict Th 2.1 to the equivalence of clauses (1), (2), (5) there; why? (1) $\Rightarrow$ (2) holds by Th 2.4 by the definitions, (2) $\Rightarrow$ (5) is immediate, and lastly (5) $\Rightarrow$ (1) holds by Th 2.3(2). A related work of Halevi, Kaplan and the author on Taylor problem for stable graphs.

The author thanks Yatir Halevi for doing much to improve the presentation and the referees for good advice.

§ 1. Structure Classes and Representations

We recall some needed definitions and properties from [CS16].

Convention 1.1. (1) The vocabulary is a set of individual constants, (partial) function symbols and relation symbols (=predicates), each with the number of places (=arity) being finite except in the models $\mathcal{M}_{\mu,\kappa}$, so for a model M, the occurance number, $\mathrm{oc}(M)$ is $\aleph_0$. Individual constants may be considered as 0-place function symbols.

Let $\mathrm{arity}_\tau(P)$ be the arity of the predicate P in the vocabulary τ, similarly for function symbols. The occurrence number of the vocabulary τ that is $\mathrm{oc}(\tau)$, is the minimal cardinal θ such that every symbol P from τ has arity $< \theta$. We shall allow function symbols F to be interpreted in a model M as partial functions but then demand that $\mathrm{dom}(F^M)$ is P^M_F for some predicate $P_F \in \tau$.

(2) A structure $\mathbf{I} = \langle \tau, I, \models \rangle$ is a triple of vocabulary, universe (=domain) and the interpretation relation for the vocabulary: let $|\mathbf{I}| = I$, $\|\mathbf{I}\|$ the cardinality of $\mathbf{I}$ and $\tau_{\mathbf{I}} = \tau$; $\mathbf{I}$ is called a τ-structure.

(3) $\mathfrak{k}$ denotes a class of structures in a given vocabulary $\tau_{\mathfrak{k}}$, so $\mathbf{I} \in \mathfrak{k} \Rightarrow \mathbf{I}$ is a $\tau_{\mathfrak{k}}$-structure.

(4) $\mathbb{L}(\tau)$ is first order logic for the vocanulary τ, $\mathbb{L}_\tau = \mathbb{L}(\tau)$ is f.o. logic in the vocabulary τ; $\mathcal{L}^\tau_{\rm qf}$ denotes all the quantifier-free formulas with terms from $\tau_{\mathfrak{k}}$. That is, finite Boolean combinations of atomic formulas, where atomic formulas (for τ) have the form $P(\sigma_0, \ldots, \sigma_{n-1})$ or $\sigma_0 = \sigma_1$ for some n-ary predicate $P \in \tau$, $\sigma_0 \ldots$ are terms, i.e. they are in the closure of the set of variables by function (and partial function) symbols.

(5) If $\mathbf{I}$ a τ-structure, $\bar a = \langle a_i : i < \alpha \rangle \in {}^\alpha|\mathbf{I}|$, then

$$\mathrm{tp}_{\rm qf}(\bar a, B, \mathbf{I}) = \left\{ \varphi(\bar x, \bar b) : \varphi(\bar x, \bar y) \in \mathcal{L}^\tau_{\rm qf} : \mathbf{I} \models \varphi(\bar a, \bar b),\ \bar b \in {}^{\ell g(\bar y)} B \right\}$$

§ 1(A). **Defining representations.** We recall the definition of a representation.

Definition 1.2. Consider a model M.

(1) For a structure $\mathbf{J}$ and a function $f : M \to |\mathbf{J}|$ is called a representation of M in $\mathbf{J}$ if

$$\mathrm{tp}_{\mathrm{qf}}(f(\bar{a}), \emptyset, \mathbf{J}) = \mathrm{tp}_{\mathrm{qf}}(f(\bar{b}), \emptyset, \mathbf{J}) \Rightarrow \mathrm{tp}(\bar{a}, \emptyset, M) = \mathrm{tp}(\bar{b}, \emptyset, M)$$

for any two sequences $\bar{a}, \bar{b} \in {}^{<\omega}M$

(2) We say that M is represented in a class of models $\mathfrak{k}$ if there exists a $\mathbf{J} \in \mathfrak{k}$ such that M is represented in $\mathbf{J}$.

(3) For two classes of structures $\mathfrak{k}_0, \mathfrak{k}$ we say that $\mathfrak{k}_0$ is represented in $\mathfrak{k}$ if every $\mathbf{I} \in \mathfrak{k}_0$ is represented in $\mathfrak{k}$.

(4) We say that a first-order theory T is represented in $\mathfrak{k}$ if the elementary class $\mathrm{EC}(T)$ of models of T is represented in $\mathfrak{k}$.

Definition 1.3. $\mathfrak{k}^{\mathrm{eq}}$ denotes the class of structures of the vocabulary $\{=\}$, where eq stands for equality.

§ 1(B). **The free algebras $\mathcal{M}_{\mu,\kappa}$.**

Definition 1.4. Let $\mu \geq \kappa = \mathrm{cf}(\kappa)$. For a given structure $\mathbf{I}$, we define the structure $\mathcal{M} = \mathcal{M}_{\mu,\kappa}(\mathbf{I})$ as the structure whose vocabulary is $\tau_{\mathbf{I}} \cup \{F_{\alpha,\beta} : \alpha < \mu, \beta < \kappa\}$, with a β-ary function symbol $F_{\alpha,\beta}$ for all $\alpha < \mu, \beta < \kappa$. (the vocabulary of $\mathbf{I}$ includes a unary relation symbol I for the structure's universe, and we will assume $F_{\alpha,\beta} \notin \tau_{\mathbf{I}}$) and we have $P_\zeta \in \tau(\mathbf{I})$ is a predicate with arity ζ and $P_\zeta^{\mathscr{M}} = {}^\zeta I$ <u>when</u> ζ is the arity of some $P \in \tau_\zeta$ or $\zeta = 1$. The universe for this structure is[1]:

$$\mathcal{M}_{\mu,\kappa}(\mathbf{I}) = \bigcup_{\gamma \in \mathrm{Ord}} \mathcal{M}_{\mu,\kappa,\gamma}(\mathbf{I})$$

Where $\mathcal{M}_\zeta = \mathcal{M}_{\mu,\kappa,\zeta}(\mathbf{I})$ is defined as follows:

- $\mathcal{M}_0(\mathbf{I}) := |\mathbf{I}|$
- For limit ζ: $\mathcal{M}_\zeta(\mathbf{I}) = \bigcup_{\xi<\zeta} \mathcal{M}_\xi(\mathbf{I})$
- For $\zeta = \gamma + 1$

$$\mathcal{M}_\zeta = \mathcal{M}_\gamma \cup \left\{F_{\alpha,\beta}(\bar{b}) : \bar{b} \in {}^\beta\mathcal{M}_\gamma,\ \alpha < \mu,\ \beta < \kappa\right\}$$

Where $F_{\alpha,\beta}(\bar{b})$ is treated as a formal object. The symbols in $\tau_{\mathbf{I}}$ have the same interpretation as in $\mathbf{I}$. In particular, α-ary functions may be interpreted as $(\alpha + 1) - ary$ relations. The β-ary function $F_{\alpha,\beta}(\overline{x})$ is interpreted as the mapping $\overline{a} \mapsto F_{\alpha,\beta}(\overline{a})$ for all $\overline{a} \in {}^\beta|\mathcal{M}_{\mu,\kappa}(I)|$, where $F_{\alpha,\beta}(\overline{a})$ on the right side of the mapping is the formal object. If $\mu = \kappa = \aleph_0$ we may omit them.

Remark 1.1. It is shown in [CS16] that $\mathcal{M}_{\mu,\kappa}(S)$ is a set (though defined as a class).

§ 1(C). **Extensions of classes of structures.**

Discussion 1.5. For a class of structures $\mathfrak{k}$, we define several classes of structures that are based on $\mathfrak{k}$.

[1] This defines a set and not a proper class by remark 1.1.

Definition 1.6. Letting $\mu \geq \kappa$, $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ is the class of structures $\mathbf{I}^+$ which, for some $\mathbf{I} \in \mathfrak{k}$ satisfy $|\mathbf{I}^+| = |\mathbf{I}|$;$\tau_{\mathbf{I}^+} = \tau_{\mathbf{I}} \cup \{P_\alpha : \alpha < \mu\} \cup \{F_\beta : \beta < \kappa\}$ for new unary relation symbols P_α and new unary function symbols F_β; such that if $\mu > 0$ then $\left\langle P^{\mathbf{I}^+}_\alpha : \alpha < \mu \right\rangle$ is a partition of $|\mathbf{I}|$, and $\left\langle F^{\mathbf{I}^+}_\beta : \beta < \kappa \right\rangle$ are **partial** unary functions.

Definition 1.7. $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k})$ is the class of structures in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ for which the closure of every element under the new functions is finite; (lf stands for "locally finite").

Definition 1.8. $\mathrm{Ex}^1_{\mu,\kappa}(\mathfrak{k})$ is the class of structures in $\mathrm{Ex}^0_{\mu,\kappa}(\mathfrak{k})$ for which $F_\beta(P_\alpha) \subseteq P_{<\alpha} := \bigcup_{\gamma<\alpha} P_\gamma$ holds for every $\alpha < \mu, \beta < \kappa$.

Definition 1.9. 1) $\mathrm{Ex}^{1.5}_{\mu,\kappa}(\mathfrak{k})$ is the class of structures of the form $\mathbf{I}^+ = \mathcal{M}_{\mu,\kappa}(\mathbf{I})$, for some $\mathbf{I} \in \mathfrak{k}$ (cf. Definition 1.4).

2) Let $\mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k})$ is the class [2] of structures $\mathscr{M}^+ = \mathscr{M}^+_{\mu,\kappa}(\mathbf{I}) which means that$ $\mathbf{I} \in \mathfrak{k}$ and the model $\mathscr{M}^+$ is $\mathscr{M} = \mathscr{M}_{\mu,\kappa}(\mathbf{I})$ expanded by $F^{\mathscr{M}^+}_{\alpha,\beta,i}$ for $\alpha < \mu, \beta < \kappa, i < \beta$ where

$F^{\mathscr{M}^+}_{\alpha,i}(a)$ is: b_i if $\mathscr{M}_{\mu,\kappa}(\mathbf{I}) \models$ "$a = F_{\alpha,\beta,i}(\bar{b})$" for some sequence $\bar{b} = \langle b_j : j < \beta \rangle$; and if there is no such sequence then $F^{\mathscr{M}^+}(a) = a$.

3) $\mathrm{Ex}^{2.5}_{\mu,\kappa}(\mathfrak{k})$ is defined as in part (2) omitting the functions $F_{\alpha,\beta}$ so consisting of $\mathscr{M}^{2.5}_{\mu,\kappa}(\mathbf{I})$ for $\mathbf{I} \in \mathfrak{k}$.

Fact 1.10. 1) If $\mathbf{I} \in \mathfrak{k}^{\mathrm{eq}}$ has cardinality $\geq \kappa$ for transparency, $\mathscr{M} = \mathscr{M}_{\mu,\kappa}(\mathbf{I})$ and $\mathscr{M}^+ \in \mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ as above so expanding $\mathscr{M}$ and $\bar{c}, \bar{d}$ realize the same qf-type in $\mathscr{M}^+$ <u>then</u> there is a permutation π of $\mathbf{I}$ such that the automorphism $\check{\pi}$ of $\mathscr{M}^+$ which it induce maps $\bar{c}$ to $\bar{d}$; recall $\kappa = \mathrm{cf}(\kappa) \leq \mu$.

2) Above, any automorphism of $\mathscr{M}_{\mu,\kappa}(\mathbf{I})$ is also an automorphism of $\mathscr{M}^+_{\mu,\kappa}(\mathbf{I})$

3) If $\mathscr{M}^+ = \mathscr{M}^+_{\mu,\kappa}(\mathbf{I})$ and $\mathscr{N}^+ \in \mathrm{Ex}^{2.5}_{\mu,\kappa}$ is the reduct of $\mathscr{M}^+$ from 1.9(3) then $\mathscr{M}^+, \mathscr{N}^+$ have the same automorphisms

Proof. Easy. $\square_{1.10}$

Convention 1.11. $\mathrm{Ex}_{\mu,\kappa}$ will denote one of the above classes.

§ 2. Superstable theories

The main theorem is

Theorem 2.1. *For a first-order, complete theory T the following are equivalent:*

(1) *T is superstable.*
(2) *T is representable in* $\mathrm{Ex}^2_{2^{|T|},\aleph_0}(\mathfrak{t}^{\mathrm{eq}})$
(3) *T is representable in* $\mathrm{Ex}^1_{2^{|T|},2}(\mathfrak{t}^{\mathrm{eq}})$ *so using unary functions only*
(4) *T is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{2^{|T|},2}(\mathfrak{t}^{\mathrm{eq}})$
(5) *T is representable in* $\mathrm{Ex}^2_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinal μ*
(6) *T is representable in* $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ *for some cardinals μ, κ.*

[2]Probably should be use also in [CS16]; also we may omit the $F_{\alpha,\beta,}$-s.

Proof. $2 \Rightarrow 5, 4 \Rightarrow 6$ are immediate.

$2 \Rightarrow 3$ is direct from [CS16, 1.30]

$3 \Rightarrow 4$ direct from [CS16, 1.24]

$5 \Rightarrow 6$

This follows since $\mathrm{Ex}^2_{\mu,\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ is qf-representable in $\mathrm{Ex}^1_{\mu,2}(\mathfrak{k}^{\mathrm{eq}})$ by [CS16, 1.30] and $\mathrm{Ex}^1_{\mu,2}(\mathfrak{k}^{\mathrm{eq}}) \subseteq \mathrm{Ex}^{0,\mathrm{lf}}_{\mu,2}(\mathfrak{k}^{\mathrm{eq}})$ by [CS16, 1.24] with 2 here standing for κ there.

The rest follows from Theorem 2.4 below giving $1 \Rightarrow 2$ and Theorem 2.3 below giving $6 \Rightarrow 1$. Together we are done proving 2.1. $\square_{2.1}$

Remark 2.2. It may be notationaly better to use $\mathfrak{k}^{=}$ instead $\mathfrak{k}^{\mathrm{eq}}$ but the later was used in [CS16].

Theorem 2.3. *1) If T is representable in $\mathrm{Ex}^{0,\mathrm{lf}}_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ for some cardinals μ, κ <u>then</u> T is superstable; moreover stable in every cardinal $\lambda \geq \mu$, recalling $\mu \geq \kappa$.*

2) If T is representable in $\mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ and $\kappa = \aleph_0$ <u>then</u> T is superstable; moreover is stable in every cardinal λ which satisfies $\lambda \geq \mu$.

Proof. Similar to the proof of Propositions [CS16, Th.2.4,2.5] but we shall elaborate on part (2).

Choose λ such that $\lambda \geq \mu$ and let M be a model of T of cardinality $\chi > \lambda$, e.g. $\chi = \lambda^+$ and $A \subseteq M$ be a set of cardinality λ . We shall prove that the set $\{\mathrm{tp}(c, A, M) : c \in M\}$ is of cardinality at most λ; this suffice.

By our assumption,

$(*)_1$ there are a structure $\mathscr{M}^\bullet$ and representation $f : M \to \mathscr{M}^\bullet$ where
- (a) for part (1), $\mathscr{M}^\bullet \in \mathrm{Ex}^{1,\mathrm{lf}}_{\mu,\kappa}(\mathbf{I})$ where $\mathbf{I} \in \mathfrak{k}^{\mathrm{eq}}$
- (b) for part (2), $\mathscr{M}^\bullet \in \mathrm{Ex}^2_{\mu,\kappa}(\mathfrak{k}^{\mathrm{eq}})$ so $\mathscr{M}^\bullet$ is the expansion $\mathscr{M}^+_{\mu,\kappa}(\mathbf{I})$ of $\mathscr{M}_{\mu,\kappa}(\mathbf{I})$ described in 1.9(2) above where $\mathbf{I} \in \mathfrak{k}^{\mathrm{eq}}$

For $a \in M$ let J_a be the closure of $\{f(a)\}$ in $\mathbf{I}$ so is a finite set in both cases and let $\langle s_{a,\ell} : \ell < n_a = n(a)\rangle$ list J_a with no repetitions. and so we can choose a term $\sigma = \sigma_a$ such that $\sigma = \sigma((x_0, \dots, x_{n(a)-1})$ belongs to $\mathbb{L}(\tau(\mathscr{M}^\bullet)$ and $\mathscr{M}^\bullet \models f(a) = \sigma(a_{s(a,0)} \dots, a_{s(a,n(a)-1)})$.

Next let $J = \cup\{J_c : c \in A\}$ so J is a subset of $\mathbf{I}$ of cardinality at most λ,

We now define an equivalence relation E on M as follows:

$(*)_2$ cEd iff the following hold:
- (a) $c, d \in M$,
- (b) $n_c = n_d$
- (c) for $\ell < n_c$ we have $s_{c,\ell} \in J$ iff $s_{d,\ell} \in J$ and if they holds then $s_{c,\ell} = s_{d,\ell}$
- (d) the sequences $\langle s_{c,\ell} : \ell < n_c\rangle$ and $\langle s_{d,\ell} : \ell < n_c\rangle$ realize the same qf-free type in $\mathbf{I}^\bullet$

Note that clause (d) actually follows from the earlier clauses because $\langle s_{b,\ell} : \ell < n_b\rangle$ is with no repetitions for every $b \in M$. Clearly:

$(*)_3$ we have
- (a) indeed E is an equivalence relation on M
- (b) the equivalence relation E has at most λ equivalence classes.

[Why? For the first clause, just read the definition of E. For the second clause, i.e. clause (b) first there are at most $\mu + \aleph_0 = \mu$ triples of the form (n_a, σ_a, u_a) for

$a \in M$ where $u_a = \{\ell < n_a : s_{a,\ell} \in J\}$. Second there are at most λ sequences of the form $\langle s_{a,\ell} : \ell \in u_a \rangle$ for $a \in M$.

Lastly

$(*)_4$ if $c, d \in M$ are E-equivalence then there is a automorphism of $\mathscr{M}^\bullet$ mapping $f(c)$ to $f(d)$ and being the identity on $\{f(a)a \in M\}$.

[Why? Let π_0 be the function with domain $J \cup \{s_{c,\ell} : \ell < n_a\}$ which is the identity on J and maps $s_{c,\ell}$ to $s_{d,\ell}$ for $\ell < n_c$. By our present assumptions it is one to one, with domain and range included in $\mathbf{I}$ and of cardinality $\leq \lambda < |\mathbf{I}|$. As J has cardinality $\leq \lambda < |\mathbf{I}|$, we can extend f_0 to a permutation π of $\mathbf{I}$. and let $\breve{\pi}$ be the automorphism of $\mathscr{M}^\bullet$ which π induce. Clearly it maps $f(c)$ to $f(d)$, so we are done proving clause (b)]

$(*)_5$ If cEd then

(a) the qf-types $\mathrm{tp}_{\mathrm{qf}}(f(c), J, \mathscr{M}^\bullet)$ and $\mathrm{tp}_{\mathrm{qf}}(f(d), J, \mathscr{M}^\bullet)$ are equal.

(b) the types $\mathrm{tp}(c, A, M)$ and $\mathrm{tp}(d, A, M)$ are equal.

[Why? For clause (a), the elements $f(c), f(d)$ realize the same qf-free type over J in $\mathbf{I}^\bullet$ by $(*)_4$.

Then clause (b) follows by the assumption on f, being a representation, see $(*)_1$ and Def 1.2]

So clearly we are done proving part (2).]

$\square_{2.3}$

Theorem 2.4. *Every superstable T is representable in* $\mathrm{Ex}^2_{2^{|T|},\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$.

Proof. Let T be superstable and $\lambda = 2^{|T|}$. Let $M \prec \mathfrak{C}_T$. We shall choose $B_n, \langle a_s, u_s : s \in S_n \rangle$ by induction on $n < \omega$ such that:

$\circledast_0$ (a) $S_n \cap S_k = \emptyset$ (for $k < n$)

(b) $\{a_s : s \in S_n\} \subseteq M$

(c) $B_n = \{a_s : s \in S_{<n}\} \subseteq M$, where $S_{<n} := \cup\{S_k : k < n\}$, as usual

(d) $\langle a_s : s \in S_n \rangle$ is without repetitions, disjoint from $\{a_s : s \in S_{<n}\}$ and independent over B_n,

(e) for all $s \in S, u_s \subseteq S_{<n}$ is finite such that $t \in u_s \Rightarrow u_t \subseteq u_s$ and $\mathrm{tp}(a_s, B_n)$ does not fork over $\{a_t : t \in u_s\}$

(f) $\langle a_s : s \in S_n \rangle$ is maximal under conditions 1-5.

Here we make a convention that u, v, w vary on $\mathscr{I}$ defined below:

$\circledast_1$ Since T is superstable, it is possible to carry the induction.

$\circledast_2$ (a) let $\mathscr{I} = \{u : u \subseteq S, u \text{ finite}\}$, where $S = \bigcup_n S_n$

(b) for $v \in \mathscr{I}$ let $\mathrm{cl}(v)$ be the minimal $u \supseteq v$ such that $u_t \subseteq u$ holds for all $t \in u$;

(c) we define $\mathscr{I}^{\mathrm{cl}} = \{u \in \mathscr{I} : u = \mathrm{cl}(u)\}$;

(d) for $s \in S$ let $u_s^+ = \mathrm{cl}(u_s) \cup \{s\}$.

$\circledast_3$ (a) if $u \in \mathscr{I}$ then $u \subseteq \mathrm{cl}(u) \in \mathscr{I}$

(b) $v \subseteq u \Rightarrow \mathrm{cl}(v) \subseteq \mathrm{cl}(u)$;

(c) $\mathrm{cl}(u_1 \cup u_2) = \mathrm{cl}(u_1) \cup \mathrm{cl}(u_2)$;

(d) $\mathrm{cl}(\{s\}) = u_s^+ = u_s \cup \{s\} = \cup\{\mathrm{cl}(\{t\}) : t \in u_s\} \cup \{s\}$;

(e) $\mathrm{cl}(\mathrm{cl}(u)) = \mathrm{cl}(u)$;
(f) $\mathrm{cl}(u) = \bigcup\{u_s^+ : s \in u\} = \cup\{\mathrm{cl}(\{t\} : t \in u_s$ for some $s \in u\}$
(g) $u_s = \mathrm{cl}(u_s), u_s^+ = \mathrm{cl}(u_s^+)$

[Why? e.g. clause (g) by $\circledast_0(e)$]

$\circledast_4$ $|M| = \{a_s : s \in S\}$

[Why? Otherwise, there exists $a \in |M| \backslash \{a_s : s \in S\}$, now we can choose (since T is superstable) a finite $v \subseteq S$ such that $\mathrm{tp}(a, \{a_s : s \in S\})$ does not fork over $\{a_s : s \in v\}$. Let $u = \mathrm{cl}(v)$, so $u \in \mathscr{I}^{\mathrm{cl}}$ and let n be such that $u \subseteq S_n$ and we get a contradiction to the maximality of $\{a_s : s \in S_n\}$.]

$\circledast_5$ Let $\langle v_\alpha : \alpha < \alpha(*)\rangle$ enumerate $\mathscr{I}$ (without repetition) such that:
(a) $v_\alpha \subseteq v_\beta \Rightarrow \alpha \leq \beta$;
(b) $\alpha < \beta \wedge v_\beta \subseteq S_{<n} \Rightarrow v_\alpha \subseteq S_{<n}$.

We choose a model M_{v_α} and set $A_{v_\alpha} \subseteq M_{v_\alpha}$ by induction on α such that:

$\circledast_6$ (a) $M_{v_\alpha} \prec \mathfrak{C}_T$ has cardinality λ;
(b) $v_\beta \subseteq v_\alpha$ implies that $\beta \leq \alpha$ and $M_{v_\beta} \prec M_{v_\alpha}$;
(c) $A_{v_\alpha} = \bigcup\{M_{v_\beta} : \beta < \alpha \wedge v_\beta \subseteq v_\alpha\} \subseteq M_{v_\alpha}$;
(d) if $s \in v_\alpha$ and $u_s \subseteq v_\alpha$ then $a_s \in M_{v_\alpha}$;
(e) $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta} : \beta < \alpha\} \cup M)$ does not fork over $B_{v_\alpha} := \cup\{M_{v_\beta} : v_\beta \subseteq v_\alpha,\ \beta < \alpha\} \cup \{a_s : \mathrm{cl}(\{s\}) \subseteq v_\alpha\}$
(f) $(M_{v_\alpha}, c)_{c \in A_{v_\alpha}}$ is saturated, hence $A_{v_\alpha} \subsetneq M_{v_\alpha}$.

[Why can we carry the induction? Arriving to the ordinal β, first, as an approximation choose a model M'_{v_β} satisfying clauses (a),(b),(c),(d). Second we choose $M''_{v_y} \prec \mathfrak{C}_T$ of cardinality λ extending M'_{v_α} such that the model $(M''_{v_\alpha}, c)_{c \in A_{v_\alpha}})$ is saturated, possible because T is stable in λ. Third and lastly choose a $\mathfrak{C}_T$-elementary mapping f_{v_α} with domain M''_{v_α} which is the identity on $B_{v_y a}$ and $\mathrm{tp}(f_{v_\alpha}(M''_{v_\alpha}), \bigcup\{M_{v_\beta} : \beta < \alpha\} \cup M)$ does not fork over B_{v_α}. Clearly $M_{v_\alpha} = f_{v_\alpha}(M''_{v_\alpha})$ is as required.]

$\circledast_7$ (a) $\alpha < \beta \Rightarrow M_{v_\alpha} \neq M_{v_\beta}$.
(b) $v \subsetneq u \Rightarrow M_v \subsetneq M_u$
(c) $M \subseteq \bigcup\{M_u : u \in \mathscr{I}\}$
(d) the set $M_u \setminus A_u$ has cardinality λ

[Why? e.g. clause (a) holds by $\circledast_6$(c),(f)]

A major point is

$\circledast_8$ $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over $A_{v_\alpha} := \cup\{M_{v_\beta} : v_\beta \subsetneq v_\alpha\}$.

[Why? If $v_\alpha = \emptyset$ this is trivial so assume $v_\alpha \neq \emptyset$.

Let n be such that $v_\alpha \subseteq S_{\leq n}, v_\alpha \nsubseteq S_{<n}$ and

$\circledast_{8.1}$ let $\langle t_\ell : \ell < k\rangle = \langle t_\ell^\alpha : \ell < k_\alpha\rangle$ list $\{s \in v_\alpha : s \notin S_{<n}$ and $\mathrm{cl}(\{s\}) \subseteq v_\alpha\}$.

First, assume $k = 0$. So if $s \in v_\alpha$ and $\mathrm{cl}(\{s\}) \subseteq v_\alpha$ then $s \in v_\alpha \cap S_{<n}$, this implies that $u_s \cup \{s\} = \mathrm{cl}(\{s\}) \subseteq S_{<n}$, hence by $\circledast_6(d)$, $a_s \in M_{v_\alpha \cap S_{<n}} \subseteq A_{v_\alpha}$ because $v_\alpha \subsetneq$. This implies that $B_{v_\alpha} \subseteq A_{v_\alpha}$ (in fact equal - see their definitions in $\circledast_6(e), \circledast_8$ resp,). Now $\circledast_6(e)$ says that $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over B_{v_α}, so by monotonicity of non-forking and the last sentence, it does not fork over A_{v_α} as desired.

Second, assume $k = 1$ and $(\forall \beta < \alpha)\,(\mathrm{cl}(\{t_0\}) \nsubseteq v_\beta)$. Hence necessarily $v_\alpha = cl(\{t_0\})$ so $B_{v_\alpha} = A_{v_\alpha} \cup \{a_{t_0}\}$ hence by $\circledast_0$ the type $\mathrm{tp}(a_{t_0}, \{a_s : s \in S_{\le n} \backslash \{t_0\}\})$ does not fork over $\{a_s : s \in u_{t_0}\}$. Next note that for $\beta \le \alpha$ the type $\mathrm{tp}(a_{t_0}, \{a_s : s \in S_{\le n} \backslash \{t_0\}\} \cup \bigcup \{M_\gamma : \gamma < \beta\})$ does not fork over $\{a_s : s \in u_{t_0}\}$, this is proved by induction on β. But clearly $\{a_s : s \in u_{t_0}\} \subseteq A_{v_\alpha}$ hence clearly $\mathrm{tp}(a_{t_0}, \cup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over A_{v_α}, together with $\circledast_6(e)$ we get that $\mathrm{tp}(M_{v_\alpha}, \cup\{M_{v_\beta} : \beta < \alpha\})$ does not fork over A_{v_α}, as desired in $\circledast_8$.

Third, assume $k = 1, \beta < \alpha$ and $\mathrm{cl}(\{t_0\}) \subseteq v_\beta$. Without loss of generality β is minimal with these properties, so necessarily $v_\beta = \mathrm{cl}(\{t_0\})$ and so again, $B_{v_\alpha} = A_{v_\alpha}$ and we continue as in "First" above.

Fourth, assume $k \ge 2$. In this case, for each $\ell < k, \mathrm{cl}(\{t_\ell\})$ is $v_{\beta(\ell)}$ for some unique $\beta(\ell) < \alpha$, so $a_{t_\ell} \in M_{v_{\beta(\ell)}} \subseteq A_{v_\alpha}$, hence, $B_{v_\alpha} \subseteq A_{v_\alpha}$ (in fact equal) and again $\circledast_6(e)$ gives the desired conclusion.]

Together we have finished proving $\oplus_8$.

Now define

$\circledast_9$ (a) the sequence $\langle M_v : v \in \mathscr{I}\rangle$ is a stable system, of models (defined in [She90, Ch.XII, 2.1]holds by ,[She90, Ch.XII, 2.3(1)] and $\circledast_7(c) + \circledast_8$)
(b) let $M_\bullet = \bigcup\{M_u : u \in \mathscr{I}\}$, so $M_\bullet \prec \mathfrak{C}_T$ and it suffice to prove that this model has a representation as promised.

A major point is (but we need only clause (a) for which the saturation is not necessary)

$\boxplus_1$ for every $v \in \mathscr{I}$ and finite sequence $\bar b \in {}^{\omega>}(M_v \setminus A_v))$ we can find a finite sequence $\bar c_{\bar b} \in {}^{\omega>}(A_v)$ such that:

(a) $\mathrm{tp}(\bar b, A_v)$ has a unique extension to a complete types over $\bigcup\{M_u : v \subsetneq u, u \in \mathscr{I}\}$ which does not fork over A_v
(b) $\mathrm{tp}(\bar b, A_v)$ is stationary over $\bar c_{\bar b}$ which means that: $\mathrm{tp}(\bar b, A_v)$ does not fork over $\bar c_{\bar b}$
(c) $\mathrm{tp}(\bar b, A_v)$ is the unique extension of $\mathrm{tp}(\bar b, \bar c_{\bar b})$ in $\mathbf{S}^{\ell g(\bar b}(A_v)$ which do not fork over $\bar c_{\bar b}$
(d) moreover $\mathrm{tp}(\bar b, \bigcup\{M_u : u \in \mathscr{I}\})$ is stationary over $\bar c_{\bar b}$ that is: is the unique extension of $\mathrm{tp}(\bar b, \bar c_{\bar b})$ in $\mathbf{S}^{\ell g(\bar b)}(\bigcup\{M_u : u \in \mathscr{I}, v \subsetneq u\})$ which does not fork over $\bar c_{\bar b}$.

[Why? For clause (a) it follows from clauses (b),(c),(d); (alternatively it suffice to recall that for every sequence $\bar d$ from $\bigcup\{M_u : v \subsetneq v, v \in \mathscr{I}\}$ the types $\mathrm{tp}(\bar d, M_v)$ is finitely satisfiable in A_v, see [She90, Ch.XII, 2.5]). Clauses (b),(c) hold by [She90, Ch.XII,3.5,pag.608] recalling $\langle M_u : u \in \mathscr{I}\rangle$ is a stable system, $\kappa(T) = \aleph_0$ and each M_u is saturated. Clause (d) follows by the properties of stable systems of $\aleph_\varepsilon$-saturated models, see [She90, Ch.XII, 2.12].]

We let

$\boxplus_2$ (a) $<_S$ be a linear order of S
(b) $\mathscr{F} = \{f : f$ is a finite order preserving function from S to $S\}$

Now $\boxplus_3$ by induction on n we choose $\langle \bar b_u : u \in [S]^n\rangle$ and $\langle \pi_f : f \in \mathscr{F}\rangle$ such that:

(a) $\bar b_u = \langle b_{u,\alpha} : \alpha < \lambda\rangle$ list $M_v \setminus A_u$ without repetitions for $u \in \mathscr{I}$
(b) π_f is a $\mathfrak{C}_T$-elementary mapping for $f \in \mathscr{F}$
(c) $\mathrm{dom}(\pi_f) = \bigcup\{M_u : u \subseteq \mathrm{dom}(f)\} = \{b_{u,\alpha} : u \subseteq \mathrm{Dom}(f), \alpha < \lambda\}$
(d) if $f \in \mathscr{F}, v_1 \subseteq \mathrm{dom}(f), v_2 = f"(v_1)$ and $\alpha < \lambda$ then f maps $b_{v_1,\alpha}$ to $b_{v_2,\alpha}$

(e) (follows) if $f \in \mathscr{F}$ and $u \subseteq \mathrm{dom}(f)$ then $\pi_{f\restriction u} \subseteq \pi_u$
(f) (follows) if $g = f^{-1} \in \mathscr{F}$ then $\pi_g = (\pi_f)^{-1}$

[Why? by $\boxplus_0$, (alternatively by uniqueness claims on stable systems in [She90, Ch.XII, 2.5]).]

$\boxplus_4$ for every $u \in [S]^n$ the type $\mathrm{tp}(\bar{b}_u, \bigcup\{M_v : v \in S_{\leq n}, v \neq u\})$ does not fork over A_u and is the unique extension to a complete type over $\bigcup\{M_v : v \in S_{\leq n}, v \neq u\}$ extending $\mathrm{tp}(\bar{b}_u, A_u)$ not forking over A_u.

[Why? By $\boxplus_1$]

$\boxplus_5$ we choose a tuple $(I, \mathbf{I}, \mathscr{M}, \mathscr{M}^+, f)$ witnessing f is a representation of M as promised, by:

(a) $\mathbf{I} \in \mathfrak{k}^{\mathrm{eq}}$ has universe $I = S$
(b) $\mathscr{M} = \mathscr{M}_{\lambda,\aleph_0}(\mathbf{I})$
(c) f is a function with domain $M_\bullet = \bigcup\{M_u : u \in \mathscr{I}\}$
(d) for $n < \omega, s_0 <_S \cdots <_S s_{n-1}$ listing $u \in [S]^n$ and $\alpha < \lambda$ we let $f(b_{u,\alpha}) = F^{\mathscr{M}}_{\alpha,n}(s_0, \ldots, s_{n-1})$
(e) $\mathscr{M}^+$ is the expansion of $\mathscr{M}$ as in Def 1.9(2), in fact we need only the following (well, after renaming)
  (α) $P^{\mathscr{M}^+}_{\zeta,n} = \{F^{\mathscr{M}}_{\zeta,n}(s_0, \ldots, s_{n-1}) : s_0, \ldots, s_{n-1} \in S\}$
  (β) $F^{\mathscr{M}^+}_{\zeta,n,i}$ is the partial function mapping $F^{\mathscr{M}}_{\zeta,n}(s_0, \ldots, s_{n-1})$ to s_i for $i < n < \omega, \alpha < \lambda$

[Why are $(I, \mathbf{I}, \mathscr{M}, \mathscr{M}^+, f)$ as required? easy to check recalling $\boxplus_4$. Note that the range of the function f is not preserved under automorphisms of $\mathscr{M}$, but this is permissible] $\square_{2.4}$

Discussion 2.5. For superstable T, we may wonder about whether "the cardinal $2^{|T|}$ is optimal". Really, $\lambda(T)$ is sufficient where

$(*)_{1.1}$ $\lambda(T) = \min\{\lambda : T \text{ is stable in } \lambda\}$.

Recall that (see [She90])

$(*)_{1.2}$ If T is countable then $\lambda(T) = \aleph_0$ is equivalent to T is $\aleph_0$-stable and
$(*)_{1.3}$ if T is countable and $\lambda(T) > \aleph_0$ then $\lambda(T) = 2^{\aleph_0}$.

Theorem 2.6. *In Theorem 2.4,* $\mathrm{Ex}^2_{\lambda(T),\aleph_0}(\mathfrak{k}^{\mathrm{eq}})$ *suffice.*

Proof. We repeat the proof of Theorem 2.4 with minor changes. We just choose $\lambda = \lambda(T)$ instead $\lambda = 2^{|T|}$. $\square_{2.6}$

References

[Bal88] John Baldwin, *Fundamentals of stability theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1988.

[CS16] Moran Cohen and Saharon Shelah, *Stable theories and representation over sets*, MLQ Math. Log. Q. **62** (2016), no. 3, 140–154, arXiv: 0906.3050. MR 3509699

[KS14] Itay Kaplan and Saharon Shelah, *Examples in dependent theories*, J. Symb. Log. **79** (2014), no. 2, 585–619, arXiv: 1009.5420. MR 3224981

[Shea] Saharon Shelah, *Introduction and Annotated Contents*, arXiv: 0903.3428 introduction of [Sh:h].

[Sheb] ______, *On spectrum of κ-resplendent models*, arXiv: 1105.3774 Ch. V of [Sh:e].

[She90] ______, *Classification theory and the number of nonisomorphic models*, second ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990. MR 1083551

[She14] ______, *Strongly dependent theories*, Israel J. Math. **204** (2014), no. 1, 1–83, arXiv: math/0504197. MR 3273451

[Shear] ______, *Divide and Conquer: Dividing lines on universality*, Theoria (to appear).

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at