LF GROUPS, AEC AMALGAMATION, FEW AUTOMORPHISMS SH1098

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Abstract. We deal mainly with $\mathbf{K}^{\rm lf}_\lambda$, the class of locally finite groups of cardinality λ, in particular $\mathbf{K}^{\rm exlf}_\lambda$, the class of existentially closed locally finite groups. In §3 we prove that for almost every cardinal λ "every locally finite G of cardinality λ can be extended to an existentially closed complete group of cardinality λ which moreover is so called (λ, θ)-full; note that §3 which do not rely on §1,§2. (in earlier results G has cardinality $< \lambda$ and also λ was restricted).

In §1 we deal with amalgamation bases, for the class of lf (= locally finite) groups, and general suitable classes, we define when it has the (λ, κ)-amalgamation property which means that "many" models $M \in K^{\mathfrak{k}}_\lambda$ are amalgamation bases and get more than expected. In this case, we deal with a general frame - so called a.e.c., abstract elementary class. In §2 we deal with weak definability of $a \in N \backslash M$ over M, for = existentially closed lf group.

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The author thanks Alice Leonhardt for the beautiful typing. First typed February 18, 2016. This is paper number 1098 in the author list of publication. In References [She17, 0.22=Lz19] means [She17, 0.22] has label z19 there, L stands for label; so will help if [She17] will change. The reader should note that the version in my website is usually more updated than the one in the mathematical archive.

Annotated Content

[Consider an a.e.c. $\mathfrak{k}$, e.g. the class of locally finite groups, $\mathbf{K}_{\rm lf}$. We define $\mathrm{AM}_{\mathfrak{k}} = \{(\lambda,\kappa) : \lambda \ge \kappa = \mathrm{cf}(\kappa), \lambda \ge \mathrm{LST}_{\mathfrak{k}}$ and the κ-majority of $M \in K^{\mathfrak{k}}_\lambda$ are amalgamation bases$\}$, on "κ-majority" see below. What pairs have to be there? That is, for all a.e.c. $\mathfrak{k}$ with $\mathrm{LST}_{\mathfrak{k}} < \lambda$. One case is when $M \in K^{\mathfrak{k}}_\lambda$ is $(<\kappa)$-existentially closed and some $\sigma \in [\mathrm{LST}^+_{\mathfrak{k}}\kappa, \lambda]$ is a compact cardinal or just satisfies what is needed for M. This implies $(\lambda,\kappa) \in \mathrm{AM}_{\mathfrak{k}}$. A similar argument gives "κ weakly compact $> \mathrm{LST}_{\mathfrak{k}} \Rightarrow (\kappa,\kappa) \in \mathrm{AM}_{\mathfrak{k}}$". Those results are naturally expected but surprisingly there are considerably more cases: if λ is strong limit singular of cofinality κ and κ is a measurable cardinal $> \mathrm{LST}_{\mathfrak{k}}$ then $(\lambda,\kappa) \in \mathrm{AM}_{\mathfrak{k}}$. Moreover if also $\theta \in (\mathrm{LST}_{\mathfrak{k}}, \lambda]$ is a measurable cardinal then $(\lambda,\theta) \in \mathrm{AM}_{\mathfrak{k}}$.]

[For an a.e.c. $\mathfrak{k}$, we may say b_1 is $\mathfrak{k}$-definable in N over M when $M \le_{\mathfrak{k}} N, b_1 \in N\backslash M$ and for no N_*, b_1, b_2 do we have $M \le_{\mathfrak{k}} N_*, b_1 \ne b_2 \in N_*$ and $\mathrm{ortp}(b_\ell, N, N_*) = \mathrm{ortp}(b, M, N)$, equality of orbital types; there are other variants. We clarify the situation for $\mathbf{K}_{\rm lf}$.]

[Our aim is to find out when for $\mu \le \lambda$ (or even $\mu = \lambda$) every $G \in \mathbf{K}^{\rm lf}_\mu$ can be extended to a complete $H \in \mathbf{K}^{\rm exlf}_\lambda$, i.e. ones for which every automorphism is an inner automorphism. We demand that moreover (λ,σ)-full, a strong form of being existentially closed. We prove this for almost all λ's. A major new point is that we allow $\mu = \lambda$.]

§ 0. Introduction

§ 0(A). **Review.**

We deal mainly with the class $\mathbf{K}_{\rm lf}$ of locally finite groups so the reader may consider only this case ignoring the general frame. We continue [She17], see history there; in it we find many definable types for the class of locally finite groups parallel to the ones for stable theories; this will have central role here in the construction of complete existentially closed locally finite groups, in §3.
We wonder:

Question 0.1. 1) May there be a universal $G \in \mathbf{K}^{\rm lf}_\lambda$, e.g. for $\lambda = \aleph_1 < 2^{\aleph_0}$, i.e. consistently?
2) Is there a universal $G \in \mathbf{K}^{\rm lf}_\lambda$, e.g. for $\lambda = \beth_\omega$? Or just λ strong limit of cofinality $\aleph_0$ (which is not above a compact cardinal)?

On 0.1(2) see [Shec]. This leads to questions on the existence of amalgamation bases. We give general claims on existence of amalgamation bases in §1.
That is, we ask:

Question 0.2. For an a.e.c. $\mathfrak{k}$ or just a universal class (justified by §(0C)) we ask:
1) For $\lambda \ge {\rm LST}_{\mathfrak{k}}$, are the amalgamation bases (in $K^{\mathfrak{k}}_\lambda$) dense in $K^{\mathfrak{k}}_\lambda$? (Amalgamation basis under $\le_{\mathfrak{k}}$, of course, see 0.7, 1.6).
2) For $\lambda \ge {\rm LST}_{\mathfrak{k}}$ and $\kappa = {\rm cf}(\kappa)$ are the κ-majority of $M \in K^{\mathfrak{k}}_\lambda$ amalgamation bases? (On κ-majority, see 1.6(3A)). The set of such pairs (λ, κ) is called ${\rm AM}_{\mathfrak{k}}$.

Using versions of existentially closed models in $K^{\mathfrak{k}}_\lambda$, for λ weakly compact we get $(\lambda, \lambda) \in {\rm AM}_{\mathfrak{k}}$; also if $(\exists \sigma)[(\sigma$ a compact cardinal $) \wedge {\rm LST}_{\mathfrak{k}} < \sigma \le \kappa \le \lambda) \Rightarrow (\lambda, \kappa) \in {\rm AM}_{\mathfrak{k}}$, by [GS83]. But surprisingly there are other cases: (λ, κ) when λ is strong limit singular, with ${\rm cf}(\lambda) > {\rm LST}_{\mathfrak{k}}$ measurable and $\kappa = {\rm cf}(\lambda)$ or just $\lambda > \kappa > {\rm LST}_{\mathfrak{k}}$ and κ is measurable.
This is the content of §1.
In §2 we deal with the number of $a \in G_2$ definable over $G_1 \subseteq G_2$ in the orbital sense and find a ZFC bound for $\mathbf{K}_{\rm lf}$.
We consider in §3:

Question 0.3. For which pair (λ, μ) with $\lambda \ge \mu + \aleph_1$ or even cardinals $\lambda = \mu \ge \aleph_1$, does every $G \in \mathbf{K}^{\rm lf}_{\le \mu}$ have a complete extension in $\mathbf{K}^{\rm exlf}_\lambda$? That is, one for which every automorphism is an inner automorphism.

We prove that e.g.(to restrict relying on [Shee] in 3.9, we may restrict ourselves to cardinals λ which are successor of regular, still there are many such cardinals; also ignoring $\aleph_1$ is not a real lose):

Theorem 0.4. *If $\lambda \ge \beth_\omega \vee \lambda = \lambda^{\aleph_0}$ <u>then</u>every $G \in \mathbf{K}^{\rm lf}_{\le \lambda}$ can be extended to a complete existentially closed $H \in \mathbf{K}^{\rm lf}_{\le \lambda}$.*

The earlier results assume more than $\lambda > \mu$, i.e. $\lambda = \mu^+ \wedge \mu^{\aleph_0} = \mu$ or $(\lambda, \mu) = (\aleph_1, \aleph_0)$; see [She17] with history; earlier [Hic78], [Tho86]; [GS84], [SZ79].
Note that for $\mathbf{K}_{\rm lf}$, the statement is stronger when, fixing λ we increase μ (because every $G_1 \in \mathbf{K}^{\rm lf}_\mu$ has an extension in $\mathbf{K}^{\rm lf}_\lambda$ when $\lambda \ge \mu$). We shall deal in §3 with proving it for most pairs $\lambda \ge \mu + \aleph_1$, even when $\lambda = \mu$. Note that if $\lambda = \mu^+$ and

we construct a sequence $\langle G_i : i < \lambda\rangle$ of members from $\mathbf{K}^{\rm lf}_\mu$ increasing continuous, $G_0 = G$ with union of cardinality λ then any automorphism π of $H = \bigcup\{G_i : i < \lambda\}$ satisfies $\{\delta < \lambda : \pi$ maps G_δ onto $G_\delta\}$ is a club, this helps. But as we like to have $\lambda = \mu$ we can use only $\langle G_i : i < \theta\rangle$, with $\theta = {\rm cf}(\theta) \in [\aleph_1, \lambda)$, to be chosen appropriately. We still like to have, as above, "every $\pi \in {\rm aut}(H)$ maps G_i onto G_i for a club of $i < \theta$". Generally this fail. However, we have a substitute: if for unboundedly many $i < \theta, \theta$ the group G_i is θ-indecomposable (see Definition0.13) and $\theta = {\rm cf}(\theta) > \aleph_0$, then for any automorphism π of $G_\theta = \bigcup\{G_i : i < \theta\}$ the set $E = \{\delta < \theta : \pi(G_\delta) = G_\delta\}$ is a club of θ. On indecomposability, see Shelah-Thomas [ST97, §(3A)] phrased there as ${\rm CF}(G)$, the cofinality spectrum of G.

An additional point is that we like our H to be "more" than existentially closed, this is interpreted as being (λ, θ)-full. A central set theoretic point is that we also need to have a list of λ countable subsets which is dense enough, for this we use $\lambda = \lambda^{\aleph_0}$ or just $\lambda = \lambda^{\langle \theta;\aleph_0\rangle}$, see below, so the RGCH (from [She00]) is relevant. In earlier version of this paper [Shee], [Shec] were included.

§ 0(B). **Amalgamation Spectrum.** On a.e.c. see [Shea], [Shef], [Bal09]. We note below that the versions of the amalgamation spectrum are the same (fixing $\lambda \geq \kappa$) for:

(∗) (a) all a.e.c. $\mathfrak{k}$ with $\kappa = {\rm LST}_{\mathfrak{k}}, \lambda = \kappa + (\tau_{\mathfrak{k}})$;
(b) all universal $\mathbf{K}$ with $\kappa = \sup\{\|N\| : N \in \mathbf{K}$ is f.g.$\}, \lambda = \kappa + |\tau_{\mathfrak{k}}|$;

Why? Recall (universal classes are defined in 0.6).

The Representation Theorem 0.5. *Let $\lambda \geq \kappa \geq \aleph_0$.*
1) For every a.e.c. $\mathfrak{k}$ with $|\tau_{\mathfrak{k}}| \leq \lambda$ and ${\rm LST}_{\mathfrak{k}} \leq \kappa$ there is $\mathbf{K}$ such that:

(a) (α) *$\mathbf{K}$ is a universal class;*
(β) *$|\tau_{\mathbf{K}}| \leq \lambda, \tau_{\mathbf{K}} \supseteq \tau_{\mathfrak{k}}, |\tau_{\mathbf{K}} \backslash \tau_{\mathfrak{k}}| \leq \kappa$;*
(γ) *any f.g. member of $\mathbf{K}$ has cardinality $\leq \kappa$.*

(b) *$K_{\mathfrak{k}} = \{N{\restriction}\tau_{\mathfrak{k}} : N \in \mathbf{K}\}$, moreover:*

(b)$^+$ *if (α) and (β), then (γ), where:*
(α) *I is a well founded partial order such that $s_1, s_2 \in I$ has a mlb (= maximal lower bound) called $s_1 \cap s_2$;*
(β) *$\bar{M} = \langle M_s : s \in I\rangle$ satisfies $s \leq_I t \Rightarrow M_s \leq_{\mathfrak{k}} M_t$ and $M_{s_1} \cap M_{s_2} = M_{s_1 \cap s_2}$;*
(γ) *there is $\bar{N}$ such that:*
- *$\bar{N} = \langle N_s : s \in I\rangle$;*
- *$N_s \in \mathbf{K}$ expand M_s;*
- *$s \leq_I t \Rightarrow N_s \subseteq N_t$.*

(b)$^{++}$ *Moreover, in clause (b)$^+$, if $I_0 \subseteq I$ is downward closed and $\bar{N}^0 = \langle N^0_s : s \in I_0\rangle$ is as required in (b)$^+$ on $\bar{N}{\restriction}I_0$, then we can demand there that $\bar{N}{\restriction}I_0 = \bar{N}^0$.*

Proof. By [Shea]. $\square_{0.5}$

Definition 0.6. 1) We say $\mathbf{K}$ is a universal class <u>when</u>:

(a) for some vocabulary $\tau, \mathbf{K}$ is a class of τ-models;
(b) $\mathbf{K}$ is closed under isomorphisms;
(c) for a τ-model $M, M \in \mathbf{K}$ iff every finitely generated submodel of M belongs to $\mathbf{K}$.

Claim 0.7. *For $\mathfrak{k}, \mathbf{K}$ as in 0.5 and see Definition 1.6.*
1) If $N \in \mathbf{K}_{\lambda_0}, M = N{\restriction}\tau_{\mathfrak{k}}$, then: N is a (λ_1, λ_2)-amalgamation base in $\mathbf{K}$ iff M is a (λ_1, λ_2)-amalgamation base in $\mathfrak{k}$.
2) $\mathbf{K}$ has $(\lambda_0, \lambda_1, \lambda_2)$-amalgamation iff $\mathfrak{k}$ has $(\lambda_0, \lambda_1, \lambda_2)$-amalgamation.
3) $\mathrm{AM}_{\mathbf{K}} = \mathrm{AM}_{\mathfrak{k}}$ see Definition 1.6(5).

Observation 0.8. *If $\mathbf{K}$ is a universal class, $\kappa \geq \sup\{\|N\| : N \in \mathbf{K}$ is finitely generated$\}, \lambda \geq \kappa + |\tau_{\mathbf{K}}|$, then $\mathfrak{k} = (\mathbf{K}, \subseteq)$ and $\mathbf{K}$ are as in the conclusion of 0.5.*

§ 0(C). **Preliminaries on groups.**

Notation 0.9. 1) For a group G and subset A let $\mathrm{sb}_G(A) = \mathrm{sb}(A, G)$ be the subgroup of G generated by A.
2) Let $\mathbf{C}_G(A) := \{g \in G : ag = ga$ for every $a \in G\}$; this is the cetralizer of the set A inside the group G.

The following will be used in §(3).

Definition 0.10. Let $\lambda \geq \theta \geq \sigma$.
1) Let $\lambda^{[\theta;\sigma]} = \min\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^\sigma$ and for every $u \in [\lambda]^\theta$ we can find $\bar{u} = \langle u_i : i < i_* \rangle$ such that $i_* < \sigma, \cup\{u_i : i < i_*\} = u$ and $[u_i]^\sigma \subseteq \mathscr{P}\}$; if $\lambda = \lambda^\sigma$ then $\mathscr{P} = [\lambda]^\sigma$ witness $\lambda = \lambda^{[\theta;\sigma]}$ trivially.
2) Let $\lambda^{\langle\theta;\sigma\rangle} = \min\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^\sigma$ and for every $u \in [\lambda]^\theta$ there is $v \in [u]^\sigma$ which belongs to $\mathscr{P}\}$.
3) Let $\lambda^{(\theta;\sigma)} = \min\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^\sigma$ and for every $u \in [\lambda]^\theta$ there is $v \in \mathscr{P}$ such that $|v \cap u| = \sigma\}$.
4) For $\lambda \geq \mu \geq \theta \geq \nu$ let $\mathrm{cov}(\lambda, \mu, \theta, \sigma) = \min\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{<\mu}$ and every $u \in [\lambda]^{<\theta}$ is included in the union of $< \sigma$ members of $\mathscr{P}\}$.

Fact 0.11. 1) If $\mu = \beth_\omega$ or just $\mu > \aleph_0$ is strong limit, then for every $\lambda \geq \mu$, for every large enough $\theta < \mu$ we have $\sigma \leq \theta \Rightarrow \lambda^{[\theta;\sigma]} = \lambda$ (hence $\sigma \leq \theta \Rightarrow \lambda^{\langle\theta;\sigma\rangle} = \lambda^{(\theta;\sigma)} = \lambda$).
2) If $\mu^+ < \lambda$ and no cardinal in the interval (μ^+, λ) is a fix point then for some regular $\sigma \leq \theta \in (\mu, \lambda)$ we have $\lambda^{(\theta,\sigma)} = \lambda$.
3) If $\sigma \leq \theta \leq \lambda$ then$\lambda = \lambda^\theta \Rightarrow \lambda^{[\theta;\sigma]} = \lambda$ and $\lambda = \lambda^\sigma \Rightarrow \lambda^{(\theta,\sigma)} = \lambda$.
4) If $\theta \leq \lambda < \theta^{+\omega}$ then $\lambda^{(\theta;\theta)} = \lambda$.

Proof. By [She94], [She00], and see [She06] gives an alternative simpler proof. $\square_{0.11}$

Remark 0.12. As far as we know, possibly, e.g. $\lambda \geq \aleph_\omega \Rightarrow (\forall^\infty n)(\forall \ell > n)[\lambda^{(\aleph_n:\aleph_\ell)} = \lambda)$ and even $\lambda \geq \aleph_\omega \Rightarrow (\exists n)[\lambda = \mathrm{cov}(\lambda, \aleph_\omega, \aleph_\omega, \aleph_n)]$. See the works of Gitik on consistency results.

Definition 0.13. 1) We say M is θ-decomposable (called $\theta \in \mathrm{CF}(M)$ in [ST97]) when: θ is regular and if $\langle M_i : i < \theta \rangle$ is $\subseteq$-increasing with union M, then $M = M_i$ for some i.
2) We say M is Θ-indecomposable when it is θ-indecomposable for every $\theta \in \Theta$.
3) We say M is $(\neq \theta)$-indecomposable when: θ is regular and if $\sigma = \mathrm{cf}(\sigma) \neq \theta$ then M is σ-indecomposable.
4) We say $\mathbf{c} : [\lambda]^2 \to S$ is θ-indecomposable when: if $\langle u_i : i < \theta \rangle$ is $\subseteq$-increasing with union λ then $S = \{\mathbf{c}\{\alpha, \beta\} : \alpha \neq \beta \in u_i\}$ for some $i < \theta$; similarly for the other variants.
5) If we replace $\subseteq$ by $\leq_{\mathfrak{k}}$, $\mathfrak{k}$ an a.e.c., then we write $\mathrm{CF}_{\mathfrak{k}}(M)$ or "$\theta{-}\mathfrak{k}$-indecomposable".

Definition 0.14. We say G is θ-indecomposable inside G^+ when:

(a) $\theta = \mathrm{cf}(\theta)$;
(b) $G \subseteq G^+$;
(c) if $\langle G_i : i \leq \theta \rangle$ is $\subseteq$-increasing continuous and $G \subseteq G_\theta = G^+$ then for some $i < \theta$ we have $G \subseteq G_i$.

Claim 0.15. *1) Assume I is a linear order or just a set, and $\mathbf{c} : [I]^2 \to \mathscr{X}$ is θ-indecomposable, $G_1 \in \mathbf{K}_{\mathrm{lf}}$ and $a_i \in G_1 (i \in J$ are[1] pairwise commuting and each of order 2.*
Then there is G_2 such that:

(a) *$G_2 \in \mathbf{K}_{\mathrm{lf}}$ extends G_1;*
(b) *G_2 is generated by $G_1 \cup \bar{b}$ where $\bar{b} = \langle b_s : s \in I \rangle$;*
(c) *b_s commutes with G_1 and has order 2 for $s \in I$*
(d) *if $s_1 \neq s_2$ are from I then [2] $[b_{s_1}, b_{s_2}] = a_{\mathbf{c}\{s_1,s_2\}}$;*
(e) *G_2 is generated by $G_1 \cup \bar{b}$ freely except the equations implicit in clauses (a),(c),(d) above;*
(f) *$\mathrm{sb}(\{a_i : i \in \mathscr{X}\}, G_1)$ is θ-indecomposable inside G_2; see Definition 0.14, in fact it is θ-indecomposable even as semi-group.*

2) Assume $G_1 \in \mathbf{K}_{\mathrm{lf}}$ and I a linear order which is the disjoint union of $\langle I_\alpha : \alpha < \alpha_* \rangle, u_\alpha \subseteq \mathrm{Ord}$ has cardinality θ_α and $\mathbf{c}_\alpha : [I_\alpha]^2 \to J_\alpha \cup \{0\}$ is θ_α-indecomposable for $\alpha < \alpha_*, \langle J_\alpha : \alpha < \alpha_* \rangle$ is a sequence of sets with union J or $J \cup \{0\}$ and $0 \in Jsdsy \notin u$ and $a_\varepsilon \in G_1$ for $\varepsilon \in J$ and a_ε, a_ζ commute for $\varepsilon, \zeta \in J_\alpha, \alpha < \alpha_*$ and each a_ε has order 2 except for $\varepsilon = 0$, and we assume $a_0 = e$.
Let $\mathbf{c} : [I]^2 \to J$ extends each $\mathbf{c}_\alpha$ and is zero otherwise.
Then there is G_2 such that:

(a)-(e) *as above*
(f) *if $\alpha < \alpha_*$ then $\mathrm{sb}(\{a_{\alpha,\varepsilon} : \varepsilon < J_\alpha\}, G_2)$ is θ_α-indecomposable inside G_2.*

3) If in part (1) we omit the assumption "$\mathbf{c}$ is θ-indecomposable" (but retain $\mathbf{c} : [I]^2 \to \theta$) then still clauses (a)-(e) of part (1) holds.

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[1]The demand "the a_i's commute in G_1" is used in the proof of $(*)_8$, and the demand "a_{β_i} has order 2" is used in the proof of $(*)_7$.

[2]Mote that as $a \in \mathscr{X} \Rightarrow a = a^{-1}$ and $[b_{s_2}.b_{s_1}] = ([b_{s_1}.b_{s_2}])^{-1}$ the order between $s_1, s)2$ is irrelavant; if $a \in \mathscr{X}$ has a differnet order we would have to be more careful.

4) If $X_i \subseteq G_1 \subseteq G_2$ for $i < i_*$ and $\mathrm{sb}(X_i, G_1)$ is indecomposable in G_2 and $X = \bigcup\{X_i : i < i_*\}$ then$\mathrm{sb}(X, G_1)$ is indecomposable in G_2.

Proof. By [Shee, =Lb15]. $\square_{0.15}$

Claim 0.16. *If $G_1 \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ then for some $G_2 \in \mathbf{K}^{\mathrm{lf}}_{\lambda}$ extending G_1 and $a^\ell_\alpha \in G_2$ for $\ell \in \{1,2\}, \alpha < \lambda$ we have:*

$\oplus$ (a) *$\mathrm{sb}(\{a^\ell_\alpha : \ell \in \{1,2\}, \alpha < \lambda\}, G_2)$ includes G_1*
(b) *if $\ell \in \{1,2\}$ then $\langle a^\ell_\alpha : \alpha < \lambda\rangle$ is a sequence of pairwise distinct commuting elements of G_2 of order 2*
(c) *G_2 is generated by $\cup\{a^\ell_\alpha : \alpha < \lambda, \ell \in \{1,2\}\}$*
(d) *the elements $a^{\ell(1)}_{\alpha(1)}, a^{\ell(2)}_{\alpha(2)}$ commute when $\alpha(1) \neq \alpha(2)$.*

Proof. y [Shee, 1.6=Lb24] $\square_{\mathrm{B}}$

Definition 0.17. 1) Let $\mathbf{K}^{\mathrm{lf}}_{\lambda,\mu}$ be the class of pairs (G_1, G_1^+) such that:

(a) $G_1 \subseteq G_1^+ \in \mathbf{K}_{\mathrm{lf}}$;
(b) G_1, G_1^+ is of cardinality λ, μ respectively

2) Let $(G_1, G_1^+) \leq^{\mathrm{lf}}_{\lambda,\mu} (G_2, G_2^+)$ means:

(a) $(G_\ell, G_\ell^+) \in \mathbf{K}^{\mathrm{lf}}_{\lambda,\mu}$ for $\ell = 1, 2$
(b) $G_2 \subseteq G_2$
(c) $G_1^+ \subseteq G_2^+$.

3) We say $(G, G^+) \in \mathbf{K}^{\mathrm{lf}}_{\lambda,\mu}$ is Θ-indecomposable when Θ is a set of regular cardinals and for every $\theta \in \Theta, G$ is θ-indecomposable inside G^+.

§ 1. Amalgamation Bases

We try to see if there are amalgamation bases $(K^{\mathfrak{k}}_\lambda, \le_{\mathfrak{k}})$ and if they are dense in a strong sense: determine for which regular κ, the κ-majority of $M \in K^k_\lambda$ are amalgamation bases.

Another problem is $\mathrm{Lim}_{\mathfrak{k}} = \{(\lambda,\kappa)$: there is a medium limit model in $K^{\mathfrak{k}}_\lambda\}$, see [Shea]. This seems close to the existence of (λ,κ)-limit models, see [She15], [She11] and [She14]. In particular, can we get the following:

Question 1.1. If the set of $M \in \mathbf{K}_\lambda$, which are an amalgamation base, is dense in $(\mathbf{K}_\lambda, \subseteq)$, <u>then</u> in $(\mathbf{K}_\lambda,\subseteq)$ there is a $(\lambda,\aleph_0)$-limit model.

We shall return to this in §(3C).

Convention 1.2. 1) $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ is an a.e.c. but for simplicity we allow an empty model, which is $\le_{\mathfrak{k}}$ than anybody else.
2) $\mathbf{K} = K_{\mathfrak{k}}$, but we may write $\mathbf{K}$ instead of $\mathfrak{k}$ when not said otherwise.

Definition 1.3. 1) For $M \in K_{\mathfrak{k}}$ and $\mu \ge \mathrm{LST}_{\mathfrak{k}}$ and ordinal ε we define an equivalence relation $E_{M,\mu,\varepsilon} = E^M_{\mu,\varepsilon} = E^M_\varepsilon = E_\varepsilon$ by induction on ε.

<u>Case 1</u>: $\varepsilon = 0$.

E^M_ε is the set of pairs $(\bar a_1, \bar a_2)$ such that: $\bar a_1, \bar a_2 \in {}^{\mu>}M$ have the same length and realize the same quantifier free type, moreover, for $u \subseteq \ell g(\bar a_1)$ we have $M{\restriction}(\bar a_1{\restriction}u) \le_{\mathfrak{k}} M \Leftrightarrow M{\restriction}(\bar a_2{\restriction}u) \le_{\mathfrak{k}} M$.

<u>Case 2</u>: ε is a limit ordinal.

$E_\varepsilon = \cap\{E_\zeta : \zeta < \varepsilon\}$.

<u>Case 3</u>: $\varepsilon = \zeta + 1$.

$\bar a_1 E^M_\varepsilon \bar a_2$ <u>iff</u> for every $\ell \in \{1,2\}, \alpha < \mu$ and $\bar b_\ell \in {}^\alpha M$ there is $\bar b_{3-\ell} \in {}^\alpha M$ such that $(\bar a_1{}^\frown \bar b_1) E_\zeta (\bar a_2{}^\frown \bar b_2)$.

Definition 1.4. For $\mu > \mathrm{LST}_{\mathfrak{k}}$ and ordinal ε we define $K_{\mathfrak{k},\varepsilon} = \mathbf{K}_\varepsilon, K_{\mathfrak{k},\mu,\varepsilon} = \mathbf{K}_{\mu,\varepsilon}$ by induction on ε by (well the notation $\mathbf{K}_\varepsilon$ from here and $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$ are in conflict, but usually clear from the context):

- (a) $\mathbf{K}_\varepsilon = \mathbf{K}_{\mathfrak{k}}$ for $\varepsilon = 0$;
- (b) for ε a limit ordinal $\mathbf{K}_\varepsilon = \cap\{\mathbf{K}_\zeta : \zeta < \varepsilon\}$;
- (c) for $\varepsilon = \zeta + 1$, let $\mathbf{K}_\varepsilon$ be the class of $M_1 \in \mathbf{K}_\zeta$ such that: if $M_1 \subseteq M_2 \in \mathbf{K}_\zeta, \bar a_1 \in {}^{\mu>}M_1, \bar b_2 \in {}^{\mu>}(M_2)$ <u>then</u> for some $b_1 \in {}^{\mu>}M_1$ we have $\bar a{}^\frown \bar b_1 E^{M_1}_\zeta \bar a{}^\frown \bar b_2$.

Claim 1.5. *For every ε:*

- (a) *for every $M_1 \in \mathbf{K}_{\mathfrak{k}}$ there is $M_2 \in \mathbf{K}_\varepsilon$ extending H;*
- (b) *E^M_ε has $\le \beth_{\varepsilon+1}(\mu)$ equivalence classes, hence in clause (a) we can[3] add $\|M_2\| \le \|M_1\| + \beth_{\varepsilon+1}(\mu)$;*
- (c) *$M_1 \in \mathbf{K}_{\mu,\varepsilon}$ <u>when</u> $\mathbf{K}_\varepsilon$ has amalgamation and $M_1 \subseteq M_2, M_2 \in \mathbf{K}_\varepsilon$ implies:*
    - *if $\zeta < \varepsilon, \bar a \in {}^{\mu>}(M_1), \bar b_2 \in {}^{\mu>}(M_2)$ then there is $\bar b_1 \in {}^{\ell g(\bar b)}(M_1)$ such that $\bar a{}^\frown \bar b_1 E^{M_2}_{\mu,\zeta} \bar a{}^\frown \bar b_2$;*

[3]We can improve the bound a little, e.g.if $\mu = \chi^+$ then $\beth_{\varepsilon+1}(\chi)$ suffices.

(d) *if I is a $(< \mu)$-directed partial order and $M_s \in \mathbf{K}_\varepsilon$ is $\subseteq$-increasing with $s \in I$, then $M = \bigcup_s M_s \in \mathbf{K}_\varepsilon$;*

(e) *if $H_1 \subseteq H_2$ are from $\mathbf{K}_\varepsilon$ then $H_1 \prec_{\mathbb{L}_{\infty,\mu,\varepsilon(\mathfrak{k})}} H_2$;*

(f) *if $\varepsilon = \mu, \mu = \mathrm{cf}(\mu)$ or $\varepsilon = \mu^+$, and $H_1 \subseteq H_2$ are from $\mathbf{K}_{\mu,\varepsilon}$, then $H_1 \prec_{\mathbb{L}_{\mu,\mu}} H_2$.*

Proof. We can prove this by induction on ε. The details should be clear. $\square_{1.5}$

Definition 1.6. 1) We say $M_0 \in \mathbf{K}_\lambda$ is a $\bar{\chi}$-amalgamation base when: $\bar{\chi} = (\chi_1, \chi_2)$ and $\chi_\ell \geq \|M\|$ and if $M_0 \leq_{\mathfrak{k}} M_\ell \in \mathbf{K}_{\chi_\ell}$ for $\ell = 1, 2$, then for some $M_3 \in \mathbf{K}_{\mathfrak{k}}$ which $\leq_{\mathfrak{k}}$-extend M, both M_1 and M_2 can be $\leq_{\mathfrak{k}}$-embedded into M_3 over M_0.
2) We may replace "χ_ℓ" by "$< \chi_\ell$" with obvious meaning (so $\chi_\ell > \|M_0\|$). If $\chi_1 = \chi_2$ we may write χ_1 instead of (χ_1, χ_2). If $\chi_1 = \chi_2 = \lambda$ we may write "amalgamation base".
3) We say $\mathbf{K}_{\mathfrak{k}}$ has $(\bar{\chi}, \lambda, \kappa)$-amalgamation bases when the κ-majority of $M \in \mathbf{K}_\lambda$ is a $\bar{\chi}$-amalgamation base where:
3A) We say that the κ-majority of $M \in \mathbf{K}_\lambda$ satisfies ψ when some F witnesses it, which means:

$(*)$
- (a) F is a function with[4] domain $\{M \in \mathbf{K}_{\mathfrak{k}} : M$ has universe an ordinal $\in [\lambda, \lambda^+)\}$;
- (b) if $M \in \mathrm{Dom}(F)$ then $M \leq_{\mathfrak{k}} F(M) \in \mathrm{Dom}(F)$;
- (c) if $\langle M_\alpha : \alpha \leq \kappa \rangle$ is increasing continuous, $M_\alpha \in \mathrm{Dom}(F)$ and $M_{2\alpha+2} = F(M_{2\alpha+1})$ for every $\alpha < \kappa$, then M_κ is a $\bar{\chi}$-amalgamation base.

4) We say the pair (M, M_0) is an (χ, μ, κ)-amalgamation base (or amalgamation pair) when: $M \leq_{\mathfrak{k}} M_0 \in \mathbf{K}_{\mathfrak{k}}, \|M\| = \kappa, \|M_0\| = \mu$ and if $M_0 \leq_{\mathfrak{k}} M_\ell \in \mathbf{K}_{\leq \chi}$ for $\ell = 1, 2$, then for some M_3, f_1, f_2 we have $M_0 \leq_{\mathfrak{k}} M_3 \in \mathbf{K}_{\mathfrak{k}}$ and f_ℓ $\leq_{\mathfrak{k}}$-embeds M_ℓ into M_3 over M_0.
5) Let $\mathrm{AM}_{\mathbf{K}} = \mathrm{AM}_{\mathfrak{k}}$ be the class of pairs (λ, κ) such that $\mathbf{K}$ has $((\lambda, \lambda), \lambda, \kappa)$-amalgamation bases.

Definition 1.7. 1) For $\mathfrak{k}, \bar{\chi}, \lambda, \kappa$ as above and $S \subseteq \lambda^+$ (or $S \subseteq \mathrm{Ord}$ but we use $S \cap \lambda^+$) we say $\mathfrak{k}$ has $(\bar{\chi}, \lambda, \kappa, S)$-amalgamation bases when there is a function F such that:

$(*)_F$
- (a) F is a function with domain $\{\bar{M} : \bar{M}$ is a $\leq_{\mathfrak{k}}$-increasing continuous sequence of members of $\mathbf{K}_{\mathfrak{k}}$ each with universe an ordinal $\in [\lambda, \lambda^+)$ and length $i + 1$ for some $i \in S\}$;
- (b) if $\bar{M} = \langle M_i : i \leq j \rangle \in \mathrm{Dom}(F)$ then:
  - (α) $F(\bar{M}) \in \mathbf{K}_{\mathfrak{k}}$;
  - (β) $M_j \leq_{\mathfrak{k}} F(\bar{M})$;
  - (γ) $F(\bar{M})$ has universe an ordinal $\in [\lambda, \lambda^+]$;
- (c) if $\delta = \sup(S \cap \delta) < \lambda^+$ has cofinality κ and $\bar{M} = \langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and for every $j < \kappa$ we have $j \in S \Rightarrow \bar{M}_{j+1} = F(\bar{M} \restriction (j+1))$ hence $\bar{M} \restriction (j+1) \in \mathrm{Dom}(F)$ then M_δ is a $\bar{\chi}$-amalgamation base.

[4]We may use F with domain $\{\bar{M} : M = \langle M_i : i < j \rangle$ is increasing, each $M_i \in \mathbf{K}$ has universe an ordinal $\alpha \in [\lambda, \lambda^+)\}$; see [Sheb].

2) We say $\mathfrak{k}$ has weak $(\bar{\chi}, \lambda, \kappa, S)$-amalgamation bases <u>when</u> above we replace clause (c) by:

(c)$'$ if $\langle M_i : i < \lambda^+ \rangle$ is $\leq_{\mathfrak{k}}$-increasing and $j \in S \cap \lambda^+ \Rightarrow M_{j+1} = F(\bar{M} \restriction (j+1))$ <u>then</u> for some club E of λ^+ we have $\delta \in E$ and $\mathrm{cf}(\delta) = \kappa \Rightarrow M_\delta$ is a $\bar{\chi}$-amalgamation base.

3) We say $\mathfrak{k}$ has $(\bar{\chi}, \lambda, W, S)$-amalgamation bases <u>when</u> $W \subseteq \lambda^+$ is stationary and in part (2) we replace (in the end of $(c)'$, "$\delta \in E$ and $\mathrm{cf}(\delta) = \kappa$" by "$\delta \in E \cap W$".

Proof. Easy. $\square_{1.19}$

Claim 1.8. *1) If $\lambda = \kappa > \mathrm{LST}_{\mathfrak{k}}$ is a weakly compact cardinal and $M \in \mathbf{K}_{\kappa,1}$, see Definition 1.4 <u>then</u> M is a κ-amalgamation base.*
2) If κ is compact cardinal and $\lambda = \lambda^{<\kappa}$ and $M \in \mathbf{K}_{\kappa,1}$ has cardinality λ, <u>then</u> M is a $(< \infty)$-amalgamation base; so $\mathfrak{k}$ has $(< \infty, \lambda, \geq \kappa)$-amalgamation bases.
3) In part (2), κ has to satisfy only: if Γ is a set $\leq \lambda$ of sentences from $\mathbb{L}_{\mathrm{LST}(\mathfrak{k})^+, \aleph_0}$ and every $\Gamma' \in [\Gamma]^{<\kappa}$ has a model, then Γ has a model.

Proof. Use the representation theorem for a.e.c. from [Shea, §1] which is quoted in 0.5 here and the definitions. $\square_{1.8}$

Conclusion 1.9. *If the pair (λ, κ) is as in 1.8, <u>then</u> $\mathfrak{k}$ has (λ, κ)-amalgamation bases; see 1.6(3).*

Claim 1.10. *If $\mathfrak{k}, \mathbf{K}$ are as in 0.5 and the universal class $\mathbf{K}$, i.e. $(\mathbf{K}, \subseteq)$ have $(\bar{\chi}, \lambda, \kappa)$-amalgamation and $\lambda \geq \mathrm{LST}(\mathfrak{k})$, <u>then</u> so does $\mathfrak{k}$.*

Proof. Easy. $\square_{1.10}$

A surprising result says that in some singular cardinals we have "many" amalgamation bases.

Claim 1.11. *If μ is a strong limit cardinal and $\mathrm{cf}(\mu) > \mathrm{LST}_{\mathfrak{k}}$ is a measurable cardinal (so μ is measurable or μ is singular but the former case is covered by 1.8(1)) <u>then</u> $\mathfrak{k}$ has $(\mu, \mathrm{cf}(\mu))$-amalgamation bases.*

Proof. By 1.10 without loss of generality $\mathfrak{k}$ is a universal class $\mathbf{K}$. Without loss of generality μ is a singular cardinal (otherwise the result follows by Claim 1.8). Let $\kappa = \mathrm{cf}(\mu), D$ a normal ultrafilter on κ and let $\langle \mu_i : i < \kappa \rangle$ be an increasing sequence of cardinals with limit μ such that $\mu_0 \geq \mathrm{LST}_{\mathfrak{k}} + \kappa$.

We choose $\mathbf{u}$ such that:

$(*)_1$ (a) $\mathbf{u} = \langle \bar{u}_\alpha : \alpha < \mu^+ \rangle$;
(b) $\bar{u}_\alpha = \langle u_{\alpha,i} : i < \kappa \rangle$;
(c) $u_{\alpha,i} \in [\alpha]^{\mu_i}$ is $\subseteq$-increasing with i;
(d) $\alpha = \bigcup_{i<\kappa} u_{i,\kappa}$;
(e) if $\alpha < \beta < \mu^+$, <u>then</u> $u_{\alpha,i} \subseteq \alpha_{\beta,i}$ for every $i < \kappa$ large enough

For transparency we allow $=^M$ to be non-standard, i.e. just a congruence relation on M.

We now choose functions F, G by:

$(*)_2$ (a) $\mathrm{dom}(F) = \{M \in \mathbf{K}_{\mathfrak{k}} : M$ has universe some $\alpha \in [\mu, \mu^+)\}$;

(b) for $\alpha \in [\mu, \mu^+)$ let $\mathscr{M}_\alpha = \{M \in \mathbf{K} : M$ has universe $\alpha\}$

(c) for $M \in \mathscr{M}_\alpha, u \subseteq \alpha$ let $M[u] = M{\restriction}\mathrm{sb}(u, M)$ and let $M^{[i]} = M[u_{\alpha,i}]$, hence $u \subseteq \alpha \Rightarrow M[u] \leq_{\mathfrak{k}} M$; recall that $\mathrm{sb}(u, M) \subseteq M$ is well defined and belongs to $\mathbf{K}$ because $\mathbf{K}$ is a universal class

(d) if $M \in \mathrm{dom}(F)$ has universe α <u>then</u> $M^+ = F(M)$ satisfies:

  (α) $M \subseteq M^+ \in \mathscr{M}_{\alpha+\lambda}$ (equivalently $M \leq_{\mathfrak{k}} M^+ \in \mathscr{M}_{\alpha+\lambda}$)

  (β) if $i < \kappa$ and $M[u_{\alpha,i}] \subseteq N \in \mathbf{K}_{\mu_i}$, then exactly one of the following occurs:
  - there is an embedding of N into M^+ over $M[u_{\alpha,i}]$
  - there is no $M' \in \mathbf{K}$ extending M^+ and an embedding of N into M' over $M^{[i]}$

This is straightforward. It is enough to prove that F witnesses that $\mathbf{K}$ has (μ, κ)-amalgamation bases, i.e. using $F(\langle M_i : i \leq j\rangle) = F(M_j)$.

For this it suffices:

$(*)_3$ M^1, M^2 can be amalgamated over M_κ (in $\mathbf{K}$) <u>when</u>:

  (a) $\langle M_i : i \leq \kappa\rangle$ is $\subseteq$-increasing continuous;

  (b) $M_i \in \mathbf{K}_\mu$ has universe α_i

  (c) $F(M_{2i+1}) = M_{2i+2}$;

  (d) $M_\kappa \subseteq M^1 \in \mathbf{K}_\mu$ and $M_\kappa \subseteq M^2 \in \mathbf{K}_\mu$.

We can find an increasing (not necessarily continuous) sequence $\langle \varepsilon(i) : i < \kappa\rangle$ of ordinals $< \kappa$ such that $i < j < \kappa \Rightarrow u_{\alpha_{\varepsilon(i)},j} \subseteq u_{\alpha_{\varepsilon(j)},j}$ and so $u_i := u_{\alpha_{\varepsilon(i)},i}$ is $\subseteq$-increasing.

Without loss of generality M^1, M^2 has universe $\beta = \alpha_\kappa + \mu$.

Now,

$(*)$ let $\langle u^*_i : i < \kappa\rangle$ be $\subseteq$-increasing with union β such that: $i < \kappa \Rightarrow u_i \subseteq u^*_i$.

Notice that:

$\boxplus$ it suffices to prove that: for every $i < \kappa$, $M^1[u^*_i], M^2[u^*_i]$ can be $\subseteq$-embedded into M_κ over $M_\kappa[u_i]$ (you can use its closure); say h^ι_i is a $\subseteq$-embedding of $M^\iota[u^*_i]$ into M_κ over $M_\kappa[u_i]$.

It suffices to prove $\boxplus$ by taking ultra-products, i.e. let N_i be $(\mu^+, M_\kappa, M^\iota, M^\iota[u_\iota]u_i, h^\iota_i)_{\iota=1,2}$ and let D be a normal ultrafilter on κ and "chase arrows" in $\prod\limits_{i<\kappa} N_i/D$. It is possible to prove $\boxplus$ by the choice of F so we are done. $\square_{1.11}$

Claim 1.12. *1) Assume $\kappa > \theta > \mathrm{LST}_{\mathfrak{k}}, \theta$ is a measurable cardinal and κ is weakly compact. <u>then</u> $\mathfrak{k}$ has (κ, θ)-amalgamation bases.*
2) Assume κ, θ are measurable cardinals $> \mathrm{LST}_{\mathfrak{k}}$ and $\mu > \kappa + \theta$ is strong limit singular of cofinality κ. <u>Then</u> $\mathfrak{k}$ has (μ, θ)-amalgamation bases.
3) If $\kappa > \theta > \mathrm{LST}_{\mathfrak{k}}, \theta$ is a measurable cardinal and $\{M \in \mathbf{K}^{\mathfrak{k}}_\kappa : M$ is a (χ_1, χ_2)-amalgamation base$\}$ is $\leq_{\mathfrak{k}}$-dense in $\mathbf{K}^{\mathfrak{k}}_\kappa$, <u>then</u> $\mathfrak{k}$ has $(\chi_1, \chi_2, \kappa, \theta)$-amalgamation bases.

Proof. 1) As $\mathfrak{k}$ has (κ, κ)-amalgamation bases by 1.8(1) we can apply part (3) of 1.12 with $(\kappa, \kappa, \kappa, \theta)$ here standing for $(\chi_1, \chi_2, \kappa, \theta)$ there.
2) Similarly to part (1) using 1.11 instead of 1.8(1).
3) Similar to the proof of 1.11, that is, we replace $\boxplus$ by Claim 1.13 and $(*)_2$ by:

- $(*)^1_2$ if $M \in \mathbf{K}_\alpha$, then $F(M)$ is a member of $K_{\mathfrak{k}}$ which is a $\bar{\chi}$-amalgamation base and $M \leq_{\mathfrak{k}} F(M)$.

$\square_{1.12}$

We finish the section with some comments; we actually proved:

Claim 1.13. *Assume κ is a measurable cardinal, $\bar{M} = \langle M_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{k}}$-increasing (not necessarily continuous) and $M_\kappa := \bigcup\limits_{i<\kappa} M_i$ is of cardinality $\leq \min\{\chi_1, \chi_2\}$ and each M_i is a $\bar{\chi}$-amalgamation base. Then M_κ is a $\bar{\chi}$-amalgamation base.*

Claim 1.14. *1) In 1.11, we can replace "$(\mu, \mathrm{cf}(\mu))-$amalgamationbase" by "$(\mu, \mathrm{cf}(\mu), S)$-amalgamation base" for any unbounded subset S of S.
2) Similarly in 1.12.*

Question 1.15. 1) What can $\mathrm{AM}_{\mathfrak{k}} = \{(\lambda, \kappa) : \mathfrak{k}$ has (λ, κ)-amalgamation, $\lambda > \mathrm{LST}_{\mathfrak{k}}\}$ be?
2) What is $\mathrm{AM}_{\mathfrak{k}}$ for $\mathfrak{k} = \mathbf{K}_{\mathrm{exlf}}$?
3) Suppose we replace κ by stationary $W \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$. How much does this matter?

Discussion 1.16. 1) May be helpful for analyzing $\mathrm{AM}_{\mathbf{K}_{\mathrm{lf}}}$ but also of self interest is analyzing $\mathfrak{S}_{k,n}[\mathbf{K}]$ with k, n possibly infinite, see [She17, §4].
2) In fact for 1.15(3) we may consider Definition 1.17.

Definition 1.17. For a regular θ and $\mu \geq \alpha$ fixing $\mathfrak{k}$ let:

- (A) $\mathrm{Seq}^0_{\mu,\alpha}$ is in the class of $\bar{N}$ such that:
  - (a) $\bar{N} = \langle N_i : i \leq \alpha \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous
  - (b) $i \neq 0 \Rightarrow \|N_i\| = \mu$;
- (B) $\mathrm{Seq}^1_{\mu,\alpha} = \{\mathbf{n} = (\bar{N}^1, \bar{N}^2) : \bar{N}^\iota \in \mathrm{Seq}^0_{\mu,\alpha+1}$ and $\beta \leq \alpha \Rightarrow N^1_\beta = N^2_\beta$ so let $N_\beta = N_{\mathbf{n},\beta} = N^1_\beta$;
- (C) we define the game $\partial_{\bar{N},\mathbf{n}}$ for $\mathbf{n} \in \mathrm{Seq}^1_{\mu,\alpha}$;
  - (a) a play last $\alpha + 1$ moves and is between AAM and AM;
  - (b) during a play a sequence $\langle (M_i, M'_i, f_i) : i \leq \alpha \rangle$ is chosen such that:
    - (α) $M_i \in \mathbf{K}_\lambda$ is $\leq_{\mathfrak{k}}$-increasing continuous;
    - (β) f_i is a $\leq_{\mathfrak{k}}$-embedding of $N_{\mathbf{n},i}$ into M_i and even M'_i;
    - (γ) f_i is increasing continuous for limit i, $f_\delta = \bigcup\limits_{i<\delta} f_i$ and f_0 is empty;
    - (δ) $M_i \leq_k M'_{i+1} \leq M_{i+1}$ and for i limit or zero $M'_i = M_i$;
  - (c)
    - (α) if $i = 0$ in the i-th move <u>first</u> AM chooses M_0 and <u>second</u> AAM chooses $f_0 = \emptyset, M'_0 = M_0$;
    - (β) if $i = j+1$, in the i-th move <u>first</u> AM chooses f_i, M'_i and <u>second</u> AAM chooses M_i;
    - (γ) if i is a limit ordinal: M_i, f_i, M'_i are determined;

(δ) if $i = \alpha + 1$, first AAM chooses $N_i \in \{N^1_\alpha, N^2_\alpha\}$ and then this continues as above;

(d) the player AMM wins when AM has no legal move;

(D) let $\mathrm{Seq}_{\mathfrak{k}}$ be the set of λ, μ, θ such that there is $\mathbf{n}$ satisfying:

(a) $\mathbf{n} \in \mathrm{Seq}^1_{\mu,\theta}$;

(b) $N^1_{\mathbf{n},\theta+1}, N^2_{\mathbf{n},\theta+2}$ cannot be amalgamated over $N_{\mathbf{n},\theta} (= N^\iota_{\mathbf{n},\theta}, \iota = 1)$;

(c) in the game $\partial_{\mathbf{n}}$, the player AM has a winning strategy.

Question 1.18. 1) What can be $\mathrm{Seq}_{\mathfrak{k}}$ for $\mathfrak{k}$ an a.e.c. with $\mathrm{LST}_{\mathfrak{k}} = \chi$?
2) What is $\mathrm{Seq}_{\mathbf{K}_{\mathrm{lf}}}$?

Claim 1.19. *Let S be the class of odd ordinals.*
1) If $\mathfrak{k}$ has $(\bar{\chi}, \lambda, \kappa, S)$-amalgamation then $\mathfrak{k}$ has $(\bar{\chi}, \lambda, \kappa)$-amalgamation.
2) If $\lambda = \lambda^{<\kappa}$ then also the inverse holds.

Proof. Should be clear. $\square_{1.19}$

§ 2. Definability

The notion of "$a \in M_2 \backslash M_1$ is definable over M_1" is clear for first order logic, $M_1 \prec M_2$. But in a class like $\mathbf{K}_{\rm lf}$ we may wonder. We can also consider the general case of an a.e.c.,e, see 2.1, but we shall concentrate on lf groups.

Claim 2.1. *Below (i.e. in 2.3 - 2.6) we can replace $\mathbf{K}_{\rm lf}$ by:*

- $(*)$ *$\mathfrak{k}$ is a a.e.c. and one of the following holds:*
  - (a) *$\mathfrak{k}$ is a universal, so $\mathbf{k}_1 = \mathfrak{k}{\restriction}\{M \in K_{\mathfrak{k}} : M$ is finitely good$\}$ determine $\mathfrak{k}$;*
  - (b) *like (a) but $\mathfrak{k}_1$ is closed under products;*
  - (c) *like (a), but in addition:*
    - (α) *$0_{\mathfrak{k}} = 0_{\mathfrak{k}_1}$ is an individual constant;*
    - (β) *if $M_1, M_2 \in K_{\mathfrak{k}_1}$ then $N = M_1 \times M_2 \in K_{\mathfrak{k}_1}$; moreover $f_\ell : M_\ell \to N$ is a $\le_{\mathfrak{k}_1}$-embedding for $\ell = 1, 2$ where:*
      - *$f_1(a_1) = (a_1, 0_{M_2})$;*
      - *$f_2(a_2) = (0_{M_1}, a_2)$.*

Discussion 2.2. Can we in (c) define types as in 2.3 such that they behave suitably (i.e. such that 2.5, 2.6 below works?) We need $c\ell(A, M)$ to be well defined.

Definition 2.3. 1) For $G \subseteq H \in \mathbf{K}_{\rm lf}$ we let ${\rm uniq}(G, H) = \{x \in H$: if $H \subseteq H^+ \in \mathbf{K}_{\rm lf}, y \in H^+$ and ${\rm tp}_{\rm bs}(y, G, H^+) = {\rm tp}_{\rm bs}(x, G, H)$ then $y = x\}$.
1A) Above we let ${\rm uniq}_\alpha(G, H) = {\rm uniq}^1_\alpha(G, H) = \{\bar{x} \in {}^\alpha H$: if $H \subseteq H^+ \in \mathbf{K}_{\rm lf}$, then no $\bar{y} \in {}^\alpha(H^+)$ realizes ${\rm tp}_{\rm bs}(\bar{x}, G, H)$ in H^+ and satisfies ${\rm Rang}(\bar{y}) \cap {\rm Rang}(\bar{x}) \subseteq G\}$.
1B) Let ${\rm uniq}^2_\alpha(G, H)$ be defined as in (1A) but in the end "${\rm Rang}(\bar{x}) = {\rm Rang}(\bar{y})$".
1C) Let ${\rm uniq}^3_\alpha(G, H)$ be defined as in (1A) but in the end "$\bar{x} = \bar{y}$".
2) For $G_1 \subseteq G_2 \subseteq G_3 \in \mathbf{K}_{\rm lf}$ let ${\rm uniq}(G_1, G_2, G_3) = \{x \in G_2$: if $G_3 \subseteq G \in \mathbf{K}_{\rm lf}$ then for no $y \in G \backslash G_2$ do we have ${\rm tp}_{\rm bs}(y, G_1, G) = {\rm tp}_{\rm bs}(x, G_1, G_2)$.

Question 2.4. 1) Given λ, can we bound $\{|{\rm uniq}(G, H)| : G \subseteq H \in \mathbf{K}_{\rm lf}$ and $|G| \le \lambda\}$.
2) Can we use the definition to prove "no $G \in \mathbf{K}^{\rm lf}_{\sqsupseteq_\omega}$ is universal"?

To answer 2.4(1) we prove 2^λ is a bound and more; toward this:

Claim 2.5. *If (A) then (B), where:*

- (A)
  - (a) *$G_n \in \mathbf{K}_{\rm lf}$ for $n < n_*$; n_* may be any ordinal but the set $\{G_n : n < n_*\}$ is finite;*
  - (b) *$h_{\alpha,n} : I \to G_n$ for $\alpha < \gamma_*, n < n_*$;*
  - (c) *if $s \in I$, <u>then</u> the set $\{(G_n, h_{\alpha,n}(s)) : \alpha < \gamma_*$ and $n < n_*\}$ is finite;*
- (B) *there is $(H, \bar{a})$ such that:*
  - (a) *$H \in \mathbf{K}_{\rm lf}$;*
  - (b) *$\bar{a} = \langle a_s : s \in I\rangle$ generates H;*
  - (c) *if $s_0, \dots, s_{k-1} \in I$ <u>then</u> ${\rm tp}_{\rm at}(\langle a_{s_\ell} : \ell < k\rangle, \emptyset, H) = \bigcap\limits_{n,\alpha} {\rm tp}_{\rm at}(\langle h_{\alpha,n}(s_0), \dots, h_{\alpha,n}(s_{k-1})\rangle, \emptyset, G_n)$;*
  - (d) *the mapping $b_s \to a_s$ for $s \in I_*$ embeds H_* into H <u>when</u> :*

(∗) *$H_* \subseteq G_n$ for $n < n_*, I_* \subseteq I, \langle b_s : s \in I_*\rangle$ list the elements of H_* (or just a sequence of elements which generates it) and $\alpha < \gamma_* \wedge s \in I_* \wedge n < n_* \Rightarrow h_{\alpha,n}(s) = b_s$.*

Proof. Note that:

$(*)_1$ there are H and $\bar{a}$ such that:

- (a) H is a group;
- (b) $\bar{a} = \langle a_s : s \in I\rangle$;
- (c) $a_s \in H$;
- (d) for any finite $u \subseteq I$ and atomic formula $\varphi(\bar{x}_{[u]})$ we have $H \models \varphi(\bar{a}_{[u]})$ <u>iff</u> for every $n < n_*$ and $\alpha < \gamma_*$ we have $G_n \models \varphi[\ldots, h_{\alpha,n}(s), \ldots]_{s \in u}$.

[Why? Let $G_{\alpha,n} = G_n$ for $\alpha < \gamma_*, n < n_*$ and let $H' = \Pi\{G_{\alpha,n} : n < n_*, \alpha < \gamma_*\}$ and let $a_s = \langle h_{\alpha,n}(s) : (\alpha, n) \in (\gamma_*, n_*)\rangle$ for $s \in I$ and, of course, $\bar{a} = \langle a_s : s \in I\rangle$.]

$(*)_2$ Without loss of generality $\bar{a}$ generates H.

[Why? Just read $(*)_1$ and replace H by the subgroup of H generated by $\bar{a}$.]

$(*)_3$ If $u \subseteq I$ is finite, then $\mathrm{sb}(\bar{a}_{[u]}, H)$ is finite (and for 2.1 it belongs to $K_{\mathfrak{k}}$)

[Why? By Clause (A)(c) of Claim 2.5; and for the generalization in 2.1 recalling 2.1(d).]

$(*)_4$ $H \in \mathbf{K}_{\rm lf}$ (i.e. (B)(a) holds).

[Why? By $(*)_2 + (*)_3$; for 2.1 use also 2.1(d).]

$(*)_5$ Clause (B)(c) holds.

[Why? By $(*)_1(d)$.]

$(*)_6$ Clause (B)(d) holds.

[Why? Follows from our choices.] $\square_{2.5}$

Claim 2.6. *If $G_1 \in \mathbf{K}^{\rm lf}_{\le\lambda}$ and $G_1 \subseteq G_2 \in \mathbf{K}_{\rm lf}$ has cardinality $\le \mu = \mu^\lambda$ (e.g. $G_1 \subseteq G_2 \in \mathbf{K}^{\rm lf}_\lambda, \mu = 2^\lambda$), <u>then</u> for some pair (G_3, X) we have:*

⊕ (a) $G_2 \subseteq G_3 \in \mathbf{K}^{\rm lf}_\mu$
- (b) *$X \subseteq G_3$ has cardinality $\le 2^\lambda$*
- (c) *if $c \in G_3$, then exactly one of the following occurs:*
  - (α) *$c \in X$ and $\{b \in G_3 : \mathrm{tp}_{\rm at}(b, G_1, G_3) = \mathrm{tp}_{\rm at}(c, G_1, G_3)\}$ is a singleton and moreover this holds also in G_4 whenever $G_3 \subseteq G_4 \in \mathbf{K}_{\rm lf}$;*
  - (β) *there are $\|G_3\|$ elements of G realizing $\mathrm{tp}_{\rm bs}(a, G_1, G_3)$;*
- (d) *if $\alpha < \lambda^+, \bar{a} \in {}^\alpha(G_3)$ and $p(\bar{x}_{[\alpha]}) = \mathrm{tp}_{\rm at}(\bar{a}, G, G_3)$, $p'(\bar{x}_{[\alpha]}) = \mathrm{tp}_{\rm bs}(\bar{a}, G, G_3)$, <u>then</u> for some non-empty $\mathscr{P} \subseteq \mathscr{P}(\alpha)$ closed under the intersection of 2 to which α belongs we have:*
  - (α) *if $\bar{a}', \bar{a}'' \in {}^\alpha(G_3)$ realizes $p(\bar{x}_{[\alpha]})$ <u>then</u> $u := \{\beta < \alpha : (a'_\beta = a''_\beta)\} \in \mathscr{P}$;*

(β) *if* $u \in \mathscr{P}$ *then we can find* $\langle \bar a_\varepsilon : \varepsilon < \|G_3\|\rangle$ *a* Δ*-system with heart* u *(i.e.* $\bar a_{\varepsilon_1,\beta_1} = \bar a_{\varepsilon_2,\beta_2} \Leftrightarrow ((\varepsilon_1,\beta_1) = (\varepsilon_2,\beta_2)) \vee (\beta_1 = \beta_2 \in u)$*), each* $\bar a_\varepsilon$ *realizing* $p(\bar x_{[\alpha]}$ *and even* $p'(\bar x_{[\alpha]})$*).*

Remark 2.7. 1) Can we generalize the (weak) elimination of quantifiers in modules?
2) An alternative presentation is to try $G^I_D/\mathscr{E}$, where:

- $\mathscr{E} \subseteq \{E : E$ is an equivalence relation on I such that I/E is finite$\}$ and $(\mathscr{E} \ge)$ is directed;
- G^I_D is $G^I{\restriction}\{f : f + G$ and there is $E \in \mathscr{E}$ such that $sEt \Rightarrow f(s) = f(t)\}$.

3) For suitable $(I, D, \mathscr{E})$ we have: if p is a set of $\le \mu$ basic formulas with parameters from $G_1 = G^I_D/\mathscr{E}$ we have: p is realized in G_1 iff every $\varphi_1,\dots,\varphi_n,\neg\varphi_i \in p, \varphi_\ell$ atomic is realized in G_1.

Proof. We can easily find G_3 such that:

$(*)_1$ (a) $G_2 \subseteq G_3 \in \mathbf{K}^{\rm lf}_\mu$;
(b) if $G_3 \subseteq H \in \mathbf{K}_{\rm lf}, \gamma < \lambda^+, \bar a \in {}^\gamma H$ and $u = \{\alpha < \gamma : a_\alpha \in G_3\}$, then there are $\bar a^\varepsilon \in {}^\gamma(G_3)$ for $\varepsilon < \mu$ such that:
(α) $\mathrm{tp}_{\rm bs}(\bar a^\varepsilon, G_1, G_3) = \mathrm{tp}_{\rm bs}(\bar a, G_1, G_3)$;
(β) if $\varepsilon,\zeta < \mu$ and $\alpha,\beta < \gamma$ and $a^\varepsilon_\alpha = a^\zeta_\beta$ then $((\varepsilon,\alpha) = (\zeta,\beta)) \vee (\alpha = \beta \in u \wedge a^\varepsilon_\alpha = a_\alpha = a^\zeta_\alpha)$.

We shall prove that

$(*)_2$ G_3 is as required in $\oplus$.

Obviously this suffices. Clearly clause $\oplus(a)$ holds and clauses $\oplus(b) + (c)$ follows from clause $\oplus(d)$.

[Why? Without loss of generality $G_1 = \mathbf{K}^{\rm lf}_\lambda$, let $\langle a_\beta : \beta < \lambda\rangle$ list the elements of G_1. For $c \in G_3$ let $\bar a_c = \langle a_\beta : \beta < \lambda\rangle{}^\frown\langle c\rangle$ and applying clause (d) we get $\mathscr{P}_c \subseteq \mathscr{P}(\lambda+1)$ as there. We finish letting $X := \{c \in G_3 : \lambda \notin \mathscr{P}_c\}$.]

Now let us prove clause $\oplus(d)$, so let $\alpha < \lambda^+, \bar a \in {}^\alpha(G_3)$ and $p(\bar x_{[\alpha]}) = \mathrm{tp}_{\rm at}(\bar a, G_1, G_3)$ and $p'(\bar x_{[\alpha]}) = \mathrm{tp}_{\rm bs}(\bar a, G_1, G_3)$; without loss of generality $\bar a$ is without repetitions but this is not used.

Define:

$(*)_3$ $\mathscr{P} = \{u \subseteq \alpha$: there are $\bar a', \bar a'' \in {}^\alpha(G_3)$ realizing $p(\bar x_{[\alpha]})$ such that $u = (\forall \beta < \alpha)(\beta \in u \equiv a'_\beta = a''_\beta)\}$.

Now

$(*)_4$ $\alpha \in \mathscr{P}$.

[Why? Let $\bar a' = \bar a'' = \bar a$.]

$(*)_5$ if $u_1, u_2 \in \mathscr{P}$, then $u_1 \cap u_2 \in \mathscr{P}$.

[Why? Let $\bar a'_\ell, \bar a''_\ell$ witness that $u_\ell \in \mathscr{P}$, i.e. both $\bar a'_\ell, \bar a''_\ell$ realize $p(\bar x_{[\alpha]})$ in G_3 and $u_\ell = \{\beta < \alpha : a'_{\ell,\beta} = a''_{\ell,\beta}\}$.

Let $I = I_* + \sum\limits_{\varepsilon<\mu} I_\varepsilon$ be linear orders (so $I_*, I_\varepsilon(\varepsilon < \mu)$ are pairwise disjoint), where we chose the linear orders such that $I_\varepsilon \cong \alpha$ for $\varepsilon < \mu$ and let $s_{\varepsilon,\beta}$ be the

β-th member of I_ε and I_* has cardinality λ and let $\langle c_s : s \in I_* \rangle$ list G_3 such that $c_{s(*)} = e_{G_3}$ and $s(*) \in I_*$.

We shall now apply 2.5, so let

- (a) $\gamma_* = 1 + \alpha + \alpha$ and $n_* = 1$
- (b) for $\varepsilon < \mu, \gamma < \gamma_*$ let $\langle h_{\gamma,0}(s_{\varepsilon,\beta}) : \beta < \alpha \rangle$ be equal to:
  - $\bar{a}$ <u>if</u> $\gamma = 0$;
  - $\bar{a}'_1$ <u>if</u> $\gamma \in \{1 + \zeta : \zeta < \varepsilon\}$;
  - $\bar{a}''_1$ <u>if</u> $\gamma \in [1 + \zeta : \zeta \in [\varepsilon, \alpha)\}$;
  - $\bar{a}'_2$ <u>if</u> $\gamma \in \{1 + \alpha + \zeta : \zeta < \varepsilon\}$;
  - $\bar{a}''_2$ <u>if</u> $\gamma \in \{1 + \alpha + \zeta : \zeta \in [\varepsilon, \alpha)\}$;
- (c) $h_{\gamma,0}(s) = c_s$ for $s \in I, \gamma < \gamma_*$;
- (d) G_3, G_3, I, I_* here stand for G_0, H_*, I, I_* there.

We get $(H, \bar{a}^*)$ as there, so by (B)(d) there essentially $G_3 \subseteq H$ and by (B)(c) there the $\bar{a}^* {\restriction} I_\varepsilon$ realizes $p(\bar{x}_{[\alpha]})$; moreover, realizes $p'(\bar{x}_{[\alpha]})$; also $\langle \bar{a}^* {\restriction} I_\varepsilon : \varepsilon < \mu \rangle$ is a Δ-system with heart u.

The rest should be clear; we do not need to extend G_3 by $(*)_1$.] $\square_{2.6}$

§ 3. Density of being Complete in $\mathbf{K}_\lambda^{\rm lf}$

We prove here that for almost all cardinals λ, the complete $G \in \mathbf{K}_\lambda^{\rm exlf}$ are dense in $(\mathbf{K}_\lambda^{\rm exlf}, \mathbf{c})$;

Discussion 3.1. 1) We would like to prove for as many cardinals $\mu = \lambda$ or at least pairs $\mu \le \lambda$ of cardinals that $(\forall G \in \mathbf{K}_\mu^{\rm lf})(\exists H \in \mathbf{K}_\lambda^{\rm exlf})(G \subseteq H \wedge H$ complete). We necessarily have to assume $\lambda \ge \mu + \aleph_1$. So far we have known it only for $\lambda = \mu^+, \mu = \mu^{\aleph_0}$, (and $\lambda = \aleph_1, \mu = \aleph_0$, see the introduction of [She17]). We would like to prove it also for as many pairs of cardinals as we can and even for $\lambda = \mu$.
2) Given $G_1 \in \mathbf{K}_\lambda^{\rm lf}$ we shall find $\mathbf{m}$ consisting of:

- $\bar G = \langle G_i : i \le \theta\rangle$, increasing continuous, $G_{2+i} \in \mathbf{K}_{\le\lambda}^{\rm lf}$
- for unboundedly many $i < \theta$, we make a step toward G_θ being in $K_{\rm exlf}$, by realizing all suitably definable complet qf types on G_i, formally $p \in \mathbf{S}_{\mathfrak{S}}(G_i)$ in G_{i+1} but not to lose control, we like to combine those types "nicely", as in [She17, §3]
- for unboundedly many $i < \theta, G_i$ is θ-indecomposable inside G_{i+3}.
- also $G_1 \le_{\mathfrak{S}} G_\theta$, see 3.2(3).

This will imply that any automorphism π of G_θ maps G_i onto G_i for a club of i's. This replaces "if $G_\alpha \in \mathbf{K}_{\le\lambda}^{\rm lf}$ is $\subseteq$-increasing continuous for $\alpha < \lambda^+$ any automorphism π of $G = \bigcup\{G_\alpha : \alpha < \lambda\}$ maps G_δ onto G_δ for a club of $\delta < \lambda^+$" which was used in earlier proofs. The present construction rely on §(0C) (so on [Shee], [She17]).
3) We shall use $\lambda = \lambda^{(\theta_0;\aleph_0)}$; how does this help? We ask, given $\pi \in {\rm aut}(G_\theta)$ whether for every $i < \theta$, on the centralizer $\mathbf{C}(G_i, G_\theta)$ of G_i in G_θ, the automorphism is not the identity.

The proof split, in the first case the answer is yes. Let $c_i \in \mathbf{C}(G_i, G_\theta)$ witness it. If we assume $\lambda = \lambda^{\langle\theta;\aleph_0\rangle}$ we may (without loss of generality the set of elements of G_δ be λ), have an a priori list of λ countable sets in which a countable subset of $\{c_i : i < \theta\}$ necessarily appear; in fact, many as we can consider any $\{c_i : i \in v\}, v \in [\theta]^\theta$. To finish, we use on the one hand, G_θ is "nicely" constructed over G_1 and on the other hand the $\mathbf{c}$'s in $\mathbf{m}$ to be derived for a witness of ${\rm Pr}_*(\lambda, \lambda, \lambda, \aleph_0)$.

The second case is when the answer to the question is no, so for some $i < \theta$ this fails, then we shall prove that for every $j, \pi{\restriction}G_j$ is induced by an inner automorphism (as G_j a conjugate in $\mathbf{C}(G_i, G_\theta)$), so we need just no θ-branch is the natural tree.

In this section, in particular in 3.2(3) we rely on [She17].

Hypothesis 3.2. 1) $\lambda > \theta = {\rm cf}(\theta) > \aleph_0$ but there is no μ such that $\lambda = \mu^+ \wedge \mu > {\rm cf}(\mu) = \theta$, this[5] exclude very few pairs.
2) $\mathbf{K} = \mathbf{K}_{\rm lf}$.
3) $\mathfrak{S}$ is a set of schemes (for $\mathbf{K}_{\rm lf}$, see [She17, Def.0.9=La14], there are $\le 2^{\aleph_0}$ ones) consisting of all of them or is just of cardinality $\le \lambda$, is dense and containing enough of those mentioned in [She17, §2]

Also $c\ell(\mathfrak{S}) = \mathfrak{S}$, i.e. $\mathfrak{S}$ is closed, see [She17, 1.6=La21,1.8=La22] hence by [She17] there is such countable $\mathfrak{S}$. Recall that $G \le_{\mathfrak{S}} H$ means that $G \subseteq H$ and for every $\bar b \in {}^{\omega>}H$ for some $\bar a \in {}^{\omega>}G$ and $\mathfrak{s} \in c\ell(\mathfrak{S})$ we have ${\rm tp}_{\rm bs}(\bar b, G, H) = q_{\mathfrak{s}}(\bar a, G)$.

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[5] We can exclude more but immaterial here.

4) $\bar{S} = \langle S_1, S_2, S_3 \rangle$ is a partition of $\theta \backslash \{0\}$ to stationary subsets, such that $S_3 \subseteq S^\theta_{\aleph_0} := \{\delta < \theta : \mathrm{cf}(\delta) = \aleph_0\}$ and for every $i \in S_2$ there is j such that $i \in \{j, j+1, j+2\} \subseteq S_2$ but $j+3 \notin S_2$ and $\omega^2 | j$; we may let $S_0 = \{0\}$ and $S_1^{\mathrm{limit}} = \{i \in S_1 : i$ is a limit ordinal $\}$.

Definition 3.3. Let $\mathbf{M}_1 = \mathbf{M}^1_{\lambda,\theta,\bar{S}}$ be the class of objects $\mathbf{m}$ which consists of:

- (a) $G_i = G_{\mathbf{m},i}$ for $i \le \theta$ is increasing continuous, G_0 is the trivial group with universe $\{0\}, G_1 \in \mathbf{K}_{\le\lambda}$ has universe $\{\theta\alpha : \alpha < |G_1|\}$, and for $i \in (\theta + 1)\backslash\{0, 1\}$ the group $G_i \in \mathbf{K}_\lambda$ has universe $\{\theta\alpha + j : \alpha < \lambda$ and $j < 1 + i\}$ and so $e_{G_i} = 0$;
- (b) if $i < \theta$, <u>then</u> we have:
  - (α)
    - sequences $\mathbf{b}_i = \langle \bar{b}_{i,s} : s \in I_i \rangle, \mathbf{a}_i = \langle \bar{a}_{i,s} : s \in J_i \rangle$;
    - each $\bar{a}_{i,s}$ is a finite sequence from G_i;
    - each $\bar{b}_{i,s}$ is a finite sequence from G_{i+1};
    - I_i is a linear order of cardinality λ with a first element;
    - J_i is a set or linear order of cardinality $\le \lambda$;
    - if $i = 0$ then $J_i \subseteq \lambda, I_i \subseteq \lambda$ and $\langle \bar{b}_{i,s} = \langle b_{i,s} \rangle : s \in I_i \rangle$ lists the members of G_1 possibly with repetitions and $\bar{a}_{i,s} = \langle\rangle$;
    - if $\ell g(\bar{a}_{i,s}) = 1$ then let $\bar{a}_{i,s} = \langle a_{i,s} \rangle$ and similarly for the $b_{i.s}$-s;
    - $\langle I_i : i < \theta \rangle$ are pairwise disjoint, and so are the $I_{i,\alpha}$ when defined, also $s \in I_i \Rightarrow s \in \lambda$ for transparency. Similarly concerning $\langle J_i : i < \theta \rangle$
  - (β) G_{i+1} is generated by $\cup\{\bar{b}_{i,s} : s \in I_i\} \cup G_i$;
  - (γ) $\bar{a}_{i,\min(J_i)} = e_{G_i}$;
  - (δ) $\mathbf{c}_i : [I_i]^2 \to \lambda$;
- (c) [toward being in $\mathbf{K}_{\mathrm{exlf}}$] if $i \in S_1$, <u>then</u> $J_i = I_i$ and we also have $\langle \mathfrak{s}_{i,s} : s \in I_i \rangle$ such that:
  - (α) $\mathfrak{s}_{i,s} \in \mathfrak{S}$;
  - (β) $\mathrm{tp}_{\mathrm{bs}}(\bar{b}_{i,s}, G_i, G_{i+1}) = q_{\mathfrak{s}_{i,s}}(\bar{a}_{i,s}, G_i)$ so $\ell g(\bar{b}_{i,s}) = n(\mathfrak{s}_{i,s})$ and $\ell g(\bar{a}_{i,s}) = k(\mathfrak{s}_{i,s})$;
  - (γ) if $s_0 <_{I_i} \ldots <_{I_i} s_{n-1}$ then $\mathrm{tp}_{\mathrm{bs}}(\bar{b}_{i,s_0}\hat{\ } \ldots \hat{\ }\bar{b}_{i,s_{n-1}}, G_i, G_{i+1})$ is gotten from $(\mathfrak{s}_{i,s_0}, \bar{a}_{i,s_0}), \ldots, (\mathfrak{s}_{i,s_{n-1}}, \bar{a}_{i,s_{n-1}})$ by one of the following two ways:
    <u>Option 1</u>: we use the linear order I_i on λ so $\mathrm{tp}_{\mathrm{qf}}(\bar{b}_{i,s}, G_{i,s}, G_{i,t})$ is equal to $q_{\mathfrak{s}_{i,s}}(\bar{a}_{i,s}, G_{i,s})$ where $G_{i,s}$ is the subgroup of G_{i+1} generated by $G_i \cup \{\bar{b}_{i,t} : t <_{I_i} s\}$, see [She17, §(1C),1.28=La58];
    <u>but</u>[6] we choose:
    <u>Option 2</u>: intersect the atomic types over all orders on $\{\alpha_0, \ldots, \alpha_{n-1}\}$ each gotten as in Option 1, so I_i can be a set of cardinality λ, see [She17, §3]; so clause (b)(γ) is the only use of "I is a linear order".
  - (δ) $\mathbf{c}_i$ is constantly zero;
- (d) [toward indecomposability] if $i \in S_2$ <u>then</u>:
  - (α) $J_i \subseteq \lambda$ and $J_i = \bigcup\{J_{i,\alpha} : \alpha < \lambda\}$ disjoint union
  - (β) $\langle I_{i,\alpha} : \alpha < \lambda \rangle$ is a partition of $I_i \subseteq \lambda$;

[6]Option 1 is useful in some generalizations to $K_{\mathfrak{k}}$ not closed under products.

(γ) $\bar{a}_{i,\alpha} = \langle a_{i,\alpha}\rangle, \bar{b}_{i,s} = \langle b_{i,s}\rangle$ and $a_{i,0} = e_{G_i}$;

(δ) if $i \in S_2^{\rm limit}$ then G_i is generated by $\{a_{s,\alpha} : s \in J_i$;

(ε) G_{i+1} is generated by $G_i \cup \{b_{i,s} : s \in I_i\}$

(ζ) $(I_i, \mathbf{c}_i, G_{i+1}, G_i, \langle b_{i,s} : s \in I_i\rangle, \langle a_{i,s} : s \in J_i\rangle, \langle I_{i,\alpha} : \alpha < \lambda\rangle, \langle J_{i,\alpha} : \alpha < \lambda\rangle)$ is like $(I, \mathbf{c}, G_2, G_1, \langle b_s, c_{\ell,s} :, s \in I\rangle, \langle a_s : s \in J\rangle, \langle I_{i,\alpha} : \alpha < \lambda\rangle, \langle J_{i.\alpha} : \alpha < \lambda\rangle)$ in 0.15(2)

(η) assume $i \in \{j, j+1, j+2\} \subseteq S_1$,
$\bullet_1$ if $i = j$ then we apply 0.16, i.e. 0.15(2), with for transparency $I_i, J_i \subseteq \lambda$, $I_i = \{2\alpha, 2\alpha+1 : \alpha \in J_i\}$, and $\mathbf{c}_i$ being zero except for the pairs $(2\alpha, 2\alpha+1)$ for $\alpha \in J_i$
$\bullet_2$ if $\ell \in \{1,2\}$ and $i = j + \ell$ then we apply 0.15(1) and $J_i = J_j$ and $a_{i,\alpha} = b^\ell_{j,\alpha}$

(e) [against external automorphism] for $i \in S_3$ the triple $(barj, \bar{I}_i, \bar{J}_i)$ satisfies (recalling $i \in S_3 \Rightarrow \mathrm{cf}(i) = \aleph_0$):

(α) $\bar{j}_i = \langle j_{i,n} : n < \omega\rangle$ is increasing with limit i;

(β) $\bar{I}_i = \langle I_{i,\alpha} : \alpha < \lambda\rangle$ is a partition of I_i; for $s \in I_i$ let $\alpha_i(s)$ be the $\alpha < \lambda$ such that $s \in I_{i,\alpha}$ and let $\mathbf{c}_{i,\alpha} = \mathbf{c}_i{\restriction}[I_{i,\alpha}]^2$;

(γ) $\langle J_{i,\alpha} : \alpha < \lambda\rangle$ is a partition of J_i and $J_{i,\alpha} = \{\omega\alpha_\ell : \ell < \omega\}$

(δ) $a_{i,\omega\alpha+\ell} \in G_{j_{i,\ell+1}}$ commutes with $G_{j_{i,\ell}}$ and if $\ell \neq 0$ then it has order 2, and $\notin G_{j_{i,\ell}}$ and $a_{i,\omega\alpha} \equiv e_{G_i}$; moreover:
- for some infinite $v \subseteq \omega\backslash\{0\}$ we[7] have $\ell \in \omega\backslash v \Rightarrow a_{i,\omega\alpha+\ell} = e_{G_i}, \ell \in v \Rightarrow a_{i,\omega\alpha+\ell} \in \mathbf{C}(G_{j[i,\omega\alpha+\ell]}, G_{j[i,\omega\alpha+\ell]+1})$, where:
- $j[i, \omega\alpha+\ell) \in [j_{i,\ell}, j_{i,\ell+1})$;

(ϵ) if $s, t \in I_{i,\alpha}$ then $[b_{i,s}, b_{i,t}] = a_{i,\mathbf{c}_i\{s,t\}}$ and $\mathbf{c}_i\{s,t\} \in \{\omega\alpha+\ell : \ell < \omega\}$;

(ζ) if $s, t \in I_i$ and $\alpha_i(s) \neq \alpha_i(t)$ then $[b_{i,s}, b_{i,t}] = e_{G_i}$

(η) $b_{i,s}$ commutes with G_i.

Convention 3.4. If the identity of $\mathbf{m}$ is not clear, we may write $G_{\mathbf{m},i}$, etc., but if it is clear from the context we may not add it.

Definition 3.5. 1) We shall say that $\mathbf{s} = (\lambda, \theta, \bar{I}, \bar{J}, \bar{\mathfrak{s}}, \bar{j}, \bar{\mathbf{c}})$ is a legal parameter when it is as in Def 3.3, ignoring the $G_i, \bar{a}_{i,s}, \bar{b}_{i,s}$-s; but we usually omit λ, θ as they are clear from the context.

2) We say $\mathbf{s}$ is a short parameter when we replace $\bar{\mathbf{c}}$ by $\mathbf{c} : [\lambda]^2 \to \lambda$. the $\mathbf{c}_i$ and $\mathbf{c}_{i,\alpha}$ are the restrictions of $\mathbf{c}$ to the suitable sets, except that when the value is "illegal" i.e. not in the required set it is corrected to be zero; illegal values are when for $\beta, \gamma \in I_i$ the value is not in $J_{i,\alpha} \cup \{0\}$ or as demanded in 3.3(d)(ζ), 3.7(d)(θ) and 3.3(e)(δ).

2A) We shall say that the legal parameter $\mathbf{s}$ is derived from the short parameter when they are as above; we may not pedantically distinguish between them.

3) We say that $\mathbf{m} \in \mathbf{M}_1$ satisfies the legal/short parameter $\mathbf{s}$ when it satisfies $\mathbf{s}$.

4) We shall say that the legal parameter $\mathbf{s}$ is θ-indecomposable <u>when</u>for every $j \in S_2^{limit}$ the function $\mathbf{c}_{j+i} : [I_i]^2 \to J_i$ is θ-indecomposable.

[7] An alternative is $v = \omega\backslash\{0\}, a_{i,\omega\alpha+\ell} \in \mathbf{C}(G_{j_{i,\ell}}, G_{j_{i,\ell+1}})$. In this case in 3.7($e$)($\varepsilon$) we naturally have $c_\varepsilon \in \mathbf{C}(G_{i_\varepsilon}, G_{i_{\varepsilon+1}})$ and $\ell_0 = 1, \ell_1 = 2, \ldots$. But then we have to be more careful in 3.10, e.g. in 3.10(1) if we assume, e.g. $\lambda = \lambda^{\langle\theta;\theta\rangle}$ and $\theta > \aleph_1$ all is O.K. (recalling we have guessing clubs on $S^\theta_{\aleph_0}$). However, using $\mathfrak{s}_{\rm cg}$, see ([She17, 2.17=Lc50]), the present is enough here.

Claim 3.6. *1) If* $\mathbf{s}$ *is a legal parameter and* G_1 *is a group of cardinality* $|J_{\mathbf{s},1}|$ *<u>then</u>there is* $\mathbf{m} \in \mathbf{M}_1$ *which satisfies this parameter.*
2) If $\mathbf{s}$ *is a short parameter <u>then</u> there is a unique legal parameter derived form it.*

Definition 3.7. 1) Let $\mathbf{M}_2 = \mathbf{M}^2_{\lambda,\theta,\bar{S}}$ be the set of $\mathbf{m} \in \mathbf{M}_1$ satisfying the following additions to Definition 3.3:

(c) (ε) if $\mathfrak{s} \in \mathfrak{S}, i \in S_1, \bar{a} \in {}^{n(\mathfrak{s})}(G_i)$ and $k = k(\mathfrak{s})$, <u>then</u> for λ elements $s \in I_i$ we have $(\mathfrak{s}_{i,s}, \bar{a}_{i,s}) = (\mathfrak{s}, \bar{a})$;

(d) (θ) if $\{j, j+1, j+2\} \subseteq S_2$ then
- $\bullet_1$ if $i = j$ then $\{a_{i,\alpha} : \alpha \in J_i\}$ generates G_{i+1} and of course $\bar{a}_{i,\alpha} = \langle a_{i,\alpha} \rangle$
- $\bullet_2$ if $\ell \in \{1,2\}$ and $i = j + \ell$ then $\mathbf{c}_i$ if θ-indecomposable.

(*e*) (ζ) if $\langle i_\varepsilon : \varepsilon < \theta \rangle$ is increasing continuous and $i_\varepsilon < \theta$ and $c_\varepsilon \in \mathbf{C}(G_{i_\varepsilon}, G_{i_\varepsilon+1})$ has order 2 and for transparency $c_\varepsilon \notin G_{i_\varepsilon}$ then for some $(i, \alpha, v, \ell_0, \ell_1, \ldots, \varepsilon_0, \varepsilon_1, \ldots)$ we have:
- $\bullet_1$ $i < \theta, \alpha < \lambda$ and $v \subseteq w \backslash \{0\}$ is infinite;
- $\bullet_2$ $\varepsilon_0 < \varepsilon_1 < \ldots < \theta$ and $1 \leq \ell_0 < \ell_1 < \ldots$;
- $\bullet_3$ $i = \cup \{\varepsilon_n : n < \omega\}$;
- $\bullet_4$ $j_{i,\omega\alpha+\ell_n} \leq i_{\varepsilon_n} < j_{i,\theta,\alpha+\ell_{n+1}}$ and $a_{\mathbf{m},i,\omega\alpha+\ell_n} = c_{\varepsilon_n}$;

1A) Let $M_{1.5} = \mathbf{M}^{1.5}_{\lambda,\theta,\bar{S}}$ be the set of $\mathbf{m} \in \mathbf{M}_1$ as it satisfies (c) of part (1).
2) Let $\mathbf{M}_4 = \mathbf{M}^4_{\lambda,\theta,\bar{S}}$ be the class of $\mathbf{m} \in \mathbf{M}_2$ such that in addition:

(f) there is a short parameter $\mathbf{s}$ of $\mathbf{m}$ such that $\mathbf{c}$ is a witness of $\mathrm{Pr}_0(\lambda, \lambda, \lambda, \aleph_0)$; see Definition 3.8(1) below.

3) $\mathbf{M}_3 = \mathbf{M}^3_{\lambda,\theta,\bar{S}}$ means $\mathbf{m} \in \mathbf{M}_2$ satisfies

(f)′ there is a legal parameter $\mathbf{s}$ of $\mathbf{m}$ such that $(\mathbf{c}, \bar{I}^3, \bar{I}^2)$ is a witness of $\mathrm{Pr}_*(\lambda, \lambda, \aleph_0, \aleph_0, \theta)$; see Definition 3.8(2) below; where $\bar{I}^\ell = \langle I_i : i \in S_\ell \rangle$.

4) Let $\mathbf{M}_{2.5} = \mathbf{M}^{2.5}_{\lambda,\theta,\bar{S}}$ be the class of $\mathbf{m} \in \mathbf{M}_{1.5}$ such that in addition:

(f) as in part (2).

The following definition 3.8(1) of Pr_0 is just a sufficient condition for what we need to get many cardinals. Then 3.8(2) give a replacement of Pr_0 which is sufficient for our purposes, not the best we can get.

Definition 3.8. Assume $\lambda \geq \mu \geq \sigma + \theta_0 + \theta_1, \bar{\theta} = (\theta_0, \theta_1)$; if $\theta_0 = \theta_1$ we may write θ_0 instead of $\bar{\theta}$.
1) Let $\mathrm{Pr}_0(\lambda, \mu, \sigma, \bar{\theta})$ mean that there is $\mathbf{c} : [\lambda]^2 \to \sigma$ witnessing it which means:

$(*)_{\mathbf{c}}$ if (a) then (b) where:

(*a*) (α) for $\iota = 0, 1$ and $\alpha < \lambda$ we have $\bar{\zeta}^\iota = \langle \zeta^\iota_{\alpha,i} : \alpha < \mu, i < \mathbf{i}_\iota \rangle$, a sequence without repetitions of ordinals $< \lambda$

(β) $\mathbf{i}_0 < \theta_0, \mathbf{i}_1 < \theta_1$;

(γ) $h : \mathbf{i}_0 \times \mathbf{i}_1 \to \sigma$

(*b*) for some $\alpha_0 < \alpha_1 < \mu$ we have:
- if $i_0 < \mathbf{i}_0$ and $i_1 < \mathbf{i}_1$ then $\mathbf{c}\{\zeta^0_{\alpha_0,i_0}, \zeta^1_{\alpha_1,i_1}\} = h(i_0, i_1)$.

1A) We define $\mathrm{Pr}_1(\lambda, \mu, \sigma, \theta)$ similalry except that in clause (a)(γ) we demand that the function h is constant.
2) Let $\mathrm{Pr}_*(\lambda, \mu, \sigma, \partial, \theta)$ mean that $\theta = \mathrm{cf}(\theta), \lambda \geq \mu, \sigma, \partial, \theta$ and some pair $(\mathbf{c}, \bar{W})$ witness it, which means (if $\lambda = \mu$ we may omit λ, if $\sigma = \partial \wedge \lambda = \mu$ then we can omit σ, λ):

- (a) $\bar{W}_\ell = \langle W^\ell_i : i < \mu\rangle$ for $\ell = 1, 2$ and $\bar{W}_1{}^\frown\bar{W}_2$ is a sequence of pairwise disjoint subsets of λ; but we may replace μ but a set of cardinality μ, even using two different such sets.
- (b) $\mathbf{c} : [\lambda]^2 \to \sigma$;
- (c) if $i \in W_1$ and $\varepsilon \in u_\varepsilon \in [\lambda]^{<\partial}$ for $\varepsilon \in W_i$ and $\gamma < \sigma$ <u>then</u> for some $\varepsilon < \zeta < \lambda$ we have:
  - (α) $\varepsilon \notin u_\zeta, \zeta \notin u_\varepsilon$;
  - (β) $\mathbf{c}\{\varepsilon, \zeta\} = \gamma$;
  - (γ) if $\xi_1 \in u_\zeta \backslash u_\varepsilon$ and $\xi_2 \in u_\varepsilon \backslash u_\zeta$ and $\{\xi_1, \xi_2\} \neq \{\varepsilon, \zeta\}$ <u>then</u> $\mathbf{c}\{\xi_1, \xi_2\} = 0$;
  - (δ) optional (u_ε, u_ζ) is a Δ-system pair (see proof);
- (d) if $\langle \mathscr{U}_\zeta : \zeta < \theta\rangle$ is $\subseteq$-increasing with union W_i where $i \in W_2$ then for some $\zeta < \theta$ we have $\mathrm{Rang}(\mathbf{c}{\restriction}[\mathscr{U}_\zeta]^2) = \sigma$.

3) We will say that the legal parameter $\mathbf{s}$ witness $\mathrm{Pr}_*(\lambda, \mu, \sigma, \partial, \theta)$ when $(\bar{\mathbf{c}}, \bar{I}_i, \bar{J}_i)$ witness it, (so $\bar{I}_i = \langle I_{\mathbf{s},i} : i \in S_3\rangle$ and $\bar{J}_i = \langle I_{\mathbf{s},i} : i \in S_3\rangle$).

Fact 3.9. 1) If $\lambda = \mu = \sigma$ is successor of regular and $\partial^+ = \theta^+ < \lambda$ then the property $\mathrm{Pr}_0(\lambda, \mu, \sigma, \partial, \theta)$ holds
2) There is a θ-indecomposable colouring $\mathbf{c} : [\lambda]^2 \to \theta$
3) If $(\lambda, \theta$ are as in Hyp 3.2(1) and) $\mu = \lambda, \sigma^+ < \lambda, \partial = \aleph_0$ then we can find a legal parameter $\mathbf{s}$ such that for every $i \in S_2 \setminus S_2^{\mathrm{limit}}$ the function $\mathbf{c}_i$ is θ-indecomposable, <u>but</u> do we have some freedom left for $i \in S_2$?..
4) If $(\lambda, \theta$ are as in Hyp 3.2(1) and) $\mu = \lambda, \sigma^+ < \lambda, \partial = \aleph_0$ then we can find a legal parameter $\mathbf{s}$ which witness $P\,\mathrm{Pr}_*(\lambda, \lambda, \aleph_0, \aleph_0, \theta)$

Proof. 1) By [Shed] and see history there.
2) Follows from part (1),
3) If part (1) apply then this follows, using a short parameter using such colouring. Otherwise Choose $\mathbf{s}$ as in 3.6(1) such that for every non limit $i \in S_2$, the function $\mathbf{c}_i$ is a θ-indecomposable function from $[I_i]^2$ onto J_i, this is possible by part (1) or directly by part (3).
4) By the recent version of [Shee], we can get more.

$\square_{3.9}$

Claim 3.10. *1) Assume $\theta = \mathrm{cf}(\theta) \in (\aleph_0, \lambda), \lambda = \lambda^{\langle \theta; \aleph_0\rangle}$ or just $\lambda = \lambda^{\langle \theta, \aleph_0\rangle}$, see Definition 0.10 recalling (see 3.2). If $G \in \mathbf{K}_{\leq \lambda}$, <u>then</u> there is $\mathbf{m} \in \mathbf{M}^2_{\lambda,\theta,\bar{S}}$ such that $G_{\mathbf{m},1} \cong G$.*
1A) If in part (1), in addition $\mathrm{Pr}_0(\lambda, \lambda, \lambda, \aleph_0)$ or just $\mathrm{Pr}_0(\lambda, \lambda, \aleph_0, \aleph_0)$ <u>then</u> we can add $\mathbf{m} \in \mathbf{M}^4_{\lambda,\theta,\bar{S}}$
1B) If in part (1) , in addition $\mathrm{Pr}_*(\lambda, \lambda, \aleph_0, \aleph_0, \theta)$ then we can add $\mathbf{m} \in \mathbf{M}^3_{\lambda,\theta.\bar{S}}$; (but here this always holds).
2) If $\lambda \geq 2^{\aleph_0}$ <u>then</u> in part (1) we can strengthen Definition 3.7 adding in clause $(e)(\varepsilon)\bullet_1, \bullet_2$ that $v = \omega\backslash\{0\}$ hence $\ell_0 = 1, \ell_1 = 2, \ldots$.
3) In part (2), if in addition $\mathrm{Pr}_0(\lambda, \lambda, \aleph_0, \aleph_0)$ then we can add $\mathbf{m} \in \mathbf{M}^{2.5}_{\lambda,\theta,5}$.

4) If $\lambda \geq \mu := \beth_\omega$ (or just μ strong limit) then for every large enough regular $\theta < \mu$, the assumption of part (1) holds.
5) If above $\theta = \aleph_1 < \lambda = \lambda^\theta$, then the assumption of part (1) holds.

Proof. 1) We us Claim 3.9(2),(3) still we have freedom in choosing the $\bar{j}$-s the $\bar{j}$-s., see below; then we shall choose $\mathbf{m} \in \mathbf{M}_2$ accordingly.

Case 1: $\lambda = \lambda^{\langle \theta, \aleph_0 \rangle}$, see §(0C).

Let $\mathscr{P}$ be a subset of $[\lambda]^{\aleph_0}$ of cardinality λ witnessing $\lambda = \lambda^{(\theta;\aleph_0)}$, so

- $(*)_1$ if $u \subseteq [\lambda]^\theta$ then $[u]^{\aleph_0} \cap \mathscr{P} \neq \emptyset$.

Without loss of generality $v \in \mathscr{P} \Rightarrow \text{otp}(v) = \omega$.

Hence

- $(*)_2$ if $\bar{\alpha} \in {}^\theta\lambda$ is increasing then $S_{\bar{\theta}} = \{\delta < \theta : \text{cf}(\delta) = \aleph_0$ and for some increasing $\bar{\varepsilon} \in {}^\omega\delta$ with limit δ we have $\{\varepsilon_n : n < \omega\} \in \mathscr{P}\}$ is stationary
- $(*)_3$ there is a stationary $S_2 \subseteq \{\delta < \theta : \text{cf}(\delta) = \aleph_0$ is stationary.

[Why? If $\theta > \aleph_1$ trivially, if not increasing $\mathscr{P}$ by decreasing using a pairing function.]

Now use 3.9(2)

Case 2: $\lambda = \lambda^{\langle \theta;\aleph_0 \rangle}$

Now we choose G_i and if $i < \theta$ also $\mathbf{a}_i, \mathbf{b}_i$ as required; but anyhow we are concentrating on the case $\lambda \geq 2^{\aleph_0}$, and then the two cases are equivalent.
1A) Similarly using 3.9(1)
1B) Similalry using 3.9(4)
2) Should be similar.
3) Straightforward.
4) By [She00] or see [She06, §1].
5) Check the definitions and 0.16. $\square_{3.10}$

Note that 3.11)(2),(3) is not used here but will help later,

Claim 3.11. *Let $\mathbf{m} \in \mathbf{M}_1$.*
1) If $i < j \leq \theta$ and $i \notin S_2^{\text{limit}}$ then $G_{\mathbf{m},i} \leq_{\mathfrak{S}} G_{\mathbf{m},j}$, see 3.2(3).
2) For every finite $A \subseteq G_{\mathbf{m},\theta}$ there is a sequence $\bar{u} = \langle u_i : i \in v \rangle$ such that:

- $(*)^1_{\bar{u}}$ *for $i \in S_2$*
  - (a) *$v \subseteq \theta$ is finite and $0 \in v$ for notational simplicity;*
  - (b) *$u_i \subseteq I_i$ is finite[8] for $i \in v$;*
  - (c) *if $i \in v$, then $\text{tp}_{\text{qf}}(\langle \bar{b}_{i,s} : s \in u_i \rangle, G_i, G_\theta)$ does not split over $\cup\{\bar{a}_{j,s} : j \in v \cap i$ and $s \in u_j\}$;*
  - (d) *if $i \in S_1$ and $s \in u_i$ then $\bar{a}_{i,s} \subseteq \text{sb}(\{\bar{b}_{j,s} : j \in v \cap i, s \in u_j\}, G_i)$;*
  - (e) *if $i \in S_2 \cup S_3$ and $s, t \in u_i$ then $\bar{a}_{i,\mathbf{c}\{s,t\}} \subseteq \text{sb}(\{\bar{b}_{j,s} : j \in v \cap i, s \in u_j\}, G_i)$;*
  - (f) *if $A \subseteq G_{\mathbf{m},i}$ and $i \in (0, \theta)$ then $v \subseteq i$;*
- $(*)_2$ *A is included in $\text{sb}(\{\bar{b}_{i,s} : i \in v, s \in u_i\}, G_\theta)$.*

3) We have $\bar{u} = \langle u^1_i \cup u^2_i : i \in v \rangle$ satisfies $(*)_1$, i.e. $(*)^1_{\bar{u}}$ from part (2) holds when:

[8]Note that in 3.11(2) we allow "u_i is empty".

$\oplus$ (a) $\bar{u}_\ell = \langle u^\ell_i : i \in v\rangle$ *for* $\ell = 1, 2$;

(b) *we have* $(*)^1_{\bar{u}_\ell}$ *for* $\ell = 1, 2$;

(c) *if* $i \in v, s_1 \in u^1_i \backslash u^2_i$ *and* $s_2 \in u^2_i \backslash u^1_i$ *then* $\mathbf{c}_i\{s_1, s_2\} = 0$.

3A) If $v_1 \subseteq v_2, \bar{u}^2 = \langle u_i : i \in v_2\rangle, \bar{u}^1 = \bar{u}^2 \restriction v_1$ *and* $i \in v_2 \backslash v_1 \Rightarrow u_i = \emptyset$ *then* $(*)^1_{\bar{u}^1} \Leftrightarrow (*)^1_{\bar{u}^2}$.
4) The type $\mathrm{tp}_{\mathrm{qf}}(\langle \bar{b}^\ell_{i,s} : s \in u^\ell_i, \ell \in \{1,2\}\rangle, G_i, G_{i+1})$ *does not split over* $\{\bar{b}^\ell_{j,s} : j \in v \cap i, s \in u^\ell_j, \ell \in \{1,2\}\} \cup \{a_{i,\alpha}\}$ *when:*

(a) $\bar{u}_\ell = \langle u^\ell_j : j \in v\rangle$;

(b) $(*)^1_{\bar{u}_\ell}$ *holds for* $\ell = 1, 2$;

(c) $i \in S_3 \cap v$;

(d) $s_* \in u^1_i \backslash u^2_i, t_* \in u^2_i \backslash u^1_i$;

(e) $\alpha = \mathbf{c}_i\{s_*, t_*\}$;

(f) *clause (c) from part (3) holds when* $\{s_1, s_2\} \neq \{s_*, t_*\}$.

Proof. 1) By part (2) recalling the assumptions on $\mathfrak{S}$.
2) By induction on $\min\{j < \theta : A \subseteq G_{\mathbf{m},j}\}$. Note that for $A \subseteq G_1$ clause $(*)^1_{\bar{u}}(c)$ is trivial.
3),4) Easy, too. $\square_{3.11}$

Main Claim 3.12. *If* $\mathbf{m} \in \mathbf{M}_2$, *then* $G_{\mathbf{m},\theta} \in \mathbf{K}^{\mathrm{exlf}}_\lambda$ *is complete and is* $(\lambda, \theta, \mathfrak{S})$*-full, (see* [She17, 1.15=La33]*) and extend* $G_{\mathbf{m},1}$.

Proof. Being in $\mathbf{K}^{\mathrm{lf}}_\lambda$ is obvious as well as extending $G_{\mathbf{m},1}$; being $(\lambda, \theta, \mathfrak{S})$-full is witnessed by $\langle G_{\mathbf{m},i} : i < \theta\rangle, S_1$ being unbounded in θ and clauses 3.3(c), 3.7(c)(ε) so far $\mathbf{m} \in \mathbf{M}_{1.5}$ is sufficient.

The main point is proving $G_{\mathbf{m},\theta}$ is complete, so assume π is an automorphism of $G_{\mathbf{m},\theta}$.

Now

$(*)_1$ if $i \in S^{\mathrm{limit}}_2$ thenG$_i$ is θ-indecomposable in G_{i+3}.

[Why? By 3.7(d)(θ) .]

So $\langle \pi(G_{\mathbf{m},i}) : i < \theta\rangle$ is $\leq_{\mathbf{K}_{\mathrm{lf}}}$-increasing with union $G_{\mathbf{m},\theta}$ hence by $(*)_1$ above, if $i \in S6[\mathrm{limit}]_2$ is a limit ordinal then $(\forall^\infty j < \theta)(G_{\mathbf{m},i} \subseteq \pi(G_{\mathbf{m},j}))$. The parallel statement holds for π^{-1} hence E is a club of θ where $E := \{i < \theta : i$ is a limit ordinal, hence $i = \sup(S_1 \cap i)$ and π maps $G_{\mathbf{m},i}$ onto $G_{\mathbf{m},i}\}$; note that by 3.7(c)(ε) and the middle demand, $i \in E \Rightarrow G_i \in \mathbf{K}_{\mathrm{exlf}}$.

Next we define:

$(*)_2$ $S^\bullet$ is the set of $i \in E \cap S_1$ such that π is not the identity on $\mathbf{C}(G_{\mathbf{m},i}, G_{\mathbf{m},i+\omega})$.

The proof now split to two cases. Case 1: $S^\bullet$ is unbounded in θ

So for $i \in S^\bullet$ choose $c_i \in \mathbf{C}(G_{\mathbf{m},i}, G_{\mathbf{m},i+\omega})$ such that $\pi(c_i) \neq c_i$. Without loss of generality c_i has order 2, because the set of elements of order 2 from $\mathbf{C}(G_{\mathbf{m},i}, G_{\mathbf{m},i+\omega})$ generates it, see [She17, 4.1=Ld36,4.10=Ld93]. Choose $\langle \mathbf{i}_\varepsilon = \mathbf{i}(\varepsilon) : \varepsilon < \theta\rangle$ increasing, $\mathbf{i}_\varepsilon \in S^\bullet$ and so as $\mathbf{i}_\varepsilon + \omega \leq \mathbf{i}_{\varepsilon+1} \in E$ clearly $\pi(c_\varepsilon) \in G_{\mathbf{m},\mathbf{i}(\varepsilon+1)}$. Now we apply 3.7(e), 3.8(1) and get contradiction by 3.11(4) recalling 3.7(2)(h) and 3.3(e); but we elaborate.

Now shall we apply 3.7(1)(e), (indirectly 3.10(1), 0.10). So there are $(i, \alpha, v, \ell_0, \ell_1, \ldots, \varepsilon_0, \varepsilon_1, \ldots)$ as there, in particular $i \in S_3$ and here $v = \omega \backslash \{0\}$. Now for every $s \in I_{i,\alpha}$ we apply 3.11(2), getting $\bar{u}_s = \langle u_{s,\iota} : \iota \in v_s \rangle$ and let ℓ_s be such that $v_s \subseteq j_{i,\omega\alpha+\ell_s}$, without loss of generality $i \in v_s, s \in u_{s,i}$.

Now consider the statement:

$(*)_3$ there are $s_1 \neq s_2 \in I_{i,\alpha}$ and k such that:

(a) $\mathbf{c}\{s_1, s_2\} = \ell_k$;

(b) $\ell_k > \ell_{s_1}, \ell_{s_2}$;

(c) if $t_1 \in \cup\{u_{s_1,\iota} : \iota \in v_{t_1} \backslash i\}, t_2 \in \cup\{u_{s_2,\iota} : \iota \in v_{t_2} \backslash i\}$ and $\{t_1, t_2\} \neq \{s_1, s_2\}$ <u>then</u> $\mathbf{c}\{t_1, t_2\} = 0$;
or for later proofs:

(c)′ (α) if $t_1 \in u_{s_1,i} \backslash u_{s_2,i}$ and $t_2 \in u_{s_2,i} \backslash u_{s_1,i}$ and
- $\{t_1, t_2\} \neq \{s_1, s_2\}$ then $\mathbf{c}\{t_1, t_2\} = 0$, <u>or</u> just
- $t_1, t_2 \in I_{i,\alpha} \Rightarrow \mathbf{c}\{t_1, t_2\} < \ell_k$;
- $t_1, t_2 \in I_{i,\beta}, \beta < \lambda; \beta \neq \alpha$ then $j_{i,\omega\beta+\mathbf{c}\{t_1,t_2\}} < j_{i,\omega\alpha+\ell(k)}$ (we use $j_{i,\omega\alpha+\ell} \in (j^*_{i,\ell}, j^*_{i,\ell+1})$ - check);

(β) if $\iota \in v_1 \cap v_2$ and $\iota > i, (\iota \in S_3), \beta < \lambda$ and $t_1 \in v_{s_1,\iota}, t_2 \in v_{s_2,\iota}$ then $\mathbf{c}\{t_1, t_2\} = 0$.

Now why is $(*)_3$ true? This is by the choice of $\mathbf{c}$, that is, as $\mathbf{c}$ witnesses $\mathrm{Pr}_0(\lambda, \lambda, \lambda, \aleph_0)$

Now to get a contradiction we would like to prove:

$(*)_4$ the type $\mathrm{tp}((\pi(b_{s_1}), \pi(b_{s_2})), G_{\mathbf{m},i}, G_{\mathbf{m},\theta})$ does not split over $G_{\mathbf{m},j_{i,\omega\alpha+\ell(k)}} \cup \{c_{\mathbf{i}(\varepsilon_k)}\}$ hence over $G_{\mathbf{m},\mathbf{i}(\varepsilon_k)} \cup \{c_{\mathbf{i}(\varepsilon(k))}\}$.

It follows from $(*)_4$ that $\mathrm{tp}((b_{s_1}, b_{s_2}), \pi^{-1}(G_{\mathbf{m},i}), \pi^{-1}(G_{\mathbf{m},\theta}))$ does not split over $\pi^{-1}(G_{m,\mathbf{i}(\varepsilon_k)}) \cup \{\pi^{-1}(c_{\mathbf{i}(\varepsilon)})\})$. But $i(\varepsilon_k), i \in E$ have it follows that $\pi(G_{m,i}) = G_{\mathbf{m},i}$ and $\pi^{-1}(G_{\mathbf{i}(\varepsilon_k)} = G_{\mathbf{i}(\varepsilon_k)})$ has $\mathrm{tp}((b_{s_1}, b_{s_2}), G_{\mathbf{m},i}, G_{\mathbf{m},\theta})$ does not split over $G_{\mathbf{i}(\varepsilon_k)} \cup \{\pi^{-1}(c_{\mathbf{i}(\varepsilon_k)})\}$.

Now $[b_{s_1}, b_{s_2}] = \pi^1([b_{s_1}, b_{s_2}]) = \pi^{-1}(c_{\mathbf{i}(\varepsilon_k)})$ which is $\neq c_{i(\varepsilon_k)}$. But as $c_{\mathbf{i}(\varepsilon_k)} \in \mathbf{C}(G_{\mathbf{m},\mathbf{i}(\varepsilon_k)}, G_{\mathbf{m},\theta})$ clearly also $\pi^{-1}(c_{\mathbf{i}(\varepsilon_k)})$ belongs to it, hence it follows that $\pi^{-1}(c_{\mathbf{i}(\varepsilon_k)}) \in \mathrm{sb}(\{c_{\mathbf{i}(\varepsilon_k)}\}; G_\theta)$, but as $c_{\mathbf{i}(\varepsilon_k)}$ has order two, the latter belongs to $\{c_{\mathbf{i}(\varepsilon_k)}, e_{G_\sigma}\}$.

However $\pi^{-1}(c_{\mathbf{i}(\varepsilon_k)})$ too has order 2 hence is equal to $c_{\mathbf{i}(\varepsilon_k)}$; applying π we get $c_{\mathbf{i}(\varepsilon_k)} = \pi(c_{\mathbf{i}(\varepsilon_k)})$ a contradiction to the choice of the c_i's.

<u>Case 2</u>: $i_* = \sup(S^\bullet) + 1$ is $< \theta$.

Now for any $i \in S' := E \cap S_1 \backslash i_*$ by [She17, 2.18=Lc62] there is $g_i \in G_{\mathbf{m},i+1}$ such that $\square^{g_i}(G_{\mathbf{m},i}) \subseteq \mathbf{C}(G_{\mathbf{m}_i}, G_{\mathbf{m},i+1})$. So if $a \in G_{\mathbf{m},i}$ then $g_i^{-1} a g_i \in \mathbf{C}(G_{m,i}, G_{\mathbf{m},i+1})$ and $a = g_i(g_i^{-1} a g_i) g_i^{-1}$ hence $\pi(a) = \pi(g_i)\pi(g_i^{-1} a g_i)\pi(g_i^{-1}) = \pi(g_i)(g_i^{-1} a g_i)\pi(g_i)^{-1}$ recalling $i \notin S^\bullet$ being $\geq i_*$ hence $\pi(a) = (g_i \pi(g_i)^{-1})^{-1} a g_i \pi(g_i^{-1})$. If for some g the set $\{i \in S' : g_i = g\}$ is unbounded in θ we are easily done, so toward contradiction assume this fails.

But for every $\delta \in \mathrm{acc}(E) \cap S_1 \backslash i_*$, we can by 3.11(1) choose a finite $\bar{a}_\delta \subseteq G_\delta$ and $\mathfrak{s}_\delta \in \mathfrak{S}$ such that $\mathrm{tp}_{\mathrm{bs}}(\pi(g_\delta) g_\delta^{-1}, G_\delta, G_\theta) = q_{\mathfrak{s}_\delta}(\bar{a}_\delta, G_\delta)$ and let $i(\delta) \in E \cap \delta$ be such that $\bar{a}_\delta \subseteq G_{i(\delta)}$.

Clearly:

$\circledast$ if $d_1, d_2 \in G_\delta, d_2 \neq \pi(d_1)$ then $\mathrm{tp}_{\mathrm{bs}}(\langle d_1, d_2 \rangle, \bar{a}_\delta, G_\delta) \neq \mathrm{tp}_{\mathrm{bs}}(\langle d_1, \pi(d_1) \rangle, \bar{a}_\delta, G_\delta)$.

[Why? Because $\pi(d_1) = \pi(g_\delta)g_i^{-1}d_1 g_i \pi(g_\delta)^{-1}$ and the choice of $\bar{a}_\delta$.]

Hence for some group term $\sigma_{d_1}(\bar{x}_{1+\ell g(\bar{b}_\delta)})$ we have $\pi(d_1) = \sigma_{d_1}^{G_\delta}(d_1, \bar{a}_\delta)$ and σ_{d_1} depends only on $\mathrm{tp}_{\mathrm{bs}}(d_1, \bar{a}_\delta, G_\delta)$. By Fodor Lemma for some $i(*)$ the set $S = \{\delta : \delta \in \mathrm{acc}(E) \cap S_1 \backslash i_* \text{ and } i(\delta) = i(*)\}$ is a stationary subset of θ.

Now we can finish easily, e.g. as G_δ for $\delta \in S$ belongs to $\mathbf{K}_{\mathrm{exlf}}$ and we know that it can be extended to a complete $G' \in \mathbf{K}_{\mathrm{exlf}}$ or just see that all the definitions in $\circledast$ agree and should be one conjugation. $\square_{3.12}$

Conclusion 3.13. *1) Assume $\lambda > \beth_\omega$ and $G \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ and $\theta = \mathrm{cf}(\theta) \in (\aleph_0, \beth_\omega)$ is large enough and $\mathfrak{S}$ is as in 3.2(3).*

Then there is a complete $(\lambda, \theta, \mathfrak{S})$-full $H \in \mathbf{K}^{\mathrm{exlf}}_\lambda$ extending G.
2) Instead $\lambda > \beth_\omega$ we can assume $\lambda = \lambda^{\aleph_0} > \aleph_1$.

Proof. 1) Fixing λ and θ and it suffices to find $\mathbf{m} \in \mathbf{M}^3_{\lambda,\theta}$ such that $G_{\mathbf{m},1} = G$. As $\lambda \geq \beth_\omega$, the assumption of 3.10(1) holds for every sufficiently large $\theta < \beth_\omega$; hence there is $\mathbf{m} \in \mathbf{M}^2_{\lambda,\theta,\bar{S}}$ such that $G_{\mathbf{m},1}$ is isomorphic to G and $\bar{S}$ as there.

As λ is a successor of a regular, the assumption of 3.10(1A) holds (by 3.8(1) hence $\mathbf{m} \in \mathbf{M}^3_{\lambda,\theta,\bar{S}}$. So by 3.12 we indeed are done. $\square_{3.13}$

Remark 3.14. The assumption "$\lambda > \beth_\omega$" comes from quoting 3.10(2) hence it is "hard" for $\lambda < \beth_\omega$ to fail. Similarly below.

Of course we have:

Observation 3.15. *If $\mathbf{m} \in \mathbf{M}_{1.5}$ then $G_{\mathbf{m},\theta}$ is $(\lambda, \theta, \mathfrak{S})$-full and extends $G_{\mathbf{m},0}$.*

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