MORE ON WEAK DIAMOND

SAHARON SHELAH

Abstract. We deal with the combinatorial principle Weak Diamond, showing that we always either a local version is not saturated or we can increase the number of colours. Then we point out a model theoretic consequence of Weak Diamond.

0. Basic definitions

In this section we present basic notations, definitions and results.

The paper was circulated (including the math arXive) and accepted to the East-West Journal of Math around 2000, but due to some problems between the editors has not appeared. Meanwhile Aspero, Larson and Moore [**?**] with a related result has appeared.

Notation 0.1.
(1) $\kappa, \lambda, \theta, \mu$ will denote cardinal numbers and α, β, δ, ε, ξ, ζ, γ will be used to denote ordinals.
(2) Sequences of ordinals are denoted by ν, η, ρ (with possible indexes).
(3) The length of a sequence η is $\ell g(\eta)$.
(4) For a sequence η and $\ell \le \ell g(\eta)$, $\eta{\restriction}\ell$ is the restriction of the sequence η to ℓ (so $\ell g(\eta{\restriction}\ell) = \ell$). If a sequence ν is a proper initial segment of a sequence η then we write $\nu \lhd \eta$ (and $\nu \trianglelefteq \eta$ has the obvious meaning).
(5) For a set A and an ordinal α, α_A stands for the function on A which is constantly equal to α.
(6) For a model M, $|M|$ stands for the universe of the model.
(7) The cardinality of a set X is denoted by $\|X\|$. The cardinality of the universe of a model M is denoted by $\|M\|$.

Definition 0.2. Let λ be a regular uncountable cardinal and θ be a cardinal number.

(1) *A* (λ,θ)*–colouring* is a function $F : \mathrm{DOM} \longrightarrow \theta$, where DOM is either ${}^{<\lambda}2 = \bigcup\limits_{\alpha<\lambda} {}^{\alpha}2$ or $\bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$. In the first case we will write $\mathrm{DOM}_\alpha = {}^{1+\alpha}2$, in the second case we let $\mathrm{DOM}_\alpha = {}^{1+\alpha}(\mathcal{H}(\lambda))$ (for $\alpha \le \lambda$).

Date: September 26, 2020.

The research partially supported by The National Science Foundation Grant No. 144–EF67. Publication 638.

If λ is understood we may omit it; if $\theta = 2$ then we may omit it too (thus *a colouring* is a $(\lambda, 2)$–colouring).

(2) For a (λ, θ)–colouring F and a set $S \subseteq \lambda$, we say that a function $\eta \in {}^S\theta$ is *an F–weak diamond sequence for S* if for every $f \in \mathrm{DOM}_\lambda$ the set
$$\{\delta \in S : \eta(\delta) = F(f{\restriction}\delta)\}$$
is stationary.

(3) WDmId_λ is the collection of all sets $S \subseteq \lambda$ such that for some colouring F there is no F–weak diamond sequence for S.

Remark 0.3. In the definition of WDmId_λ (0.2(3)), the choice of DOM (see 0.2(1)) does not matter; see [She98, AP, §1], remember that $\|\mathcal{H}(\lambda)\| = 2^{<\lambda}$.

Theorem 0.4 (Devlin Shelah [DS78]; see [She98, AP, §1] too). *Assume that $2^\theta = 2^{<\lambda} < 2^\lambda$ (e.g. $\lambda = \mu^+$, $2^\mu < 2^\lambda$). Then for every λ–colouring F there exists an F–weak diamond sequence for λ. Moreover,* WDmId_λ *is a normal ideal on λ (and $\lambda \notin \mathrm{WDmId}_\lambda$).*

Remark 0.5. One could wonder why the weak diamond (and WDmId_λ) is interesting. Below we list some of the applications, limitations and related problems.

(1) Weak diamond is really weaker than diamond, but provably (in ZFC) it holds true for some cardinals λ. Note that under GCH, $\diamondsuit_{\mu^+}$ holds true for each $\mu > \aleph_0$, so the only interesting case then is $\lambda = \aleph_1$.

(2) Original interest in this combinatorial principle comes from Whitehead groups:

if G is a strongly λ–free Abelian group and $\Gamma(G) \notin$ WDmId_λ
then G is Whitehead.

(3) A related question was: can we have stationary subsets $S_1, S_2 \subseteq \omega_1$ such that $\diamondsuit_{S_1}$ but $\neg\diamondsuit_{S_2}$? (See [She77].)

(4) Weak diamond has been helpful particularly in problems where we have some uniformity, e.g.:

$(*)_1$ *Assume $2^\lambda < 2^{\lambda^+}$. Let $\psi \in \mathbb{L}_{\lambda^+,\omega}$ be categorical in λ, λ^+. Then $(\mathrm{MOD}_\psi, \prec_{\mathrm{Frag}(\psi)})$ has the amalgamation property in λ.*

$(*)_2$ *If G is an uncountable group then we can find subgroups G_i of G (for $i < \lambda$) non-conjugate in pairs (see* [She87b]*).*

(5) One may wonder if assuming $\lambda = \mu^+$, $2^\lambda > 2^\mu$ (and e.g. μ regular) we may find a regular $\sigma < \mu$ such that
$$\{\delta < \lambda : \mathrm{cf}(\delta) = \sigma\} \notin \mathrm{WDmId}_\lambda(\lambda).$$
Unfortunately, this is not the case (see [She85] even for $\mu = \aleph_1$).

(6) We would like to prove

(a) WDmId_λ is not λ^+–saturated or

(b) a strengthening, e.g. weak diamond for more colours.

We will get (a variant of) a local version of the disjunction, where we essentially fix F. There are two reasons for interest in **(a)**: understanding λ^+–saturated normal ideals (e.g. we get more information on the case CH + "$\mathcal{D}_{\omega_1}$ is $\aleph_2$–saturated"; see also Zapletal Shelah [SZ99]), and non λ^+–saturation helps in "non-structure theorems" (see [She83], [She01]). That is, having $2^\mu < 2^{\mu^+} < 2^{\mu^{++}}$ and some "bad" (i.e. "nonstructure") properties for models in μ we get $2^{\mu^{++}}$ models in μ^{++} when $\mathrm{WDmId}_{\lambda^+}$ is not λ^{++}–saturated (and using the local version does not hurt).

(7) Note that for $S \notin \mathrm{WDmId}_\lambda$ we have a weak diamond sequence $f \in {}^S2$ such that the set of "successes"(=equalities) is stationary, but it does not have to be in $(\mathrm{WDmId}_\lambda)^+$. We would like to start and end in the same place: being positive for the same ideal. Also, in **(b)** above the set of places we guess was stationary, when we start with $S \in (\mathrm{WDmId}_\lambda)^+$.

Note that it may well be that $\lambda \in \mathrm{WDmId}_\lambda$ (if $(\exists\theta < \lambda)(2^\theta = 2^\lambda)$ this holds), but some "local" versions may still hold. E.g. in the Easton model, we have F–weak diamond sequences for all F which are reasonably definable (see [She98, AP, §1]; define

$$F(f) = 1 \;\Leftrightarrow\; \mathrm{L}[X, f] \models \varphi(X, f)$$

for a fixed first order formula φ, where $X \subseteq \lambda$ depends on F only). So the case $\mathrm{WDmId}_\lambda = \mathcal{P}(\lambda)$ has some interest.

We would like to thank Andrzej Rosłanowski for mathematical comments and improving the presentation.

1. When colourings are almost constant

Definition 1.1. Let λ be a regular uncountable cardinal.

(1) Let $S \subseteq \lambda$ and let F be a (λ, θ)–colouring. We say that *a sequence $\eta \in {}^S\theta$ is coded by F* if there exists $f \in \mathrm{DOM}_\lambda$ such that

$$\alpha \in S \quad \Leftrightarrow \quad \eta(\alpha) = F(f{\restriction}(1+\alpha)).$$

We let

$$\mathfrak{B}(F) \stackrel{\text{def}}{=} \{\eta \in {}^\lambda\theta : \eta \text{ is coded by } F\}.$$

(2) For a family $\mathcal{A}$ of subsets of λ let $\mathrm{ideal}_\lambda(\mathcal{A})$ be the λ–complete normal ideal on λ generated by $\mathcal{A}$ (i.e. it is the closure of $\mathcal{A}$ under unions of $< \lambda$ elements, diagonal unions, containing singletons, and subsets).

[Note that $\mathrm{ideal}_\lambda(\mathcal{A})$ does not have to be a proper ideal.]

(3) For a λ–colouring F (so $\theta = 2$) we define by induction on α:

$$\mathrm{ID}_0^-(F) = \emptyset, \qquad \mathrm{ID}_0(F) = \{S \subseteq \lambda : S \text{ is not stationary }\},$$

for a limit α

$$\mathrm{ID}_\alpha^-(F) = \bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F), \qquad \mathrm{ID}_\alpha(F) = \mathrm{ideal}_\lambda(\bigcup_{\beta<\alpha} \mathrm{ID}_\beta(F)),$$

and for $\alpha=\beta+1$

$$\begin{aligned}\mathrm{ID}^-_\alpha(F)= &\ \big\{S\subseteq\lambda: \text{for each } S^*\subseteq S \text{ there is } f\in\mathrm{DOM}_\lambda \text{ such that}\\ &\quad \{\delta<\lambda:\delta\in S^*\Leftrightarrow F(f\restriction\delta)=0\}\in\mathrm{ID}_\beta(F)\big\};\\ \mathrm{ID}_\alpha(F)= &\ \mathrm{ideal}_\lambda\big(\mathrm{ID}^-_\alpha(F)\big).\end{aligned}$$

Finally we let $\mathrm{ID}(F)=\bigcup\limits_\alpha \mathrm{ID}_\alpha(F)$.

(4) We say that F is *rich* if $\mathrm{DOM}(F)=\bigcup\limits_{\alpha<\lambda}{}^\alpha\mathcal{H}(\lambda)$, and for every function $f\in\mathrm{DOM}_\lambda$ and $\alpha<\lambda$ and a set $A\subseteq\alpha$ there is $f'\in\mathrm{DOM}_\lambda$ such that

$$(\forall i<\lambda)(f(1+i)=f'(1+i)\ \&\ F(f\restriction(\alpha+i))=F(f'\restriction(\alpha+i)))$$

and $(\forall j<\alpha)(F(f'\restriction j)=1 \ \Leftrightarrow\ j\in A)$.

Definition 1.2. Let λ be a regular uncountable cardinal and let F be a λ–colouring.

(1) $\mathrm{WDmId}_\lambda(F)$ is the family of all sets $S\subseteq\lambda$ with the property that for every $S^*\subseteq S$ there is $f\in\mathrm{DOM}_\lambda$ such that the set

$$\{\delta\in S:\delta\in S^*\ \Leftrightarrow\ F(f\restriction\delta)=1\}$$

is not stationary.

(2) $\mathfrak{B}^+(F)$ is the closure of

$$\mathfrak{B}(F)\cup\{S\subseteq\lambda: S \text{ is not stationary }\}$$

under unions of $<\lambda$ sets, complement and diagonal unions (here, in $\mathfrak{B}(F)$, we identify a subset of λ with its characteristic function).

(3) $\mathrm{ID}^1(F)\stackrel{\text{def}}{=}\{S\subseteq\lambda:(\exists X\in\mathfrak{B}^+(F))(S\subseteq X\ \&\ \mathcal{P}(X)\subseteq\mathfrak{B}^+(F))\}$.

(4) $\mathrm{ID}^2(F)$ is the collection of all $S\subseteq\lambda$ such that for some $X\in\mathfrak{B}^+(F)$ we have: $S\subseteq X$ and there is a partition X_0,X_1 of X such that
 (α) $\mathcal{P}(X_\ell)=\{Y\cap X_\ell: Y\in\mathfrak{B}^+(F)\}$ for $\ell=0,1$, and
 (β) there is no $Y\in\mathfrak{B}^+(F)$, $\ell<2$ satisfying

$$Y\setminus X_\ell\in\mathrm{ID}^1(F)\quad\&\quad Y\notin\mathrm{ID}^1(F).$$

Proposition 1.3. *Assume λ is a regular uncountable cardinal and F is a λ–colouring.*

(1) *If $\mathcal{A}$ is a family of subsets of λ such that*
 $(\circledast_{\mathcal{A}})$ *if $S_0\subseteq S_1$ and $S_1\in\mathcal{A}$ and $A\in[\lambda]^{<\lambda}$ then $S_0\cup A\in\mathcal{A}$,*
then $\mathrm{ideal}_\lambda(\mathcal{A})$ is the collection of all diagonal unions $\mathop{\nabla}\limits_{\xi<\lambda}A_\xi$ such that $A_\xi\in\mathcal{A}$ for $\xi<\lambda$.

(2) *The condition $(\circledast_{\mathrm{ID}^-_\alpha(F)})$ (see above) holds true for each α. Consequently, if $\alpha=\beta+1$ then $\mathrm{ID}_\alpha(F)=\{\mathop{\nabla}\limits_{i<\lambda}A_i:\langle A_i:i<\lambda\rangle\subseteq \mathrm{ID}^-_\alpha(F)\}$, and if α is limit then $\mathrm{ID}_\alpha(F)=\{\mathop{\nabla}\limits_{i<\lambda}A_i:\langle A_i:i<\lambda\rangle\subseteq\bigcup\limits_{\beta<\alpha}\mathrm{ID}_\beta(F)\}$.*

(3) $\mathrm{ID}(F)$ *and* $\mathrm{ID}_\alpha(F)$ *are* λ*–complete normal ideals on* λ *extending the ideal of non-stationary subsets of* λ *(but they do not have to be proper). For* $\alpha<\gamma$ *we have* $\mathrm{ID}_\alpha(F)\subseteq \mathrm{ID}_\gamma(F)$ *and hence* $\mathrm{ID}(F)=\mathrm{ID}_\alpha(F)$ *for every large enough* $\alpha<(2^\lambda)^+$.

(4) *Suppose* $\bar B=\langle B_\ell:\ell\le m\rangle$, *where* $B_\ell\subseteq B_{\ell+1}$ *(for* $\ell<m$*) and* $B_m\in \mathrm{ID}(F)$. *Then* $\bar B$ *has an* F*–representation, which means that there are a well founded tree* $T\subseteq {}^{\omega>}\lambda$, *sequences* $\langle B^\ell_\eta:\eta\in T,\ \ell\le\ell_\eta\rangle$, *and* $\langle f^k_\eta:\eta\in T,\ k\le k_\eta\rangle$ *such that* $k_\eta\le\ell_\eta+1$ *and*

 (a) $B^\ell_{\langle\rangle}=B$, $\ell_{\langle\rangle}=m$, $B^\ell_\eta\subseteq B^{\ell+1}_\eta\subseteq\lambda$, $f^\ell_\eta\in{}^\lambda 2$,

 (b) $(\forall\eta\in T\setminus\max(T))(\forall i<\lambda)(\eta^\frown\langle i\rangle\in T)$,

 (c) *for each* $\eta\in T\setminus\max(T)$ *there is* $\alpha_\eta<\lambda$ *such that for all* $\delta\in\lambda\setminus\alpha_\eta$

  $(\oplus)$ $\delta\in B^\ell_\eta$ *iff*

   $(\exists i<\delta)(\delta\in B^\ell_{\eta^\frown\langle i\rangle})$ *or*

   $F(f^\ell_\eta\restriction\delta)=1\ \&\ \neg(\exists i<\delta)(\exists k)(\delta\in B^k_{\eta^\frown\langle i\rangle})$,

 (d) *for each* $\eta\in\max(T)$, B_η *is a bounded subset of* λ *with* $\min(B_\eta)>\max(\{\eta(n):n<\ell g(\eta)\})$.

(5) *If for some* $f^*\in{}^\lambda 2$ *we have* $(\forall\alpha<\lambda)(F(f^*\restriction\alpha)=0)$ *then in part (4) above we can demand that* $k_\eta=\ell_\eta+1$.

(6) *If* F *is rich then in part (4) above we can add*

 (e) α_η*=0 for* $\eta\in T\setminus\max(T)$ *and* $B_\eta=\emptyset$ *for* $\eta\in\max(T)$.

(7) $\mathrm{ID}(F)$ *is the minimal normal filter on* λ *such that there is no* $S\in(\mathrm{ID}(F))^+$ *satisfying*

$$(\forall S^*\subseteq S)(\exists A\in\mathfrak{B}(F))(S^*\mathbin{\triangle} A\in\mathrm{ID}(F)).$$

Proof. (1)–(2) Should be clear.

(3) By induction on $\gamma<\lambda$ and then by induction on $\alpha<\gamma$ we show that $(\forall\gamma<\lambda)(\forall\alpha<\gamma)(\mathrm{ID}_\alpha(F)\subseteq\mathrm{ID}_\gamma(F))$. If $\gamma=1$ then this follows immediately from definitions; similarly if γ is limit. So suppose now that $\gamma=\gamma_0+1$ and we proceed by induction on $\alpha\le\gamma_0$. There are no problems when $\alpha=0$ nor when α is limit. So suppose that $\alpha=\beta+1<\gamma$ (so $\beta<\gamma_0$). By the inductive hypothesis we know that $\mathrm{ID}_\beta(F)\subseteq\mathrm{ID}_{\gamma_0}(F)$. Let $A\in\mathrm{ID}_{\beta+1}(F)$. By (2) there are $A_\xi\in\mathrm{ID}^-_{\beta+1}$ (for $\xi<\lambda$) such that $A=\mathop{\nabla}\limits_{\xi<\lambda}A_\xi$. Now look at the definition of $\mathrm{ID}^-_{\beta+1}(F)$: since $\mathrm{ID}_\beta(F)\subseteq\mathrm{ID}_{\gamma_0}(F)$ we see that $A_\xi\in\mathrm{ID}^-_{\gamma_0+1}(F)$. Hence $A\in\mathrm{ID}_\gamma$.

(4) By induction on α we show that if $\bar B=\langle B_\ell:\ell\le m\rangle$, where $B_\ell\subseteq B_{\ell+1}$ (for $\ell<m$) and $B_m\in\mathrm{ID}_\alpha(F)$ then $\bar B$ has an F–representation.

Case 1: $\alpha=0$.

Thus the set B_m is not stationary and we may pick up a club E of λ disjoint from B_m. Let $E=\{\alpha_\zeta:\zeta<\lambda\}$ be the increasing enumeration. Put $T=\{\langle\rangle\}\cup\{\langle i\rangle:i<\lambda\}$, $\alpha_{\langle\rangle}=1$, $\ell_{\langle\rangle}=\ell_{\langle i\rangle}=m$, $B^\ell_{\langle\rangle}=B_\ell$ and $B^\ell_{\langle i\rangle}=B_\ell\cap\alpha_{i+1}$. Now check.

CASE 2: α is limit.
It follows from (2) that $B_\ell = \mathop{\nabla}\limits_{i<\lambda} B_{\ell,i}$ for some $B_{\ell,i} \in \bigcup\limits_{\beta<\alpha} \mathrm{ID}_\beta(F)$. Let $B'_{\ell,i}$ be defined as follows:

if $i = (m+1)j + t$, $\ell < t \leq m$ then $B'_{\ell,i} = \emptyset$,
if $i = (m+1)j + t$, $t \leq m$, $t \leq \ell$ then $B'_{\ell,i} = B_{\ell,i}$.

Then for each i, ℓ we may find $\langle B^{i,\ell}_\eta, f^{i,\ell'}_\eta, \alpha^i_\eta : \eta \in T_i,\ \ell < \ell^{i,1}_\eta,\ \ell' < \ell^{i,2}_\eta \rangle$ satisfying clauses (a)–(d) and such that $\langle B^{\ell,i,k}_{\langle\rangle} : k \leq k^1_\eta \rangle = \langle B'_{\ell,i} : \ell \leq m \rangle$ (by the inductive hypothesis). Put

$$\begin{aligned}
&T = \{\langle\rangle\} \cup \{\langle i\rangle^\frown\eta : \eta \in T_i\},\\
&\ell_{\langle\rangle} = m, \quad \ell'_{\langle\rangle} = 0, \quad \ell_{\langle i\rangle^\frown\eta} = \ell^{i,1}_\eta, \quad \ell_{\langle i\rangle^\frown\eta} = \ell^{i,2}_\eta\\
&B^\ell_{\langle\rangle} = B_\ell, \quad B^\ell_{\langle i\rangle^\frown\eta} = B^{i,\ell}_\eta, \quad f^{\ell'}_{\langle i\rangle^\frown\eta} = f^{i,\ell'}_\eta,\\
&\alpha_{\langle\rangle} = \omega, \quad \alpha_{\langle i\rangle^\frown\eta} = \alpha^i_\eta.
\end{aligned}$$

Checking that $\langle B^\ell_\eta, f^{\ell'}_\eta, \alpha_\eta : \eta \in T,\ \ell \leq \ell_\eta,\ \ell' \leq \ell'_\eta \rangle$ is as required is straightforward.
CASE 3: $\alpha = \beta + 1$.
By (2) above and the proof of Case 2 we may assume that $B_m \in \mathrm{ID}^-_\alpha(F)$. It follows from the definition of $\mathrm{ID}^-_\alpha(F)$ that there are $f_\ell \in {}^\lambda 2$ (for $\ell \leq m$) such that

$$B^\oplus_\ell \stackrel{\mathrm{def}}{=} \{\delta < \lambda : \delta \text{ is limit and } F(\eta{\restriction}\delta) = 0 \Leftrightarrow \delta \in B_\ell\} \in \mathrm{ID}_\beta(F),$$

and hence $B^\oplus \stackrel{\mathrm{def}}{=} \bigcup\limits_{\ell\leq m} B^\oplus_\ell \in \mathrm{ID}_\beta(F)$. Therefore $B^*_\ell \stackrel{\mathrm{def}}{=} B_\ell \cap B^\oplus \in \mathrm{ID}_\beta(F)$. Now apply the inductive hypothesis for β and $\bar{B}^* = \langle B^*_\ell : \ell \leq m \rangle$ to get the sequences $\langle B^{\ell,*}_\eta, f^{k,*}_\eta : \eta \in T^*,\ \ell \leq \ell^*_\eta,\ k \leq k^*_\eta \rangle$ satisfying clauses (a)–(d) and such that $\langle B^{\ell,*}_{\langle\rangle} : \ell \leq \ell^*_\eta \rangle = \langle B^*_\ell : \ell \leq m \rangle$. Put

$$\begin{aligned}
&T = \{\langle\rangle\} \cup \{\langle i\rangle : i < \lambda\} \cup \{\langle 0\rangle^\frown\eta : \eta \in T^*\},\\
&\ell_{\langle 0\rangle^\frown\eta} = \ell^*_\eta, \quad k_{\langle\rangle} = m+1, \quad k_{\langle 0\rangle^\frown\eta} = k_\eta,\\
&B^\ell_{\langle 0\rangle^\frown\eta} = B^{\ell,*}_\eta, \quad B^\ell_{\langle 0\rangle^\frown\langle i\rangle} = B_\ell \cap (i+\omega),\\
&f^k_{\langle\rangle} = f_k, \quad f^k_{\langle 0\rangle^\frown\eta} = f^{k,*}_\eta,\\
&\alpha_{\langle\rangle} = \omega, \quad \alpha_{\langle 0\rangle^\frown\eta} = \alpha^*_\eta.
\end{aligned}$$

(5) If f^ℓ_η is not defined then choose f^* as it. $\square$

Remark 1.4. Note that it may happen that $\lambda \in \mathrm{ID}(F)$. However, if $\eta \in {}^\lambda 2$ is a weak diamond sequence for F then the set $\{\gamma < \lambda : \eta(\gamma) = 0\}$ witnesses $\lambda \notin \mathrm{ID}^-_1(F)$. And conversely, if $\lambda \notin \mathrm{ID}^-_1(F)$ and $S^* \subseteq \lambda$ witnesses it, then the function $0_{S^*} \cup 1_{\lambda\setminus S^*}$ is a weak diamond sequence for F.

Definition 1.5. For a λ–colouring F we define λ–colourings $F^\oplus$ and $F^\otimes$ as follows.

(1) A function $g \in {}^{\gamma}(\mathcal{H}(\lambda))$ is called $F^{\oplus}$–standard if there is a tuple $(T, \bar{f}, \bar{\alpha}, \bar{A})$ (called a witness) such that
  (i) $T \subseteq {}^{\omega>}\gamma$ is a well founded tree (so $\langle\rangle \in T$, $\nu \triangleleft \eta \in T \Rightarrow \nu \in T$ and T has no ω–branch);
  (ii) $\bar{f} = \langle f^{\ell}_{\eta} : \eta \in T,\ \ell \le k_{\eta}\rangle$, where $f^{\ell}_{\eta} \in \mathrm{DOM}(F) \cap {}^{\gamma}(\mathcal{H}(\lambda))$;
  (iii) $\bar{\alpha} = \langle \alpha_{\eta} : \eta \in T\rangle$, where $\alpha_{\eta} < \lambda$;
  (iv) $\bar{A} = \langle A^{\ell}_{\eta} : \eta \in T,\ \ell \le \ell_{\eta}\rangle$, where $A^{\ell}_{\eta} \subseteq \alpha_{\eta}$;
  (v) $g(\beta) = (T \cap {}^{\omega>}\beta, \langle f^{\ell}_{\eta}{\restriction}\beta : \eta \in T \cap {}^{\omega>}\beta,\ \ell < k_{\eta}\rangle, \langle \alpha_{\eta} : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle A^{\ell}_{\eta} : \eta \in T \cap {}^{\omega>}\beta,\ \ell \le \ell_{\eta}\rangle)$ for each $\beta < \gamma$.

(2) $\mathrm{DOM}(F^{\oplus}) = \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ and for $g \in {}^{\gamma}(\mathcal{H}(\lambda))$:
  $(\oplus)_{\alpha}$ if $\gamma = 0$ then $F^{\oplus}(g) = 0$,
  $(\oplus)_{\beta}$ if $\gamma > 0$ and g is not standard then $F^{\oplus}(g) = 0$,
  $(\oplus)_{\gamma}$ if $\gamma > 0$ and g is standard as witnessed by $\langle \bar{T}, \bar{f}, \bar{\alpha}, \bar{A}\rangle$ then $F^{\oplus}(g) = \mathbf{t}^{0}_{F,g}(\langle\rangle)$, where $\mathbf{t}^{\ell}_{F,g}(\eta) \in \{0,1\}$ (for $\eta \in T$, $\ell = 0,1$) are defined by downward induction as follows.
    If $\eta \in \max(T)$ then $\mathbf{t}^{\ell}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,
    if $\eta \in T \setminus \max(T)$, $\gamma < \alpha_{\eta}$ then $\mathbf{t}^{\ell}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,
    if $\eta \in T \setminus \max(T)$, $\gamma \ge \alpha_{\eta}$ then

$$\begin{array}{lll} \mathbf{t}^{1}_{F,g}(\eta) = 1 & \text{iff} & F(f_{\eta}) = 1 \ \text{ or } \ (\exists i < \gamma)(\mathbf{t}^{1}_{F,g}(\eta^\frown\langle i\rangle) = 1), \\ \mathbf{t}^{0}_{F,g}(\eta) = 1 & \text{iff} & (\exists i < \gamma)(\mathbf{t}^{0}_{F,g}(\eta^\frown\langle i\rangle) = 1) \ \text{ or} \\ & & F(f'_{\eta}) = 1 \ \& \ (\forall i < \gamma)(\mathbf{t}^{1}_{F,g}(\eta^\frown\langle i\rangle) = 0). \end{array}$$

(3) A function $g \in {}^{\gamma}(\mathcal{H}(\lambda))$ is called $F^{\otimes}$–standard if there is a tuple $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ (called a witness) such that
  (i) $T \subseteq {}^{\omega>}\gamma$ is a well founded tree;
  (ii) $\bar{f} = \langle f_{\eta} : \eta \in T\rangle$, where $f_{\eta} \in \mathrm{DOM}(F) \cap {}^{\gamma}(\mathcal{H}(\lambda))$;
  (iii) $\bar{\ell} = \langle \ell_{\eta} : \eta \in T\rangle$, where $\ell_{\eta} : {}^{3}\{0,1\} \longrightarrow \{0,1\}$;
  (iv) $\bar{\alpha} = \langle \alpha_{\eta} : \eta \in T\rangle$, where $\alpha_{\eta} < \lambda$;
  (v) $\bar{A} = \langle A_{\eta} : \eta \in T\rangle$, where $A_{\eta} \subseteq \alpha_{\eta}$;
  (vi) $g(\beta) = (T \cap {}^{\omega>}\beta, \langle f_{\eta}{\restriction}\beta : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle \ell_{\eta} : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle \alpha_{\eta} : \eta \in T \cap {}^{\omega>}\beta\rangle, \langle A_{\eta} : \eta \in T \cap {}^{\omega>}\beta\rangle)$ for each $\beta < \gamma$.

(4) $\mathrm{DOM}(F^{\otimes}) = \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ and for $g \in {}^{\gamma}(\mathcal{H}(\lambda))$:
  $(\otimes)_{\alpha}$ if $\gamma = 0$ then $F^{\otimes}(g) = 0$,
  $(\otimes)_{\beta}$ if $\gamma > 0$ and g is not $F^{\otimes}$–standard then $F^{\otimes}(g) = 0$,
  $(\otimes)_{\gamma}$ if $\gamma > 0$ and g is $F^{\otimes}$–standard as witnessed by $\langle \bar{T}, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A}\rangle$ then $F^{\otimes}(g) = \mathbf{t}_{F,g}(\langle\rangle)$, where $\mathbf{t}_{F,g}(\eta) \in \{0,1\}$ (for $\eta \in T$) are defined by downward induction as follows.
    If $\eta \in \max(T)$ then $\mathbf{t}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,
    if $\eta \in T \setminus \max(T)$, $1 + \gamma < \alpha_{\eta}$ then $\mathbf{t}_{F,g}(\eta) = 1$ iff $\gamma \in A_{\eta}$,
    if $\eta \in T \setminus \max(T)$, $1 + \gamma \ge \alpha_{\eta}$ then

$$\mathbf{t}_{F,g}(\eta) = \ell_{\eta}(F(f_{\eta}), \max\{\mathbf{t}_{F,g}(\eta^\frown\langle\beta\rangle) : \beta < \gamma\}, \min\{\mathbf{t}_{F,g}(\eta^\frown\langle\beta\rangle) : \beta < \gamma\}).$$

Proposition 1.6. *Let F be a λ–colouring. Then $F^{\oplus}$ is a λ–colouring and*

(a) *if $S \in \mathrm{ID}(F)$ then $0_S \cup 1_{\lambda \setminus S} \in \mathfrak{B}(F^{\oplus})$ and $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^{\oplus})$,*

(b) $\mathrm{ID}(F) \subseteq \mathrm{ID}_1(F^{\oplus}) = \mathrm{ID}_1^-(F^{\oplus}) = \mathrm{ID}(F^{\oplus})$,

Proof. (a) Check.
(b) $\mathrm{ID}(F) \subseteq \mathrm{ID}_1(F^{\oplus})$.
Suppose that $B \in \mathrm{ID}(F)$. We are going to show that then $B \in \mathrm{ID}_1^-(F^{\oplus})$. So suppose that $B' \subseteq B$. We want to find $g \in \mathrm{DOM}_\lambda(F^{\oplus})$ such that the set

$$\{\delta < \lambda : \delta \text{ is limit and } F(g{\restriction}\delta) = 0 \Leftrightarrow \delta \in B'\}$$

is in $\mathrm{ID}_0(F^{\oplus})$ (what just means that it is non-stationary). Since $B \in \mathrm{ID}(F)$ we have $B' \in \mathrm{ID}(F)$, so by 1.3(4) we may find $\langle B^\ell_\eta, f^k_\eta, \alpha_\eta : \eta \in T,\ \ell \leq \ell_\eta,\ k < k_\eta\rangle$ such that the clauses (a)–(d) of 1.3(4) are satisfied with $\ell_{\langle\rangle} = 0$, $B' = B^0_{\langle\rangle}$. Define g as follows. For $\beta < \lambda$ let $T_\beta = T \cap {}^{\omega >}\beta$ and

$$g(\beta) = \big(T_\beta,\ \langle f^k_\eta : \eta \in T_\beta, k \leq k_\eta\rangle,\ \langle \alpha_\eta : \eta \in T_\beta\rangle,\ \langle B^\ell_\eta \cap \alpha_\eta : \ell \leq \ell_\eta, \eta \in T_\beta\rangle\big).$$

Now look at the demands in 1.5(2) – they are exactly what 1.3(4) guarantees us. $\square$

Definition 1.7. Let F_1, F_2 be λ–colourings (with $\mathrm{DOM}(F_\ell)$ being either ${}^{\lambda >}2$ or $\bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$, see 0.2(1)).

(1) We say that $F_1 \leq F_2$ if there is $h : \mathrm{DOM}(F_1) \longrightarrow \mathrm{DOM}(F_2)$ such that
 (a) $\eta \trianglelefteq \nu \quad \Rightarrow \quad h(\eta) \trianglelefteq h(\nu)$,
 (b) $h(\eta) = \lim\limits_{\alpha<\delta} h(\eta \restriction \alpha)$, for every $\eta \in {}^{\delta}2$, δ a limit,
 (c) $(\forall \eta \in \mathrm{DOM}(F_1))(0 < \ell g(\eta) = \ell g(h(\eta)) \ \Rightarrow\ F_1(\eta) = F_2(h(\eta)))$.

(2) We say that $F_1 \leq^* F_2$ if there is $h : \mathrm{DOM}(F_1) \longrightarrow \mathrm{DOM}(F_2)$ such that the clauses (a)–(c) above hold but
 (d) if $\eta \in \mathrm{DOM}_\lambda(F_1)$ and $\lim\limits_{\alpha<\lambda} h(\eta{\restriction}\alpha)$ has length $< \lambda$ then $F_1(\eta{\restriction}\alpha) = 0$ for every large enough α.

Proposition 1.8. (1) *$\leq^*$ and $\leq$ are transitive relations on λ–colourings, $\leq^* \subseteq \leq$.*
(2) *$\leq$ is λ^+ directed.*

Proposition 1.9. (1) *For every colouring $F_1 : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$ there is a colouring $F_2 : {}^{\lambda>}2 \longrightarrow 2$ such that $F_1 \leq F_2 \leq^* F_1$.*
(2) *For every λ–colouring $F_2 : {}^{\lambda>}2 \longrightarrow 2$ there is a λ–colouring $F_1 : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda))$ such that $F_2 \leq F_1 \leq^* F_2$.*

Proof. 1) Let $F_1 : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$. Let h_0 be a one-to-one function from $\mathcal{H}(\lambda)$ to ${}^{\lambda>}2$, say $h_0(\eta) = \langle \ell_{\eta,i} : i < \ell g(h_0(\eta))\rangle$. Define a function

$h_1 : \mathcal{H}(\lambda) \longrightarrow {}^{\lambda>}2$ by:

$$\begin{aligned}&\ell g(h_1(\eta)) = \ell g(h_0(\eta)) + 2,\\ &h_1(\eta)(2i) = h_0(\eta)(i), \quad h_1(\eta)(2i+1) = 0 \quad \text{for } i < \ell g(h_0(\eta)), \quad \text{and}\\ &h_1(\eta)(2\ell g(h_0(\eta))) = h_1(\eta)(2\ell g(h_0(\eta)+1)) = 1.\end{aligned}$$

Next, by induction on $\ell g(\eta)$, we define a function $h^+ : \bigcup\limits_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow {}^{\lambda>}2$ as follows:

$$h^+(\langle\rangle) = \langle\rangle, \qquad h^+(\eta^\frown\langle x\rangle) = h^+(\eta)^\frown h_1(x).$$

Finally we define a colouring $F_2 : {}^{\lambda>}2 \longrightarrow 2$ by

$$F_2(\nu) = \begin{cases} F_1(\eta) & \text{if } \nu = h^+(\eta),\\ 0 & \text{if } \nu \notin \operatorname{rng}(h^+).\end{cases}$$

$\square$

Proposition 1.10. *Assume that F_1, F_2 are λ–colourings such that $F_1 \leq F_2$, or just $F_1 \leq^* F_2$. Then:*

1. *For every $\eta \in {}^{\lambda}2$ there are $\nu \in {}^{\lambda}2$ and a club E of λ such that*
$$(\forall\delta \in E)(F_1(\eta\restriction\delta) = F_2(\nu\restriction\delta)).$$
2. $\mathrm{ID}_\alpha(F_1) \subseteq \mathrm{ID}_\alpha(F_2)$, $\mathrm{ID}^-_\alpha(F_1) \subseteq \mathrm{ID}^-_\alpha(F_2)$*; hence* $\mathrm{ID}(F_1) \subseteq \mathrm{ID}(F_2)$ *and* $\mathfrak{B}^+(F_1) \subseteq \mathfrak{B}^+(F_2)$.
3. *For every colouring F there is a colouring F' such that $F \leq F'$ and* $\mathrm{ID}^2(F) \subseteq \mathrm{ID}(F')$.

Proof. Straightforward. $\square$

Conclusion 1.11. Assume that λ is a regular uncountable cardinal and $F : {}^{\lambda>}2 \longrightarrow 2$ is a λ–colouring. Let

$$F^{\otimes} : \bigcup_{\alpha<\lambda} {}^{\alpha}(\mathcal{H}(\lambda)) \longrightarrow 2$$

be the colouring defined for F in Definition 1.5(4). Then:

(a) $F \leq F^{\otimes}$.
(b) $\mathrm{ID}(F^{\otimes})$ is a normal ideal on λ.
(c) $\mathfrak{B}(F) \subseteq \mathfrak{B}(F^{\otimes})$ and $\mathrm{ID}(F) \subseteq \mathrm{ID}(F^{\otimes}) = \mathrm{WDmId}_\lambda(F^{\otimes})$.
(d) $F^{\otimes}$ relates to itself as it relates to F, i.e. *if* $\alpha^* < \lambda^+$, $\langle S_\alpha : \alpha < \alpha^*\rangle$ is increasing continuous modulo $\mathrm{ID}(F^{\otimes})$, $S_{\alpha+1} = S_\alpha \cup A_\alpha$ mod $\mathrm{ID}(F^{\otimes})$, $A_\alpha \in \mathfrak{B}(F^{\otimes})$, $\ell_\alpha \in 2$,

then for some $f \in {}^{\lambda}(\mathcal{H}(\lambda))$
$$\{\alpha < \lambda : F(f\restriction\alpha) = 1\}/\mathcal{D}_\lambda$$
is, in $\mathcal{P}(\lambda)/\mathcal{D}_\lambda$, the least upper bound of the family $\{(A_\alpha \setminus S_\alpha)/\mathcal{D}_\lambda : \ell_\alpha = 1\}$ (where $\mathcal{D}_\lambda$ stands for the club filter).
(e) The family $\mathfrak{B}(F^{\otimes})$ is closed under complements, unions and intersections of less than λ sets, diagonal unions and diagonal intersections and it includes bounded subsets of λ. Moreover $\mathfrak{B}^+(F^{\otimes}) = \mathfrak{B}(F^{\otimes})$.

(f) If $\mathcal{P}(\lambda)/\mathrm{ID}(F^{\otimes})$ is λ^+–saturated then
for every set $X \subseteq \lambda$ there are sets $A, B \in B(F^{\otimes})$ such that
(α) $A \subseteq X \subseteq B$,
(β) for every $Y \in \mathfrak{B}(F^{\otimes})$ one of the following occurs:
(i) the sets $(X \setminus A) \cap Y$, $(X \setminus A) \setminus Y$, $(B \setminus X) \cap Y$, $(B \setminus X) \setminus Y$ are[1] not in $\mathrm{ID}(F^{\otimes})$,
(ii) $Y \cap (B \setminus A) \in \mathrm{ID}(F^{\otimes})$,
(iii) $(B \setminus A) \setminus Y \in \mathrm{ID}(F^{\otimes})$.
In the situation as above we denote $A = \max_{F^{\otimes}}(X)$, $B = \min_{F^{\otimes}}(X)$ (note that these sets are unique modulo $\mathrm{ID}(F^{\otimes})$). Moreover

(g) if $A \subseteq \min_{F^{\otimes}}(B)$ then $\min_{F^{\otimes}}(A) \subseteq \min_{F^{\otimes}}(B) \mod \mathrm{ID}(F^{\otimes})$.

(h) If $X \subseteq \lambda$, $X \notin \mathrm{ID}(F^{\otimes})$ then for some $Y_1, Y_2 \subseteq X$ which are not in $\mathrm{ID}(F^{\otimes})$ we have

$$\max_{F^{\otimes}}(Y_1) = \max_{F^{\otimes}}(Y_2) = \emptyset \quad \text{and} \quad \min_{F^{\otimes}}(Y_1) = \min_{F^{\otimes}}(Y_2) \notin \mathrm{ID}(F^{\otimes}).$$

Proof. Clauses (a) and (b): Should be clear.

Clause (e): Note that as $\theta = 2$ we identify a sequence $\eta \in {}^{\lambda}2$ with $\{i < \lambda : \eta(i) = 1\}$.

$\mathfrak{B}(F^{\otimes})$ **is closed under complementation.**
Suppose that $A \in \mathfrak{B}(F^{\otimes})$. If A is bounded then let g, $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ be as in 1.5(3) with $T = \{\langle\rangle\} \cup \{\langle i \rangle : i < \lambda\}$, $A_{\langle\rangle} = \alpha_{\langle\rangle} \setminus A$, $\alpha_{\langle\rangle} > \sup(A)$, $\ell_{\langle\rangle}$ constantly 1. Then $(\forall \alpha < \lambda)(F^{\otimes}(g{\restriction}(1+\alpha)) = 1 \Leftrightarrow \alpha \in A)$, so F codes $\lambda \setminus A$. So suppose that $\sup(A) = \lambda$. Pick g such that

$$(\forall \alpha < \lambda)(F^{\otimes}(g{\restriction}(1+\alpha)) = 1 \Leftrightarrow \alpha \in A).$$

By our assumption, for arbitrarily large $\beta < \lambda$ we have $F^{\otimes}(g{\restriction}\beta) = 1$, so $g(\beta)$ is

$$\big(T_\beta, \langle f^\beta_\eta : \eta \in T_\beta\rangle, \langle \alpha^\beta_\eta : \eta \in T_\beta\rangle, \langle \ell^\beta_\eta : \eta \in T_\beta\rangle, \langle \alpha^\beta_\eta : \eta \in T_\beta\rangle, \langle A^\beta_\eta : \eta \in T_\beta\rangle\big)$$

and it is as in 1.5(3). If $\beta_1 < \beta_2$ then the two values necessarily cohere, in particular $T_{\beta_1} = T_{\beta_2} \cap {}^{\omega>}(\beta_1)$. Consequently there is $(T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ such that $T = \bigcup\limits_{\beta<\lambda} T_\beta \subseteq {}^{\omega>}\lambda$ is closed under initial segments and is well founded (as T_β increase with β and $\mathrm{cf}(\lambda) > \aleph_0$). Thus we have proved

($\boxtimes$) if $A \subseteq \lambda$ is unbounded and $F^{\otimes}$ coded by g then there is $\mathbf{p} = (T, \bar{f}, \bar{\ell}, \bar{\alpha}, \bar{A})$ such that the clauses (i)–(vi) of 1.5(3) hold for $\gamma = \lambda$ and $g(\beta) = \mathbf{p}{\restriction}\beta$.

Now define $\mathbf{p}'$ like $\mathbf{p}$ (with the same T etc) except that $\ell^{\mathbf{p}'}_{\langle\rangle} = 1 - \ell^{\mathbf{p}}_{\langle\rangle}$ and $A^{\mathbf{p}'}_{\langle\rangle} = A^{\mathbf{p}}_{\langle\rangle}$.

$\mathfrak{B}(F^{\otimes})$ **contains all bounded subsets of** λ.
By the first part of the arguments above all co-bounded subsets of λ are in $\mathfrak{B}(F^{\otimes})$, so (by the above) their complements are there too.

[1] hence none of $X \setminus A$, $B \setminus A$ includes (modulo $\mathrm{ID}(F^{\otimes})$) a member of $\mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}(F^{\otimes})$

$\mathfrak{B}(F^{\otimes})$ **is closed under unions of length** $<\lambda$.
Let $B=\bigcup\limits_{i<\alpha} B_i$ where $\alpha<\lambda$ and $B_i\in\mathfrak{B}(F^{\otimes})$. Let $w=\{i<\alpha:\sup(B_i)=\lambda\}$ and for $i\in w$ let B_i be represented by $g_i\in{}^{\lambda}(\mathcal{H}(\lambda))$ which, by ($\boxtimes$), comes from $\mathbf{p}^i=(T^i,\bar f^i,\bar\ell^i,\bar\alpha^i,\bar A^i)$. We may assume that $w=\beta\le\alpha$. Let

$$\begin{aligned}
&T=\{\langle\rangle\}\cup\{\langle i\rangle : i<\lambda\}\cup\{\langle i\rangle^\frown\eta:\eta\in T^i,\ i<\beta\},\\
&f_{\langle i\rangle^\frown\eta}=f^i_\eta,\ \text{etc}\\
&\alpha_{\langle\rangle} \text{ is the first } \gamma\ge\omega \text{ such that } \gamma\ge\alpha \ \&\ (\forall i\in[\beta,\alpha))(B_i\subseteq\gamma),\\
&B_{\langle i\rangle}=\emptyset \quad \text{if } i\ge\beta,\\
&A_{\langle\rangle}=\bigcup_{i<\alpha} B_i\cap\alpha_{\langle\rangle},\\
&\ell_{\langle\rangle}(i_0,i_1,i_2)=i_1.
\end{aligned}$$

Checking is straightforward.

$\mathfrak{B}(F^{\otimes})$ **is closed under diagonal unions.**
Let $B=\mathop{\nabla}\limits_{i<\lambda} B_i$, where each $B_i\in\mathfrak{B}(F^{\otimes})$ is represented by $g_i\in{}^{\lambda}(\mathcal{H}(\lambda))$ which, by ($\boxtimes$), comes from $\mathbf{p}^i=(T^i,\bar f^i,\bar\ell^i,\bar\alpha^i,\bar A^i)$. Let $T=\{\langle\rangle\}\cup\{\langle i\rangle^\frown\eta: \eta\in T_i,\ i<\lambda\}$, $f_{\langle i\rangle^\frown\eta}=f^i_\eta$, etc, $\alpha_{\langle\rangle}=\omega$, $B_{\langle\rangle}=B\cap\omega$ and $\ell_{\langle\rangle}(i_0,i_1,i_2)=i_1$.

Clause (c): First note that $\mathfrak{B}(F)\subseteq\mathfrak{B}(F^{\otimes})$ as $\mathfrak{B}(F)\subseteq\mathfrak{B}^+(F)\subseteq\mathfrak{B}^+(F^{\otimes})=\mathfrak{B}(F^{\otimes})$ (the second inclusion by (a) and 1.10, the last equality by (e)). Next note that

$$\mathrm{WDmId}_\lambda(F^{\otimes})\subseteq\mathrm{ID}_1^-(F^{\otimes})\subseteq\mathrm{ID}_1(F^{\otimes})\subseteq\mathrm{ID}(F^{\otimes}).$$

Now by induction on α we are proving that $\mathrm{ID}_\alpha(F^{\otimes})\subseteq\mathrm{WDmId}_\lambda(F^{\otimes})$. So suppose that we have arrived to a stage α.
If $\alpha=0$ then we use the fact that every non-stationary subset of λ is in $\mathfrak{B}(F^{\otimes})$ (by (e)).
If α is limit then, by the induction hypothesis, $\mathrm{ID}^-_\alpha(F^{\otimes})\subseteq\mathfrak{B}(F^{\otimes})$ and hence $\mathrm{ID}_\alpha\subseteq\mathfrak{B}(F^{\otimes})$ (as $gB(F^{\otimes})$ is closed under diagonal unions by (e); remember 1.3(3)).
So suppose that $\alpha=\beta+1$ and $B\in\mathrm{ID}_\alpha(F^{\otimes})$. Suppose $B'\subseteq B$ (so $B'\in\mathrm{ID}^-_\alpha(F^{\otimes})$). There is $B''\in\mathfrak{B}(F)$ such that $B''\triangle B'\in\mathrm{ID}_\beta(F)$. By the first part we know that $B''\in\mathfrak{B}(F^{\otimes})$ and by the induction hypothesis $B'\triangle B''\in\mathfrak{B}(F^{\otimes})$. Consequently $B'\in\mathfrak{B}(F^{\otimes})$.

Together we have proved that $\mathrm{ID}(F^{\otimes})=\mathrm{WDmId}_\lambda(F^{\otimes})$. The inclusion $\mathrm{ID}(F)\subseteq\mathrm{ID}(F^{\otimes})$ is easy. $\square$

Proposition 1.12. *Let λ be a regular uncountable cardinal and F be a λ–colouring.*

1. *If $\mathrm{ID}_\alpha(F)$ is λ^+–saturated then for some $\beta<\lambda^+$ we have $\mathrm{ID}_{\alpha+\beta}(F)=\mathrm{ID}(F)$.*
2. $\mathrm{ID}_\alpha(F)\subseteq\mathrm{WDmId}_\lambda$.
3. *If $\mathrm{ID}_\alpha(F)$ is λ^+–saturated and $\lambda\notin\mathrm{WDmId}_\lambda$ then $\mathrm{WDmId}_\lambda=\mathrm{ID}_1(F')$ for some λ–colouring F'.*

(4) $\mathrm{ID}^2(F)$ *is a normal ideal, and* $\mathrm{ID}^1(F) \subseteq \mathrm{ID}^2(F) \subseteq \mathrm{WDmId}_\lambda$.
(5) $\mathrm{ID}^1(F^\otimes) = \mathrm{WDmId}_\lambda(F^\otimes)$.

Proof. 1) It follows from 1.3(3) that $\mathrm{ID}_\gamma(F)$ increases with γ, so the assertion should be clear.

2) By 1.11(c).

3) Assume that $\mathrm{ID}_\alpha(F)$ is λ^+–saturated and $\lambda \notin \mathrm{WDmId}_\lambda$. By induction on $\beta < \lambda^+$ we try to define colourings F_β such that

(a) $\mathrm{ID}_\alpha(F) \subseteq \mathrm{ID}(F_0)$,
(b) if $\beta < \gamma$ then $\mathrm{ID}(F_\beta) \subseteq \mathrm{ID}(F_\gamma)$,
(c) $\mathrm{ID}(F_\beta) \neq \mathrm{ID}(F_{\beta+1})$.

So we let $F_0 = F$. If β is limit then we use 1.9(2) to choose F_β so that $(\forall \gamma < \beta)(F_\gamma \leq F_\beta)$. Finally, if $\beta = \gamma + 1$ then we let $F'_\beta = (F_\gamma)^\otimes$ (so $\mathrm{ID}(F_\gamma) \subseteq \mathrm{ID}_1(F'_\beta) = \mathrm{ID}(F'_\beta) \subseteq \mathrm{WDmId}_\lambda$). If $\mathrm{ID}(F'_\beta) \neq \mathrm{WDmId}_\lambda$ then we choose a set $A \in \mathrm{WDmId}_\lambda \setminus \mathrm{ID}(F'_\beta)$ and F^*_β witnessing $A \in \mathrm{WDmId}_\lambda$. We may assume that $(\forall \alpha \in \lambda \setminus A)(\forall \eta \in {}^\alpha 2)(F^*_\beta(\eta) = 0)$. Now take a colouring F_β such that $F'_\beta, F^*_\beta \leq F_\beta$.

After carrying out the construction choose $S^0_\beta \in \mathrm{ID}(F_{\beta+1}) \setminus \mathrm{ID}(F_\beta)$ (for $\beta < \lambda^+$) and let $S_\beta = S^0_\beta \setminus \mathop{\nabla}\limits_{\gamma<\beta} S^0_\gamma$. Then $\langle S_\beta : \beta < \lambda^+\rangle$ is a sequence of pairwise disjoint members of $\mathcal{P}(\lambda) \setminus \mathrm{ID}(F_0) \subseteq \mathcal{P}(\lambda) \setminus \mathrm{ID}_\alpha(F)$, contradicting our assumptions. $\square$

For the rest of this section we will assume the following

Hypothesis 1.13. Assume that

(a) λ is a regular uncountable cardinal,
(b) F is a λ–colouring,
(c) $\lambda \notin \mathrm{ID}(F^\otimes)$, and
(d) $\mathrm{ID}(F^\otimes)$ is λ^+–saturated, that is there is no sequence $\langle A_\alpha : \alpha < \lambda^+\rangle$ such that for each $\alpha < \beta < \lambda^+$
$$A_\alpha \notin \mathrm{ID}(F^\otimes) \quad \text{and} \quad \|A_\alpha \cap A_\beta\| < \lambda.$$

For each limit ordinal $\alpha \in [\lambda, \lambda^+)$ fix an enumeration $\langle \varepsilon^\alpha_i : i < \lambda\rangle$ of α.

Construction 1.14. Fix a sequence $\eta \in {}^\lambda 2$ for a moment. We define a sequence
$$\langle S_\alpha[\eta], A_\alpha[\eta], B_\alpha[\eta], \ell_\alpha[\eta], m_\alpha[\eta], f_\alpha[\eta] : \alpha < \alpha^*[\eta]\rangle$$
as follows. By induction on $\alpha < \lambda^+$ we try to choose $S_\alpha[\eta] = S_\alpha$, $A_\alpha[\eta] = A_\alpha$, $B_\alpha[\eta] = B_\alpha$, $\ell_\alpha[\eta] = \ell_\alpha$, $m_\alpha[\eta] = m_\alpha$, $f_\alpha[\eta] = f_\alpha$ such that

(a) $S_\alpha, A_\alpha, B_\alpha \subseteq \lambda$, $\ell_\alpha, m_\alpha \in \{0,1\}$, $f_\alpha \in {}^\lambda 2$,
(b) $A_\alpha \notin \mathrm{ID}(F^\otimes)$, $A_\alpha \cap S_\alpha = \emptyset$,
(c) $S_{\alpha+1} = S_\alpha \cup A_\alpha$; if $\alpha < \lambda$ is limit then $S_\alpha = \bigcup\limits_{\beta<\alpha} S_\alpha$; if $\alpha \in [\lambda, \lambda^+)$ is limit then $S_\alpha = \{\gamma < \lambda : (\exists i < \gamma)(\gamma \in S_{\varepsilon^\alpha_i})\}$, $S_0 = \emptyset$,

(d) $B_\alpha \in \mathrm{ID}(F^{\otimes})$,

(e) for every $\delta \in \lambda \setminus (S_\alpha \cup B_\alpha)$

$$\eta(\delta) = m_\alpha \qquad \Rightarrow \qquad F(f_\alpha{\restriction}\delta) = \ell_\alpha,$$

(f) $A_\alpha = \{\delta \in \lambda \setminus S_\alpha : F(f_\alpha{\restriction}\delta) = 1 - \ell_\alpha\}$.

It follows from 1.13 that at some stage $\alpha^* = \alpha^*[\eta] < \lambda^+$ we get stuck (remember clause (b) above). Still, we may define then S_{α^*} as in the clause (c).

Proposition 1.15. *Assume 1.13. Then:*

(1) *There exists $\eta \in {}^{\lambda}2$ such that*

$$\lambda \setminus S_{\alpha^*[\eta]}[\eta] \notin \mathrm{ID}(F^{\otimes}).$$

(2) *If $S \in \mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}(F^{\otimes})$ then we can demand $S \subseteq S_{\alpha^*[\eta]}[\eta]$.*

Proof. Assume not. Then for each $\eta \in {}^{\lambda}2$ the set $B_{\alpha^*}[\eta] \stackrel{\mathrm{def}}{=} \lambda \setminus S_{\alpha^*[\eta]}$ is in $\mathrm{ID}(F^{\otimes})$. Now,

$$\{\alpha \in B_{\alpha^*}[\eta] : \eta(\alpha) = 1\} \in \mathrm{ID}(F^{\otimes}) \subseteq \mathfrak{B}(F^{\otimes})$$

(see 1.6).

Claim 1.15.1. *For each α, $S_\alpha \in \mathfrak{B}(F^{\otimes})$.*

Proof of the claim. We show it by induction on α. If $\alpha = 0$ then $S_\alpha = \emptyset \in \mathfrak{B}(F^{\otimes})$ (see 1.11(c)). If $\alpha < \lambda$ is a limit ordinal then $S_\alpha = \bigcup\limits_{\beta<\alpha} S_\beta$ and by the inductive hypothesis $S_\beta \in \mathfrak{B}(F^{\otimes})$, so by 1.11(e) we are done (as $\mathfrak{B}(F^{\otimes})$ is closed under unions of $< \lambda$ elements). If $\alpha \in [\lambda, \lambda^+)$ is limit then we use the fact that $\mathfrak{B}(F^{\otimes})$ is closed under diagonal unions. If $\alpha = \beta + 1$ then $A_\beta \in \mathfrak{B}(F)$ or $\lambda \setminus A_\beta \in \mathfrak{B}(F)$ and hence we may conclude that $A_\beta \in \mathfrak{B}(F^{\otimes})$ (remember 1.11(e)). Since $\mathfrak{B}(F^{\otimes})$ is closed under unions of length $< \lambda$ we are done. $\square$

Claim 1.15.2. *For each α, $Y_\alpha \stackrel{\mathrm{def}}{=} \{\beta < \lambda : \eta(\beta) = 1\} \cap S_\alpha \in \mathfrak{B}(F^{\otimes})$.*

Proof of the claim. We prove it by induction on α. If $\alpha = 0$ then $Y_\alpha = \emptyset$ and there is nothing to do. The case of limit α is handled like that in the proof of 1.15.1. So suppose that $\alpha = \beta + 1$. It suffices to show that the set $Y_\alpha \cap (S_\alpha \setminus S_\beta)$ is in $\mathfrak{B}(F)$, what means that $Y_\alpha \cap A_\alpha$ is there (remember clauses (e) and (f)). Note that if $\delta \in A_\alpha \setminus B_\alpha$ then $F(f_\alpha{\restriction}\delta) = 1 - \ell_\alpha \neq \ell_\alpha$ and hence $\eta(\delta) \neq m_\alpha$ so $\eta(\delta) = 1 - m_\alpha$. Consequently $Y_\alpha \cap (A_\alpha \setminus B_\alpha) \in \{A_\alpha \setminus B_\alpha, \emptyset\}$. But $\mathcal{P}(B_\alpha) \subseteq \mathfrak{B}(F^{\otimes})$ so together we are done. $\square$

It follows from 1.15.1, 1.15.2 that

$$\{\beta : \eta(\beta) = 1\} \cap S_{\alpha^*[\eta]}[\eta] \in \mathfrak{B}(F^{\otimes}).$$

But $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \in \mathrm{ID}(F^{\otimes})$, so $\mathcal{P}(\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \subseteq \mathfrak{B}(F^{\otimes})$ so we get a contradiction. $\square$

Conclusion 1.16. Assume 1.13. Let $\eta \in {}^{\lambda}2$, $X_\ell[\eta] = (\lambda \setminus S_{\alpha^*[\eta]}[\eta]) \cap \eta^{-1}(\{\ell\})$ (for $\ell = 0, 1$). Then one of the following occurs:

(A) $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \in \mathrm{ID}(F^{\otimes})$,

(B) $X_0[\eta], X_1[\eta] \notin \mathrm{ID}(F^{\otimes})$, and $X_0[\eta] \cup X_1[\eta] \in \mathfrak{B}(F^{\otimes})$, $X_0[\eta] \cap X_1[\eta] = \emptyset$, and for every $f \in {}^{\lambda}2$,

either the sequence $\langle F(f\restriction\delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}[\eta])\rangle$ is $\mathrm{ID}(F^{\otimes})$–almost constant
or both sequences $\langle F(f\restriction\delta) : \delta \in X_0[\eta]\rangle$ and $\langle F(f\restriction\delta) : \delta \in X_1[\eta]\rangle$ are not $\mathrm{ID}(F^{\otimes})$–almost constant.

Proof. Assume that the first possibility fails, so $\lambda \setminus S_{\alpha^*[\eta]}[\eta] \notin \mathrm{ID}(F^{\otimes})$.

Assume $X_0[\eta] \in \mathrm{ID}(F^{\otimes})$. Take any $f_{\alpha^*[\eta]} \in {}^{\lambda}2$ and choose $\ell_{\alpha^*[\eta]} \in \{0, 1\}$ so that

$$\{\delta \in \lambda \setminus S_{\alpha^*[\eta]}[\eta] : F(f_{\alpha^*[\eta]}\restriction\delta) = 1 - \ell_{\alpha^*[\eta]}\} \notin \mathrm{ID}(F^{\otimes}).$$

Putting $m_{\alpha^*[\eta]} = 0$ and $B_{\alpha^*[\eta]} = X_0[\eta]$ we get a contradiction with the definition of $\alpha^*[\eta]$. Similarly one shows that $X_1[\eta] \notin \mathrm{ID}(F^{\otimes})$.

Suppose now that $f \in {}^{\lambda}2$ and the sequence $\langle F(f\restriction\delta) : \delta \in (\lambda \setminus S_{\alpha^*[\eta]}[\eta])\rangle$ is not $\mathrm{ID}(F^{\otimes})$–almost constant but, say, the sequence $\langle F(f\restriction\delta) : \delta \in X_0[\eta]\rangle$ is $\mathrm{ID}(F^{\otimes})$–almost constant (and let the constant value be $\ell_{\alpha^*[\eta]}$). Let $m_{\alpha^*[\eta]} = 0$, $B_{\alpha^*[\eta]} = \{\delta \in X_0[\eta] : F(f\restriction\delta) = 1 - \ell_{\alpha^*[\eta]}\}$. Then $B_{\alpha^*[\eta]} \in \mathrm{ID}(F^{\otimes})$ and since necessarily

$$\{\delta \in X_0[\eta] \cup X_1[\eta] : F(f\restriction\delta) = 1 - \ell_{\alpha^*[\eta]}\} \notin \mathrm{ID}(F^{\otimes}),$$

we immediately get a contradiction. Similarly in the symmetric case. $\square$

Remark 1.17. Note that if $S \in \mathfrak{B}(F^{\otimes}) \setminus \mathrm{ID}(F^{\otimes})$ then there is $\eta \in {}^{\lambda}2$ such that $\eta^{-1}[\{0\}] \supseteq \lambda \setminus S$ and above $X_0, X_1 \subseteq S$ and possibility (A) fails.

Proposition 1.18. *Assume 1.13.*

(1) *We can find $S^* = S^*_F$, S^*_0 and S^*_1 such that:*
  (a) $S^* \in \mathfrak{B}(F^{\otimes})$,
  (b) $S^* = S^*_0 \cup S^*_1$, $S^*_0 \cap S^*_1 = \emptyset$,
  (c) *if $S^* \neq \lambda$ then* $\mathrm{ID}^2(F^{\otimes})\restriction\mathcal{P}(\lambda \setminus S^*) = \mathrm{WDmId}_\lambda(F^{\otimes})\restriction\mathcal{P}(\lambda \setminus S^*)$, $\lambda \setminus S^* \notin \mathrm{ID}^2(F^{\otimes})$.
  (d) *if $S^* \neq \emptyset$ then $S^* \notin \mathrm{ID}(F^{\otimes})$ and*
  $$\{(S^*_0 \cap F^{\otimes}(f)/\mathrm{ID}(F^{\otimes}), S^*_1 \cap F^{\otimes}(f)/\mathrm{ID}(F^{\otimes})) : f \in \mathrm{DOM}_\lambda\}$$
  is an isomorphism from $\mathcal{P}(S^*_0)/\mathrm{ID}(F^{\otimes})$ onto $\mathcal{P}(S^*_1)/\mathrm{ID}(F^{\otimes})$.

(2) *If in 1.16, $S_F \subseteq S_{\alpha^*[\eta]}[\eta] \mod \mathrm{ID}(F)$ then we can add*
  ($\circledast$) *for some $\rho \in {}^{X_1}2$ for every $f \in {}^{\lambda}2$ we have*
  $$\{\delta \in X_1 : F(f\restriction\delta) = \rho(\delta)\} \neq \emptyset \mod \mathrm{ID}(F^{\otimes}).$$

Proof. 1) We try to choose by induction on $\alpha < \lambda^+$ sets $S_\alpha, S_{\alpha,0}, S_{\alpha,1}$ such that

(a) $S_\alpha \subseteq \lambda$,

(b) $S_\alpha = S_{\alpha,0} \cup S_{\alpha,1}$, $S_{\alpha,0} \cap S_{\alpha,1} = \emptyset$,
(c) if $\beta < \alpha$ and $\ell < 2$ then

$$S_\beta \subseteq S_\alpha \mod \mathrm{ID}(F^\otimes) \quad \text{ and } \quad S_{\beta,\ell} \subseteq S_{\alpha,\ell} \mod \mathrm{ID}(F^\otimes),$$

(d) the sets $S_{\alpha,0}, S_{\alpha_1}$ witness that $S \in \mathrm{ID}^2(F^\otimes)$ (see 1.2(4)).

At some stage $\alpha < \lambda^+$ we have to be stuck (as $\mathrm{ID}(F^\otimes)$ is λ^+–saturated) and then $(S_\alpha, S_{\alpha,0}, S_{\alpha,1})$ can serve as (S^*_F, S^*_0, S^*_1).

2) By the choice of S_F, for some $\ell < 2$ we have

$$\mathcal{P}(X_\ell) \neq \{F^\otimes(f) \cap X_\ell : f \in {}^\lambda\},$$

so let $Y \subseteq X_\ell$ be such that $Y \notin \{F^\otimes(f) \cap X_\ell : f \in {}^\lambda\}$. Let $\rho = 0_Y \cup 1_{X_\ell \setminus Y}$. Since without loss of generality $\ell = 1$, we are done. $\square$

Remark 1.19. (1) If $\lambda \notin \mathrm{WDmId}_\lambda$ ten $S^* \neq \lambda$.
(2) Recall: $\mathrm{ID}^1(F^\otimes) = \mathrm{ID}(F^\otimes) = \mathrm{WDmId}_\lambda(F^\otimes)$ is a normal ideal and $\mathrm{ID}^2(F^\otimes)$ is a normal ideal extending it.

2. Weak diamond for more colours

In this section we deduce a weak diamond for, say, three colours, assuming the weak diamond for two colours and assuming that a certain ideal is saturated.

Proposition 2.1. *Assume that λ is a regular uncountable cardinal and $\mu \leq 2^{<\lambda}$. Let $F_i : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be λ–colourings for $i < \mu$. Then there is a colouring $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ such that $F_i \leq F$ for every $i < \mu$.*

Proof. Case 1. $\mu \leq 2^{\|\alpha\|}$ for some $\alpha < \lambda$.
Let $\rho_i \in {}^\alpha 2$ for $i < \mu$ be distinct. For $\eta \in {}^{\lambda>}2$ let $h_i(\eta) = \rho_i {}^\frown \eta$. Define F by:

$$F(\nu) = \begin{cases} 0 & \text{if } \ell g(\nu) < \alpha, \text{ or } \ell g(\nu) \geq \alpha \\ & \text{but } \nu \restriction \alpha \notin \{\rho_i : i < \nu\}, \\ F_i(\langle \nu(\alpha+\varepsilon) : \varepsilon < \ell g(\nu) - \alpha\rangle) & \text{if } \ell g(\nu) \geq \alpha \text{ and } \nu \restriction \alpha = \rho_i. \end{cases}$$

It is easy to see that $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ and h_i exemplifies that $F_i \leq F$.

Case 2. $\mu = \lambda$.
For $\eta \in {}^{\lambda>}2$, $i < \mu$ and $\gamma < \lambda$ let

$$h_i(\eta)(\gamma) = \begin{cases} 0 & \text{if } \gamma < i, \\ 1 & \text{if } \gamma = i, \\ \eta(\gamma - (i+1)) & \text{otherwise.} \end{cases}$$

Next, for $\nu \in {}^{\lambda>}2$ define:

$$F(\nu) = \begin{cases} F_i(\langle \nu(i+1+\gamma) : \gamma < \ell g(\nu) - (i+1)\rangle) & \text{if } i = \min\{j : \nu(j) = 1\} \\ 0 & \text{if there is no such } i. \end{cases}$$

Now check.

CASE 3. Otherwise, for each $\alpha<\lambda$ choose $F^\alpha : {}^{\lambda>}2 \longrightarrow \{0,1\}$ such that $(\forall i<2^{\|\alpha\|})(F_i \leq F^\alpha)$ (exists by Case 1). Let $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be such that $(\forall \alpha<\lambda)(F^\alpha \leq F)$ (exists by Case 2).

The proposition follows. $\square$

Theorem 2.2. *Assume that λ is a regular uncountable cardinal. Let $F^{\rm tr} : {}^{\lambda>}2 \longrightarrow 3$. For $i<3$ let $F_i : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be such that*

$$F_i(\eta)=\begin{cases} 1 & \text{if } F^{\rm tr}(\eta)=i,\\ 0 & \text{otherwise,}\end{cases}$$

and let $F : {}^{\lambda>}2 \longrightarrow \{0,1\}$ be such that $(\forall i<3)(F_i \leq F)$. Assume that $\lambda \notin {\rm ID}^2(F^\otimes)$ (remember 1.10(3)), and ${\rm ID}(F^\otimes)$ is λ^+–saturated, i.e. there is no sequence $\langle A_\alpha : \alpha<\lambda^+\rangle$ such that

$$(\forall \alpha<\beta<\lambda^+)(A_\alpha \notin {\rm ID}(F) \quad \& \quad \|A_\alpha \cap A_\beta\|<\lambda).$$

Then there is a weak diamond sequence for $F^{\rm tr}$, even for every $S \in \mathfrak{B}(F^\otimes)\setminus {\rm ID}^2(F^\otimes)$.

Proof. Let S^*_F be as in 1.18. Since $\lambda \notin {\rm ID}^2(F^\otimes)$ necessarily $\lambda\setminus S^*_F \notin {\rm ID}(F^\otimes)$. Recall that ${\rm ID}^2(F^\otimes)={\rm ID}(F)+S_F$.

It follows from 1.15 and 1.16 that there are disjoint sets $X_0,X_1\subseteq\lambda$ (even disjoint from S^*_F from 1.18) such that $X_0,X_1\notin {\rm ID}(F^\otimes)$, $X_0\cup X_1 \in \mathfrak{B}(F^\otimes)$ and for every $f\in{}^\lambda 2$ we have one of the following:

(a) the sequence $\langle F(f{\restriction}\delta) : \delta\in X_0\cup X_1\rangle$ is ${\rm ID}(F^\otimes)$–almost constant, or
(b) both sequences $\langle F(f{\restriction}\delta) : \delta\in X_0\rangle$ and $\langle F(f{\restriction}\delta) : \delta\in X_1\rangle$ are not ${\rm ID}(F^\otimes)$–almost constant.

It follows from 1.18(2) that we may assume that there is $\eta\in{}^{X_1}2$ such that for every $f\in{}^\lambda 2$ the set

$$\{\delta\in X_1 : F(f{\restriction}\delta)=\eta(\delta)\}$$

is stationary. Define a function $\rho\in{}^\lambda 2$ as follows:

$$\rho(\alpha)=\begin{cases} 1+\eta(\alpha) & \text{if } \alpha\in X_1,\\ 0 & \text{otherwise.}\end{cases}$$

Claim 2.2.1. *ρ is a weak diamond sequence for $F^{\rm tr}$ even on $X_0\cup X_1$.*

Proof of the claim. Let $f\in{}^\lambda 2$. If $\{\alpha\in X_0 : F^{\rm tr}(f{\restriction}\alpha)=0\}\notin {\rm ID}(F)$ then we are done (remember 1.3(3)). Otherwise, we have

$$\{\alpha\in X_0 : F_0(f\restriction\alpha)=1\}\in {\rm ID}(F).$$

For $\ell<3$ let $f_\ell\in{}^\lambda 2$ be such that the set $\{\alpha<\lambda : F_\ell(f{\restriction}\alpha)=F(f_\ell{\restriction}\alpha)\}$ contains a club of λ (exists by 1.10); we first use f_0. Then

$$\{\alpha\in X_0 : F(f_0{\restriction}\alpha)=1\}\in {\rm ID}(F^\otimes),$$

and hence, by the choice of the sets X_0,X_1,

$$\{\alpha\in X_1 : F(f_0{\restriction}\alpha)=1\}\in {\rm ID}(F^\otimes).$$

Consequently,

$$\{\alpha \in X_1 : F^{\rm tr}(f{\restriction}\alpha) = 0\} = \{\alpha \in X_1 : F_0(f{\restriction}\alpha) = 1\} \in {\rm ID}(F^{\otimes}).$$

Now we use the choice of η. We know that the set

$$Y = \{\delta \in X_1 : F(f_1{\restriction}\delta) = \eta(\delta)\}$$

is stationary. Hence for some $k \in \{0,1\}$ the set

$$Y_k = \{\delta \in X_1 : F(f_1{\restriction}\delta) = k = \eta(\delta)\}$$

is stationary, but $\{\delta \in X_1 : F(f_1{\restriction}\delta) = F_1(f{\restriction}\delta)\}$ contains a club. Hence

$$Y_k^* = \{\delta \in X_1 : F(f_1{\restriction}\delta) = k = \eta(\delta) \ \text{ and } \ F(f_1{\restriction}\delta) = F_1(f{\restriction}\delta)\}$$

is stationary. Finally note that if $k = 1$ then

$$\delta \in Y_k \quad \Rightarrow \quad F(f_1{\restriction}\delta) = \eta(\delta) = F_1(f{\restriction}\delta) = 1 \quad \Rightarrow \quad F^{\rm tr}(f{\restriction}\delta) = 1.$$

The claim and the theorem are proved. $\square$

$\square$

Theorem 2.3. *Suppose $F^{\rm tr}$ is a (λ,θ)–colouring, $\theta \le \lambda$ and F_i (for $i < \theta$) are given by*

$$F_i(f) = \begin{cases} 1 & \text{if } F(f) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F : {}^{\lambda>}2 \longrightarrow 2$ be such that $(\forall i < \theta)(F_i \le F)$ and let $F^{\otimes}$ be as in 1.5 for F. Suppose that ${\rm ID}(F^{\otimes})$ is λ^+–saturated, and $S^*_{F^{\otimes}} \ne \lambda$ (i.e. $\lambda \notin {\rm ID}^2(F^{\otimes})$). Furthermore, assume that

($\otimes$) *there are sets $Y_i \subseteq \lambda \setminus S^*_{F^{\otimes}}$ for $i < \theta$ such that*
  (a) $(\forall i < \theta)(Y_i \notin {\rm ID}(F^{\otimes}))$,
  (b) *the sets Y_i are pairwise disjoint or at least*

$$(\forall i < j < \theta)(Y_i \cap Y_j \in {\rm ID}(F^{\otimes})),$$

  (c) $\bigcap\limits_{i<\theta} \min_{F^{\otimes}}(Y_i) \notin {\rm ID}(F^{\otimes})$, *see 1.11(h).*

Then

($\bigstar$) *there is a weak diamond sequence $\eta \in {}^{\lambda}\theta$ for $F^{\rm tr}$, i.e.*

$$(\forall f \in {}^{\lambda}2)(\{\delta < \lambda : F^{\rm tr}(f{\restriction}\delta) = \eta(\delta)\} \text{ is stationary });$$

moreover

$$(\forall f \in {}^{\lambda}2)(\{\delta < \lambda : F^{\rm tr}(f{\restriction}\delta) = \eta(\delta)\} \notin {\rm ID}(F^{\otimes})).$$

Proof. We may assume that the sets $\langle Y_i : i < \theta\rangle$ are pairwise disjoint (otherwise we use $Y_i' = Y_i \setminus \bigcup\limits_{j<i} Y_j$). Let $\eta \in {}^{\lambda}\theta$ be such that $(\forall i < \theta)(\eta{\restriction}Y_i = i)$. Note that if

$$\{\delta \in Y_i : F^{\rm tr}(f{\restriction}\delta) = i\} \in {\rm ID}(F^{\otimes})$$

then we also have

$$\{\delta < \lambda : F^{\rm tr}(f{\restriction}\delta) = i\} \in \mathfrak{B}(F^{\otimes})$$

(use $F_i \le F \le F^{\otimes}$). Consequently, in this case, we have

$$\{\delta \in \min_{F^{\otimes}}(Y_i) : F^{\mathrm{tr}}(f{\restriction}\delta) = i\} \in \mathrm{ID}(F^{\otimes}).$$

If this occurs for every $i < \theta$ then

$$\{\delta \in \bigcap_{i<\theta} \min_{F^{\otimes}}(Y_i) : (\exists i < \theta)(F(f{\restriction}\delta) = i)\} \in \mathrm{ID}(F^{\otimes}),$$

but for each δ, for some $i < \theta$ we have $F(f{\restriction}\delta) = i$, a contradiction. $\square$

Proposition 2.4. *Under the assumptions of 2.2 (so the ideal* $\mathrm{ID}(F^{\otimes})$ *is* λ^+*–saturated), if* $X \subseteq \lambda \setminus S^*_{F^{\otimes}}$*,* $X \notin \mathrm{ID}(F^{\otimes})$ *then there is a partition* (X_0, X_1) *of* X *(so* $X_0 \cup X_1 = X$*,* $X_0 \cap X_1 = \emptyset$*) such that*

$$X_0, X_1 \notin \mathrm{ID}(F^{\otimes}), \quad \textit{and} \quad \min_{F^{\otimes}}(X_0) = \min_{F^{\otimes}}(X_1) = \min_{F^{\otimes}}(X).$$

Proof. Let

$$\begin{aligned}\mathcal{A}_{F^{\otimes}} \stackrel{\mathrm{def}}{=} \{Z \subseteq \lambda : \ & Z \notin \mathrm{ID}(F^{\otimes}) \text{ and there is a partition } (Z_0, Z_1) \text{ of } Z \\ & \text{such that } \min_{F^{\otimes}}(Z_1) = \min_{F^{\otimes}}(Z_2) \mod \mathrm{ID}(F^{\otimes})\}.\end{aligned}$$

Note that, by 1.11(h),

$(*)$ $\qquad (\forall Y \in \mathrm{ID}(F^{\otimes})^+)(\exists Z \in \mathcal{A}_{F^{\otimes}})(Z \subseteq Y).$

Let $X \subseteq \lambda$, $X \notin \mathrm{ID}(F^{\otimes})$ and let $\langle Z_\alpha : \alpha < \alpha^* \rangle$ be a maximal sequence such that for each $\alpha < \alpha^*$:

$$Z_\alpha \in \mathcal{A}_{F^{\otimes}}, \quad Z_\alpha \subseteq X, \quad \text{and} \quad (\forall \beta < \alpha)(Z_\alpha \cap Z_\beta \in \mathrm{ID}(F^{\otimes})).$$

Necessarily $\alpha^* < \lambda^+$, so without loss of generality $\alpha^* \le \lambda$, $\min(Z_\alpha) > \alpha$ and $Z_\alpha \cap Z_\beta = \emptyset$ for $\alpha < \beta < \alpha^*$. Let $\langle Z^0_\alpha, Z^1_\alpha \rangle$ be a partition of Z_α witnessing $Z_\alpha \in \mathcal{A}_{F^{\otimes}}$. Put

$$Z_0 \stackrel{\mathrm{def}}{=} \bigcup_{\alpha<\alpha^*} Z^0_\alpha \quad \text{and} \quad Z_1 \stackrel{\mathrm{def}}{=} \bigcup_{\alpha<\alpha^*} Z^1_\alpha.$$

Then $Z_0 \cap Z_1 = \emptyset$, $Z_0 \cup Z_1 \subseteq X$. Note that $\bigcup_{\alpha<\alpha^*} Z_\alpha$ is equal to the diagonal union and, by $(*)$ above, $X \setminus \bigcup_{\alpha<\alpha^*} Z_\alpha \in \mathrm{ID}(F^{\otimes})$. Consequently we may assume $Z_0 \cup Z_1 = \bigcup_{\alpha<\alpha^*} Z_\alpha = X$. Next, since

$$\min_{F^{\otimes}}(Z_0) \supseteq \min_{F^{\otimes}}(Z^0_\alpha) \supseteq Z^0_\alpha \cup Z^1_\alpha = Z_\alpha,$$

we get

$$\min_{F^{\otimes}}(Z_0) \supseteq \bigcup_{\alpha<\alpha^*} Z_\alpha = X = Z_0 \cup Z_1,$$

and similarly one shows that $\min_{F^{\otimes}}(Z_1) \supseteq X$. Now we use 1.11(h) to finish the proof. $\square$

Proposition 2.5. *Under the assumptions of 2.3:*

(1) *If* $2^\theta < \lambda$ *then there is a sequence* $\langle Y_i : i < \theta \rangle$ *as required in 2.3(*$\oplus$*).*
(2) *Similarly if* $\theta \le \aleph_0$*.*
(3) *In both cases, if* $S \notin \mathrm{ID}(F^{\otimes})$ *then we can demand* $(\forall i < \theta)(Y_i \subseteq S)$*.*

Proof. 1) By induction on $\alpha \le \theta$ we choose sets $X_\eta \subseteq \lambda$ for $\eta \in {}^{\alpha}2$ such that:

(i) $X_{\langle\rangle} \notin \mathrm{ID}(F^{\otimes})$,
(ii) if α is limit then $X_\eta = \bigcap\limits_{i<\alpha} X_{\eta\restriction i}$,
(iii) if $\alpha = \beta+1$, $\eta \in {}^{\beta}2$ and $X_\eta \in \mathrm{ID}(F^{\otimes})$ then $X_{\eta^\frown\langle 0\rangle} = X_\eta$, $X_{\eta^\frown\langle 1\rangle} = \emptyset$;
if $\alpha = \beta+1$, $\eta \in {}^{\beta}2$ and $X_\eta \notin \mathrm{ID}(F^{\otimes})$ then $(X_{\eta^\frown\langle 0\rangle}, X_{\eta^\frown\langle 1\rangle})$ is a partition of X_η such that $\min_{F^{\otimes}}(X_{\eta^\frown\langle 0\rangle}) = \min_{F^{\otimes}}(X_{\eta^\frown\langle 1\rangle}) = \min_{F^{\otimes}}(X_\eta)$.

It follows from 2.4 that we can carry out the construction.

Clearly $\langle X_\eta : \eta \in {}^{\theta}2\rangle$ is a partition of $X_{\langle\rangle}$, so (as $2^\theta < \lambda$ and $\mathrm{ID}(F^{\otimes})$ is λ–complete) we can find a sequence $\eta \in {}^{\theta}2$ such that $X_\eta \notin \mathrm{ID}(F^{\otimes})$. Then

$$(\forall \alpha < \theta)(X_{\eta\restriction\alpha} \notin \mathrm{ID}(F^{\otimes}))$$

(as each of these sets includes X_η). Moreover, for each $\alpha < \theta$ and for $\ell = 0, 1$ we have

$$\min_{F^{\otimes}}(X_{\eta\restriction\alpha^\frown\langle\ell\rangle}) \supseteq X_{\eta\restriction\alpha} \supseteq X_\eta.$$

Put $Y_\alpha = X_{\eta\restriction\alpha^\frown\langle 1-\eta(\alpha)\rangle}$. Then $\langle Y_\alpha : \alpha < \theta\rangle$ is a sequence of pairwise disjoint sets (as $X_{\eta\restriction\alpha^\frown\langle 0\rangle} \cap X_{\eta\restriction\alpha^\frown\langle 1\rangle} = \emptyset$) and for every $\alpha < \theta$

$$Y_\alpha \notin \mathrm{ID}(F^{\otimes}) \quad \text{and} \quad \min_{F^{\otimes}}(Y_\alpha) \supseteq X_{\eta\restriction\alpha} \supseteq X_\eta.$$

Hence $\bigcap\limits_{\alpha<\theta} \min_{F^{\otimes}}(Y_\alpha) \notin \mathrm{ID}(F^{\otimes})$. Let $Z_\alpha = Y_\alpha \cap \min_{F^{\otimes}}(X_\eta)$. Note that $\min_{F^{\otimes}}(Z_\alpha) = \min_{F^{\otimes}}(X_\eta)$ (the "$\le$" is clear; if $\min_{F^{\otimes}}(Z_\alpha) < \min_{F^{\otimes}}(X_\eta)$ then $\min_{F^{\otimes}}(X_\eta)\setminus\min_{F^{\otimes}}(Z_\alpha)$ contradicts the definition of $\min_{F^{\otimes}}(Y_\alpha)$). Thus the sequence $\langle Z_\alpha : \alpha < \theta\rangle$ is as required. Moreover

$$\min_{F^{\otimes}}(Z_\alpha) = \bigcup_{\beta} \min_{F^{\otimes}}(Z_\beta).$$

2) Let $X \subseteq \lambda$, $X \notin \mathrm{ID}(F^{\otimes})$. By induction on n we choose sets X'_n, X''_n such that $X'_n \cap X''_n = \emptyset$, $X'_n \cup X''_n \supseteq X$, and

$$\min_{F^{\otimes}}(X'_n) = \min_{F^{\otimes}}(X''_n) = \min_{F^{\otimes}}(X).$$

For $n = 0$ we use 2.4 for X to get X'_0, X''_0. For $n+1$ we use 2.4 for X''_n to get X'_{n+1}, X''_{n+1}.

Finally we let $Y_n = X''_n$ (note that $\min_{F^{\otimes}}(Y_n) = \min_{F^{\otimes}}(X)$). □

Conclusion 2.6. Assume that

(A) λ is a regular uncountable cardinal,
(B) F is a (λ, θ)–colouring such that $\lambda \notin \mathrm{ID}(F)$ and $\mathrm{ID}(F)$ is λ^+–saturated,
(C) $2^\theta < \lambda$ or $\theta = \aleph_0$,
(D) $(\exists \mu < \lambda)(2^\mu = 2^{<\lambda} < 2^\lambda)$ or at least $\lambda \notin \mathrm{WDmId}_\lambda$ or at least $\lambda \notin \mathrm{ID}^2(F)$.

Then there is a weak diamond sequence for F. Moreover, there is $\eta \in {}^{\lambda}\theta$ such that for each $f \in \mathrm{DOM}_\lambda(F)$ we have

$$\{\delta < \lambda : F(f{\restriction}\delta) = \eta(\delta)\} \notin \mathrm{ID}(F).$$

3. An application of Weak Diamond

In this section we present an application of Weak Diamond in model theory. For more on model–theoretic investigations of this kind we refer the reader to [She01] and earlier work [She87a], and to an excellent survey my Makowsky, [Mak85].

Definition 3.1. Let $\mathfrak{K}$ be a collection of models.

(1) For a cardinal λ, $\mathfrak{K}_\lambda$ stands for the collection of all members of $\mathfrak{K}$ of size λ.
(2) We say that a partial order $\leq_{\mathfrak{K}}$ on $\mathfrak{K}_\lambda$ is λ–nice if
   - (α) $\leq_{\mathfrak{K}}$ is a suborder of $\subseteq$ and it is closed under isomorphisms of models (i.e. if $M, N \in \mathfrak{K}_\lambda$, $M \leq_{\mathfrak{K}} N$ and $f : N \longrightarrow N' \in \mathfrak{K}_\lambda$ is an isomorphism then $f[M] \leq_{\mathfrak{K}} N'$),
   - (β) $(\mathfrak{K}_\lambda, \leq_{\mathfrak{K}})$ is λ–closed (i.e. any $\leq_{\mathfrak{K}}$–increasing sequence of length $\leq \lambda$ of elements of $\mathfrak{K}_\lambda$ has a $\leq_{\mathfrak{K}}$–upper bound in $\mathfrak{K}_\lambda$) and
   - (γ) if $\bar{M} = \langle M_\alpha : \alpha < \lambda\rangle$ is an $\leq_{\mathfrak{K}}$–increasing sequence of elements of $\mathfrak{K}_\lambda$ then $\bigcup\limits_{\alpha<\lambda} M_\alpha$ is the $\leq_{\mathfrak{K}}$–upper bound to $\bar{M}$ (so $\bigcup\limits_{\alpha<\lambda} M_\alpha \in \mathfrak{K}_\lambda$).
(3) Let $N \in \mathfrak{K}_\lambda$, $A \subseteq |N|$. We say that *the pair* (A, N) *has the amalgamation property in* $\mathfrak{K}_\lambda$ if for every $N_1, N_2 \in \mathfrak{K}_\lambda$ such that $N \leq_{\mathfrak{K}} N_1$, $N \leq_{\mathfrak{K}} N_2$ there are $N^* \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$–embeddings f_1, f_2 of N_1, N_2 into N^*, respectively, such that $f_1{\restriction}A = f_2{\restriction}A$. (In words: N_1, N_2 can be amalgamated over (A, N).)
(4) We say that $(\mathfrak{K}, \leq_{\mathfrak{K}})$ has *the amalgamation property for* λ if for every $M_0, M_1, M_2 \in \mathfrak{K}_\lambda$ such that $M_0 \leq_{\mathfrak{K}} M_1$, $M_0 \leq_{\mathfrak{K}} M_2$ there are $M \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$–embeddings f_1, f_2 of M_1, M_2 into M, respectively, such that
$$M_0 \leq_{\mathfrak{K}} M \quad \text{ and } \quad f_1{\restriction}M_0 = f_2{\restriction}M_0 = \mathrm{id}_{M_0}.$$

Theorem 3.2. *Assume that λ is a regular uncountable cardinal for which the weak diamond holds (i.e. $\lambda \notin \mathrm{WDmId}_\lambda$). Suppose that $\mathfrak{K}$ is a class of models, $\mathfrak{K}$ is categorical in λ (i.e. all models from $\mathfrak{K}_\lambda$ are isomorphic), it is closed under isomorphisms of models, and $\leq_{\mathfrak{K}}$ is a λ–nice partial order on $\mathfrak{K}_\lambda$ and $M \in \mathfrak{K}_\lambda$. Let $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle$ be an increasing continuous sequence of subsets of $|M|$ such that*

$$(\forall \alpha < \lambda)(\|A_\alpha\| < \lambda) \quad \textit{and} \quad \bigcup_{\alpha<\lambda} A_\alpha = M.$$

Then the set

$$S^{\bar{A}}_M \stackrel{\text{def}}{=} \big\{\alpha < \lambda \colon (A_\alpha, M) \textit{ does not have the amalgamation property}\big\}$$

is in WDmId_λ.

Proof. Assume that $S^{\bar{A}}_M \notin \mathrm{WDmId}_\lambda$.

We may assume that $|M| = \lambda$. By induction on $i < \lambda$ we choose pairs (B_η, N_η) and sequences $\langle C^\eta_j : j < \lambda \rangle$ for $\eta \in {}^i 2$ such that

(a) $\|B_\eta\| < \lambda$, $N_\eta \in \mathfrak{K}_\lambda$, $B_\eta \subseteq |N_\eta| \subseteq \lambda$,

(b) $\langle C^\eta_j : j < \lambda \rangle$ is increasing continuous, $\bigcup\limits_{j<\lambda} C^\eta_j = |N_\eta|$, $\|C^\eta_j\| < \lambda$,

(c) if $\nu \triangleleft \eta$ then $N_\nu \leq_{\mathfrak{K}} N_\eta$ and $B_\nu \subseteq B_\eta$,

(d) if $j_1, j_2 < i$ then $C^{\eta\restriction j_1}_{j_2} \subseteq B_\eta$,

(e) if the pair (B_η, N_η) does not have the amalgamation property in $\mathfrak{K}_\lambda$ then $N_{\eta^\frown\langle 0\rangle}$, $N_{\eta^\frown\langle 1\rangle}$ witness it (i.e. they cannot be amalgamated over B_η),

(f) if i is limit and $\eta \in {}^i 2$ then $B_\eta = \bigcup\limits_{j<i} B_{\eta\restriction j}$, $\bigcup\limits_{j<i} N_{\eta\restriction j} \subseteq N_\eta$.

There are no problems with carrying out the construction (remember that $\leq_{\mathfrak{K}}$ is a nice partial order), we can fix a partition $\langle D_i : i < \lambda \rangle$ of λ into λ sets each of cardinality λ, and demand that the universe of N_η is included in $\bigcup\{D_j : j < 1 + \ell g(\eta)\rangle$. Finally, for $\eta \in {}^\lambda 2$ we let $B_\eta = \bigcup\limits_{i<\lambda} B_{\eta\restriction i}$ and $N_\eta = \bigcup\limits_{i<\lambda} N_{\eta\restriction i}$. Clearly, by 3.1($2\gamma$), we have $N_\eta \in \mathfrak{K}$ and $B_\eta \subseteq |N_\eta|$ for each $\eta \in {}^\lambda 2$. Moreover,

$$|N_\eta| = \bigcup_{j<\lambda} |N_{\eta\restriction j}| = \bigcup_{j<\lambda} \bigcup_{i<\lambda} C^{\eta\restriction j}_i = \bigcup_{j^*<\lambda} \bigcup_{j_1,j_2<j^*} C^{\eta\restriction j_1}_{j_2} \subseteq \bigcup_{j^*<\lambda} B_{\eta\restriction j^*} = B_\eta,$$

and thus $B_\eta = |N_\eta|$. Since $\mathfrak{K}$ is categorical in λ, for each $\eta \in {}^\lambda 2$ there is an isomorphism $f_\eta : N_\eta \xrightarrow{\text{onto}} M$.

Fix $\eta \in {}^\lambda 2$ for a moment.
Let $E_\eta = \{\delta < \lambda : f_\eta[B_{\eta\restriction\delta}] = A_\delta = \delta\}$. Clearly, E_η is a club of λ. Note that if $\delta \in E_\eta$ then:

$$(\boxtimes) \quad \begin{aligned} \delta \in S^{\bar{A}}_M \quad &\Rightarrow \quad (A_\delta, M) \text{ does not have the amalgamation property} \\ &\Rightarrow \quad (B_{\eta\restriction\delta}, N_\eta) \text{ fails the amalgamation property} \\ &\Rightarrow \quad (B_{\eta\restriction\delta}, N_{\eta\restriction\delta}) \text{ fails the amalgamation property} \\ &\Rightarrow \quad N_{\eta\restriction\delta^\frown\langle 0\rangle},\ N_{\eta\restriction\delta^\frown\langle 1\rangle} \text{ cannot be amalgamated} \\ &\qquad\qquad \text{over } (B_{\eta\restriction\delta}, N_{\eta\restriction\delta}) \\ &\Rightarrow \quad \text{for each } \nu \in {}^\lambda 2 \text{ such that } \eta\restriction\delta^\frown\langle 1-\eta(\delta)\rangle \triangleleft \nu \\ &\qquad \text{we have } f_\nu\restriction B_{\eta\restriction\delta} \neq f_\eta\restriction B_{\eta\restriction\delta}. \end{aligned}$$

We define a colouring

$$F : \bigcup_{\alpha<\lambda} {}^\alpha(\mathcal{H}(\lambda)) \longrightarrow \{0,1\}$$

by letting, for $f \in \mathrm{DOM}_\alpha$, $\alpha < \lambda$,

$$F(f) = 1 \quad \text{iff} \quad (\exists \eta \in {}^\lambda 2)\big(\eta(\alpha) = 0 \ \&\ (\forall i < \alpha)(f(i) = (\eta(i), f^{-1}_\eta(i)))\big).$$

We have assumed $S^{\bar{A}}_M \notin \mathrm{WDmId}_\lambda$, so there is $\rho \in {}^\lambda 2$ such that for each $f \in \mathrm{DOM}_\lambda$ the set

$$S_f = \{\delta \in S^{\bar{A}}_M : \rho(\delta) = F(f{\restriction}\delta)\}$$

is stationary. Let $f \in \mathrm{DOM}_\lambda$ be defined by $f(i) = (\rho(i), f^{-1}_\rho(i))$ (for $i < \lambda$). Note that

if $\alpha \in E_\rho$, $\rho(\alpha) = 0$

then ρ is a witness to $F(f{\restriction}\alpha) = 1$ and hence $\alpha \notin S_f$.

Since S_f is stationary and E_ρ is a club of λ we may pick $\delta \in S_f \cap E_\rho$. Then $\rho(\delta) = 1$ and hence $F(f{\restriction}\delta) = 1$, so let $\eta_\delta \in {}^\lambda 2$ be a witness for it. It follows from the definition of F that then $\eta_\delta(\delta) = 0$, and $\eta_\delta{\restriction}\delta = \rho{\restriction}\delta$, and $f^{-1}_{\eta_\delta}{\restriction}\delta = f^{-1}_\rho{\restriction}\delta$. Hence $f_{\eta_\alpha}{\restriction}B_{\eta_\delta{\restriction}\delta} = f_\rho{\restriction}B_{\rho{\restriction}\delta}$, so both have range $A_\delta = \delta$ (and $\delta \in E_{\eta_\delta} \cap E_\rho \cap S^{\bar{A}}_M$). But now we get a contradiction with ($\boxtimes$). $\square$

References

[DS78] Keith J. Devlin and Saharon Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), no. 2-3, 239–247. MR 0469756

[Mak85] Johann A. Makowsky, *Compactnes, embeddings and definability*, Model-Theoretic Logics (J. Barwise and S. Feferman, eds.), Springer-Verlag, 1985, pp. 645–716.

[She77] Saharon Shelah, *Whitehead groups may be not free, even assuming CH. I*, Israel J. Math. **28** (1977), no. 3, 193–204. MR 0469757

[She83] ______, *Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel J. Math. **46** (1983), no. 4, 241–273. MR 730343

[She85] ______, *More on the weak diamond*, Ann. Pure Appl. Logic **28** (1985), no. 3, 315–318. MR 790390

[She87a] ______, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 419–497. MR 1033034

[She87b] ______, *Uncountable groups have many nonconjugate subgroups*, Ann. Pure Appl. Logic **36** (1987), no. 2, 153–206. MR 911580

[She98] ______, *Proper and improper forcing*, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998. MR 1623206

[She01] ______, *Categoricity of an abstract elementary class in two successive cardinals*, Israel J. Math. **126** (2001), 29–128, arXiv: math/9805146. MR 1882033

[SZ99] Saharon Shelah and Jindřich Zapletal, *Canonical models for $\aleph_1$-combinatorics*, Ann. Pure Appl. Logic **98** (1999), no. 1-3, 217–259, arXiv: math/9806166. MR 1696852

Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA, and Mathematics Department, University of Wisconsin – Madison, Madison, WI 53706, USA

Email address: shelah@math.huji.ac.il

URL: http://www.math.rutgers.edu/~shelah