HANF NUMBER FOR THE STRICTLY STABLE CASES SH1048

SAHARON SHELAH

ABSTRACT. Suppose $\mathbf{t} = (T, T_1, p)$ is a triple of two first order theories $T \subseteq T_1$ in vocabularies $\tau \subseteq \tau_1$ (respectively) of cardinality λ and a τ_1-type p over the empty set; the main case here is with T stable. We show that the Hanf number for the property: "there is a model M_1 of T_1 which omits p, but $M_1 \restriction \tau$ is saturated" is larger than the Hanf number of $\mathbb{L}_{\lambda^+,\kappa}$ but smaller than the Hanf number of $\mathbb{L}_{(2^\lambda)^+,\kappa}$ when T is stable with $\kappa = \kappa(T)$. In fact, surprisingly we even characterize the Hanf number of $\mathbf{t}$ when we fix (T, λ) where T is a first order complete (and stable), $\lambda \geq |T|$ and demand $|T_1| \leq \lambda$.

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§ 0. Introduction

§ 0(A). Background on Results.

This continues papers of Baldwin-Shelah, starting from a problem of Newelski [?] concerning the Hanf number described in the abstract for classes $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ (defined formally in 1.1), that is:

- for T a complete first order theory, λ an infinite cardinal $\geq |T|$ let $\mathbf{N}_{\lambda,T}$ be the class of triples $\mathbf{t} = (T, T_1, p)$ such that $T_1 \supseteq T$ is first order of cardinality $\leq \lambda$ and $p = p(x)$ a type in the vocabulary of T_1
- for $\mathbf{t} \in \mathbf{N}_{\lambda,T}, M$ is a model of $\mathbf{t}$ <u>iff</u> it is a model of T_1 (so have the same vocabulary) omitting the type p such that its restriction to the vocabulary of T is a saturated model
- the Hanf number $H(\mathbf{t})$ of $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ is the first cardinal μ such that $\mathbf{t}$ has no model of cardinality $\geq \mu$ and is infinity when there is no such bound
- the Hanf number $H(\mathbf{N}_{\lambda,T})$ of $\mathbf{N}_{\lambda,T}$ is $\sup\{H(\mathbf{t}) : \mathbf{t} \in \mathbf{N}_{\lambda,T}$ and $H(\mathbf{t}) < \infty\}$
- the Hanf number $H_{\mathbf{N}}(\lambda)$ is $\sup\{H(\mathbf{N}_{T,\lambda}) : (T, \lambda)$ as above$\}$
- note that, considering $\mathbf{N}_{\lambda,T}$ if T is unstable it is natural to assume that $\{\mu : \mu = \mu^{<\mu}\}$ is an unbounded class as otherwise for any T_1, λ we have $H((T_1, T, \lambda)) \leq \sup\{\mu^+ : \mu = \mu^{<\mu}\}$; Newelski in [?] essentially asks what is $H_{\mathbf{N}}(\lambda)$, Baldwin-Shelah [?], [?] have dealt with those numbers.

They showed in [?] that the Hanf number $H_{\mathbf{N}}(\lambda)$ is essentially equal to the Löwenheim number of second order logic using unstable T's and in [?] showed that for superstable $T, H(\mathbf{N}_{\lambda,T})$ is bigger than the Hanf number of $\mathbb{L}_{(2^\lambda)^+,\aleph_0}$ but it is smaller than $\mathbb{L}_{\beth_2(\lambda)^+,\aleph_0}$.

Our original aim was to deal with the case where T is a stable theory and concentrate on the strictly stable case (i.e. stable not superstable).

So here we are trying to sort out when cases Hanf number are manageable, as e.g. for $\mathbb{L}_{\lambda^+,\aleph_0}$ where it is $\beth_\delta, \delta < (2^\lambda)^+$ and cases it is not, say e.g. is above a compact cardinal.

From another perspective we are trying to classify T, so fixing T's, considering Hanf number of $\mathbf{t} = (T, T_1, p)$ and so $H(\mathbf{N}_{T,\lambda}) = \sup\{\text{Hanf}(\mathbf{t}) : T_{\mathbf{t}} = T, |T_1| \leq \lambda\}$ is a measure of the complexity of T. Indeed, for unstable T it is very large, for T superstable it is quite small. Our original object was to sort out the case of T strictly stable, which falls in the middle.
However, we ask a stronger question.

Question 0.1. Fix a complete first order theory T and a cardinal $\lambda \geq |T|$, what is $H(\mathbf{N}_{\lambda,T})$? recalling it is $\sup\{H(\mathbf{t}) : H(\mathbf{t}) < \infty$ and $\mathbf{t}$ as above with $T_{\mathbf{t}} = T$ and $|T_{\mathbf{t},1}| \leq \lambda$, i.e. belongs to $\mathbf{N}_{\lambda,T}$ from 1.1(1)$\}$, recalling $H(\mathbf{t})$ is the supremum of the cardinalities of models in $\text{Mod}_{\mathbf{t}}$.

Clearly this is a considerably more ambitious question. Now [?] actually determines $H(\mathbf{N}_{\lambda,T})$ when T is unstable, so we shall concentrate here on the case T is stable. We give a quite complete answer. For T strictly stable, our original case, it appears that we need only the cardinals $|T|, \kappa(T)$ and a derived Boolean Algebra $\mathbb{B}(T)$ of cardinality $|D(T)|$, and a little more where $D(T) = \cup\{D_n(T) : n < \omega\}, D_n(T)$

is the set of complete n-types realized in models of T. In fact, for any T, the little more is the truth value of ($2^{\aleph_0} > |D(T)| > |T| \wedge$ "T unstable in $|D(T)|$" $\wedge$ (T superstable).

Here the infinitary logic $\mathbb{L}_{\lambda^+,\kappa}$ is central.

A major point is to deal abstractly with what is essentially the Boolean algebra of formulas over the empty set, $\mathbb{B}_T$ (so modulo T of course). We introduce in Definition 1.6 the logics $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ where $\mathbb{B} = \mathbb{B}_T$, the members of the Boolean algebra (i.e. formulas from $\mathbb{L}(\tau_T)$) are coded by elements of the model and the union of these logics over the relevant $\mathbb{B}$'s is called $\mathbb{L}^{\rm ba}_{\lambda^+,\kappa}$, moreover $\mathbb{L}^{\rm ba}_{\lambda,\kappa}$ is equivalent to $\mathbb{L}_{\lambda,\kappa}[\mathbb{B}^{\rm fr}_\lambda]$, see 0.7(5). Then in Observation 1.9(4) we note that:

$$H(\mathbb{L}_{\lambda^+,\kappa}) \le H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]) \le H(\mathbb{L}^{\rm ba}_{\lambda^+,\kappa}) \le H(\mathbb{L}_{(2^\lambda)^+,\kappa}).$$

The main result shows that there is an exact equivalence between classes of the form $\mathbf{N}_{\lambda,T}$ and classes of the form $\text{Mod}_\psi, \psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ for $\mathbb{B}$ the Boolean Algebra formulas over the empty set in T.

Note Grossberg-Vasey [**?**, Th.4.8] proves a generalization of the superstalbe case to a.e.c. by coding that does not use Boolean Algebra.

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§ 0(B). **Description of the Proof.**

We describe the proof, concentrating on the case $\kappa > \aleph_0, T$ stable with $\kappa(T) = \kappa$. First, recalling the characterization of "$M \in \text{Mod}_T$" is saturated, it is natural when considering $\mathbf{t} \in N_\lambda$ to use $\mathbb{L}_{\lambda^+,\kappa}$ as the relevant logic. That is, if $M_2 \in \text{Mod}_{\mathbf{t}}$ and $\lambda > 2^{|T|}$ there is a sentence $\psi_1 \in \mathbb{L}_{\lambda,\kappa}(\tau_T)$ saying M is κ-saturated, hence to be saturated it is enough to have: every countable infinite indiscernible set can be extended to one of cardinality $\|M\|$. Restricting ourselves to models of cardinality $\mu = \mu^{\aleph_0}$, there is ψ_2 in $\mathbb{L}_{|T|^+,\aleph_1}(\tau_T \cup \{F, F_n : n < \omega\})$ saying this with F, F_n unary functions. This extension of τ_{T_1} is fine for our problem. This describes how starting from $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ we find $\psi \in \mathbb{L}_{\lambda^+,\kappa}(\tau_2)$ such that for $\mu = \mu^{\aleph_0} \ge \lambda$, there is $M \in \text{Mod}_{\mathbf{t}}$ of cardinality μ <u>iff</u> there is $N \in \text{Mod}_\psi$ of cardinality μ.

We need also translation in the other direction, i.e. given $\psi \in \mathbb{L}_{\lambda^+,\kappa}(\tau_\psi)$ to find suitable T. Here $\mu = \mu^{<\kappa}$ is helpful: we can replace ψ by $\mathbf{t} \in N_{\lambda,T}$ such that T_1 codes enough set theory, and the main point is that in every model M_1 of $T_{1,\mathbf{t}}$, there are relation and function pretending to code the set of sequences ${}^{\kappa>}M_1$. The main point is why every such sequence is coded, which is done using "$M_1 {\restriction} \tau_T$" is κ-saturated and $\kappa = \kappa|T|$. But why is it enough to concentrate on cardinals $\mu = \mu^{<\kappa} > 2^\lambda$? as we get the same Hanf numbers when we restrict ourselves to those cardinals, i.e. considering only the classes $\text{spec}^2_{\mathbf{t}} \backslash 2^\lambda$. So when $\lambda \ge 2^{|T|}, \kappa > \aleph_0$ we get an accurate description of the family of classes $\text{spec}^2_{\mathbf{t}}$ for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$. Allowing $\kappa = \aleph_0$ requires non-essential changes.

However, the most natural case is $\lambda = |T| = |T_1|$ so allowing $\lambda < 2^{|T|}$; e.g. T says $p_\alpha(\alpha < \lambda)$ are independent unary predicate. This requires the use of $\mathbb{B}_T$.

§ 0(C). **Preliminaries.**

Here for a first order complete T we define the relevant parameters; $\kappa(T), \mathbb{B}_T$ and quote the characterization of the existence of saturated models.

Notation 0.2. 1) τ will denote a vocabulary, $\tau_M = \tau(M)$ is the vocabulary of a model $M, |M|$ is the universe of M and $\|M\|$ its cardinality; $\mathbb{L}(\tau)$ is the first order logic for this vocabulary, i.e. the set of first order formulas in τ.
1A) T denotes a first order theory in $\mathbb{L}_{\tau(T)}, \tau_T = \tau(T)$ the vocabulary of T and T is complete and T is complete and stable if not said otherwise (but T_1 is neither necessarily complete nor necessarily stable).
2) $\bar{x}_{[u]} = \langle x_i : i \in u\rangle$, similarly $\bar{y}_{[u]}$; e.g. $\bar{x}_{[\alpha]} = \langle x_i : i < \alpha\rangle$.
3) $\mathbb{L}_{\lambda,\kappa}$ for $\lambda \geq \kappa$ is the logic where the language $\mathbb{L}_{\lambda,\kappa}(\tau)$ is the following set of formulas; it is the closure of the set of atomic formulas under negation, conjunction of the form $\bigwedge\limits_{\alpha<\gamma} \varphi_\alpha, \gamma < \lambda$ and quantification of the form $(\exists \bar{x}_{[u]})\varphi$ where $u \in [\kappa]^{<\kappa}$ (really just $(\exists \bar{x}_{[\varepsilon]})\varphi$ for $\varepsilon < \kappa$ suffice), but every formula has $< \kappa$ free variables.
4) Let $\mathbb{B}$ denote a Boolean Algebra and uf$(\mathbb{B})$ the set of ultra-filters of $\mathbb{B}$.
5) Let $\mathbf{t}$ denote an object as in Definition 1.1 below.
6) For a theory T let Mod_T be the class of models of T.

Recall

Definition 0.3. Let T be a first order complete stable theory (as usual here).
0) For a model M of T and $A \subseteq M$ let $\mathbf{S}^n(A, M)$ be the set of complete n-types over A in M, equivalently $\{\text{tp}(\bar{a}, A, N) : M \prec N \text{ and } \bar{a} \in {}^n N\}$ recalling that for $\bar{a} \in {}^n M$ and $A \subseteq M$ we let $\text{tp}(\bar{a}, A, M) = \{\varphi(\bar{x}_{[n]}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M) \text{ and } \bar{b} \in {}^{\ell g(\bar{y})}M$ and $M \models \varphi[\bar{a}, \bar{b}]\}$; if $n = 1$ then we may omit n and $\mathbf{S}^n(M) = \mathbf{S}^n(|M|, M)$ recalling $|M|$ is the universe of M.
Recall:

(a) T is stable in λ or λ-stable when for every model M of T and $A \subseteq M$ of cardinality $\leq \lambda$ the set $\mathbf{S}(A, M)$ has cardinality $\leq \lambda$
(b) T is superstable iff T is λ-stable for every λ large enough.

1) $\kappa(T)$ is the minimal κ such that: if $A \subseteq M_* \in \text{Mod}_T$ and $p \in \mathbf{S}(A, M)$ then there is $B \subseteq A$ of cardinality $< \kappa$ such that p does not fork over B, see [**?**, Ch.III].
2) Let $\kappa_r(T) = \min\{\kappa : \kappa \text{ regular } \geq \kappa(T)\}$ so $\kappa_r(T)$ is the minimal regular κ such that T is stable in λ whenever $\lambda = \lambda^{<\kappa} + 2^{|T|}$, see [**?**, Ch.III].
3) Let $\lambda(T)$ be the minimal λ such that T is stable in λ, that is $[M \models T, \|M\| \leq |T| + \aleph_0 \Rightarrow |\mathbf{S}(M)| \leq \lambda]$, see [**?**, Ch.III,§5,§6].
4) $D_m(T) = \{\text{tp}(\bar{a}, \emptyset, M) : \bar{a} \in {}^m M \text{ and } M \models T\}$ and $D(T) = \bigcup\limits_m D_m(T)$.
5) Let $\text{EQ}_T = \{\varphi(\bar{x}_{[n]}, \bar{y}_{[n]}) : n < \omega, \varphi \in \mathbb{L}(\tau_T)$ and for every model M of $T, \{(\bar{a}, \bar{b}) : \bar{a}, \bar{b} \in M \text{ and } M \models \varphi[\bar{a}, \bar{b}]\}$ is an equivalence relation on ${}^n M$ with finitely many equivalence classes$\}$.
6) M is $\aleph_\varepsilon$-saturated when for every triple (b, A, N) satisfying $A \subseteq M \prec N, b \in N, A$ finite, there is $b' \in M$ realizing the type $\{\varphi(x, b; \bar{a}) : \bar{a} \subseteq A, \varphi(x, y, \bar{a})$ is an equivalence relation with finitely many equivalence classes in $M\}$, this type is called $\text{stp}(b, A, N)$, see [**?**, Ch.III].

Remark 0.4. By [?, Ch.III,§5,§6] we have that $\kappa(T) \le |T|^+, \lambda(T) = |D(T)|^{<\kappa(T)}$ except when $|D(T)| < 2^{\aleph_0}, T$ is superstable and unstable in $|T|$. In this case $|D(T)| < 2^{\aleph_0} = \lambda(T)$ and $\lambda(T) = |D(T)|^{<\kappa(T)}$, see 0.11.

The point is that by [?, Ch.III]:

Fact 0.5. Let T be a complete first order stable theory and let $\lambda \ge \aleph_1 + |T|$ be an infinite cardinal. Then T has a saturated model of cardinality λ if and only if T is λ-stable, if and only if $\lambda = \lambda^{<\kappa(T)} + \lambda(T)$.

Note that

Observation 0.6. *For every Boolean Algebra $\mathbb{B}_1$ of cardinality $\le \lambda$ and $\kappa \le \lambda^+$ there is a Boolean Algebra $\mathbb{B}_2$ of cardinality λ such that $|\mathrm{uf}(\mathbb{B}_2)| = \Sigma\{|\mathrm{uf}(\mathbb{B}_1)|^\theta : \theta < \kappa\}$.*

Proof. Without loss of generality $|\mathbb{B}_1| = \lambda$, as otherwise we replace $\mathbb{B}$, by $\mathbb{B}_1 \oplus \mathbb{B}^0_\lambda$ where $\mathbb{B}^0_\lambda$ is the Boolean Algebra of finite and co-finite subsets of λ. If $|\mathbb{B}_1| = \lambda, \kappa = \theta^+, \theta \le \lambda$ we define the Boolean Algebra $\mathbb{B}_2$ as the free product of θ copies of $\mathbb{B}_1$.

If κ is a limit cardinal $\le \lambda, |\mathbb{B}_1| = \lambda$ let $\mathbb{B}_{2,\theta}$ be as above for $\theta < \kappa$ and $\mathbb{B}_2$ the disjoint sum of $\langle \mathbb{B}_{2,\theta} : \theta < \kappa\rangle$ so essentially except one ultrafilter, all ultrafilters on $\mathbb{B}_2$ are ultrafilters on some $\mathbb{B}_{2,\theta}$ so $\mathrm{uf}(\mathbb{B}_2) = 1 + \sum\limits_{\theta<\kappa} \mathrm{uf}(\mathbb{B}_{2,\theta})$. $\square_{0.6}$

Definition 0.7. 1) For a model M and formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$ let $\varphi(M, \bar{a}) = \{\bar{b} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{b}, \bar{a}]\}$.
2) For a model $M, \mathbb{B}_{M,m}$ is the Boolean Algebra of subsets of ${}^m M$ consisting of the sets $\{\varphi(M) : \varphi = \varphi(\bar{x}_{[m]})\}$.
2A) $\mathbb{B}_{T,m}$ is the Boolean Algebra of the formulas $\varphi(\bar{x}_{[m]}) \in \mathbb{L}(\tau_T)$ modulo equivalence over T, so $\varphi_1(\bar{x}_{[m]}) \le \varphi_2(\bar{x}_{[m]})$ iff $T \vdash$ "$\varphi_1(\bar{x}_{[m]}) \to \varphi_2(\bar{x}_{[m]})$", so the elements are actually $\varphi(\bar{x}_{[m]})/\equiv_T$.
3) Let $\bar{\mathbb{B}}_M = \langle \mathbb{B}_{M,m} : m < \omega\rangle$; abusing notation let $\mathrm{uf}(\bar{\mathbb{B}}_M) = \bigcup\limits_m \mathrm{uf}(\mathbb{B}_{M,m})$. Similarly with T instead of M, also below.
3A) Let $\mathbb{B}_M$ be the direct sum of $\bar{\mathbb{B}}_M := \langle \mathbb{B}_{M,m} : m < \omega\rangle$ so $\langle 1_{\mathbb{B}_{M,m}} : m < \omega\rangle$ be a maximal antichain of $\mathbb{B}_M, \mathbb{B}_M{\restriction}\{x \in \mathbb{B}_M : x \le 1_{\mathbb{B}_{M,m}}\} = \mathbb{B}_{M,m}$ and $\cup\{\mathbb{B}_{M,m} : m < \omega\}$ generates $\mathbb{B}_M$. Let[1] $\mathrm{tr}-\mathrm{ufil}(\mathbb{B}_M) =$ the ultrafilter of $\mathbb{B}_M$ disjoint to $\{1_{\mathbb{B}_{M,n}} : n < \omega\}$ and let $\mathrm{uf}^-(\mathbb{B}_M) = \mathrm{uf}(\mathbb{B}_M) \backslash \{\mathrm{tr}-\mathrm{ufil}(\mathbb{B}_M)\}$, ($\mathrm{tr}-\mathrm{ufil}$ stands for trivial ultra-filter).
4) Let $\lambda'(M)$ be the cardinality of $\mathrm{uf}(\mathbb{B}_M)$ and $\lambda'(T) = \lambda'(M)$ when $M \models T$.
5) Let $\mathbb{B}^{\mathrm{fr}}_\lambda$ be the Boolean algebra generated freely by $\{\mathbf{a}_\alpha : \alpha < \lambda\}$ so $\mathrm{uf}(\mathbb{B}^{\mathrm{fr}}_\lambda)$ has cardinality 2^λ.

Remark 0.8. We may be interested in the Boolean Algebra of formulas which are almost over $\emptyset$, i.e. $\varphi(\bar{x}_m, \bar{a}), \bar{a} \in {}^{\ell g(\bar{y})}M$ where $\varphi(\bar{x}_m, \bar{y}) \in \mathbb{L}(\tau_T)$ satisfies: $\varphi(\bar{x}_m, \bar{y})$ such that for some $\vartheta(\bar{x}_m, \bar{y}_m) \in \mathrm{EQ}^m_M$, see Definition 0.3(5), we have

$$M \models (\forall \bar{z})(\forall \bar{x}_m, \bar{y}_m)[\vartheta(\bar{x}_m, \bar{y}_m) \to (\varphi(\bar{x}_m, \bar{z}) \equiv \varphi_n(\bar{y}_m, \bar{z})].$$

But this is not necessary here.

[1]The point is we like to say: the set of ultrafilters of $\mathbb{B}_M$ is the union of the set of ultrafilters of $\mathbb{B}_{M,m}$ for $m < \omega$, but one ultrafilter called trivial of $\mathbb{B}_M$ does not fit, see 0.9(3). Now this justifies treating members of $\mathrm{uf}(\bar{\mathbb{B}}_M)$ from 0.7(3) as a case $\mathrm{uf}(\mathbb{B})$.

Observation 0.9. *1) $\mathbb{B}_{M,m}$ essentially depend just on* $\mathrm{Th}(M)$*, i.e. if* $T = \mathrm{Th}(M)$ *then $\mathbb{B}_{M,m}$ is isomorphic to $\mathbb{B}_{T,m}$ where an isomorphism $\mathbf{j}$ is defined as follows: $\varphi(\bar{x}_{[m]}) \in \mathbb{L}(\tau_T) \Rightarrow \mathbf{j}(\varphi(M)) = \varphi(\bar{x}_{[m]})/\equiv_T$, so $\lambda'(T)$ is well defined.*
2) Similarly for other notions from Definition 0.7.
3) $\mathrm{uf}^-(\mathbb{B}_M), \mathrm{uf}(\mathbb{B}_M)$ *have the same cardinality, in fact, there is a natural one-to-one mapping π from $\mathrm{uf}(\bar{\mathbb{B}}_M)$ onto $\mathrm{uf}^-(\mathbb{B}_M)$ such that $D \in \mathrm{uf}(\mathbb{B}_{M,m}) \Rightarrow \pi(D) = \{a \in \mathbb{B}_{M,m} : a \cap 1_{\mathbb{B}_{M,m}} \in D\}$.*

Recall that by Lemma [**?**, Ch.III,3.10]:

Fact 0.10. Let T be a stable (first order complete) theory, $\kappa = \kappa(T)$ and M is an uncountable model of T. Then M is saturated <u>iff</u>

<u>Case 1</u>: $\kappa > \aleph_0$

- (a) if $\mathbf{I} \subseteq M$ is an infinite indiscernible set <u>then</u> there is an indiscernible set $\mathbf{J} \subseteq M$ extending $\mathbf{I}$ of cardinality $\|M\|$
- (b) M is κ-saturated.

<u>Case 2</u>: $\kappa = \aleph_0$

- $(a)'$ if $A \subseteq M$ is finite and $a \in M \backslash \mathrm{acl}(A)$ <u>then</u> there is an indiscernible set $\mathbf{J}$ over A in M based on A such that $a \in \mathbf{J}$ and $\mathbf{J}$ is of cardinality $\|M\|$
- $(b)'$ M is $\aleph_\varepsilon$-saturated, see Definition 0.3(6) or [**?**].

Recall also (by [**?**, Ch.II,5.9,5.10,5.11]).

Fact 0.11. Assume T is a stable (first order complete) theory.
1) If $\kappa(T) > \aleph_0$ then $\lambda(T) = |D(T)|^{<\kappa_r(T)}$.
2) If $\kappa(T) = \aleph_0$ <u>then</u> $\lambda(T)$ is $|D(T)|$ or $(\lambda(T) = 2^{\aleph_0} + |D(T)|) \wedge (\mathrm{st})_T$ where

- $(\mathrm{st})_T$ for some finite $A \subseteq M, M \in \mathrm{Mod}_T$, the set $\{\mathrm{stp}(a, A) : a \in M\}$ has cardinality continuum.

Definition 0.12. 1) For a cardinal θ let T^{eq}_θ be the model completion of $T^{\mathrm{eq},0}_\theta$, see below.
2) Let $\tau^{\mathrm{eq}}_\theta = \{E_i : i < \theta\}, E_i$ a two-place predicate.
3) Let T^{eq}_θ be the universal theory included in $\mathbb{L}(\tau^{\mathrm{eq}}_\theta)$ such that: for a $\tau^{\mathrm{eq}}_\theta$-model $M, M \models T^{\mathrm{eq}}_\theta$ <u>iff</u> E^M_i is an equivalence relation and E^M_j refines E^M_i for $i < j < \theta$.

Claim 0.13. *(Basic properties of non-forking)*
1) $M_\delta = \bigcup\limits_{i<\delta} M_i$ is λ-saturated <u>when</u>:

- (a) *$\langle M_i : i < \delta\rangle$ is a $\prec$-increasing sequence of models of T*
- (b) *T is stable and $\kappa(T) \leq \mathrm{cf}(\delta)$*
- (c) *each M_i is λ-saturated.*

Proof. See [**?**, Ch.III]. $\square_{0.13}$

§ 1. The frame

First, we define here $\mathbf{N}_{\lambda,T}$, the set of triples $\mathbf{t}$ from the abstract when we fix T, λ and for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ we define the class of models $\mathrm{Mod}_{\mathbf{t}}$ (in 1.1, 1.2) and give easy properties (in 1.3, 1.5). Second, we deal with the logics $\mathbb{L}_{\lambda,\kappa}[\mathbb{B}]$ via which we shall characterize the Hanf number of $\mathbf{N}_{\lambda,T}$ and look at the relations among such logics (see 1.6, 1.12, 1.13). Third, we deal with representations, e.g. how $\psi \in \mathbb{L}_{\lambda^+,\kappa}$ can be translated to models of first order T, with extra demands (see 1.14 - 1.18). Lastly, we look at order between the $\mathbb{B}$'s.

Definition 1.1. 1) For T a complete first order theory and $\lambda \geq |T|$ let $\mathbf{N}_{\lambda,T}$ be the class of triples $\mathbf{t} = (T, T_1, p) = (T_{\mathbf{t}}, T_{1,\mathbf{t}}, p_{\mathbf{t}})$ such that:

- (a) $T_{\mathbf{t}} = T$
- (b) $T_1 \supseteq T$ is a first order theory and $|\tau(T_1)| \leq \lambda$
- (c) $p(x)$ is an $\mathbb{L}(\tau_{T_1})$-type, not necessarily complete.

1A) For $\mathbf{t}$ as above we say $M_1 \models \mathbf{t}$ or $M_1 \in \mathrm{Mod}_{\mathbf{t}}$ or M_1 is a model of $\mathbf{t}$ <u>when</u>:

- (a) $M_1 \models T_{1,\mathbf{t}}$ and M_1 a τ_{T_1}-model
- (b) M_1 omits the type $p_{\mathbf{t}}(x)$
- (c) $M_1 {\restriction} \tau_T$ is saturated.

1B) Let $\mathrm{Mod}^0_{\mathbf{t}} = \{M \in \mathrm{Mod}_{\mathbf{t}} : \|M\| \geq |T_{1,\mathbf{t}}| + \aleph_1\}$. Let $\mathbf{N} = \{\mathbf{t} : \mathbf{t} \in N_{\lambda,T}$ for some λ and $T\}$. For $\lambda \geq \mu$ let $\mathbf{N}_{\lambda,\mu} = \{\mathbf{t} \in \mathbf{N} : |T_{\mathbf{t}}| \leq \mu$ and $|T_{1,\mathbf{t}}| \leq \lambda\}$ and $\mathbf{N}_\lambda = \mathbf{N}_{\lambda,\lambda}$.
2) Let $\mathrm{spec}_{\mathbf{t}} = \{\|M\| : M \models \mathbf{t}\}$ for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$.
3) The Hanf number $H(\mathbf{N}_{\lambda,T})$ is the minimal μ such that: if $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ and $\mathbf{t}$ has a model of cardinality $\geq \mu$ <u>then</u> $\mathbf{t}$ has models of arbitrarily large cardinality; see 1.6(3).
3A) Equivalently, $H(\mathbf{N}_{\lambda,T}) = \sup\{H(\mathbf{t}) : H(\mathbf{t}) < \infty, \mathbf{t} \in \mathbf{N}_{\lambda,T}\}$ where $H(\mathbf{t}) = \sup\{\|M\|^+ : M \in \mathrm{Mod}_{\mathbf{t}}\}$.
4) Let $\lambda(\mathbf{t}) := \lambda(T_{\mathbf{t}}) + |T_{1,\mathbf{t}}|$ recalling Definition 0.3(3).

Convention 1.2. Below T is stable and $\mathbf{t}, T, T_1, p, \lambda$ are as in Definition 1.1 if not said otherwise and then $\kappa = \kappa_r(T)$ is as in 0.3.

Claim 1.3. *1) If $M \in \mathrm{Mod}_{\mathbf{t}}$ has cardinality $\mu > \aleph_0$ <u>then</u> $\mu = \mu^{<\kappa(T)} + |\lambda(T)|$, i.e. $\aleph_0 < \mu \in \mathrm{spec}_{\mathbf{t}} \Rightarrow \mu = \mu^{<\kappa(T)} + \lambda(T)$.*
2) If $M \in \mathrm{Mod}_{\mathbf{t}}$ and $\lambda(\mathbf{t}) \leq \mu = \mu^{<\kappa(T)} < \|M\|$ recalling 1.1(4) and $A \subseteq M$ is of cardinality μ <u>then</u> for some N we have:

- (a) *$N \in \mathrm{Mod}_{\mathbf{t}}$*
- (b) *$A \subseteq N \prec M$*
- (c) *N has cardinality μ.*

Remark 1.4. Recall that there is a countable T categorical in $\aleph_0$, superstable, not stable in $\aleph_0$ so to simplify 1.3 we ignore $\aleph_0$.

Proof. 1) By 0.5.
2) Note that also $\mu = \mu^{<\kappa_r(T)}$ by cardinal arithmetic and hence $\kappa_r(T) \leq \mu$; we choose M_i by induction on $i < \kappa_r(T)$ such that:

- (a) if i is even then $M_i \prec M$ and $\|M_i\| = \mu$

(b) if i is odd then $M_i \restriction \tau(T_{\mathbf{t}}) \prec M \restriction \tau(T_{\mathbf{t}}), \|M_i\| = \mu$ and M_i is saturated

(c) if $j < i$ then $A \cup |M_j| \subseteq |M_i|$.

There is no problem to carry the induction and then $M' = \cup\{M_{2i} : i < \kappa_r(T)\} = \cup\{M_{2i+1} : i < \kappa_i(T)\}$ is as required: $M' \prec M$ by (a)+(c) and Tarski-Vaught, $\|M'\| = \mu$ since $\mu^{<\kappa_T(T)} = \mu$ and $M' \restriction \tau(T)$ is saturated by (b) + (c) and 0.13. $\square_{1.3}$

Conclusion 1.5. *For understanding the Hanf number of* $\mathbf{t}$*, it is enough to consider cardinals* $\mu = \mu^{<\kappa(T)} \geq \lambda(\mathbf{t})$.

Now we turn to the logics of the form $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$; first we define them.

Definition 1.6. 1) Assume

(a) $\lambda \geq \kappa = \mathrm{cf}(\kappa)$

(b) $\mathbb{B}$ is a Boolean Algebra of cardinality $\leq \lambda$ and recall $\mathrm{uf}(\mathbb{B})$ is the set of ultrafilters on $\mathbb{B}$.

Then

(α) Let $\mathrm{voc}_\lambda[\mathbb{B}]$ be the class of vocabularies τ of cardinality $\leq \lambda$ such that $c_b \in \tau$, an individual constant for each $b \in \mathbb{B}$, and $P, Q \in \tau$ unary predicates and $R \in \tau$ binary predicate and τ may have additional signs.

(β) For $\tau \in \mathrm{voc}_\lambda[\mathbb{B}]$ let $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ be the set of sentences $\psi \in \mathbb{L}_{\lambda^+,\kappa}(\tau)$ but we stipulate that from ψ we can reconstruct the triple $(\lambda^+, \kappa, \mathbb{B})$ hence $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$, (e.g. demand $<_\bullet \in \tau$ is a two-place predicate and $<^M_\bullet = \{(a,b) : \mathbb{B} \models \text{“}c_a < c_b\text{”}\}$.

[Note that ψ has $\leq \lambda$ sub-formulas]:

(γ) omitting τ means $\tau = \tau_\psi$ is the minimal $\tau \in \mathrm{voc}_\lambda[\mathbb{B}]$ such that $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$.

2) For $\tau \in \mathrm{voc}_\lambda[\mathbb{B}]$ and $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ let $\mathrm{Mod}^1_\psi[\mathbb{B}]$ be the class of models M of ψ (which are τ_ψ-models if not said otherwise) such that (note: clauses (a)-(e) can be expressed in $\mathbb{L}_{\lambda^+,\aleph_0}$, but when $|\mathrm{uf}(\mathbb{B})| > \lambda$ not so clause (f)):

(a) $P^M = \{c^M_b : b \in \mathbb{B}\}$

(b) $\langle c^M_b : b \in \mathbb{B}\rangle$ are pairwise distinct

(c) $R \subseteq P^M \times Q^M$

(d) for every $a \in Q^M$ the set $\mathrm{uf}^M(a) := \{b \in \mathbb{B} : M \models c_b R a\}$ belongs to $\mathrm{uf}(\mathbb{B})$

(e) if $a_1 \neq a_2$ are from Q^M then $\mathrm{uf}^M(a_1) \neq \mathrm{uf}^M(a_2)$

(f) for every $u \in \mathrm{uf}(\mathbb{B})$ there is $a \in Q^M$ such that $M \models \bigwedge\limits_{i<\lambda} (c_b R a)^{\mathrm{if}(b \in u)}$, (by clause (e) the element a is unique).

3) Let $\mathrm{Mod}^2_\psi[\mathbb{B}]$ be the class of $M \in \mathrm{Mod}^1_\psi[\mathbb{B}]$ such that:

(f) $\|M\| = \|M\|^{<\kappa}$ and (follows) $\|M\| \geq |\mathrm{uf}(\mathbb{B})|$.

4) For $\iota = 1, 2$ and $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ let $\mathrm{spec}^\iota_\psi[\mathbb{B}] = \{\|M\| : M \in \mathrm{Mod}^\iota_\psi[\mathbb{B}]\}$.
4A) Writing $\mathrm{Mod}^\iota_\psi, \mathrm{spec}^\iota_\psi$ we mean $\iota \in \{1, 2\}$ and may omit ι when $\iota = 2$ (because this is the main case for us), see 1.9(1) below and $\mathbb{B}$ can be reconstructed from ψ.
5) Let $H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])$ be the first μ such that: if $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ and there is $M \in \mathrm{Mod}_\psi[\mathbb{B}]$ of cardinality $\geq \mu$ then $\{\|M\| : M \in \mathrm{Mod}_\psi[\mathbb{B}]\}$ is an unbounded class of cardinals.
6) Let $\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$ be $\cup\{\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}] : \mathbb{B}$ a Boolean[2] Algebra of cardinality $\leq \lambda\}$ so every sentence of $\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}(\tau)$ is a sentence in $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ for some $\mathbb{B}$ as above; so we may stipulate that the set of elements of $\mathbb{B}$ is a cardinal $\leq \lambda$ and $c_i \in \tau$ for $i < \lambda$.
7) We define $H(\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa})$ similarly; yes, this is just $\sup\{H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]) : \mathbb{B}$ as above$\}$.

Having defined the sets $(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])(\tau)$ of sentences and the relevant classes of models $\mathrm{Mod}^\iota_\psi[\mathbb{B}]$ and spectrum $\mathrm{spec}^\iota_\psi[\mathbb{B}]$ and Hanf numbers we should now try to understand the order between them.

Claim 1.7. *1) Recalling $\mathbb{B}^{\mathrm{fr}}_\lambda$ is the Boolean Algebra generated freely by λ generators:*

- (a) *for every Boolean algebra $\mathbb{B}_1$ of cardinality λ or just $\leq \lambda$ and $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}_1]$ there is $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]$ such that $\mathrm{spec}^\iota_{\psi_1} \backslash 2^\lambda = \mathrm{spec}^\iota_\psi \backslash 2^\lambda$ for $\iota = 1, 2$*
- (b) *$H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}_1]) \leq H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda])$ for $\mathbb{B}_1$ as above.*

2) If $\mathbb{B}_1, \mathbb{B}_2$ are Boolean algebras of cardinality $\leq \lambda$ and $\mathbb{B}_1$ is a homomorphic image of $\mathbb{B}_2$, then:

- (a) *for every $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}_1]$ there is $\psi_2 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}_2]$ such that $\mathrm{spec}^\iota_{\psi_1}[\mathbb{B}_1] \backslash \|\mathbb{B}_2\| = \mathrm{spec}^\iota_{\psi_2}[\mathbb{B}_1] \backslash \mathbb{B}_2$ for $\iota = 1, 2$*
- (b) *$H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}_1]) \leq H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}_2]$.*

3) For every $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ there are $\psi_2, \psi'_2, \psi''_2 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that:

- (a) *$\mathrm{spec}^1_{\psi_2}[\mathbb{B}] = \{\mu : \mu = \mu^{<\kappa} \in \mathrm{spec}^1_{\psi_1}[\mathbb{B}]\} = \mathrm{spec}^2_{\psi_1}[\mathbb{B}]$ and*
- (b) *$\mathrm{spec}^1_{\psi'_2}[\mathbb{B}] = \{\mu^{<\kappa} : \mu \in \mathrm{spec}^1_{\psi_1}[\mathbb{B}]\}$ and*
- (c) *$\mathrm{spec}^1_{\psi''_2}[\mathbb{B}] = \{\mu : \mu \geq \lambda + \|\mathbb{B}\|$ and $\mu \in \mathrm{spec}^1_{\psi_1}[\mathbb{B}]\}$.*

Remark 1.8. Concerning 1.7(3)(a),(b) recall that if $\mu > 2^{<\kappa}$ then $(\mu^{<\kappa})^{<\kappa} = \mu$, see [?].

Proof. 1) Let h be a homomorphism from $\mathbb{B}^{\mathrm{fr}}_\lambda$ onto $\mathbb{B}_1$, exists as $\mathbb{B}_1$ is a Boolean algebra of cardinality $\leq \lambda$. Now apply part (2).
2) Let $I := \mathrm{Ker}(h) := \{a \in \mathbb{B}^{\mathrm{fr}}_\lambda : h(a) = 0\}$ and let $h_1 : \mathbb{B}_1 \to \mathbb{B}_2$ be such that $a \in \mathbb{B}_1 \Rightarrow h(h_2(a)) = a$. Let $\mathbb{B}'_1$ be the Boolean Algebra with set of elements $\mathrm{Rang}(h_1)$ such that h_1 is an isomorphism from $\mathbb{B}_1$ onto $\mathbb{B}'_1$. Let ψ'_1 be like ψ_1 replacing $\mathbb{B}_1$ by $\mathbb{B}'_1$ and the predicate P by a predicate P'. The rest should be clear.
3) Should be clear but we elaborate.

Clause (a): Let $\tau_2 = \tau(\psi_1) \cup \{F_{i,j} : i < j < \kappa\}$ with $F_{i,j} \notin \tau(\psi)$ be pairwise distinct unary function.

[2] So every sentence $\psi \in \mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$ fixes a Boolean Algebra $\mathbb{B}$ as above and a vocabulary of cardinality $\leq \lambda$ from $\mathrm{voc}_\lambda[\mathbb{B}]$ as described.

Let $\psi_2 = \psi_1 \wedge \varphi_2$ where

$$\varphi_2 = \bigwedge_{0<j<\kappa} (\forall \ldots, x_i, \ldots)_{i<j} (\exists y)[\bigwedge_{i<j} F_{i,j}(y) = x_i].$$

Now think

Clause (b): Let $\tau'_2 = \tau(\psi_1) \cup \{F_{i,j} : i < j < \kappa\} \cup \{P_j : j \leq \kappa\}$ with $F_{i,j}$ as above $F_{i,j}, P_j \notin \tau(\psi_1)$ for $j \leq \kappa$ be pairwise distinct unary predicates and let $P = P_\kappa$.

Let $\psi_1^P \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ be such that for a $(\tau(\psi_1) \cup \{P\})$-model $M, M \models \psi_1^P$ iff $(M{\restriction}P^M){\restriction}\tau(\psi_1)$ is a $\tau(\psi_1)$-model and is a model of T.

Lastly, let $\psi_2 = \psi_1^p \wedge \varphi'_2$ where φ'_2 is the conjunction of:

- $\varphi_2^0 = (\forall x)(P(x) \vee \bigvee_{i<\kappa} P_i(x)) \wedge \bigwedge_{i<j\leq\kappa} \neg(\exists x)(P_i(x) \wedge P_j(x))$,
  so φ_2^0 says $\langle P^M\rangle{}^\frown\langle P_j^M : j < \kappa\rangle$ is a partition of M, the universe of the model
- $\varphi_{2,i,j}^1 = (\forall x)(P(F_{i,j}(x))$ for $i < j < \kappa$
- $\varphi_{2,j}^2 = (\forall x, y)[x \neq y \wedge P_j(x) \to \bigvee_{i<j} F_{i,j}(x) \neq F_{i,j}(y)]$
- $\varphi_{2,j}^3 = (\forall \ldots, x_i, \ldots)_{i<j}(\bigwedge_{i<j} P(x_i) \to (\exists y)(P_j(y) \wedge \bigwedge_{i<j} F_{i,j}(y) = x_i)$.

Now check.

Clause (c):

Even easier noting that "$\geq \|\mathbb{B}\|$" holds by the definitions. $\square_{1.7}$

Observation 1.9. *Let $\mathbb{B}$ be a Boolean Algebra of cardinal $\leq \lambda$ and $\kappa \leq \lambda^+$.*
0) If $\mathrm{uf}(\mathbb{B})$ has cardinality $\leq \lambda$ (hence $\mathbb{B}$ has cardinality $\leq \lambda$), then $H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]) = H(\mathbb{L}_{\lambda^+,\kappa})$.
1) In the Definition 1.6(5) of $H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])$ it does not matter if we use $\mathrm{Mod}^1_\psi[\mathbb{B}]$ or $\mathrm{Mod}^2_\psi[\mathbb{B}]$.
2) For every $\mu < H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])$ we have $2^\mu < H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])$ hence $H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])$ is a strong limit cardinal of cofinality $> \lambda$.
3) $H(\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < H(\mathbb{L}_{(2^\lambda)^+,\kappa})$.
4) We have $H(\mathbb{L}_{\lambda^+,\kappa}) \leq H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]) \leq \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda] = H(\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < H(\mathbb{L}_{(2^\lambda)^+,\kappa})$.
5) If $\mathbb{B}^{\mathrm{fr}}_\lambda$ is the free Boolean Algebra of cardinality λ from 0.7(5) and $\kappa = \aleph_0$ then $H(\mathbb{L}_{\lambda^+,\kappa}) < \beth_{(2^\lambda)^+} < H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda])$. Also for any $\kappa \geq \aleph_0$ we have $H(\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < H(\mathbb{L}_{\lambda^+,\lambda^+})$.
6) If $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ and $H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]) \leq \sup\{\|M\| : M \in \mathrm{Mod}_\psi[\mathbb{B}]\}$ then $\infty = \sup\{\|M\| : M \in \mathrm{Mod}_\psi[\mathbb{B}]\}$ hence $\mathrm{cf}(H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])) \leq 2^\lambda$).
7) Like part (6) for $\psi \in \mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$ and $\mathrm{Mod}^{\mathrm{ba}}_\psi$.

Proof. 0) Easy.
1) First, as easily the Hanf number is $> 2^\lambda \geq |\mathrm{uf}(\mathbb{B})|$, we can ignore models of cardinality $< 2^\lambda$. Second,

$(*)_1$ if $\psi_1 \in \mathbb{L}_{\lambda,\kappa}[\mathbb{B}](\tau)$ and $\sup(\mathrm{spec}^1_{\psi_1}) < \infty$ then $\sup(\mathrm{spec}^2_{\psi_1}) \leq \sup(\mathrm{spec}^1_{\psi_1}) \leq (\sup(\mathrm{spec}^2_{\psi_1}))^{<\kappa} < \infty$.

[Why? the first inequality because $\mathrm{spec}^1_\psi \supseteq \mathrm{spec}^2_\psi$; the second inequality by 1.3(2).]

We can conclude that spec^1_ψ is bounded iff spec^2_ψ is bounded and the Hanf number of the logic $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ using Mod^1_ψ is smaller or equal to the Hanf number of the logic $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ using Mod^2_ψ. The other inequality holds by 1.7(3)(b) and $(*)$.

Alternatively, if $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ then by 1.7(3)(b) there is $\psi'_2 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that $\sup(\mathrm{spec}^1_{\psi_1}) < \infty \Rightarrow \sup(\mathrm{spec}^1_{\psi_1}) \le \sup(\mathrm{spec}^2_{\psi'_2}) < \infty$, hence the Hanf number using spec^1_ψ's is $\le$ the Hanf number using spec^2_ψ's. Moreover, above we get $\sup(\mathrm{spec}^1_{\psi_1}) \le \sup(\mathrm{spec}^2_{\psi'_2}) = \sup(\mathrm{spec}^1_{\psi'_2})$ as $\mathrm{spec}^2_{\psi'_2} = \mathrm{spec}^1_{\psi'_2}$. On the other hand, by clause (a) of 1.7(3) if $\psi_1 \in \mathbb{L}_{\lambda,\kappa}[\mathbb{B}]$ then there is $\psi_2 \in \mathbb{L}_{\lambda,\kappa}[\mathbb{B}]$ such that $\mathrm{spec}^1_{\psi_2} = \mathrm{spec}^2_{\psi_1}$ so $\sup(\mathrm{spec}^2_{\psi_1}) < \infty \Rightarrow \sup \mathrm{spec}^2_{\psi_1} = \sup \mathrm{spec}^1_{\psi_2} < \infty$ so also the other inequality holds.

2) For any $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ we can find $\psi_2 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that $\tau_{\psi_1} \subseteq \tau_{\psi_2}, P_*, R_* \in \tau_{\psi_2} \backslash \tau_{\psi_1}$ are unary, binary predicates respectively and:

$(*)_1$ $M_2 \in \mathrm{Mod}^\iota_{\psi_2}[\mathbb{B}]$ iff
- $(M_2 {\restriction} P^{M_2}_* {\restriction} \tau_{\psi_1}) \in \mathrm{Mod}_{\psi_1}[\mathbb{B}]$
- $M_2 \models (\forall y,z)(\exists x)[P_*(x) \wedge (R(x,y) \equiv \neg R(x,z))]$ hence $|P^{M_2}_*| \le \|M_2\| \le 2^{|P_*(M_2)|}$.

Clearly

$(*)_2$ for every $M_1 \in \mathrm{Mod}^1_{\psi_1}[\mathbb{B}]$ and $\mu = \mu^{<\kappa} \in [\|M_1\|, 2^{\|M_1\|}]$ there is $M_2 \in \mathrm{Mod}^1_{\psi_2}[\mathbb{B}]$ of cardinality μ.

Using $(*)_2$ this clearly suffices for the first statement. The second is easy, too.

3) Let $\mathbf{K}_{\lambda^+,\kappa}$ be the class of pairs $(\psi,\mathbb{B})$ such that $\mathbb{B}$ is a Boolean Algebra of cardinality $\le \lambda, \psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$. For $(\psi,\mathbb{B}) \in \mathbf{K}_{\lambda^+,\kappa}$ let $H(\psi,\mathbb{B}) = \cup\{\mu^+ : \mu \in \mathrm{spec}^2_\psi(\mathbb{B})\}$. Clearly up to isomorphism (of vocabularies) $\mathbf{K}_{\lambda^+,\kappa}$ has cardinality $\le 2^\lambda$ and hence $\mathbf{C}_{\lambda^+,\kappa} := \{H(\psi,\mathbb{B}) : (\psi,\mathbb{B}) \in \mathbf{K}_{\lambda^+,\kappa}\}$ has cardinality $\le 2^\lambda$. So let $\langle(\psi_i,\mathbb{B}_i) : i < 2^\lambda\rangle$ be such that $(\psi_i,\mathbb{B}_i)$ is as above and $\mathbf{C}_{\lambda^+,\kappa} \backslash \{\infty\} = \{\mu_i : i < 2^\lambda\}$ where $\mu_i = H(\psi_i,\mathbb{B}_i) = \cup\{\mu^+ : \mu \in \mathrm{spec}^1_{\psi_i}[\mathbb{B}_i]\}$ for $i < 2^\lambda$. Now we can find $\psi \in \mathbb{L}_{(2^\lambda)^+,\kappa}$ such that $M \models \psi$ iff

$(*)$ $<^M$ is a linear order of $|M|$ and for arbitrarily large $a \in M$ there are $i < 2^\lambda$ and $N \in \mathrm{Mod}^2_{\psi_i}[\mathbb{B}_i]$ with universe $\{b : b <^M a\}$.

Together with part (2), clearly $\infty > \sup(\mathrm{spec}_\psi) = \max(\mathrm{spec}_\psi) = \cup\{\mu_i : i < 2^\lambda\}$ so we are done.

4) For the first inequality "$H(\mathbb{L}_{\lambda^+,\kappa}) \le H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])$", see the definitions of $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$. For the second inequality, "$H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]) \le H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda])$", use 1.7(1)(b). For the third inequality, "$H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]) = H(\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa})$", use the definition of the latter and the second inequality. For the fourth inequality, "$H(\mathbb{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < H(\mathbb{L}_{(2^\lambda)^+,\kappa})$", recall 1.9(3).

5) The first inequality "$H(\mathbb{L}_{\lambda^+,\kappa}) < \beth_{(2^\lambda)^+}$" is well known see, e.g. Theorem 5.4 and 5.5 of [?, Ch.VII,§5] recalling $\kappa = \aleph_0$. The second inequality, "$\beth_{(2^\lambda)^+} < H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda])$", holds by the equality in part (4) and part (3).

For the third inequality note that:

(∗) there is $\psi \in \mathbb{L}_{\lambda^+,\lambda^+}$ such that: $M \models \psi$ iff:

(a) P^M, Q^M, R^M are as in Definition 1.6(2)

(b) $F^M_i (i < \lambda)$ are as in 1.7(3)(a) for Q^M, i.e. $M \models (\forall \ldots, x_i, \ldots)_{i<\lambda}(\exists y)[\bigwedge\limits_{i<\lambda} Q(x_i) \rightarrow (\exists y)(Q(y) \wedge \bigwedge\limits_{i<\lambda} F_i(y) = x_i)]$

(c) $<^M$ is a well ordering of Q^M.

6) As in the end of the proof of part (3) replacing ψ_i by ψ_1, that is, we can find $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that:

(∗) $M_1 \models \psi_1$ iff for some $< \in \tau(\psi_1), <^{M_1}$ is a linear order of $|M_1|$ such that for arbitrarily large $b \in M_1, M_1 {\restriction} \{a : a <^{M_1}\} {\restriction} \tau_\psi$ is a model of ψ.

Clearly this suffice.
7) So assume $\mu < H(\mathbb{L}^{\rm ba}_{\lambda^+,\kappa})$ hence by the definition there is $\psi \in \mathbb{L}^{\rm ba}_{\lambda^+,\kappa}$ such that $\{\|M\| : M \models \psi\}$ is bounded but has a member $\geq \mu$. By the definition of $\mathbb{L}^{\rm ba}_{\lambda^+,\kappa}$ for some Boolean Algebras $\mathbb{B}$ of cardinality $\leq \lambda$ we have $\psi \in L^{\rm ba}_{\lambda^+,\kappa}[\mathbb{B}]$ and now apply part (2). $\square_{1.9}$

The following 1.11, 1.12, 1.13 is another way to represent the logic $\mathbb{L}^{\rm ba}_{\lambda^+,\kappa}$ equivalently the logic $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\rm fr}_\lambda]$, hence eventually to state the Hanf numbers.

Definition 1.10. 1) Let $\mathbb{L}^*_{\lambda^+,\kappa}$ be defined like $\mathbb{L}^{\rm ba}_{\lambda^+,\kappa}$, see 1.6(3) replacing $\langle c_b : b \in \mathbb{B}\rangle$ by $\langle c_i : i < \lambda\rangle$ and uf$(\mathbb{B})$ by $\mathscr{P}(\{c_i : i < \lambda\})$.
2) For $\psi \in \mathbb{L}^*_{\lambda^+,\kappa}$ let Mod^*_ψ be defined as in 1.6(1A),(2),(3) replacing uf$(\mathbb{B})$ by $\mathscr{P}(\lambda)$.
3) Let $H(\mathbb{L}^*_{\lambda^+,\kappa})$ be defined like $H(\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}])$ in 1.6(5).
4) For $\psi \in \mathbb{L}^*_{\lambda^+,\kappa}$ let $\mathrm{spec}^*_\psi = \mathrm{spec}^{1,*}_\psi = \{\|M\| : M \in \mathrm{Mod}^*_\psi\}$; and $\mathrm{spec}^{2,*}_\psi = \{\|M\| : M \in \mathrm{Mod}^*_\psi$ and $\|M\| = \|M\|^{<\kappa}\}$; for transparency we will stipulate that from ψ we can reconstruct $\mathbb{L}^*_{\lambda^+,\kappa}$.

Remark 1.11. The next claim essentially tells us that for determining the Hanf number of $\mathbb{L}^{\rm ba}_{\lambda^+,\kappa}$, we may use the "worst" Boolean Algebra, $\mathbb{B}^{\rm fr}_\lambda$ and $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\rm fr}_\lambda]$ is essentially equal to $\mathbb{L}^*_{\lambda^+,\kappa}$.

Parallel to 1.9, 1.7(3):

Claim 1.12. *1) In the natural definition of $H(\mathbb{L}^*_{\lambda^+,\kappa})$ it does not matter if we use $\mathrm{spec}^{1,*}_\psi$ or $\mathrm{spec}^{2,*}_\psi$ for $\psi \in \mathbb{L}^+_{\lambda^+,\kappa}$.*
2) For every $\mu < H(\mathbb{L}^*_{\lambda^+,\kappa})$ we have $2^\mu < H(\mathbb{L}^*_{\lambda^+,\kappa})$ hence $H(\mathbb{L}^*_{\lambda^+,\kappa})$ is a strong limit cardinal; moreover, of cofinality $> \lambda$.
3) $H(\mathbb{L}^*_{\lambda^+,\kappa}) < H(\mathbb{L}_{(2^\lambda)^+,\kappa})$.
4) $H(\mathbb{L}_{\lambda^+,\kappa}) < H(\mathbb{L}^*_{\lambda^+,\kappa}) < H(\mathbb{L}_{(2^\lambda)^+,\kappa})$.
5) If $\psi \in \mathbb{L}^*_{\lambda^+,\kappa}$ and $H(\mathbb{L}^*_{\lambda^+,\kappa}) \leq \sup\{\|M\| : M \in \mathrm{Mod}_\psi\}$, then $\infty = \sup\{\|M\| : M \in \mathrm{Mod}_\psi\}$.
6) For every $\psi_1 \in \mathbb{L}^*_{\lambda^+,\kappa}$ there are $\psi_2, \psi'_2, \psi''_2 \in \mathbb{L}^*_{\lambda^+,\kappa}$ such that:

(a) $\mathrm{spec}^*_{\psi_2} = \{\mu : \mu = \mu^{<\kappa} \in \mathrm{spec}^*_{\psi_1}[\mathbb{B}]\} = \mathrm{spec}^{2,*}_{\psi_1}$

(b) $\mathrm{spec}^*_{\psi'_2} = \{\mu^{<\kappa} : \mu \in \mathrm{spec}^{1,*}_{\psi_1}\}$ *and*

(c) $\mathrm{spec}^*_{\psi''_2} = \{\mu : \mu \geq \lambda$ *and* $\mu \in \mathrm{spec}^{1,*}_{\psi_1}[\mathbb{B}]\}$.

Proof. Similarly to 1.9 and 1.7(3). $\square_{1.12}$

Claim 1.13. *1) For every $\psi_1 \in \mathbb{L}^*_{\lambda^+,\kappa}$ there is $\psi_2 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\rm fr}_\lambda]$ such that $\{\|M\| : M \in \mathrm{Mod}^{\rm ba}_{\psi_1}\} = \{\|M\| : M \in \mathrm{Mod}^*_{\psi_2}[\mathbb{B}]\}$, that is $\mathrm{spec}^*_{\psi_1} = \mathrm{spec}_{\psi_2}[\mathbb{B}]$.*
2) For every $\psi_2 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\rm fr}_\lambda]$ there is $\psi_1 \in \mathbb{L}^*_{\lambda^+,\kappa}$ which are as in clause (c).

Proof. The point is that (A) implies (B) when:

- (A) assume $\mathbb{B}$ is the Boolean Algebra generated freely by $\langle b_i : i < \lambda\rangle, M$ is a model, $P^M_1 = \{b_i : i < \lambda\}, P^M_2 = \mathbb{B}, Q^M_1 = \mathscr{P}(\lambda), Q^M_2 = \mathrm{uf}(\mathbb{B}), R^M_1 = \{(c_i, u) : u \subseteq \lambda, i \in u\}$ and $R^M_2 = \{((c, D) : c \in \mathbb{B}, D \in \mathrm{uf}(\mathbb{B}))$ and $c \in D\}, c_{\bar{b}} \in \bar{c}(M)$ and $c^M_b = b$ for $b \in \mathbb{B}$
- (B) if N is a model of $\mathrm{Th}(M)$ omitting the type $p(x) = \{P(x) \wedge x \neq c_b : b \in \mathbb{B}\}$ then $(a) \Rightarrow (b)$ when:
  - (a) N satisfies the demands in Definition 1.6(2) of $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}^{\rm fr}_\lambda]$ with P_2, Q_2, R_2 here standing for P, Q, R there
  - (b) N satisfies the demands in Definition 1.11(1) of $L^*_{\lambda^+,\kappa}$ with P_1, Q_1, R_1 here standing for P, Q, R there.

$\square_{1.13}$

Next we have to connect those logics with first order T's. The easy part is to start with a Boolean Algebra $\mathbb{B}$ and construct a related T.

Claim 1.14. *1) For every Boolean Algebra $\mathbb{B}$ of cardinality $\leq \lambda$ and cardinal $\kappa \leq \lambda^+$ there is $T = T^1_{\mathbb{B},\kappa}$ such that:*

$(*)_1$
- (a) *T is a first order complete and stable*
- (b) *$|T| = \lambda$ and $\kappa(T) = \kappa$*
- (c) *$\lambda(T)$ is the cardinality of $\mathrm{uf}(\mathbb{B})$, see Definition 0.7(5), 0.2(4), in fact, $\mathbb{B}_T$ is not much more complicated than $\mathbb{B}$ but we shall not elaborate, see* **??** *below*
- (d) *T has elimination of quantifiers.*

2) For $\mathbb{B}, \lambda, \kappa$ as above there is $T = T^2_{\mathbb{B},\kappa}$ such that:

$(*)_2$
- *(a),(b) as above*
- *(c)$\lambda(T) = \lambda + 2^{\aleph_0}$.*

Proof. Easy, but we elaborate.
1) We choose τ_*, T_0 by:

$(*)'_1$
- (a) $\tau_* = \tau_{\mathbb{B},\kappa} = \{P_b : b \in \mathbb{B}\} \cup \{Q_\theta : \theta < \kappa$ is infinite$\} \cup \{E_{\theta,i} : \theta < \kappa$ is infinite, $i < \theta\}$ where P_b, Q_θ are unary predicates, $E_{\theta,i}$ a binary predicate
- (b) the universal theory $T_0 \subseteq \mathbb{L}(\tau_*)$ is such that: a τ_*-model M satisfied T_0 <u>iff</u>
  - (α) $b \mapsto P^M_b$ embeds $\mathbb{B}$ into the Boolean Algebra $\mathscr{P}(P^M_{1_\mathbb{B}})$ so $P^M_{0_\mathbb{B}} = \emptyset$
  - (β) $\langle P^M_{1_\mathbb{B}}\rangle^\frown\langle Q^M_\theta : \theta < \kappa\rangle$ are pairwise disjoint

(γ) $E^M_{\theta,i}$ is an equivalence relation on Q^M_θ so $aE^M_{\theta,i}b \Rightarrow a,b \in Q^M_\theta$
(ε) if $i < j < \theta$ then $E^M_{\theta,j}$ refines $E^M_{\theta,i}$.

So

$\oplus_1$ (a) $T_0 \subseteq \mathbb{L}(\tau_*)$ is a well defined universal theory
(b) Mod_{T_0} has amalgamation and the JEP.

Let

$\oplus_2$ $\mathbb{T}$ is the set of $\tau \subseteq \tau_*$ satisfying:
(a) $P, P_{1_\mathbb{B}}, P_{0_B} \in \tau$
(b) $E_{\theta,i} \in \tau \Rightarrow Q_\theta \in \tau$
(c) if $\mathbb{B} \models$ "$b \cap c = a \wedge -b = d$" then $\{P_b, P_c\} \subseteq \tau_1 \Rightarrow \{P_a, P_d\} \subseteq \tau$

$\oplus_3$ for $\tau \in \mathbb{T}$ let $T_{0,\tau}$ be defined like T_0 but restricting ourselves to predicates from τ.

Now

$\oplus_4$ for $\tau \in \mathbb{T}$
(a) if M is a τ-model of $T_{0,\tau}$, <u>then</u> M can be expanded to a τ_*-model of T_0
(b) $T_{0,\tau}$ has the JEP
(c) $T_{0,\tau}$ has the amalgamation property
(d) if $M_1 \subseteq M_2$ are models or $T_{0,\tau}$ and $\tau \subseteq \tau_1 \in \mathbb{T}$ and N_1 is a τ_1-model expanding M_2 <u>then</u> there is a τ_1-model N_2 expanding M_1 and extending N_1.

[Why? Easy, e.g. clause (b) by disjoint union.]

$\oplus_5$ For finite $\tau \in \mathbb{T}, T_{0,\tau}$ has a model completion called $T_{1,\tau}$ which has elimination of quantifiers.

[Why? Because τ is a relational finite vocabulary and $T_{0,\tau}$ is universal with JEP and amalgamation.]

$\oplus_6$ If $\tau_1 \subseteq \tau_2$ are from $\mathbb{T}$ <u>then</u> $T_{1,\tau_1} \subseteq T_{1,\tau}$.

[Why? By $\oplus_4(d) + \oplus_5$.]

$\oplus_7$ $T = T^1_{\mathbb{B},\kappa} := \cup\{T_{1,\tau} : \tau \in \mathbb{T}$ finite$\}$ is the model completion of T_0 and has elimination of quantifiers.

[Why? Follows from the above.]

$\oplus_8$
(a) If $\tau \in \mathbb{T}$ is finite, <u>then</u> $T_{1,\tau}$ is $\aleph_0$-categorical and $\aleph_0$-stable
(b) T is stable
(c) $\kappa(T) = \kappa$
(d) $|\lambda'(T)| = |\mathbb{B}| + \aleph_0$
(e) $\lambda(T) = \min\{\mu : \mu \geq \lambda$ and $\mu^{<\kappa} = \mu\}$.

[Why? Consider the monster $\mathfrak{C} = \mathfrak{C}_{T_{1,\tau}}$ and use automorphisms.]

So $T = T^1_{\mathbb{B},\kappa}$ from $\oplus_7$ is as promised.

2) We use T_0 such that $(*)'_2$ below holds and continue as above.

$(*)'_2$ as in $(*)'_1$ above but

- (a) we add $Q_0, E_{0,n}(n < \omega)$ with Q_0 unary and $E_{0,n}$ binary
- (b)
  - (β) also Q_0^M is disjoint to $Q_\theta^M(\theta \in [\aleph_0, \kappa))$ and to $P^M_{1_\mathbb{B}}$
  - (ζ) $E^M_{0,n}$ is an equivalence relation on P^M_0
  - (η) $E^M_{0,0}$ has one equivalence class
  - (θ) $E^M_{0,n+1}$ refines $E^M_{0,n}$ and divides each $E^M_{0,n}$ equivalence class to at most 2 equivalence classes.

$\square_{1.14}$

Discussion 1.15. 1) We would like to translate "$M \models$ "$\psi, \psi \in \mathbb{L}_{\lambda^+,\kappa}$" to "$M \in \mathrm{Mod}_\mathbf{t}$", that is, when $\kappa(T) \geq \kappa$ and, in particular, when $\kappa > \aleph_0$. However, the following is the "translation of $\psi \in \mathbb{L}_{\lambda^+,\kappa}(\tau_0)$"; i.e. it deals strictly with the logic $\mathbb{L}_{\lambda^+,\kappa}$; in particular a Boolean Algebra $\mathbb{B}$ is not present. Our aim is to do some of the work of 1.18 in which we are really interested. So 1.16 is not directly related to $\mathbf{t}$'s! as there is no saturation requirement; moreover stability appears neither in 1.16 nor in 1.18.

2) Note that in 1.16 we can let κ_1 be such that $\kappa = \kappa_1^+$ or $\kappa_1 = \kappa$ is a limit cardinal and let $\Upsilon = \kappa_1 + 1$ and omit $F_{\kappa_1}, P_{\kappa_1}$.

Theorem 1.16. *The $\mathbb{L}_{\lambda^+,\kappa}$-representation Theorem*

Assume $\psi \in \mathbb{L}_{\lambda^+,\kappa}(\tau_0)$, so of course, $|\tau_0| \leq \lambda$. Let Υ be κ if $\kappa \leq \lambda$ and $\lambda + 1$ if $\kappa = \lambda^+$.

Then we can find a tuple $(\tau_1, T_1, p(x), \bar{F}, \bar{P})$ such that (for $\bar{F}, \bar{P}$ as below):

- (A) (a) *τ_1 is a vocabulary $\supseteq \tau_0$ of cardinality λ*
  - (b) *$\bar{F}$ is a sequence of unary function symbols with no repetitions of length Υ, new (i.e. from $\tau_1 \backslash \tau_0$), let $\bar{F} = \langle F_i : i < \Upsilon \rangle$*
  - (c) *$\bar{P}$ is a sequence of unary predicates with no repetitions of length Υ and they are new (i.e. from $\tau_1 \backslash \tau_0$), let $\bar{P} = \langle P_i : i < \Upsilon \rangle$*
  - (d) *T_1 is a first order theory in the vocabulary τ_1*
  - (e) *$p(x)$ is $\{P_*(x) \wedge x \neq c_i : i < \lambda\}$, an $\mathbb{L}(\tau_1)$-type (even quantifier-free), so P_* is a unary predicate and c_i for $i < \lambda$ individual constants, all new*
- (B) *the following conditions on a τ_0-model M_0 are equivalent*
  - (a) *$M_0 \models \psi$ and $\|M_0\| = \|M_0\|^{<\kappa} + \lambda^{<\kappa}$*
  - (b) *there is a τ_1-expansion M_1 of M_0 to a model of T_1 omitting $p(x)$ such that:*
    - (α) *$\langle P_i^{M_1} : i < \Upsilon \rangle$ is a partition of $|M_1|$*
    - (β) *if $i < \Upsilon$ and $\langle a_j : j < i \rangle$ is a sequence of elements of M_1 (of length i) then for some $b \in P_i^{M_1}$ we have $j < i \Rightarrow F_j^{M_1}(b) = a_j$.*

Proof. What we shall do is essentially add Skolem functions, and coding sequences of length $< \kappa$. In particular, using the function symbols F_ε (for $\varepsilon < \kappa$) we can replace quantifying over ε-tuple $\langle x_\zeta : \zeta < \varepsilon\rangle$ by quantifying by one x.

Note that as ψ has no free variables, without loss of generality every subformula φ of ψ has a set of free variables equal to $\{x_i : i < \varepsilon\}$ for some $\varepsilon = \varepsilon_\varphi < \kappa$ such that if φ is a sub-formula of ψ and $\varphi = \bigwedge\limits_{i<j} \varphi_i$ then $\varepsilon_{\varphi_i} = \varepsilon_\varphi$.

Let Δ be the set of sub-formulas of ψ so without loss of generality (a syntactical rewriting) there is a list $\langle \varphi_i(\bar{x}_{[\varepsilon(i)]}) : i < i(*)\rangle$ for some $i(*) \le \lambda$ of Δ such that $\varepsilon(0) = 0, \varphi_0 = \psi$ and $\bar{x}_{[\varepsilon(i)]}$ is a sequence of length $< \kappa$ of variables, in fact, $\bar{x}_{[\varepsilon(i)]} = \langle x_\varepsilon : \varepsilon < \varepsilon(i)\rangle$ and $\varepsilon(i) < \kappa$.

For any τ_0-model M such that $\|M\| = \|M\|^{<\kappa} + \lambda^{<\kappa}$, we say N codes M <u>when</u>:

$(*)$ (a) N expands M

(b) $\langle F^N_i : i < \Upsilon\rangle, \langle P^N_i : i < \Upsilon\rangle$ satisfies $(B)(b)(\alpha),(\beta)$ of the theorem (with N instead of M_1)

(c) $Q^N_i = \{b \in P^N_{\varepsilon(i)} : M \models \varphi_i[\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle]\}$ for $i < i(*)$

(d) $\langle c^N_i : i < \lambda\rangle$ are pairwise distinct and $P^N_* = \{c^N_i : i < \lambda\}$

(e) if $\varphi_i(\bar{x}_{\varepsilon(i)}) = \bigwedge\limits_{j<j(i)} \varphi_{i,j}(\bar{x}_{\varepsilon(i)})$ so for some $\mathbf{i}(i,j) < i(*)$ we have $\varphi_{i,j}(\bar{x}_{\varepsilon(i)}) = \varphi_{\mathbf{i}(i,j)}(\bar{x}_{\varepsilon(\mathbf{i}(i,j))})$ and so $\varepsilon(\mathbf{i}(i,j)) = \varepsilon(i)$ <u>then</u> $F_{1,i} \in \tau(N)$ is unary and for $b \in P^N_{\varepsilon(i)}$ we have:

  (α) $N \models$ "$F_{1,i}(b) = c_j \wedge \neg\varphi_i(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$" implies $M \models \neg\varphi_{i,j}(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$ which means: if $\varphi_{i,j} = \varphi_{\mathbf{i}(i,j)}$ and $N \models$ "$\neg Q_i(b) \wedge c_j = F_{1,i}(b)$" then $M \models$ "$\neg Q_{\mathbf{i}(i,j)}[b]$" and, of course

  (β) if $M \models \varphi_i(\langle f_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$ and $j < \varepsilon(i)$ then $M \models \varphi_{i,j}(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$

(f) if $\varphi_i(\bar{x}_{\varepsilon(i)}) = (\exists \bar{x}_{[\varepsilon(i),\zeta(i))})\varphi_{j_1(i)}(\bar{x}_{\varepsilon(i)}, \bar{x}_{[\varepsilon(i),\zeta(i))})$ and $F_\varepsilon(b) = a_\varepsilon$ for $\varepsilon < \varepsilon(i)$ <u>then</u> $(\alpha) \Leftrightarrow (\beta)$ where

  (α) $M_1 \models \varphi_i[\langle a_\varepsilon : \varepsilon < \varepsilon(i)\rangle]$ equivalently $M_1 \models \varphi_1[\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle]$

  (β) $M_1 \models (\exists y)\varphi_{j_1(i)}(\langle a_\varepsilon : \varepsilon < \varepsilon(i)\rangle, \langle F_\zeta(y) : \zeta \in [\varepsilon(i), \zeta(i)]\rangle$.

Now let

$\boxplus$ (a) τ_1 is $\tau_\psi \cup \{F_\varepsilon, P_\varepsilon : \varepsilon < \Upsilon\} \cup \{Q_i : i < i(*)\} \cup \{F_{1,i} : i < i(*)$ and φ_i is a conjunction$\}$

(b) $T_1 = \cap\{\mathrm{Th}(N)$: there is M, a τ_0-model of ψ such that $\|M\| = \|M\|^{<\kappa} + \lambda$ and N code $M\}$

(c) $p(x) = \{P_*(x) \wedge x \neq c_i : i < \lambda\}$.

Now check that

$\oplus$ $(\tau_1, T_1, p(x), \bar{F}, \bar{P})$ is as required.

$\square_{1.16}$

Remark 1.17. So how does 1.16 help for our main aim? It starts to translate $\psi \in \mathbb{L}_{\lambda^+,\kappa}(\tau_0)$ to $\mathbf{t} = (\tau_1, T_1, p(x))$, so instead having blocks of quantifiers $(\exists \bar{x}_{[\varepsilon]}), \varepsilon < \kappa$ we have $(\exists x)$, i.e. by the sequence of functions $\langle F_i : i < \varepsilon \rangle$ we code any ε-tuple by one element.

This will help later to make "the $\tau(T_{\mathbf{t}})$-reduct is saturated" equivalent to the existence of suitable coding.

Recalling Definition 1.6(6) of $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$, we get the section's main result: translating from $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ to a representation, naturally more complicated than the one for $\psi \in \mathbb{L}_{\lambda^+,\aleph_0}$.

Theorem 1.18. *The $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$-representation theorem*

Assume $\mathbb{B}$ is a Boolean Algebra of cardinality $\leq \lambda$ and for notational transparency no $b \in \mathbb{B}$ is an ordinal and $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_0)$. Then we can find a tuple $(\tau_1, T_1, p(x), \bar{F}, \bar{P})$ such that (for $\bar{F}, \bar{P}$ as below):

- (A) *as in 1.16*
- (B) *the following conditions on a τ_0-model M_0 are equivalent:*
  - (a) *$M_0 \in \mathrm{Mod}^2_\psi[\mathbb{B}]$, so $M_0 \models \psi$ and $\|M_0\| = \|M_0\|^{<\kappa} + \lambda^{<\kappa}$*
  - (b) *there is a τ_1-expansion M_1 of M_0 to a model of T_1 omitting $p(x)$ such that:*
    - (α) *$\langle P_i^{M_1} : i < \kappa \rangle$ is a partition of $|M_1|$*
    - (β) *if $i < \kappa$ and $a_j \in M_1$ for $j < i$ then for some $b \in P_i^{M_1}$ we have $j < i \Rightarrow F_j^{M_1}(b) = a_j$*
    - (γ) *$c_b(b \in \mathbb{B})$ are individual constants (in $\tau_1 \backslash \tau_0$) with no repetition, $P, Q \in \tau_1$ unary, $R \in \tau_1$ binary*
    - (δ) *$P^{M_1} = \{c_b^{M_1} : b \in \mathbb{B}\}$*
    - (ε) *$R^{M_1} \subseteq P^{M_1} \times Q^{M_1}$*
    - (ζ) *for every $b \in Q^{M_1}$ the set $u(b, M_1) := \{c_b \in P^{M_1} : (c_b, b) \in R^{M_1}\}$ is an ultrafilter of $\mathbb{B}$*
    - (η) *for every ultrafilter D of the Boolean Algebra $\mathbb{B}$ there is one and only one $b \in Q^{M_1}$ such that $u(b, M_1) = D$.*

Proof. First, note that $P, Q, c_b(b \in \mathbb{B})$ are in τ_ψ as in Definition 1.6. Second, we repeat the proof of 1.16 or just quote it:

- $(*)_1$ there is $\tau_* \supset \tau_\psi, |\tau_*| = \lambda$ with $F_\varepsilon, P_\varepsilon, F_{1,\varepsilon}, c_\varepsilon, Q \in \tau_*$ as there, i.e. satisfying clauses (A)(a)-(e).

Third, we prove clause (B) of 1.18. The direction (B)(b) $\Rightarrow$ (B)(a) holds as in 1.16. For the other direction, assume $M_0 \in \mathrm{Mod}^2_\psi[\mathbb{B}]$ and we choose M_1 as in 1.16(B)(b).

Lastly, clauses (B)(b)$(\gamma) - (\eta)$ holds because $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ and M_1 expands M_0. $\square_{1.18}$

Claim 1.19. *Assume $\kappa \leq \lambda^+, \kappa$ is singular (hence $\kappa^+ \leq \lambda$). For every $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa^+}[\mathbb{B}]$ there is $\psi_2 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that $\mathrm{spec}^2_{\psi_2} = \mathrm{spec}^2_{\psi_1}$.*

Proof. Easy. $\square_{1.19}$

Remark 1.20. 1) The only non-"$\mathbb{L}_{\lambda^+,\kappa}$ demand" in clause (B) of 1.18 is in $(b)(\eta)$, the existence, this is not expressible by a sentence of $\mathbb{L}_{\lambda^+,\kappa}$, even with extra predicates.
2) As indicated above, $\mathbb{B}^{\rm fr}_\lambda$ is the "worst, most complicated Boolean Algebra" for our purpose. So it is natural to wonder about the order among the relevant Boolean Algebras which we intend to comment on elsewhere.

§ 2. Real equality for each T

§ 2(A). **Answering the Original Question and the New One.**

The original question for this work was about the strictly stable case, i.e. fixing $\kappa > \aleph_0$, dealing with $\{\mathbf{t} \in \mathbf{N}_\lambda : \kappa(T_{\mathbf{t}}) = \kappa\}$, so we deal with this case first.

In this case Theorem 2.1 tells us that for strictly stable T and $\lambda \geq |T|$, the family of classes $\mathrm{Mod}_{\mathbf{t}}$ for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ and the family of classes $\mathrm{Mod}^2_\psi[\mathbb{B}]$ for $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ where $\kappa = \kappa_r(T)$ and $\mathbb{B}$ is the Boolean algebra $\mathbb{B}_T$ from 0.7(2),(2A),(3),(3A) are very similar. How this is proved? For one direction, we start with $\mathbf{t} \in N_{\lambda,T}$; so the (essential) non-first order part of the demand $M \in \mathrm{Mod}_{\mathbf{t}}$ is "$M{\restriction}\tau(T_{\mathbf{t}})$ is saturated". At first glance we need (in addition to the first order theory and the omission of a type) to say some things on eliminating $u \in [M]^{<\|M\|}$ and relation on it, but because of T being stable it can be (see 0.10) expressed by:

- (a) $M{\restriction}\tau(T_{\mathbf{t}})$ is $\kappa_r(T)$-saturated
- (b) if $\mathbf{I} \subseteq M$ is an infinite indiscernible set in $M{\restriction}\tau(T_{\mathbf{t}}), |\mathbf{I}| = \aleph_0$ then we can find an indiscernible set $\mathbf{J} \supseteq \mathbf{I}$ in $M{\restriction}\tau(T_{\mathbf{t}})$ of cardinality $\|M\|$.

So the use of $\mathbb{L}_{\lambda^+,\kappa}$ where $\kappa = \kappa_r(T)$ is natural. If $2^{|T|} \leq \lambda$ this is obvious but otherwise we have to be more careful. We use the Boolean algebra $\mathbb{B} = \mathbb{B}_T$ and the use of $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ rather than $\mathbb{L}_{\lambda^+,\kappa}$ to express $M{\restriction}\tau(T_{\mathbf{t}})$ is $\aleph_0$-saturated, so by $\kappa_r(T)$-sequence homogeneity this is enough.

Note that on the one hand $M \in \mathrm{Mod}_{\mathbf{t}} \Rightarrow \|M\| \in \mathbf{C}_T = \{\mu : \mu = \mu^{<\kappa(T)} + \lambda(T)\}$, see 1.3 but on the other hand for $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}], M \models \psi$ does not imply it. Still we know that $\mathrm{spec}^1_\psi = \{\|M\| : M \models \psi\}$ and $\mathrm{spec}^2_\psi = \mathrm{spec}^1_\psi \cap \mathbf{C}_T$ are closed enough, see Claim 1.9, in particular 1.9(1). Recall that $\mathbb{B} = \mathbb{B}^{\mathrm{fr}}_\lambda$ is the worst case.

For superstable T (for the case we fix (λ, T)), the case, of e.g. $= \mathrm{Th}({}^\omega 2, E_n)_n, E_n = \{(\eta,\nu) : \eta,\nu \in {}^\omega 2, \eta{\restriction}n = \nu{\restriction}n\}$ makes us work somewhat more.

Theorem 2.1. *Assume T is a stable first order complete of cardinality $\leq \lambda$ and $\kappa = \kappa_r(T) = \min\{\theta : \theta$ regular and $\theta \geq \kappa(T)\}$ and $\lambda(T) = \min\{\lambda : T$ stable in $\lambda\}$, see 0.3(3), and let $\mathbb{B} = \mathbb{B}_T$, see Definition 0.7(3A).*

Assume further that $\kappa(T) > \aleph_0$ (i.e. T is not superstable).
1) We have $\{\mathrm{spec}_{\mathbf{t}} : \mathbf{t} \in \mathbf{N}_{\lambda,T}\} = \{\mathrm{spec}^2_\psi[\mathbb{B}] : \psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]\}$.
2) If $\tau_0 = \tau_T$ and $\psi_0 = \wedge\{\varphi : \varphi \in T\}$ <u>or</u> just $\tau_T \subseteq \tau_0, |\tau_0| \leq \lambda, \psi_0 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_0)$ and $M \in \mathrm{Mod}_{\psi_0}[\mathbb{B}] \Rightarrow M \models T$ <u>then</u> there is $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ such that $\mathrm{spec}^2_{\psi_0}[\mathbb{B}] = \mathrm{spec}_{\mathbf{t}}$.
3) If $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ <u>then</u> for some $\psi_1 \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_1), \tau_1 \supseteq \tau(T_2)$ and $\mathrm{spec}^1_{\psi_1}[\mathbb{B}] = \mathrm{spec}_{\mathbf{t}} = \mathrm{spec}^2_{\psi_1}[\mathbb{B}]$.

Remark 2.2. The proof gives more: that the two contexts have the same PC classes. This proof is divided to two subsections each to one direction.

Proof. 1) By parts (2),(3).
2) By §(2C) below.
3) By §(2B) below, i.e. by 2.5 noting 2.4. $\square_{2.1}$

Conclusion 2.3. *If T is first order complete stable theory, $\kappa = \kappa(T) > \aleph_0$ and $|T| \leq \lambda$ <u>then</u> $H(\mathbf{N}_{\lambda,T})$ is bigger than $H(\mathbb{L}_{\lambda^+,\kappa})$ but smaller than $H(\mathbb{L}_{(2^\lambda)^+,\kappa})$.*

Proof. First assume T is strictly stable, i.e. $\kappa(T) > \aleph_0$. The "bigger than $H(\mathbb{L}_{\lambda^+,\kappa})$" follows by 2.1(2) recalling 1.9(4), the first inequality. The "smaller than $H(\mathbb{L}_{(2^\lambda)^+,\kappa})$" follows by 2.1(3) recalling 1.9(4), the second and third inequality. We are left with the case T is superstable, but then we quote [**?**, Th.1.2], or see 2.6, 2.7 below. $\square_{2.3}$

§ 2(B). Given $\mathbf{t} \in \mathbf{N}_{\lambda,1}$.

Hypothesis 2.4. For this subsection we are given $\mathbf{t} = (T, T_1, p) \in \mathbf{N}_{\lambda,T}$ such that T is complete first order stable so $\lambda \geq |T_1| \geq |T|$ and let $\mathbb{B} = \mathbb{B}_T, \kappa = \kappa_\tau(T)$; without loss of generality:

- (a) $P, Q, R, c_b (b \in \mathbb{B})$ are <u>not</u> in $\tau(T_1)$ and with no repetition
- (b) P, Q are unary predicates, R is a binary predicate, c_b an individual constant
- (c) $\tau_2 = \tau(T_1) \cup \{P, Q, R, c_b : b \in \mathbb{B}\}$.

Claim 2.5. *Assume $\kappa > \aleph_0$. There is $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_1)$ such that* $\mathrm{Mod}_{\mathbf{t}} = \{N{\restriction}\tau(T_1) : N \models \psi$ *so* $\tau(N) = \tau(\psi) \supseteq \tau_1\}$.

Proof. Note that below proving 2.6, 2.7 we use this proof stating the changes; there $\kappa(T) = \aleph_0$, i.e. T is superstable.

<u>Stage A</u>:

Without loss of generality we can replace T by $T^{\rm eq}$ (no need for new elements: we can extend T_1 to have a copy of $M^{\rm eq}$ with new predicates and an isomorphism). The use of $T^{\rm eq}$ is anyhow just for transparency. For $\theta = \mathrm{cf}(\theta) < \kappa_r(T)$ choose a sequence $\bar{\varphi}_\theta = \langle \varphi_{\theta,i}(x, \bar{y}_{\theta,i}) : i < \theta\rangle$ witnessing $\theta < \kappa_r(T)$ equivalently $\theta < \kappa(T)$.

<u>Stage B</u>:

Let $\tau = \tau(T_1) \cup \{P, Q, R, S_{\varphi(\bar{x}_{[n]},\bar{y}_{[n]})}, G_n, c_b, Q_\theta, <_\theta, F_i, P_i, F_{1,i} : b \in \mathbb{B}, i < \kappa, \varphi(\bar{x}_n, \bar{y}_n) \in \mathrm{EQ}_T\}$, see Definition 0.3(5) on EQ_T; where the union is a disjoint union and the second set without repetitions, P_i, Q_θ unary predicates, c_b an individual constant, R binary predicate, $S_{\varphi(\bar{x}_{[n]})}$ an n-place function for $\varphi(\bar{x}_{[n]}) \in \mathbb{L}(\tau_T)$, F_i unary function for $i < \kappa$; $F_{1,n}$ is an n-place function symbol, G_n an n-place function symbol.

For awhile fix $M_1 \in \mathrm{Mod}_{\mathbf{t}}$, note that by 0.5

$(*)_1$ $\|M_1\| = \|M_1\|^{<\kappa} \geq \lambda(T)$.

Let $M = M_1{\restriction}\tau(T)$ and let $\mathscr{M}[M_1]$ be the set N of such that (for use in other places in $(*)_2$ we do not use "$\kappa > \aleph_0$"):

$(*)_2$ (a) N is a τ-expansion of M_1

(b) $P^N, Q^N, R, \langle c_b^N : b \in \mathbb{B}\rangle$ code $\mathbb{B}_T$ and $\mathrm{uf}(\mathbb{B}_T)$, see 0.7(3) and e.g. 1.18$(B)(b)(\gamma) - (\eta)$

(c) (α) letting π be the canonical isomorphism from $\mathbb{B}_T$ onto $\mathbb{B}_M$, see §(0B), if $\bar{a} \in {}^n M$ and $\varphi = \varphi(\bar{x}_{[n]}, \bar{y}_{[n]}) \in \mathrm{EQ}_n(T)$ and $\pi(c) = \varphi(M, \bar{a})$, <u>then</u> $\mathbf{S}_\varphi^N(\bar{a}) = b_c^N$

(β) $Q^N = \{d_D$: for some $m, D \in \mathrm{uf}(\mathbb{B}_{T,m})\}$ and $R^N = \{(c^N_b, d_D) : b \in \mathbb{B}_{T,m}$ and $D \in \mathrm{uf}(\mathbb{B}_{T,m}), b \in D\}$ where d_D belongs to $\cap\{\pi(c) : c \in D\}$

(d) $N{\restriction}\tau_T$ is κ-saturated

(e) (α) if $\kappa > \aleph_0$ and $\langle a_n : n < \omega\rangle$ is an indiscernible set in M then for[3] some $b, a \mapsto G^N_2(a,b)$ is a one-to-one function from M onto an indiscernible set which includes $\{a_n : n < \omega\}$

(β) if $\kappa = \aleph_0, \bar{c} \in {}^n M, b \in M$ is not algebraic over $\bar{c}$, then

- $a \mapsto G^N_{n+2}(a,b,\bar{c})$ is a one-to-one function
- $G^N_{n+2}(b,b,\bar{c}) = b$
- $\{G^N_{n+2}(a,b,\bar{c}) : a \in M\}$ is an indiscernible set over $\bar{c}$ based on $\bar{c}$, all in M.

(f) (α) $F^N_{1,m}$ is a function from ${}^m M$ to Q^N such that if $\bar{a} \in {}^m M$ then $d = F^N_{1,m}(\bar{a})$ is the member of Q^N coding $\mathrm{stp}(\bar{a}, \emptyset, M)$, i.e.

- if $D \in \mathrm{uf}(\mathbb{B}_T)$, then we have that $F_{1,m}(\bar{a}) = d_D$ if and only if $\mathrm{stp}(\bar{a}, \emptyset, M) = D$

(β) if $D \in \mathrm{uf}(\mathbb{B}_{T,m})$ and $1_{\mathbb{B}_{T,m}} \in D$ then for some $\bar{a} \in {}^m M, F^N_{1,m}(\bar{a}) = d_D$, (recall $\mathbb{B}_{T,m} = \mathbb{B}_T \restriction 1_{\mathbb{B}_{T,m}}$)

(g) for every $i < \kappa$ and $\bar{a} = \langle a_j : j < i\rangle \in {}^i M$ for some $b \in N$ we have $(\forall j < i)(F^N_j(b) = a_j)$ and $b \in P^N_i$

(h) $\langle P^N_i : i < \kappa\rangle$ is a partition of N

(i) for any regular $\theta < \kappa_r(T)$ we have:

(α) $Q^N_\theta = \cup\{P^N_i : i \le \theta\}$ and $(Q^N_\theta, <^N_\theta)$ is a partial order which is a tree with θ levels isomorphic to $({}^{\theta \ge}\|M_1\|, \triangleleft)$ say $\pi_\theta : {}^{\theta >}\|M_1\| \to Q^N_\theta$ is such an isomorphism

(β) let $\bar{a}^\theta_\eta = \langle F^N_i(\pi_\theta(\eta)) : \ell < \ell g(\bar{y}_{\theta,i})\rangle$ for $\eta \in {}^{\theta \ge}\|M_1\|$

(γ) $b_1 <^N_\theta b_2$ iff for some $i_1 < i_2 < \theta$ we have $b_1 \in P^N_{i_1}, b_2 \in P^N_{i_2}$ and $j < i_1 \Rightarrow F^N_j(b_1) = F^N_j(b_2)$

(δ) if $i < \theta, \eta \in {}^i\|M_1\|$ and $\alpha < \beta < \|M_1\|$ then $N \models \neg(\exists x)\big((\varphi_{\theta,i}(x, \bar{a}^\theta_{\eta\char`^\langle\alpha\rangle}) \wedge \varphi_i(x, \bar{a}^\theta_{\eta\char`^\langle\beta\rangle})\big)$

(ε) if $n < \omega, i_0 < \ldots < i_{n-1} < \theta, \eta_k \in {}^{(i_k)}\|M_\ell\|$ for $k < n$ and $\eta_0 \triangleleft \eta_1 \triangleleft \ldots \eta_{n-1}$ then $N \models (\exists x)(\bigwedge\limits_{k<n} \varphi_{i_k}(x, \bar{a}^\theta_{\eta_k}))$

(ζ) $F_{\theta,j,i}(\pi(\eta)) = \pi(\eta{\restriction}i)$ when $i < j \le \theta, \eta \in {}^j\|M_1\|$

(θ) for every $c \in Q^N_\theta, F^N_\theta(c)$ is $\pi_\theta(\eta)$ for some $\eta \in {}^{\theta \ge}\|M_1\|$ and letting $j_\eta = \ell g(\eta)$ we have

- if $i < j_\eta$ then $N \models \varphi_{\theta,i}[c, \bar{a}^0_{\eta\restriction\langle i_1\rangle}]$
- if $j_\eta < \theta$ then $\alpha < \|M_1\| \Rightarrow N \models \neg\varphi_{J_\eta}[c, \bar{a}_{\eta\char`^\langle\alpha\rangle}]$

(ι) $F^N_{\theta,2}$ is a binary function such that: if $\eta \in {}^{\theta >}\|M_1\|$ then $\langle F^N_{\theta,i}(c, \pi_\theta(\eta)) : c \in \|M_1\|\rangle$ list with no repetitions $\langle \pi_\theta(\eta\char`^\langle\alpha\rangle) : \alpha < \|M_1\|\rangle$

(κ) $F^N_{i,1,\theta}$ or $F^N_{\theta,1}$ is a unary function such that for every $c \in M, F_{1,\theta}(c)$ is

[3] note that when $\kappa > \aleph_0$ we can use G a two-place function symbol

- $\pi(\eta)$ for some $\eta \in {}^{\theta \geq}\|M_1\|$ and for any $i \leq \theta, \nu \in {}^i\|M_1\|$ we have c realize $\{\varphi_j(x, \bar{a}^\theta_{\nu \restriction j}) : j < i$ iff $\nu \trianglelefteq \eta\}$

(j) if $j < \kappa$ has cofinality θ and $\langle i_j(\iota) : \iota < \theta \rangle$ is an increasing sequence of ordinals with limit $j, b_i \in M_2$ for $i < j, d \in N$ and $F^N_{\theta,2}(d) \in P^N_\theta$ and $\iota < \theta \wedge i_* < i_j(\iota) \Rightarrow F^N_{i_*}(F^N_\iota(d)) = b_{i_*}$ <u>then</u> there is $d' \in P_j$ such that $i_* < j \Rightarrow F_{i_*}(d') = b_{i_*}$.

Let $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ be such that:

$(*)_3$ a τ-model N satisfies ψ <u>iff</u>: for a relevant large enough subset Λ of $\mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ of cardinality $\leq \lambda, \psi = \wedge\{\varphi \in \Lambda$: if $M_1 \in \mathrm{Mod}_{\mathbf{t}}$ and $N \in \mathscr{M}[M_1]$ then $N \models \varphi\}$; the "Λ is large enough" means that the sentences expressing "the τ-model satisfies clause (x)" belong to Λ for each of the clauses (a)-(d) below (note clause (d) means clauses (a)-(j),(e) of $(*)_2$:

(a) $N \restriction \tau_T$ is a model of T

(b) $N \restriction \tau_{T_1}$ is a model of T_1

(c) $N \restriction \tau_{T_1}$ omits p

(d) $N \in \mathscr{M}[N \restriction \tau_{T_1}]$ see $(*)_2$

Now

$(*)_4$ (a) $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ indeed

(b) every $M_1 \in \mathrm{Mod}_{\mathbf{t}}$ can be expanded to a model for Mod^*_ψ (see Definition 1.11(2); this is more than being a model of ψ!)

(c) if $N \in \mathrm{Mod}_\psi$ then $N \restriction \tau(T_1) \in \mathrm{Mod}_{\mathbf{t}}$.

[Why? For clause (a) read $(*)_3$. For clause (b) read $(*)_2 + (*)_3$.

For clause (c), first why $M_1 = N \restriction \tau_{T_1}$ is a model of T_1? Since $M_1 \in \mathrm{Mod}_t$ and $N \in \mathscr{M}[M_1]$, we have that $N \restriction \tau(T_1)$ is M_1 by $(*)_2(a)$. Second, why M_1 omit $p_{\mathbf{t}}$? Recalling clause (c) of $(*)_3$ and choice of ψ this should be clear. Third, why is $M = N \restriction \tau_T$ saturated? It realizes every $p \in D_m(T) = \mathbf{S}^m(\emptyset, M)$, by $(*)_2(e)$, it is κ-saturated by $(*)_2(d)$. By $(*)_2(e)(\alpha)$, every indiscernible subset $\mathbf{I}$ of cardinal $\aleph_0$ can be extended to one of cardinality $\|M\|$. By the last two sentences, M is saturated by Case 1 of 0.10.]

So we are done. $\square_{2.5}$

Claim 2.6. *Like 2.5, but T is superstable and $\lambda(T) \leq \lambda$.*

Proof. As in the proof of 2.5 with some changes. Here the proof "why $M = N \restriction \tau_T$ is saturated" inside the proof of $(*)_4(c)$ is different. There is a saturated $M_* \in \mathrm{Mod}_T$ of cardinality $\leq \lambda$ and we can demand on ψ that $N \models \psi$ implies M_* is elementarily embeddable into $N \restriction \tau_T$ and $N \restriction \tau_T$ is $\aleph_0$-sequence homogeneous.

Note that

$(*)$ if $M_* \prec M \in \mathrm{Mod}_T$ and M is $\aleph_0$-sequence homogeneous implies M is $\aleph_\varepsilon$-saturated, see 0.3(0).

Another difference is that $(*)_2(e)(\beta)$ of the proof of 2.5 implies M is saturated because by case 2 of 0.10

(∗) M is saturated <u>when</u>: if M is $\aleph_\varepsilon$-saturated and for every finite $A \subseteq M$ and $a \in M \backslash \mathrm{acl}(A)$ there is an indiscernible set $\mathscr{I} \subseteq M$ over A of cardinal $\|M\|$ based on A (i.e. $\mathrm{Av}(M, \mathbf{I})$ does not fork over A) to which a belongs.

$\square_{2.6}$

Claim 2.7. *1) Like 2.5 but T is superstable and $2^{\aleph_0} \leq \lambda$.*
2) Like 2.5, but T superstable and $|D(T)| > |T|$.

Proof. As the proof of 2.6 the problem is how ψ guarantees "$N{\restriction}\tau_T$ is $\aleph_\varepsilon$-saturated". As the model is $\aleph_0$-saturated it suffices to prove:

(∗) for every m and $D \in \mathrm{uf}(\mathbb{B}_{T,m+1})$ equivalently $p \in D_{m+1}(T)$ for some $\bar{a}{}^\frown\langle c\rangle \in {}^{m+1}N$ realizing p, we have: if $N{\restriction}\tau_T \prec M'$ and $c' \in M'$ realizes $\mathrm{tp}(c, \bar{a}, N{\restriction}\tau_T)$ <u>then</u> some $c'' \in N{\restriction}\tau_T$ realizes $\mathrm{stp}(c', \bar{a}, M')$ in M'.

Let $p = \mathrm{tp}(c, \bar{a}, M)$ and we let $\lambda_* = \lambda(p), \langle E_\alpha(x_0, x_1; \bar{y}_{[m]}) : \alpha < \lambda_*\rangle$ be as in [**?**, Ch.III,5.1,pg.123] and 2^{λ_*} is the cardinality of the set $\{\mathrm{stp}(c', a, M') : M \prec M', c' \in M, c \text{ realizes } \mathrm{tp}(c; \bar{a}, M')\}$ from (∗). Hence it suffices to prove $2^\lambda \leq |D(T)|$.

<u>Case 1</u>: $\lambda_* = \aleph_0$

If $2^{\aleph_0} \leq \lambda$ this is easy. If $|D(T)| > |T|$ then for some m there is an independent sequence $\langle \varphi_n(\bar{x}_{[m]}) : n < \omega\rangle$ of formulas of $\mathbb{L}(\tau_T)$ over T; (that is, if $M \in \mathrm{Mod}_T$ then any non-trivial finite Boolean combination of them is realized in M) and we continue as in the second case.

<u>Case 2</u>: $\lambda_* > \aleph_0$

In this case by [**?**, Ch.III,5.9,5.10,pg.126] there is an independent over T sequence $\langle \varphi_i(x, \bar{y}_{[m]}) : i < \lambda_*\rangle$ of formulas from $\mathbb{L}(\tau_T)$, so $\mathbb{B}^{\mathrm{fr}}_{\lambda_*}$ is embeddable into $\mathbb{B}_{T,m+1}$. Hence ψ says that the Boolean Algebra $\mathscr{P}(\lambda_*)$ is interpreted in N for every relevant λ_*, but $\lambda_* \leq |T|$.

From this it is easy to have ψ ensuring (∗). $\square_{2.7}$

§ 2(C). Coding $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}_T]$.

Hypothesis 2.8.

(*a*) T is a complete first order theory,

(*b*) $\lambda \geq |T|, \lambda^+ \geq \kappa$

(*c*) $\mathbb{B} = \mathbb{B}_T$.

Claim 2.9. *Assume $\psi \in \mathbb{L}_{\lambda^+,\kappa}[\mathbb{B}]$ and $\kappa = \kappa_r(T) < \infty$ so T is stable.*

There is $\mathbf{t} = (T, T_1, p) \in \mathbf{N}_{\lambda,T}$ such that $\tau(T_1) \supseteq \tau(\psi)$ and $\mathrm{Mod}_{\mathbf{t}} = \{N{\restriction}\tau(\psi) : N \in \mathrm{Mod}_\psi[\mathbb{B}]\}$.

Proof. We apply 1.18 to $\mathbb{B}$ and ψ and get $(\tau_1, T_1, p(*), \bar{F}, \bar{P})$ as in 1.16, 1.18 and without loss of generality $\tau_1 \cap \tau(T) = \emptyset$. Now we imitate the proof of 2.5. $\square_{2.9}$

§ 2(D). **Elaborating Case C.**

In §(2B) we treat most theories T but not all. The remaining case is

Hypothesis 2.10.

$\boxplus$ (a) T is superstable of cardinality λ
(b) $\lambda(T) > \lambda$
(c) $2^{\aleph_0} > \lambda$
(d) $\lambda \geq |D(T)|$.

Claim 2.11. *There are $m, M \in \mathrm{Mod}_T$ and $\bar{a} \in {}^m M$ such that $\{\mathrm{stp}(c, \bar{a}, M) : c \in M\}$ is of cardinality $2^{\aleph_0}$.*

Proof. By [?], but for completeness we elaborate. As $\lambda \geq |D(T)|$ there is an $\aleph_0$-saturated model M of T of cardinality λ. Moreover, without loss of generality if $A \subseteq M$ is finite and $a \in M$ is not algebraic over A, then there is $\mathbf{I} \subseteq M$ of cardinality λ which is indiscernible over A, based on A and $a \in \mathbf{I}$.

Also without loss of generality if $A \subseteq M$ is finite and $\mathscr{P}_{M,A} = \{\mathrm{stp}(c, A, M') : M \prec M' \text{ and } c \in M'\}$ has cardinality $\leq \lambda$ then all of them are realized in M.

By clause $\boxplus(b)$, M is not saturated, hence for some $\bar{a} \in {}^{\omega >}M, |\mathscr{P}_{M,\bar{a}}| \geq \lambda$, which easily implies $|\mathscr{P}_{M,\bar{a}}| \geq 2^{\aleph_0}$. If $|\mathscr{P}_{M,\bar{a}}| > 2^{\aleph_0}$ by [?, Ch.III,§5] we get a contradiction to $\boxplus(d)$. $\square_{2.11}$

Definition 2.12. For any model M and a sequence $\bar{a}$ from M (or a set $\subseteq$), let $\mathbb{B}_{M,\bar{a},m}$ be the Boolean Algebra of subsets of ${}^m M$ of the form $\varphi(M, \bar{c})$, where $\varphi(\bar{x}_{[m]}, \bar{z}) \in \mathbb{L}(\tau_M), \bar{b} \in {}^{\ell g(\bar{z})}M$ and $\varphi(\bar{x}, \bar{c})$ is almost over $\bar{a}$ which means: for some $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}) \in \mathbb{L}(\tau_M)$ we have:

- in $M, \vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{a}) \vdash \varphi(\bar{x}_{[m]}, \bar{c}) \equiv \varphi(\bar{y}_{[m]}, \bar{c})$
- $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{a})$ defines in M an equivalence relation with finitely many equivalence classes.

Claim 2.13. *For T as in 2.10, letting $M, \bar{a}, m$ be as in 2.11 and $\mathbb{B} = \mathbb{B}_{M,\bar{a},m}$ the result of 2.5 and Theorem 2.1 hold if we use $\mathbb{B}$ instead of $\mathbb{B}_T$.*

Proof. As above, really $m = 1$ suffice; in particular if $p \in \mathbf{S}(\bar{a}, M), \bar{a} \in {}^m M, M \in \mathrm{Mod}_T$ then $\lambda_*(p) \leq \aleph_0$ (otherwise by Lemma 5.9, 5.10 and 5.11 [?, Ch.III] we have $|\mathbf{S}^{2m}(\bar{a}, m)| \geq 2^{\lambda_*(p)} > \lambda$, contradiction). $\square_{2.13}$

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and, Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at