NICE $\aleph_1$ GENERATED NON-P-POINTS, I

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ABSTRACT. We define a family of non-principal ultrafilters on $\mathbb{N}$ which are, in a sense, very far from P-points. We prove the existence of such ultrafilters under reasonable conditions. In subsequent articles we shall prove that such ultrafilters may exist while no P-point exists. Though our primary motivations came from forcing and independence results, the family of ultrafilters introduced here should be interesting from combinatorial point of view too.

1. INTRODUCTION

One of the important notions in general topology and set theory of the reals is that of a P-point. Recall that *a P-point* is a non-principal ultrafilter D on $\mathbb{N}$ with the property that for any countable family $\mathscr{A} \subseteq D$ there is a $B \in D$ almost (modulo finite) included in all $A \in \mathscr{A}$ (see Definition 3.6). Concerning these and other special ultrafilters on $\mathbb{N}$, their history and basic applications we refer the reader to the survey article by Blass [1].

In many applications it is important to preserve P-points by specific forcing notions and by a forcing iterated with countable supports. Recall that *preservation of an ultrafilter* means that the ultrafilter from the ground model $\mathbf{V}$ generates an ultrafilter in the generic extension $\mathbf{V}[G]$ (see [7, Chapter VI]). We have a very good understanding of these questions and many relevant results have been presented in the literature. From our point of view the P-points are tractable for independence results because of the following fact:

Fact 1.1 (Nice properties of P-points). (A) there are quite many forcing notions preserving P-points,

(B) a proper forcing notion $\mathbb{Q}$ which preserves "D is an ultrafilter" preserves its being a P-point,

(C) the preservation of P-points is preserved in limits of CS iterations.

(D) We can destroy a P-point by forcing, i.e., ensure it has no extension to a P-point (and consequently we may prove the consistency of "there are no P-points"),

(E) moreover, we can "split hairs", i.e., destroy some P-points while preserving other, so we can have unique P-point up to isomorphisms.

(Already the properties (A,B,C) give a well controlled way to have ultrafilters generated by $\aleph_1 < 2^{\aleph_0}$ sets).

For more details we refer the reader to [7, Ch.VI and Ch.XVIII,§4].

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We may wonder if the theory developed for P-points can be repeated for other ultrafilters. We may ask:

Question 1.2. Are there other types of ultrafilters preserved by CS iterations of suitable forcing notions?

In particular, we are interested in preservation of our ultrafilters at limit stages of CS iterations: for a limit ordinal δ, having been preserved by $\mathbb{P}_\alpha$ for $\alpha < \delta$, does this hold for $\mathbb{P}_\delta$ when $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is a CS iteration of proper forcing notions?

We suggested this problem in [6, 3.13] and we speculated about it there. Note that ultrafilters as in Question 1.2 for CS iterations are naturally generated by $\aleph_1$ sets; moreover CS iterations are mainly interesting when we start with CH, and "preserve an ultrafilter" is meaningful only when we add reals, naturally $\aleph_2$ ones. We suspect this direction is related to the question on the existence of a point of van Douwen [3] (see Question 1.3 below), but at present we do not know neither if they are related nor how to answer it. Other specific problems that we have in mind when developing the theory for Question 1.2 are a problem of Nyikos and a problem of Dow:

Question 1.3. [E. van Douwen] Is it consistent that: there is no ultrafilter D on $\mathbb{Q}$ such that every $A \in D$ contains a member of D which is a closed set with no isolated points?

Question 1.4. [P. Nyikos] Is it consistent to have some ultrafilter $D \in \beta^*(\mathbb{N}) \setminus \mathbb{N}$ of character $\aleph_1$, but no P-point?

Question 1.5. [A. Dow] Is it consistent to have $\mathfrak{u} = \aleph_1$, there is a P-point D, but no P-point D with $\chi(D) = \aleph_1$?

In the series of papers started here the main points are:

(A) we have an involved family of sets (really well founded trees) appearing in the definition,
(B) each ultrafilter has no P-point as a quotient,
(C) they are related to a game,
(D) such systems exists assuming, e.g., $\diamondsuit_{\aleph_1}$,
(E) enough relevant forcing notions preserve such systems, in particular, some serving 1.1(C), so answering the first question in 1.2,
(F) we have a preservation theorem for such ultrafilters under CS iterations.
(G) As an application, we will solve Nyikos' problem 1.4.

So problems 1.2 and 1.4 will be resolved by the methods we start developing here, but presently not 1.3 and 1.5 (a problem of van Douwen and a problem of Dow).

In the present article we define ultrafilters analogous to P-points but with no P-point as a quotient; this is done in Sections 2 and 3. In the fourth section we deal with basic connections to forcing that we will use in the independence results in subsequent papers.

In the second paper of the series (still "work in progress") we present these ultrafilters in a more general framework and deal with sufficient conditions for such an ultrafilter to generate an ultrafilter in a suitable generic extension. For the limit case we continue the proof of preservation theorems in [7], in particular [7, Ch.VI,1.26,1.27] and Case A with transitivity of [7, Ch.XVIII,§3]. For the successor case we need that the relevant forcing preserves our ultrafilters. We will conclude with the proof for CON($\mathfrak{u} = \aleph_1 +$ no P-points).

Noting that the ultrafilters so far were really analogous to selective (i.e., Ramsey) ultrafilters we plan to give a more general framework which also includes P-points in a planned third part.

Remark 1.6. There may be P-point while $\mathfrak{d} > \aleph_1$, see Blass and Shelah [2] and references there, but the existence of ultrafilters in the direction here, far from P-point, implies $\mathfrak{d} = \aleph_1$, see the survey of Blass [1]. But note that the ultrafilter may be $\aleph_1$-generated in a different sense: union of $\aleph_1$ families of the form $\text{fil}(B) \cap \mathscr{P}(\max(B))$.

Note that it may be harder (than in the P-point case) to build such ultrafilters as here which are μ-generated instead of $\aleph_1$-generated because of the unbounded countable depth involved. We have not looked at this as well as at the natural variants of our definition (not to speak of generalization to reasonable ultrafilters, see [9] and Rosłanowski and Shelah [4, 5]).

2. System of filters using well founded trees

Notation 2.1. Here, $M = (M, <_M)$ is a partial order and B is a subset of M inheriting its order.

For $\eta \in B$ we let $B_{\geq \eta} = \{\nu \in B : \eta \leq_M \nu\}$ and similarly $B_{>\eta}$. We also define

$$\text{suc}_B(\eta) = \{\nu \in B : \eta <_M \nu \text{ but for no } \rho \in B \text{ do we have } \eta <_M \rho <_M \nu\}$$

and $\max(B) = \{\nu \in B : B \cap M_{>\nu} = \emptyset\}$.

We say that Y is *a front of* $B \subseteq M$ iff: $Y \subseteq B$ and every branch (maximal chain) of B meets Y and the members of Y are pairwise $<_M$-incomparable.

Definition 2.2. Let $M = (M, <_M)$ be a partial order. A set $B \subseteq M$ is a *countable well-founded sub-tree of* M if the following conditions (a)–(f) are satisfied.

(a) The set B is a countable subset of M.
(b) The set B has a $<_M$-minimal member called its root, $\text{rt}(B)$.
(c) The structure B (i.e., $(B, <_M \restriction B)$) is a tree with $\leq \omega$ levels and no ω-branch (so all chains in B are finite).
(d) For each $\nu \in B$ the set $\text{suc}_B(\nu)$ is either empty or infinite.
(e) If $\eta, \nu \in B$ are $<_M$–incomparable, then they have no common $\leq_M$–upper bound (i.e., they are incompatible not only in B but even in M). We abbreviate this as $\eta \parallel_M \nu$.
(f) If $\nu \in B \setminus \max(B)$ and $F \subseteq M \setminus M_{\leq \nu}$ is finite, then for infinitely many $\varrho \in \text{suc}_B(\nu)$ we have $(\forall \rho \in F)(\rho \parallel_M \varrho)$.

The family of all countable well-founded sub-trees of M is denoted by $\text{CWT}(M)$.

We will define a natural filter on the set of maximal nodes of every countable well-founded tree B; this filter will naturally induce Rudin-Keisler images on each front of B.

Definition 2.3. For $B \in \text{CWT}(M)$ let $\text{frt}(B)$ be the set of all fronts of B, which in this case means the family of all maximal sets of pairwise incomparable members of B.

For antichains Y_1, Y_2 of M we say that Y_2 is *above* Y_1 iff:

$$(\forall \eta \in Y_2)(\exists \nu \in Y_1)[\nu \leq_M \eta].$$

This will be used mainly for $Y_1, Y_2 \in \mathrm{frt}(B)$, $B \in \mathrm{CWT}(M)$.

For Y_1, Y_2 as above let the projection h_{Y_1,Y_2} be the unique function $h : Y_2 \longrightarrow Y_1$ such that $h(\eta) \leq_M \eta$ for $\eta \in Y_2$.

If $Y_1, Y_2 \in \mathrm{frt}(B)$ *then* Y_2 *is almost above* Y_1 iff:

for some $B' \in \mathrm{sb}(B)$, see 2.4 below, $B' \cap Y_2$ is above $B' \cap Y_1$.

We also define the projection h_{Y_1,Y_2} as above, but its domain is not Y_2 but the set $\{\eta \in Y_2 : (\exists \nu \in Y_1)(\nu \leq_M \eta)\}$.

The default value of $Y \in \mathrm{frt}(B)$ is $\max(B) = \{\nu \in B : \nu \text{ is } <_M\text{-maximal in } B\}$.

We now define two notions of largeness for subtrees. *Exhaustive* subtrees correspond to filter sets or "measure 1" sets, *positive* subtrees will correspond to the notion "positive modulo a filter" or "not in the ideal dual to the filter".

Definition 2.4. Let $B \in \mathrm{CWT}(M)$. We call B' is an *exhaustive subtree* of B iff:

(a) $B' \in \mathrm{CWT}(M)$, $B' \subseteq B$,
(b) $\mathrm{rt}(B') = \mathrm{rt}(B)$,
(c) for all $\nu \in B'$ we have: $\mathrm{suc}_{B'}(\nu) \subseteq \mathrm{suc}_B(\nu)$ and $\mathrm{suc}_B(\nu) \backslash \mathrm{suc}_{B'}(\nu)$ is finite.

We let $\mathrm{sb}(B)$ be the set of all exhaustive subtrees B' of B, and we say f *witnesses* "$B' \in \mathrm{sb}(B)$" if $f : B' \backslash \max(B) \longrightarrow [B]^{<\aleph_0}$ satisfies

$$\nu \in B' \backslash \max(B) \quad \Rightarrow \quad \mathrm{suc}_B(\nu) \backslash \mathrm{suc}_{B'}(\nu) \subseteq f(\nu).$$

Note that for f being a witness only $f{\restriction}B'$ matters; in fact only the restriction $f{\restriction}\{\nu \in B' \mid \exists \eta \in Y : \nu \leq \eta\}$ matters when we are interested in $D_{B,Y}$.

For $B \in \mathrm{CWT}(M)$ and $Y \in \mathrm{frt}(B)$ let $E_{B,Y}$ be the filter on Y generated by the family

$$\{Y \cap B' : B' \text{ is an exhaustive subtree of } B, \text{ i.e., } B' \in \mathrm{sb}(B)\}.$$

For $B \in \mathrm{CWT}(M)$ let $\mathrm{psb}_M(B)$ ("p" stands for positive) be the set of *positive subtrees* B' of B which means (a),(b) as above and

(c)' if $\nu \in B' \backslash \max(B)$, then $\mathrm{suc}_{B'}(\nu)$ is an infinite subset of $\mathrm{suc}_B(\nu)$.

Definition 2.5. An antichain $Y \subseteq M$ is an *almost front of* B if for some $B' \in \mathrm{sb}(B)$ the intersection $Y \cap B'$ is a front of B'. Let $\text{alm-frt}(B) = \text{alm-frt}_M(B)$ denote the set of all almost fronts of B.

For $Y \in \text{alm-frt}_M(B)$ let

$$\mathrm{fil}_M(Y, B) = \{X \subseteq Y : \text{ for some } B' \in \mathrm{sb}(B) \text{ we have } X \supseteq B' \cap Y\}.$$

Definition 2.6. Let $\leq^*_M$ be the following two-place relation (actually a partial order) on $CWT(M)$:

$B_1 \leq^*_M B_2$ *iff* ($B_1, B_2 \in \mathrm{CWT}(M)$,
$\mathrm{rt}(B_1) = \mathrm{rt}(B_2)$,
and for some $B'_2 \in \mathrm{sb}(B_2)$, we have
– $B'_2 \cap B_1 \in \mathrm{psb}_M(B_1)$, and
– every almost front of $B'_2 \cap B_1$ is an almost front of B_2.

The tree B'_2 as above will be called *a witness for* $B_1 \leq^*_M B_2$.

For $B \in \mathrm{CWT}(M)$, *the depth of* B is defined recursively by

$$\mathrm{Dp}(B) = \sup\{\mathrm{Dp}(B_{\geq\eta}) + 1 : \eta \in B \backslash \{\mathrm{rt}(B)\}\}.$$

Remark 2.7. If $B, B' \in \mathrm{CWT}(M)$, $B' \subseteq B$ and $\nu \in B'$, then $\mathrm{suc}_B(\nu) \cap B' \subseteq \mathrm{suc}_{B'}(\nu)$, but the two sets do not have to be equal. Note that in the definitions of both $B' \in \mathrm{sb}(B)$ and $B' \in \mathrm{psb}_M(B)$ we do require that

$$\big(\forall \nu \in B'\big)\big(\mathrm{suc}_B(\nu) \cap B' = \mathrm{suc}_{B'}(\nu)\big)$$

This condition implies that if $Y \subseteq B$ is a front of B, then $Y \cap B'$ is a front of B'.

Observation 2.8. Let M be a partial order and $B, B_1, B_2 \in \mathrm{CWT}(M)$.

(1) We have that $B_1 \leq^*_M B_2$ if and only if every almost front of B_1 is an almost front of B_2.
(2) The relation $\leq^*_M$ is a partial order on $\mathrm{CWT}(M)$.
(3) If $B_2 \in \mathrm{psb}_M(B_1)$, *then* $B_1 \leq^*_M B_2$ and $\mathrm{psb}_M(B_2) \subseteq \mathrm{psb}_M(B_1)$.
(4) If $B_2 \in \mathrm{sb}(B_1)$, *then* $B_2 \in \mathrm{psb}(B_1)$, $\mathrm{sb}(B_2) \subseteq \mathrm{sb}(B_1)$ and $B_1 \leq^*_M B_2 \leq^*_M B_1$.
(5) For $B \in \mathrm{CWT}(M)$, $\max(B)$ is a front of B and also $\{\mathrm{rt}(B)\}$ is. If $B \neq \{\mathrm{rt}(B)\}$, *then* $\mathrm{suc}_B(\mathrm{rt}(B))$ is a front of B.
(6) Every front of $B \in \mathrm{CWT}(M)$ is an almost front of B.
(7) If $B \in \mathrm{CWT}(M)$ *then* $\mathrm{Dp}(B)$ is a countable ordinal and $B_{\geq \eta} \in \mathrm{CWT}(M)$ for all $\eta \in B$.
(8) If $Y \subseteq B \setminus \{\mathrm{rt}(B)\}$ is a front of B, and $\eta \in \mathrm{suc}_B(\mathrm{rt}(B))$, *then* $Y \cap B_{\geq \eta}$ is a front of $B_{\geq \eta}$.
(9) If Y is an almost front of B and an antichain Z is an almost front of $B_{\geq \eta}$ for every $\eta \in Y \cap B$, *then* Z is an almost front of B.
(10) If $B_1 \leq^*_M B_2$ and Y is a front of B_1, *then* there is $B_2' \in \mathrm{sb}(B_2)$ such that $Y \cap B_2'$ is a front of B_2' and $(B_1)_{\geq \eta} \leq^*_M (B_2')_{\geq \eta}$ for all $\eta \in Y \cap B_2'$.

Proof. Straightforward. □

Definition 2.9. Let $\mathbf{K}$ be the class of the objects $\mathbf{x} = \langle M_{\mathbf{x}}, <_{M_{\mathbf{x}}}, \bar{\mathscr{A}}_{\mathbf{x}}, \mathscr{A}_{\mathbf{x}}, \mathscr{B}_{\mathbf{x}}, \leq_{\mathbf{x}} \rangle$ satisfying the following properties (a)–(h).

(a) The structure $(M_{\mathbf{x}}, <_{M_{\mathbf{x}}}) = (M, <)$ is a partial order with the smallest element $\mathrm{rt}_{\mathbf{x}} = \mathrm{rt}(\mathbf{x})$. Let $M^-_{\mathbf{x}} = M_{\mathbf{x}} \setminus \{\mathrm{rt}_{\mathbf{x}}\}$,
(b) $\bar{\mathscr{A}}_{\mathbf{x}} = \bar{\mathscr{A}} = \langle \mathscr{A}_\eta : \eta \in M \rangle = \langle \mathscr{A}^{\mathbf{x}}_\eta : \eta \in M_{\mathbf{x}} \rangle$ and $\mathscr{A}_{\mathbf{x}} = \bigcup \{\mathscr{A}_\eta : \eta \in M^-_{\mathbf{x}}\}$,
(c) $\mathscr{A}_\eta \subseteq \mathrm{CWT}(M)$, let $\mathscr{A}^-_\eta = \mathscr{A}_\eta \setminus \{\{\eta\}\}$,
(d) $\mathrm{rt}(B) = \eta$ for every $B \in \mathscr{A}_\eta$,
(e) $\mathscr{A}_\eta$ is not empty, in fact $\{\eta\} \in \mathscr{A}_\eta$,
(f) $\mathscr{B}_{\mathbf{x}} = \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}_{\mathbf{x}}} \setminus \big\{\{\mathrm{rt}_{\mathbf{x}}\}\big\}$ and $\leq_{\mathbf{x}}$ is a directed partial order on $\mathscr{B}_{\mathbf{x}}$,
(g) $B_1 \leq_{\mathbf{x}} B_2$ implies $B_1 \leq^*_M B_2$, see Definition 2.6 and, of course, $B_1, B_2 \in \mathscr{B}_{\mathbf{x}}$,
(h) if $\nu \in B \in \mathscr{A}_\eta$ then $B \cap M_{\geq \nu} \in \mathscr{A}_\nu$.

When dealing with $M_{\mathbf{x}}, \bar{\mathscr{A}}_{\mathbf{x}}$ etc we may omit $\mathbf{x}$ when clear from the context.

Definition 2.10. Let $\mathbf{x} \in \mathbf{K}$ and $\eta \in M_{\mathbf{x}}$.

(1) Let $\mathrm{frt}(\eta) = \mathrm{frt}_{\mathbf{x}}(\eta) = \{Y : Y$ is a front of B for some $B \in \mathscr{A}^{\mathbf{x}}_\eta\}$ and instead of $\mathrm{frt}(B)$ (see Definition 2.3) we may write also $\mathrm{frt}_{\mathbf{x}}(B)$. We let

$$\mathrm{frt}^-(\eta) = \{Y \in \mathrm{frt}(\eta) : Y \neq \{\eta\}\}.$$

Omitting η means $\eta = \mathrm{rt}_{\mathbf{x}}$.
(2) Similarly, using Definition 2.5, we define $\text{alm-frt}_{\mathbf{x}}(\eta)$ (and $\text{alm-frt}_{\mathbf{x}}$).

(3) Let $B \in \mathscr{A}^{\mathbf{x}}_\eta$. We define

$$\mathrm{Fin}(B) = \big\{f : \ f \text{ is a function with domain } B \setminus \max(B) \text{ such that } f(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0} \text{ for all } \nu \in B \setminus \max(B)\big\},$$

and for $f \in \mathrm{Fin}(B)$ we set

$$A_f = A_{B,f} = \big\{\eta \in B : \big(\forall \rho \in B \setminus \max(B)\big)\big(\forall \varrho \in \mathrm{suc}_B(\rho)\big)\big(\varrho \leq_M \eta \ \Rightarrow \ \varrho \notin f(\rho)\big)\big\}.$$

(Recall Definition 2.4.)

(4) Assume that $Y \in \text{alm-frt}_{\mathbf{x}}$. We let $D_Y = D^{\mathbf{x}}_Y$ be the family

$$\big\{\, Z \subseteq Y : \text{for some } B \in \mathscr{B}_{\mathbf{x}} \text{ and } B' \in \mathrm{sb}(B) \text{ we have } Y \in \text{alm-frt}(B) \text{ and } B' \cap Y \subseteq Z \,\big\}.$$

(5) If $B \in \mathscr{B}_{\mathbf{x}}$, then $D_{\mathbf{x}}(B) = D^{\mathbf{x}}_{\max(B)}$.

(6) We let $\mathrm{Dp}_{\mathbf{x}}(\eta) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathscr{A}^{\mathbf{x}}_\eta\}$ (recall Definition 2.6).

If $\mathbf{x}$ is clear from the context, then we may omit the subscript/superscript $\mathbf{x}$ in the objects defined above.

Let us recall the definition of the Rudin–Keisler order on ultrafilters.

Definition 2.11. Let D_ℓ be an ultrafilter on $\mathscr{U}_\ell$ for $\ell = 1, 2$. We say $D_1 \leq_{\mathrm{RK}} D_2$ *iff* there is a function h whose domain and range are subsets of $\mathscr{U}_2$, $\mathscr{U}_1$, respectively, such that

$$\forall A \subseteq \mathscr{U}_1 : \quad A \in D_1 \ \Leftrightarrow \ \{a \in \mathrm{Dom}(h) : h(a) \in A\} \in D_2$$

Observation 2.12. Assume $\mathbf{x} \in \mathbf{K}$ and let $B, B_1, B_2 \in \mathscr{B}_{\mathbf{x}}$.

(1) The singleton $\{\mathrm{rt}_{\mathbf{x}}\}$ is in $\mathrm{frt}_{\mathbf{x}}$ and $D^{\mathbf{x}}_{\{\mathrm{rt}_{\mathbf{x}}\}} = \big\{\{\mathrm{rt}_{\mathbf{x}}\}\big\}$.

(2) If $B_1 \leq_{\mathbf{x}} B_2$, $f \in \mathrm{Fin}(B_1)$ and $Y \in \text{alm-frt}(B_1)$, *then* $Y \in \text{alm-frt}(B_2)$ and there is $g \in \mathrm{Fin}(B_2)$ such that $Y \cap A_{B_2,g} \subseteq Y \cap A_{B_1,f}$.

(3) If $Y \in \text{alm-frt}(B_\ell)$, $f_\ell \in \mathrm{Fin}(B_\ell)$ (for $\ell = 1, 2$), *then* there are $B^* \in \mathscr{B}_{\mathbf{x}}$ and $g \in \mathrm{Fin}(B^*)$ such that $B_1 \leq_{\mathbf{x}} B^*$, $B_2 \leq_{\mathbf{x}} B^*$ and

$$Y \cap A_{B^*,g} \subseteq Y \cap A_{B_1,f_1} \cap A_{B_2,f_2}.$$

(4) If $Y \in \text{alm-frt}_{\mathbf{x}}$, *then* $D^{\mathbf{x}}_Y$ is a filter on Y.

(5) If $B_1 \leq_{\mathbf{x}} B_2$, $Y_1 \in \text{alm-frt}(B_1)$, and $Y_2 = Y_1 \cap B_2$ (hence $Y_2 \in \text{alm-frt}(B_2)$), *then* $Y_2 \in D^{\mathbf{x}}_{Y_1}$ and $D^{\mathbf{x}}_{Y_2} = D^{\mathbf{x}}_{Y_1} \restriction Y_2$.

(6) Assume that $Y_1, Y_2 \in \mathrm{frt}(B)$ and Y_2 is above Y_1. Let $h : Y_2 \xrightarrow{\text{onto}} Y_1$ be the projection, i.e.,

$$h(\nu_2) = \nu_1 \quad \Leftrightarrow \quad \nu_1 \in Y_1 \ \wedge \ \nu_2 \in Y_2 \ \wedge \ \nu_1 \leq_{M_{\mathbf{x}}} \nu_2.$$

Then $h(D_{Y_2}) = D_{Y_1}$, i.e., $D_{Y_1} = \big\{A \subseteq Y_1 : h^{-1}[A] \in D_{Y_2}\big\}$ (so h witnesses $D_{Y_1} \leq_{\mathrm{RK}} D_{Y_2}$).

(7) If $B_1 \leq_{\mathbf{x}} B_2$ and $Y_\ell = \mathrm{suc}_{B_\ell}(\mathrm{rt}_{\mathbf{x}})$ for $\ell = 1, 2$, *then*:
  (a) Y_ℓ is a front of B_ℓ and Y_1 almost above Y_2, see Definition 2.3,
  (b) if Y is a front of B_ℓ and it is not $\{\mathrm{rt}_{\mathbf{x}}\}$, then Y is above Y_ℓ.

(8) The set $\max(B)$ is the maximal front of B which means that it is above any other.

(9) If $\mathbb{Q}$ is an ${}^\omega\omega$-bounding forcing and $B \in \mathscr{B}_{\mathbf{x}}$, *then* for any $B' \in \mathrm{sb}(B)^{\mathbf{V}[\mathbb{Q}]}$ there is $B'' \in (\mathrm{sb}(B))^{\mathbf{V}}$ such that $B'' \subseteq B'$.

(10) If F is a finite subset of $M^-_{\mathbf{x}}$, $B \in \mathscr{B}_{\mathbf{x}}$, *then* there is a branch (i.e., a maximal chain) $C \subseteq B$ such that

$$(\forall \rho \in F)(\forall \sigma \in C)(\rho \not\leq_M \sigma).$$

(11) If $B \in \mathscr{A}_\eta$ and $\nu \in B \setminus \max(B)$, *then* $\mathrm{id}_{\mathbf{x}}(\nu, B)$ is a proper ideal ideal on $\mathrm{suc}_B(\nu)$.

Proof. Straightforward. □

Definition 2.13. (1) For an (infinite) cardinal κ let $\mathbf{K}_{<\kappa}$ be the class of $\mathbf{x} \in \mathbf{K}$ such that $\|\mathbf{x}\| := |M_{\mathbf{x}}| + \sum\{|\mathscr{A}^{\mathbf{x}}_\eta| : \eta \in M_{\mathbf{x}}\} < \kappa$, similarly $\mathbf{K}_{\leq\kappa}$.

(2) The relation $\leq_{\mathbf{K}}$ is the following two-place relation on $\mathbf{K}$ (it is a partial order, see Observation 2.14 below): $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ if and only if
   (a) $M_{\mathbf{x}} \subseteq M_{\mathbf{y}}$ (as partial orders) and, moreover, for any $\eta, \nu \in M_{\mathbf{x}}$ we have

$$\nu \parallel_{M_{\mathbf{x}}} \eta \quad \text{if and only if} \quad \nu \parallel_{M_{\mathbf{y}}} \eta,$$

   and
   (b) $\eta \in M_{\mathbf{x}} \quad \Rightarrow \quad \mathscr{A}^{\mathbf{x}}_\eta \subseteq \mathscr{A}^{\mathbf{y}}_\eta$, and
   (c) $\mathrm{rt}_{\mathbf{y}} = \mathrm{rt}_{\mathbf{x}}$ (actually follows from (2d)), and
   (d) $\leq_{\mathbf{x}} = \leq_{\mathbf{y}} \restriction \mathscr{B}_{\mathbf{x}}$.

(3) If $\langle \mathbf{x}_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbf{K}}$-increasing sequence we define $\mathbf{x}_\delta = \bigcup\{\mathbf{x}_\alpha : \alpha < \delta\}$, the union of the sequence, by $M_{\mathbf{x}_\delta} = \bigcup\{M_{\mathbf{x}_\alpha} : \alpha < \delta\}$ as partial orders and $\mathscr{A}^{\mathbf{x}_\delta}_\eta = \bigcup\{\mathscr{A}^{\mathbf{x}_\alpha}_\eta : \alpha < \delta$ satisfies $\eta \in M_{\mathbf{x}_\alpha}\}$ and $\leq_{\mathbf{x}_\delta} = \bigcup\{\leq_{\mathbf{x}_\alpha} : \alpha < \delta\}$.

Observation 2.14. (1) It is easy to see that the relation $\leq_{\mathbf{K}}$ is really a partial order.

(2) Moreover, this order is closed under chains, i.e.:
Whenever $\langle \mathbf{x}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathbf{K}}$–increasing, we can define $\mathbf{x}_\delta$ as the union of the sequence. It is then clear that $\mathbf{x}_\delta$ is a $\leq_{\mathbf{K}}$–lub of the sequence and $\|\mathbf{x}_\delta\| \leq \sum \|\{\|\mathbf{x}_\alpha\| : \alpha < \delta\}$.

Definition 2.15. Let $\mathbf{x} \in \mathbf{K}$. We say that $\mathbf{x}$ is:

fat iff: if $B \in \mathscr{B}_{\mathbf{x}}$ and $B' \in \mathrm{sb}(B)$, then there is $B'' \in \mathrm{sb}(B')$ such that $B'' \in \mathscr{B}_{\mathbf{x}}$ and $B \leq_{\mathbf{x}} B''$;

big iff: if $B \in \mathscr{B}_{\mathbf{x}}$ and $\mathbf{c} : \max(B) \longrightarrow \{0,1\}$, then for some $B' \in \mathscr{B}_{\mathbf{x}}$ we have that

$$B' \in \mathrm{psb}_{M_{\mathbf{x}}}(B) \cap \mathscr{B}_{\mathbf{x}}, \quad B \leq_{\mathbf{x}} B', \quad \text{and} \quad \mathbf{c}\restriction \max(B') \text{ is constant,}$$

large iff: whenever $B \in \mathscr{B}_{\mathbf{x}}$ and $\mathbf{c}$ is a function with domain $\max(B)$, then for some $B' \in \mathrm{psb}_{M_{\mathbf{x}}}(B) \cap \mathscr{B}_{\mathbf{x}}$ and a front Y of B' we have $B \leq_{\mathbf{x}} B'$ and

$$\big(\forall \eta, \nu \in \max(B')\big)\big(\mathbf{c}(\eta) = \mathbf{c}(\nu) \;\Leftrightarrow\; (\exists \rho \in Y)(\rho \leq_{M_{\mathbf{x}}} \eta \wedge \rho \leq_{M_{\mathbf{x}}} \nu)\big),$$

full iff: whenever $B \in \mathscr{A}^{\mathbf{x}}_\eta$, $\eta \neq \mathrm{rt}_{\mathbf{x}}$ and $B' \in \mathrm{psb}_{M_{\mathbf{x}}}(B)$, then $B' \in \mathscr{A}^{\mathbf{x}}_\eta$.

3. Construction of ultra-systems

Lemma 3.1. *The set* $\mathbf{K}_{\leq\aleph_0}$ *is non-empty.*

Proof. Define $\mathbf{x}$ so that $M_{\mathbf{x}} = \{\eta_*\}$, $\mathscr{A}^{\mathbf{x}}_{\eta_*} = \{\{\eta_*\}\}$, $\mathrm{rt}_{\mathbf{x}} = \eta_*$. Now it is easy to check. □

Lemma 3.2. *If* $\mathbf{x} \in \mathbf{K}$ *and* $\eta \in M_{\mathbf{x}}$ *satisfies* $|\mathscr{A}^{\mathbf{x}}_\eta| = 1$, *i.e.,* $\mathscr{A}^{\mathbf{x}}_\eta = \big\{\{\eta\}\big\}$, then *for some* $\mathbf{y} \in \mathbf{K}$ *we have* $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$, $|\mathscr{A}^{\mathbf{y}}_\eta| > 1$ *and* $\|\mathbf{y}\| \leq \|\mathbf{x}\| + \aleph_0$.

Proof. Let $\langle \eta_n : n < \omega \rangle$ be pairwise distinct objects not belonging to $M_{\mathbf{x}}$. We define $\mathbf{y}$ by:

(a) $M_{\mathbf{y}}$ has set of elements $M_{\mathbf{x}} \cup \{\eta_n : n < \omega\}$,
(b) $\nu <_{M_{\mathbf{y}}} \rho$ if and only if $\nu <_{M_{\mathbf{x}}} \rho$ or $\nu \leq_{M_{\mathbf{x}}} \eta \ \wedge\ (\exists n)(\rho = \eta_n)$,
(c) $\mathscr{A}^{\mathbf{y}}_\nu$ is defined by a case distinction:
  – If $\nu \in M_{\mathbf{x}} \setminus \{\eta\}$, then $\mathscr{A}^{\mathbf{y}}_\nu := \mathscr{A}^{\mathbf{x}}_\nu$.
  – If $\nu = \eta$, then $\mathscr{A}^{\mathbf{y}}_\nu := \{\{\eta\}, \{\eta_n : n < \omega\} \cup \{\eta\}\}$.
  – If $\nu = \eta_n$, then $\mathscr{A}^{\mathbf{y}}_\nu := \{\{\eta_n\}\}$.
(d) the order $\leq_{\mathbf{y}}$ is $\leq_{\mathbf{x}}$ if $\eta \neq \mathrm{rt}_{\mathbf{x}}$, and it is determined by: $\{\eta\} \leq_{\mathbf{y}} \{\eta_n : n < \omega\} \cup \{\eta\}$ if $\eta = \mathrm{rt}_{\mathbf{x}}$.

Now check. □

Lemma 3.3. *(1) If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$* then *for some $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and in $\mathscr{B}_{\mathbf{y}}$ there is a $\leq_{\mathbf{y}}$–maximal member.*
(2) If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$ and some $B \in \mathscr{B}_{\mathbf{x}}$ is $\leq_{\mathbf{x}}$–maximal then *for some $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ and $B' \in \mathscr{B}_{\mathbf{y}}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and $B <_{\mathbf{y}} B'$.*
(3) If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$, $\eta \in M_{\mathbf{x}}$, $B_1 \in \mathscr{A}^{\mathbf{x}}_\eta$, $B_2 \in \mathrm{psb}_{M_{\mathbf{x}}}(B_1)$ and

$$\eta = \mathrm{rt}_{\mathbf{x}} \ \Rightarrow\ B_1 \textit{ is } \leq_{\mathbf{x}}\textit{–maximal},$$

then *there is $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ such that $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and $B_2 \in \mathscr{A}^{\mathbf{y}}_\eta$.*
(4) If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$, $B_1 \in \mathscr{B}_{\mathbf{x}}$ and $B_2 \in \mathrm{sb}(B_1)$, then *there is $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ such that $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and $B_2 \in \mathscr{B}_{\mathbf{y}}$.*

Proof of (1). If in $(\mathscr{B}_{\mathbf{x}}, \leq_{\mathbf{x}})$ there is a maximal member then we let $\mathbf{y} = \mathbf{x}$. Otherwise, as it is directed (see clause (f) of Definition 2.9) and $\|\mathbf{x}\| \leq \aleph_0$ (because $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$), there is a strictly $\leq_{\mathbf{x}}$–increasing cofinal sequence $\langle B_n : n < \omega \rangle$. Let $Y_n = \mathrm{suc}_{B_n}(\mathrm{rt}_{\mathbf{x}})$.

Note that for each $m_1 < m_2$, the set $Y_{m_1} \cap B_{m_2}$ is an almost front of B_{m_2} (so also it is almost above Y_{m_2}). Hence for $m_1 < m_2 \leq n$ we have that $Y_{m_1} \cap B_n$ is an almost front of B_n which is almost above $Y_{m_2} \cap B_n$. Consequently we may choose $B^*_n \in \mathrm{sb}(B_n)$ such that each $Y_\ell \cap B^*_n$ is a front of B^*_n and $Y_\ell \cap B^*_n$ is above $Y_{\ell+1} \cap B^*_n$ (for all $\ell < n$). Moreover, we may also require that

(3.1) $$\text{for each } \ell < n \text{ and } \eta \in Y_\ell \cap B^*_n \text{ we have } (B_\ell)_{\geq\eta} \leq^*_{M_{\mathbf{x}}} (B^*_n)_{\geq\eta}$$

(remember Observation 2.8(10)).

Fix a list $\langle \rho_\ell : \ell < \omega \rangle$ of all members of $M_{\mathbf{x}}$ (possibly with repetitions). By induction on $n < \omega$ choose ν_n such that

(3.2) $$\nu_n \in Y_n \cap B^*_n = \mathrm{suc}_{B^*_n}(\mathrm{rt}_{\mathbf{x}})$$

(3.3) $$\text{if } \ell < n, \text{ then } \nu_n, \nu_\ell \text{ are } <_{M_{\mathbf{x}}}\text{–incompatible (i.e., } \nu_\ell \parallel_{M_{\mathbf{x}}} \nu_n),$$

(3.4) $$\text{if } \ell < n \text{ and } \rho_\ell \neq \mathrm{rt}_{\mathbf{x}}, \text{ then } \rho_\ell \parallel_{M_{\mathbf{x}}} \nu_n.$$

[Why is the choice possible? By the demand (f) of Definition 2.2 applied to $\nu = \mathrm{rt}_{\mathbf{x}}$ and $F = \{\nu_\ell, \rho_\ell : \ell < n\} \setminus \{\mathrm{rt}_{\mathbf{x}}\}$.]

We define

$$B^* = \{\mathrm{rt}_{\mathbf{x}}\} \cup \bigcup \{B^*_n \cap (M_{\mathbf{x}})_{\geq\nu_n} : n < \omega\}.$$

This set B^* is clearly a countable well-founded tree, $B^* \in \mathrm{CWT}(M_{\mathbf{x}})$ with root $\mathrm{rt}_{\mathbf{x}}$ and $\mathrm{suc}_{B^*}(\mathrm{rt}_{\mathbf{x}}) = \{\nu_n : n < \omega\}$.

[Why? It should be clear that conditions (a)–(d) of Definition 2.2 hold, $\mathrm{rt}(B^*) = \mathrm{rt}_{\mathbf{x}}$ and $\mathrm{suc}_{B^*}(\mathrm{rt}_{\mathbf{x}}) = \{\nu_n : n < \omega\}$. To verify clause (e) suppose $\eta, \nu \in B^*$ are $<_{M_{\mathbf{x}}}$–incomparable. Then both $\eta \neq \mathrm{rt}_{\mathbf{x}}$ and $\nu \neq \mathrm{rt}_{\mathbf{x}}$, so $\eta, \nu \in \bigcup\limits_{n<\omega} (B^*)_{\nu_n}$. If, for some n, we have $\eta, \nu \in B^*_n \cap (M_{\mathbf{x}})_{\geq \nu_n}$, then they are $<_{M_{\mathbf{x}}}$–incompatible as $B^*_n \subseteq B_n$ and B_n satisfies 2.2(e). Otherwise, for some distinct ℓ, n we have $\eta \in B^*_\ell \cap (M_{\mathbf{x}})_{\geq \nu_\ell}$ and $\nu \in B^*_n \cap (M_{\mathbf{x}})_{\geq \nu_n}$. Now, if we could find $\rho \in M_{\mathbf{x}}$ such that $\rho \geq_{M_{\mathbf{x}}} \eta$ and $\rho \geq_{M_{\mathbf{x}}} \nu$, then ν_ℓ, ν_n would be compatible contradicting (3.3), so B^* indeed satisfies clause (e) of Definition 2.2. Finally, to verify (f) suppose $\nu \in B^* \setminus \max(B^*)$ and $F \subseteq M_{\mathbf{x}} \setminus (M_{\mathbf{x}})_{\leq \nu}$ is finite. If $\nu_n \leq_{M_{\mathbf{x}}} \nu$ for some n, then the properties of B^*_n apply. So suppose $\nu = \mathrm{rt}_{\mathbf{x}}$. Choose m so that $F \subseteq \{\rho_\ell : \ell < m\}$ and use condition (3.4) to argue that for all $n \geq m$ and $\rho \in F$ we have $\nu_n \parallel_{M_{\mathbf{x}}} \rho$.]

Also:

$$B \leq^*_{M_{\mathbf{x}}} B^* \text{ for all } B \in \mathscr{B}_{\mathbf{x}}.$$

[Why? Since $\leq^*_{M_{\mathbf{x}}}$ is a partial order and by the choice of B_n, it is enough to show that for each $n < \omega$ we have $B_n \leq^*_{M_{\mathbf{x}}} B^*$, i.e., that every almost front of B_n is an almost front of B^*. To this end suppose that $Z \subseteq B_n$ is an almost front of B_n for some $n < \omega$. If $Z = \{\mathrm{rt}_{\mathbf{x}}\}$, then there is nothing to do, so suppose $Z \subseteq B_n \setminus \{\mathrm{rt}_{\mathbf{x}}\}$, i.e., $Z \subseteq \bigcup\{(B_n)_{\geq \rho} : \rho \in Y_n\}$. Plainly, the set

$$X = \{\rho \in Y_n : Z \text{ is not an almost front of } (B_n)_{\geq \rho}\}$$

is finite and hence for some $m > n$ we have $X \subseteq \{\rho_\ell : \ell < m\}$. Then for every $k > m$ we have:

(a) The element ν_k is incompatible with every $\nu \in X$,
(b) The set $Y_n \cap (B^*_k)_{\geq \nu_k}$ is a front of $(B^*_k)_{\geq \nu_k}$,
(c) $(B_n)_{\geq \eta} \leq^*_{M_{\mathbf{x}}} (B^*_k)_{\geq \eta}$ for every $\eta \in Y_n \cap (B^*_k)_{\geq \nu_k}$ (by (3.1)),
(d) The set $Z \cap (B_n)_{\geq \eta}$ is an almost front of $(B_n)_{\geq \eta}$ for every $\eta \in Y_n \cap (B^*_k)_{\geq \nu_k}$, and thus
(e) The set $Z \cap (B^*_k)_{\geq \eta}$ is an almost front of $(B^*_k)_{\geq \eta}$ for every $\eta \in Y_n \cap (B^*_k)_{\geq \nu_k}$.
(f) Finally, Z is an almost front of $(B^*_k)_{\geq \nu_k}$ (by Observation 2.8(9) and (b)+(e)).

Since $\mathrm{suc}_{B^*}(\mathrm{rt}_{\mathbf{x}}) = \{\nu_k : k < \omega\}$, we know that $\{\nu_k : m < k < \omega\}$ is an almost front of B^*. Therefore, by Observation 2.8(9) and (f), we conclude that Z is an almost front of B^*.]

Lastly, we define $\mathbf{y}$:

- $(M_{\mathbf{y}}, <_{M_{\mathbf{y}}}) = (M_{\mathbf{x}}, <_{M_{\mathbf{x}}})$,
- $\mathscr{A}^{\mathbf{y}}_\nu = \mathscr{A}^{\mathbf{x}}_\nu$ iff: $\nu \in M_{\mathbf{x}} \setminus \{\mathrm{rt}_{\mathbf{x}}\}$, and $\mathscr{A}^{\mathbf{y}}_{\mathrm{rt}_{\mathbf{x}}} = \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}_{\mathbf{x}}} \cup \{B^*\}$,
- $B_1 \leq_{\mathbf{y}} B_2$ if and only if $B_1 \leq_{\mathbf{x}} B_2$ or $B_1 \in A^{\mathbf{y}}_{\mathrm{rt}_{\mathbf{x}}} \wedge B_2 = B^*$.

It should be clear that $\mathbf{y} \in \mathbf{K}_{\leq \aleph_0}$ is as required. □

Proof of Lemma 3.3(2),(3),(4). Straightforward; see also Lemmas 3.4, 3.5 below. □

Lemma 3.4. *Assume that $\mathbf{x} \in \mathbf{K}_{\leq \aleph_0}$ and $B \in \mathscr{B}_{\mathbf{x}}$ is $\leq_{\mathbf{x}}$–maximal. Then for some $\mathbf{y} \in \mathbf{K}_{\leq \aleph_0}$ and $B' \in \mathscr{B}_{\mathbf{y}}$ we have*

(a) $\mathbf{x} \leq \mathbf{y}$, $M_{\mathbf{x}} = M_{\mathbf{y}} = M$, and
(b) $B' \in \mathscr{B}_{\mathbf{y}}$ is $\leq_{\mathbf{y}}$–maximal,
(c) if $\nu \in B' \setminus \max(B')$ and $\rho \in M \setminus M_{\leq \nu}$, then for all but finitely many $\varrho \in \mathrm{suc}_{B'}(\nu)$ we have $\rho \parallel_M \varrho$.

Proof. Fix a list $\langle \rho_\ell : \ell < \omega \rangle$ of all members of $M_{\mathbf{x}}$ (possibly with repetitions). For each $\eta \in B \setminus \max(B)$ by induction on $n < \omega$ we choose $\nu_{\eta,n}$ such that

- $\nu_{\eta,n} \in \mathrm{suc}_B(\eta)$,
- $\nu_{\eta,n} \neq \nu_{\eta,k}$ for $k < n$ (and hence $\nu_{\eta,n} \parallel \nu_{\eta,k}$ for $k < n$),
- if $k < n$ and $\rho_k \notin M_{\leq \eta}$, then $\rho_k \parallel \nu_{\eta,n}$.

Next, by downward induction on $\eta \in B$ we define

$$B_\eta = \bigcup \{B_{\nu_{\eta,n}} : n < \omega\} \cup \{\eta\}.$$

Lastly we define $\mathbf{y}$ so that:
$(M_{\mathbf{y}}, <_{\mathbf{y}}) = (M_{\mathbf{x}}, <_{\mathbf{x}})$,
$\mathscr{A}^{\mathbf{y}}_\eta = \mathscr{A}^{\mathbf{x}}_\eta$ if $\eta \in M_{\mathbf{x}}$ but $\eta \notin B \setminus \max(B)$, and
$\mathscr{A}^{\mathbf{y}}_\eta = \mathscr{A}^{\mathbf{x}}_\eta \cup \{B_\eta\}$ if $\eta \in B \setminus \max(B)$,
$\mathscr{B}_{\mathbf{y}} = \mathscr{B}_{\mathbf{x}} \cup \{B_{\mathrm{rt}_{\mathbf{x}}}\}$ and for $B', B'' \in \mathscr{B}_{\mathbf{y}}$ we let:

$B' \leq_{\mathbf{y}} B''$ if and only if $B' \leq_{\mathbf{x}} B''$ or $B'' = B_{\mathrm{rt}_{\mathbf{x}}}$. □

Lemma 3.5. *(1) If $\mathbf{x} \in \mathbf{K}_{\leq \aleph_0}$, $Y \in$ alm-frt$_{\mathbf{x}}$ and $Z \subseteq Y$ then for some $\mathbf{y} \in \mathbf{K}_{\leq \aleph_0}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and either $Z \in D^{\mathbf{y}}_Y$ or $(Y \setminus Z) \in D^{\mathbf{y}}_Y$.*

(2) Moreover, if h is a function with domain Y, then above we can demand that for some $B \in \mathscr{B}_{\mathbf{y}}$, $Y \cap B$ is a front of B and for some front Y' of B which is below Y and a one-to-one function h' with domain Y' we have

$$\rho \in Y' \wedge \varrho \in Y \cap B \wedge \rho \leq_{M_{\mathbf{y}}} \varrho \quad \Rightarrow \quad h(\rho) = h'(\varrho).$$

(Note that possibly $Y' = \{\mathrm{rt}_{\mathbf{y}}\}$ and then $h \restriction (Y \cap B)$ is constant.)

Proof of (1). By Lemma 3.3(1) without loss of generality there is $B \in \mathscr{B}_{\mathbf{x}}$ such that B is $\leq_{\mathbf{x}}$-maximal in $\mathscr{B}_{\mathbf{x}}$; clearly $Y \cap B$ is an almost front of B and so without loss of generality $Y \subseteq B$.

We know that $B[\leq Y] := \{\rho \in B : (\exists \nu)[\rho \leq_{M_{\mathbf{x}}} \nu \in Y]\}$ has no ω–branch, so by $<_{M_{\mathbf{x}}}$–downward induction on $\nu \in B[\leq Y]$ we choose $(\mathbf{t}_\nu, Y_\nu)$ such that (where $M = M_{\mathbf{x}}$, of course):

(a) $\mathbf{t}_\nu \in \{0, 1\}$ and:
if $\mathbf{t}_\nu = 1$, then $Y_\nu \subseteq M_{\geq \nu} \cap Z$,
if $\mathbf{t}_\nu = 0$, then $Y_\nu \subseteq M_{\geq \nu} \cap (Y \setminus Z)$,
(b) $Y_\nu = \max(B'_\nu)$ for some $B'_\nu \in \mathrm{psb}_M(B_{\geq \nu})$,
(c) if $\nu \in Y$ then $Y_\nu = \{\nu\}$ and $\mathbf{t}_\nu =$ (the truth value of $\nu \in Z$),
(d) if $\nu \in B[\leq Y] \setminus Y$ then: for every finite set $F \subseteq M \setminus M_{\leq \nu}$ there are infinitely many $\varrho \in \mathrm{suc}_B(\nu)$ such that $(\forall \rho \in F)(\rho \parallel \varrho)$ and $\mathbf{t}_\varrho = \mathbf{t}_\nu$, $Y_\nu = \bigcup \{Y_\varrho : \varrho \in \mathrm{suc}_B(\nu)$ and $\mathbf{t}_\varrho = \mathbf{t}_\nu\}$.

This is easily done and so $\mathbf{t}_{\mathrm{rt}_{\mathbf{x}}}$ is well defined. For $\nu \in B[\leq Y]$ we let

$$B^*_\nu = \{\rho \in B_{\geq \nu} : \text{ for some } \varrho \in Y_\nu \text{ we have } \varrho \leq_M \rho \vee \rho \leq_M \varrho\}.$$

Now define $\mathbf{y}$ by adding B^*_ν to $\mathscr{A}^{\mathbf{x}}_\nu$ for every $\nu \in B[\leq Y]$, and check. □

Proof of (2). First note that by Lemmas 3.3(1) and 3.4 we may assume that there is $B \in \mathscr{B}_{\mathbf{x}}$ such that B is $\leq_{\mathbf{x}}$–maximal, the set Y is a front of B, and:

if $\nu \in B \setminus \max(B)$ and $\rho \in M \setminus M_{\leq \nu}$,
then for all but finitely many $\varrho \in \mathrm{suc}_B(\nu)$ we have $\rho \parallel_M \varrho$.

Now note: if $h' : Y' \longrightarrow A$, $Y' \in \mathrm{frt}(B')$, $Z = \{\eta \in B' : \mathrm{suc}_{B'}(\eta) \subseteq Y'\}$ is a front of B' and $h' \restriction \mathrm{suc}_{B'}(\eta)$ is one-to-one for all $\eta \in Z$, *then* we can find $B'' \in \mathrm{psb}_M(B)$ such that $h' \restriction B'' \cap Y'$ is one-to-one. So we may follow similarly as in (1). $\square$

Let us recall the following definition.

Definition 3.6 (P-points and Q-points)**.** Let D be a nonprincipal ultrafilter on a countable set $\mathrm{Dom}(D)$.

We say D is a *Q*-point if: whenever f is a finite-to-one function with domain $\mathrm{Dom}(D)$, then $f \restriction A$ is one-to-one for some $A \in D$.

We say that D is a P-point if: for each sequence $\langle A_n : n < \omega\rangle$ of sets from D there is an $A \in D$ such that $A \setminus A_n$ is finite for each $n < \omega$.

We can conclude the main result of this section.

Theorem 3.7. *Assume CH. There is a* $\mathbf{x} \in \mathbf{K}$ *such that:*

(a) (α) $\mathscr{A}^{\mathbf{x}}_\eta \neq \{\{\eta\}\}$ *for* $\eta \in M_{\mathbf{x}}$,
  (β) $\mathscr{B}_{\mathbf{x}} = \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})} \setminus \{\{\mathrm{rt}_{\mathbf{x}}\}\}$ *is* $\aleph_1$*–directed under* $\leq_{\mathbf{x}}$,

(b) if $Y \in \mathrm{frt}^-_{\mathbf{x}}$, *then*
  (α) $D^{\mathbf{x}}_Y$ *is a non-principal ultrafilter on* Y, *and*
  (β) $D^{\mathbf{x}}_Y$ *is a Q-point, see Definition 3.6,*

(c) if $B_1 \in \mathscr{B}_{\mathbf{x}}$, then *for some* $B_2 \in \mathscr{B}_{\mathbf{x}}$ *we have* $B_1 \leq_{\mathbf{x}} B_2$ *and* $B_1 \cap \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}) = \emptyset$, *moreover*[1]

$$(\forall \varrho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}))(\exists^\infty \rho \in \mathrm{suc}_{B_1}(\mathrm{rt}_{\mathbf{x}}))[\varrho \leq_{M_{\mathbf{x}}} \rho].$$

(d) $\mathbf{x}$ *is (see Definition 2.15): fat, big, large, and full.*

Proof. We choose $\mathbf{x}_\alpha \in \mathbf{K}_{\leq\aleph_0}$ by induction on $\alpha < \aleph_1$ so that

(i) if $\beta < \alpha < \aleph_1$, then $\mathbf{x}_\beta \leq_{\mathbf{K}} \mathbf{x}_\alpha$,
(ii) for each successor α, there is a $\leq_{\mathbf{x}_\alpha}$–maximal element in $\mathscr{B}_{\mathbf{x}_\alpha}$.

We use a bookkeeping device to ensure largeness and bigness and

- for $\alpha = 0$ we use Lemma 3.1,
- for α limit we use Definition 2.13(3) and Observation 2.14(2),
- if $\alpha = \beta + 1$, β is limit, then we use Lemma 3.5(1) (and the instructions from our bookkeeping device) to take care of the bigness,
- if $\alpha = \beta + 2$, β is limit, then we use Lemma 3.5(2) (and the instructions from our bookkeeping device) to take care of the largeness,
- if $\alpha = \beta + 3$, β is limit, then we use Lemma 3.3(3,4) (and the instructions from our bookkeeping device) to ensure that at the end $\mathbf{x}$ is fat and full,
- if $\alpha = \beta + k$, β is limit, $4 \leq k < \omega$, then we ensure clause (d).

In the end we let $\mathbf{x} = \bigcup\limits_{\alpha<\aleph_1} \mathbf{x}_\alpha$. Then $\mathbf{x}$ is fat, big, large and $\mathscr{B}_{\mathbf{x}}$ is $\aleph_1$–directed. Note that clause (b)(β) follows from the largeness. $\square$

Definition 3.8. (1) We say that $\mathbf{x} \in \mathbf{K}$ is *nice* if it satisfies conditions (a)–(d) of Theorem 3.7. The class of all nice $\mathbf{x}$ is denoted by $\mathbf{K}_{\mathrm{n}}$.
(2) An $\mathbf{x} \in \mathbf{K}$ is *reasonable* if it satisfies (a), (c) of Theorem 3.7. Let $\mathbf{K}_{\mathrm{r}}$ be the set of all $\mathbf{x} \in \mathbf{K}$ which are reasonable.

[1] Not a serious addition. As always, the number of $\varrho \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}})$ failing this is finite.

(3) Let $\mathbf{K}_{\rm u}$ be the set of $\mathbf{x} \in \mathbf{K}_{\rm r}$ for which clause (b)(α) of Theorem 3.7 holds.
(4) For $\mathbf{x} \in \mathbf{K}$ we say that $\mathscr{I} \subseteq \mathscr{A}_{\mathbf{x}}$ (see Definition 2.9(b)) is $\mathbf{x}$–*dense* iff: for every $B_1 \in \mathscr{B}_{\mathbf{x}}$ there is B_2 such that
  (α) $B_1 \leq_{\mathbf{x}} B_2 \in \mathscr{B}_{\mathbf{x}}$, and
  (β) if $A \subseteq M_{\mathbf{x}} \setminus \{{\rm rt}_{\mathbf{x}}\}$ is finite, then for some ν we have

$$\nu \in {\rm suc}_{B_2}({\rm rt}_{\mathbf{x}}), \quad (B_2)_{\geq \nu} \in \mathscr{I}, \quad \text{and} \quad (\forall \rho \in A)(\rho \parallel \nu).$$

(5) For $\mathbf{x} \in \mathbf{K}$ we say $\mathscr{I}$ is $\mathbf{x}$–*open* if $\mathscr{I} \subseteq \mathscr{A}_{\mathbf{x}}$ and if $B_1 \in \mathscr{I}$ then ${\rm sb}(B_1) \cap \mathscr{A}_{\mathbf{x}} \subseteq \mathscr{I}$.
(6) Let $\mathbf{K}_{\rm g}$ be the class of $\mathbf{x} \in \mathbf{K}_{\rm r}$ which are *good*, which means: if $\mathscr{I}$ is $\mathbf{x}$–dense, $\mathbf{x}$–open and $B_1 \in \mathscr{B}_{\mathbf{x}}$ *then* for some $B_2 \in \mathscr{B}_{\mathbf{x}}$ we have $B_1 \leq_{\mathbf{x}} B_2$ and $(B_2)_{\geq \eta} \in \mathscr{I}$ for all but finitely many $\eta \in {\rm suc}_{B_2}({\rm rt}_{\mathbf{x}})$.
(7) We say that $\mathbf{x} \in \mathbf{K}$ is *ultra* if it is both nice and good. Let $\mathbf{K}_{\rm ut}$ be the class of $\mathbf{x}$ which are ultra, i.e., $\mathbf{K}_{\rm ut} = \mathbf{K}_{\rm g} \cap \mathbf{K}_{\rm n}$.

Theorem 3.9. *Assume $\diamondsuit_{\aleph_1}$. Then there exists an ultra $\mathbf{x} \in \mathbf{K}$.*

Proof. We repeat the proof of Theorem 3.7 but at limit stages $\delta < \aleph_1$ we use additionally $\diamondsuit_{\aleph_1}$ to take care of the additional demand $\mathbf{x} \in \mathbf{K}_{\rm g}$ here.

So we are given: a limit ordinal $\delta < \aleph_1$ and a set $\mathscr{J} \subseteq \mathscr{A}_{\mathbf{x}_\delta}$ such that for some $\mathbf{y} \in \mathbf{K}$ with $\mathbf{x}_\delta \leq \mathbf{y}$ and some $\mathscr{I} \subseteq \mathscr{A}_{\mathbf{y}}$ we have

> The set $\mathscr{I}$ is dense open in $\mathscr{A}_{\mathbf{y}}$, satisfies $\mathscr{J} = \mathscr{I} \cap \mathscr{A}_{\mathbf{x}_\delta}$, and moreover: There is a countable elementary submodel $N \prec \mathscr{H}(\aleph_2)$ with $(\mathbf{y}, \mathscr{I}) \in N$ and $(\mathbf{x}_\delta, \mathscr{J}) = (\mathbf{y}{\restriction}N, \mathscr{I} \cap N)$, so $M_{\mathbf{x}_\delta} = M_{\mathbf{y}}{\restriction}N$, etc.

Let $\langle B^0_\ell : \ell < \omega \rangle$ be an increasing cofinal subset of $(\mathscr{B}_{\mathbf{x}_\delta}, \leq_{\mathbf{x}_\delta})$. For every ℓ there is $B^1_\ell \in \mathscr{B}_{\mathbf{x}_\delta}$ such that $B^0_\ell \leq_{\mathbf{x}_\delta} B^1_\ell$, and for every finite $A \subseteq M_{\mathbf{x}_\delta} \setminus \{{\rm rt}(\mathbf{x}_\delta)\}$ there is $\nu \in {\rm suc}_{B^1_\ell}({\rm rt}(\mathbf{x}_\delta))$ such that

$$(\forall \rho \in A)(\rho \parallel \nu) \quad \text{and} \quad (B^1_\ell)_{\geq \nu} \in \mathscr{I}.$$

Clearly, for every ℓ for some $k(\ell) > \ell$ we have $B^1_\ell \leq_{\mathbf{x}_\delta} B^0_{k(\ell)}$. We can choose $\langle \ell_n : n < \omega \rangle$ so that $k(\ell_n) < \ell_{n+1}$. Let $B_n = B^1_{\ell_n}$. We continue as in Lemma 3.3(1) using the $\langle B_n : n < \omega \rangle$ and, when choosing ν_n, demanding additionally that $(B_n)_{\geq \nu_n} \in \mathscr{I}$. (Note that $(B_n)_{\geq \nu_n} \in \mathscr{I}$ implies $(B^*_n)_{\geq \nu_n} \in \mathscr{I}$ for B^*_n as there.) $\square$

Proposition 3.10. *Assume $\mathbf{x} \in \mathbf{K}_{\rm n}$.*

(i) If $B \in \mathscr{B}_{\mathbf{x}}$ and $Y_1, Y_2 \in {\rm frt}(B)$ and Y_2 is above Y_1, then $h^{\mathbf{x}}_{Y_2,Y_1}$ exemplifies $D^{\mathbf{x}}_{Y_1} \leq_{\rm RK} D^{\mathbf{x}}_{Y_2}$.
(ii) The family $\{D^{\mathbf{x}}_Y : Y \in {\rm frt}^-_{\mathbf{x}}\}$ is $\geq_{\rm RK}$–directed (even $\aleph_1$ directed).
(iii) If $Y \in \text{alm-frt}^-_{\mathbf{x}}$, then $\leq_{\rm RK}$–below $D^{\mathbf{x}}_Y$ there is no P-point.

Proof. (i) Follows from Observation 2.12(6).
(ii) By (i) and the directedness of $\mathscr{B}_{\mathbf{x}}$.
(iii) Let $B_1 \in \mathscr{B}_{\mathbf{x}}$ be such that $B_1 \cap Y$ is an almost front of B_1. Suppose that $h : Y \longrightarrow \mathbb{N}$ is such that $h^{-1}[\{n\}] = \emptyset \mod D^{\mathbf{x}}_Y$ for every n, hence there is $A_n \in \mathscr{B}_{\mathbf{x}}$ which witnesses this. Assume towards contradiction that $h(D^{\mathbf{x}}_Y)$ is a P-point; without loss of generality h is onto $\mathbb{N}$. As $\mathscr{B}_{\mathbf{x}}$ is $\aleph_1$–directed we may pick $B_2 \in \mathscr{B}_{\mathbf{x}}$ such that $A_n \leq_{\mathbf{x}} B_2$ (for all $n < \omega$) and $B_1 \leq_{\mathbf{x}} B_2$.

As $\mathbf{x}$ is large, we may apply the Definition 2.15 of large to the pair (B_2, h') where $h'(\eta) = h(\nu)$ when $\nu \leq_{M_{\mathbf{x}}} \eta \in \max(B)$ and zero if there is no such ν. So there are B_3, Y_3 such that

- $B_2 \leq_{\mathbf{x}} B_3$,
- Y_3 is a front of B_3 below $Y \cap B_3$,
- for $\eta, \nu \in Y \cap B_3$ we have: $h(\eta) = h(\nu) \Leftrightarrow (\exists \rho \in Y_3)(\rho \leq_{M_{\mathbf{x}}} \eta \wedge \rho \leq_{M_{\mathbf{x}}} \nu)$.

Let $Z = \mathrm{suc}_{B_3}(\mathrm{rt}_{\mathbf{x}})$. If $Y_3 = \{\mathrm{rt}_{\mathbf{x}}\}$, then for some n we have $h^{-1}[\{n\}] \in D^{\mathbf{x}}_Y$, a contradiction. Therefore $Y_3 \neq \{\mathrm{rt}_{\mathbf{x}}\}$ and thus $\mathrm{rt}_{\mathbf{x}} \notin Y_3$, so Y_3 is above Z. Clearly, $D^{\mathbf{x}}_Z \leq_{\mathrm{RK}} h(D^{\mathbf{x}}_Y)$ and hence $D^{\mathbf{x}}_Z$ is a P-point.

By clauses (c) and (d) of Theorem 3.7 there is $B_4 \in \mathscr{B}_{\mathbf{x}}$ such that $B_3 \leq_{\mathbf{x}} B_4$, $B_4 \cap Z$ is a front of B_4 and

$$(\forall \varrho \in \mathrm{suc}_{B_4}(\mathrm{rt}_{\mathbf{x}}))(\exists^\infty \rho \in \mathrm{suc}_{B_3}(\mathrm{rt}_{\mathbf{x}}))[\varrho \leq_{M_{\mathbf{x}}} \rho].$$

For each $\varrho \in \mathrm{suc}_{B_4}(\mathrm{rt}_{\mathbf{x}})$ let $Z_\varrho = \{\rho \in Z : \varrho \leq_{M_{\mathbf{x}}} \rho\}$, so $\langle Z_\varrho : \rho \in \mathrm{suc}_{B_4}(\mathrm{rt}_{\mathbf{x}})\rangle$ is a partition of Z, and $Z_\varrho = \emptyset \mod D^{\mathbf{x}}_Z$ for each ϱ. But clearly there is no $Z' \in D^{\mathbf{x}}_Z$ such that $Z' \cap Z_\varrho$ is finite for every $\varrho \in \mathrm{suc}_{B_4}(\mathrm{rt}_{\mathbf{x}})$, contradiction to "$D^{\mathbf{x}}_Z$ is a P-point". $\square$

4. Basic connections to forcing

Definition 4.1. For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define $\partial^{\mathrm{sb}}_p = \partial^{\mathrm{sb}}_{\mathbb{Q},p}$, the strong bounding game between the null player NU and the bounding player BND as follows:

- A play last ω moves, and
- in the n-th move:
  * first the NU player gives a (non-empty) tree $\mathscr{T}_n$ with ω levels and no maximal node and a $\mathbb{Q}$-name $\underset{\sim}{F}_n$ of a function with domain $\mathscr{T}_n$ such that

$$\eta \in \mathscr{T}_n \quad \Rightarrow \quad p \Vdash_{\mathbb{Q}} \text{“ } \underset{\sim}{F}_n(\eta) \in \mathrm{suc}_{\mathscr{T}_n}(\eta) \text{ ”},$$

  * then BND player chooses $\eta_n \in \mathscr{T}_n$.
- In the end, the BND player wins the play $\langle \mathscr{T}_n, \eta_n : n < \omega\rangle$ *iff* there is $q \in \mathbb{Q}$ above p forcing that

$$\big(\forall n < \omega\big)\big(\exists k < \mathrm{level}(\eta_n)\big)\big(\underset{\sim}{F}_n(\eta_n{\restriction}k) \leq_{\mathscr{T}_n} \eta_n \wedge k \text{ is even }\big),$$

  where $\eta_n{\restriction}k$ is the unique $\nu \leq_{\mathscr{T}_n} \eta_n$ of level k.

Omitting p means NU chooses it in his first move. The game $\partial^{\mathrm{sb}}_{\mathbb{Q}}$ (without a parameter $p \in \mathbb{Q}$ is defined similarly, but here the first player NU also chooses a condition p in the first move.

Definition 4.2. A forcing notion $\mathbb{Q}$ is *strongly bounding* if for every condition $p \in \mathbb{Q}$ player BND has a winning strategy in the game $\partial^{\mathrm{sb}}_{\mathbb{Q},p}$.

Definition 4.3. (1) We say $\mathscr{P} \subseteq [\mathbb{N}]^{\aleph_0}$ is big iff: for every $\mathbf{c} : \mathbb{N} \to \{0,1\}$ there is $A \in \mathscr{P}$ such that $\mathbf{c}{\restriction}A$ is constant.

(2) For $B \in \mathrm{CWT}({}^{\omega>}\omega, \triangleleft)$ we say that a family $\mathscr{B} \subseteq \mathrm{psb}(B)$ is big (in B) iff: for every $\mathbf{c} : \max(B) \longrightarrow \{0,1\}$ there is $B' \in \mathscr{B}$ such that $\mathbf{c}{\restriction}\max(B')$ is constant.

(3) For $B \in \mathrm{CWT}({}^{\omega>}\omega, \triangleleft)$ we say that a family $\mathscr{B} \subseteq \mathrm{psb}(B)$ is large (in B) iff for every function $\mathbf{c}$ with domain $\max(B)$ there is $B' \in \mathscr{B}$ and front Y of B' such that

for every $\eta, \nu \in \max(B')$ we have
$$\mathbf{c}(\eta) = \mathbf{c}(\nu) \;\Leftrightarrow\; (\exists \rho \in Y)(\rho \leq_B \nu \;\wedge\; \rho \leq_B \eta).$$

Theorem 4.4. *Assume that:*

(a) $B \in \mathrm{CWT}(M)$ *for a partial order* M*,* without loss of generality $M = ({}^{\omega>}\omega, \triangleleft)$*,*
(b) The forcing notion $\mathbb{Q}$ *is strongly bounding.*
(c) (α) *forcing with* $\mathbb{Q}$ *preserves some non-principal ultrafilter on* $\mathbb{N}$*,* **or just**
(β) $([\mathbb{N}]^{\aleph_0})^{\mathbf{V}}$ *is big in* $\mathbf{V}^{\mathbb{Q}}$*, see Definition 4.3,*
(d) $p \Vdash$ *"*$\underset{\sim}{A} \subseteq \max(B)$*".*

Then there are $B' \in \mathrm{psb}(B)$ *and* $q \in \mathbb{Q}$ *such that* $p \leq q$ *and*
$$q \Vdash \text{“} \max(B') \subseteq \underset{\sim}{\mathcal{I}} \text{”} \quad \textbf{or} \quad q \Vdash \text{“} \max(B') \subseteq \max(B) \backslash \underset{\sim}{\mathcal{I}} \text{”}.$$

Proof. We prove this by induction on $\mathrm{Dp}(B)$ (see Definition 2.6), for all such B's. Let $\eta = \mathrm{rt}(B)$.

Case 1: $\mathrm{Dp}(B) = 0$

Trivial, as then $B = \{\eta\}$, i.e., B is a singleton so $B' = B$ can serve.

Case 2: $\mathrm{Dp}_{\mathbf{x}}(B) = 1$

Then $\mathrm{Dp}(B_{\geq \nu}) = 0$ for all $\nu \in B \backslash \{\eta\}$. Now, $|B\backslash\{\eta\}| = \aleph_0$ and we just need to find $p' \in \mathbb{Q}$ above p such that $\{\nu \in B : \nu \neq \eta$ and p' forces $\nu \in \underset{\sim}{A}$ or forces $\nu \notin \underset{\sim}{A}\}$ is infinite. As $\Vdash_{\mathbb{Q}}$ "$\big([\mathbb{N}]^{\aleph_0}\big)^{\mathbf{V}}$ is big in $\mathbf{V}^{\mathbb{Q}}$" (see clause (c) of our assumptions) this is possible.

Case 3: $\alpha = \mathrm{Dp}(B) > 1$

Let $Y = \mathrm{suc}_B(\eta)$. Then for $\nu \in Y$ we have $\mathrm{Dp}(B_{\geq \nu}) < \alpha$, hence the induction hypothesis applies to $B_{\geq \nu}$. We may assume that if ρ is not below η then for all but finitely many $\nu \in Y$ we have $\nu \parallel \rho$ (cf. the proof of Lemma 3.4). Let $\langle \nu_n : \nu \in \mathbb{N} \rangle$ list Y.

We simulate a play of $\partial^{\mathrm{sb}}_{\mathbb{Q},p}$ in which the BND player uses a winning strategy and the NU player acts so that in the n-th move:

- $\mathscr{T}_n = \big\{\langle B_0, \dots, B_{k-1}\rangle : k \in \mathbb{N},\, B_\ell \in \mathrm{psb}(B_{\geq \nu_n})$ for $\ell < k$ and $B_{\ell+1} \subseteq B_\ell$ if $\ell + 1 < k\big\}$,
- the relation $<_{\mathscr{T}_n}$ is being an initial segment,
- $\underset{\sim}{F}_n(\langle B_0, \dots, B_{k-1}\rangle)$ is $\langle B_0, \dots, B_{k-1}, B'\rangle$ for some $B' \in \mathrm{psb}(B_{k-1}) \cap \mathbf{V}$ such that
$$\text{either} \quad \max(B') \subseteq \underset{\sim}{A} \quad \text{or} \quad \max(B') \cap \underset{\sim}{A} = \emptyset.$$

There is such a function $\underset{\sim}{F}_n$ because of the induction hypothesis.

Clearly we can do this. As the player BND has used a winning strategy, BND has won the play so there is $q \in \mathbb{Q}$ stronger than p and such that $q \Vdash$ "for every n for some even $k <$ $\mathrm{level}_{\mathscr{T}_n}(\eta_n)$ we have $\underset{\sim}{F}_n(\eta_n{\restriction}k) \leq_{\mathscr{T}_n} \eta_n$".

Hence by the choice of $(\mathscr{T}_n, \underset{\sim}{F}_n)$, letting $\eta_n = \langle B_{n,0}, \dots, B_{n,k(n)}\rangle$ we have: for some $\langle \underset{\sim}{\mathbf{t}}_n : n \in \mathbb{N}\rangle$

- $B_{n,k(n)} \in \mathrm{psb}(B_{\geq \nu_n})$,
- $\underset{\sim}{\mathbf{t}}_n$ is a $\mathbb{Q}$-name of the truth value,
- $q \Vdash$ "if $\underset{\sim}{\mathbf{t}}_n = 1$, then $\max(B_{n,k(n)}) \subseteq \underset{\sim}{A}$,
- if $\underset{\sim}{\mathbf{t}}_n = 0$ then $\max(B_{n,k(n)}) \cap \underset{\sim}{A} = \emptyset$".

Now by clause (c) of our assumptions

> there is an infinite $\mathscr{U} \subseteq \mathbb{N}$, a truth value $\mathbf{t}$ and a condition r such that $q \leq_{\mathbb{Q}} r$ and $r \Vdash$ "$\underset{\sim}{\mathbf{t}}_n = \mathbf{t}$ for $n \in \mathscr{U}$".

Lastly, let $B_* = \bigcup\{B_{n,k(n)} : n \in \mathscr{U}\} \cup \{\eta\}$ and clearly B_*, r are as required. $\square$

Remark 4.5. In the assumption (b) of Theorem 4.4 it is enough that the BND player does not lose the game $\Game^{\rm sb}_{\mathbb{Q}}$, i.e., the NU player has no winning strategy.

Theorem 4.6. *Assume that*

(a) $\mathbb{Q}$ *is an* ${}^\omega\omega$*-bounding proper forcing notion,*
(b) forcing with $\mathbb{Q}$ *preserves some* P*-point, and*
(c) $B \in \mathrm{CWT}({}^{\omega>}\omega, \vartriangleleft)$.

Then $(\mathrm{psb}(B))^{\mathbf{V}}$ *is big in* $\mathbf{V}^{\mathbb{Q}}$*; see Definition 4.3(2).*

Proof. Let D be a P-point ultrafilter such that $\Vdash_{\mathbb{Q}}$" D generates an ultrafilter " and $p \in \mathbb{Q}$. Suppose that $p \Vdash$"$\underset{\sim}{c} : \max(\underset{\sim}{B}) \longrightarrow \{0,1\}$ ". Let χ be a large enough regular cardinal and $N \prec (\mathscr{H}(\chi), \in)$ be a countable model with $B, \mathbb{Q}, p, \underset{\sim}{c}, \ldots \in N$. Let $q \in \mathbb{Q}$ be such that

- $p \leq_{\mathbb{Q}} q$,
- q is $(N, \mathbb{Q})$–generic,
- for some $g \in \left({}^\omega\omega\right)^{\mathbf{V}}$ we have $q \Vdash$" if $\underset{\sim}{f} \in {}^\omega\omega \cap N$, then $\underset{\sim}{f} <_{J^{\rm bd}_\omega} g$ ",
- for some $A \in D$ we have $q \Vdash$" if $\underset{\sim}{B} \in \tilde{D} \cap N$, then $A \subseteq^* \underset{\sim}{B}$ ".

From (g, A) we can compute $\mathbf{c}$ and $B' \in \left(\mathrm{psb}(B)\right)^{\mathbf{V}}$ such that $q \Vdash$" $\underset{\sim}{c}{\restriction}B'$ is constantly $\mathbf{c}$ ", so we are done. $\square$

Theorem 4.7. *Assume that* $\mathbf{x} \in \mathbf{K}$ *and*

(A) The forcing notion $\mathbb{Q}$ *is a proper forcing notion,*
(B) the set D_* *is a Ramsey ultrafilter in* $\mathbf{V}$,
(C) $\Vdash_{\mathbb{Q}}$ "$\mathrm{fil}(D_*)$ *is a Ramsey ultrafilter",*
(D) $B \in \mathscr{B}_{\mathbf{x}}$.

Then $(\mathrm{psb}(B))^{\mathbf{V}}$ *is large in* $\mathbf{V}^{\mathbb{Q}}$ *(see Definition 4.3).*

Proof. We prove this by induction on $\mathrm{Dp}(B)$ for $B \in \mathscr{B}_{\mathbf{x}}$. Let $\mathbf{c} : \max(B) \longrightarrow \mathbb{N}$ be from $\mathbf{V}^{\mathbb{Q}}$ and we should find (B', Y) as promised. We shall work in $\mathbf{V}^{\mathbb{Q}}$.

If $\mathrm{Dp}(B) = 0$, i.e., $|B| = 1$ this is trivial.

If $\mathrm{Dp}(B) = 1$ let $\langle \eta_n : \eta \in \mathbb{N}\rangle \in \mathbf{V}$ list $\mathrm{suc}_B(\mathrm{rt}_{\mathbf{x}})$: by assumption (C) in $\mathbf{V}^{\mathbb{Q}}$, for some $A \in \mathrm{fil}(D_*)$ the sequence $\langle \mathbf{c}(\eta_n) : n \in A\rangle$ is constant or without repetitions. Without loss of generality $A \in D_* \subseteq \mathbf{V}$ and then $\{\mathrm{rt}_{\mathbf{x}}\} \cup \{\eta_n : n \in A\}$ is as required.

So assume $\mathrm{Dp}(B) > 1$. Without loss of generality $0 \notin \mathrm{Rang}(\mathbf{c})$. For $\nu \in B \setminus \max(B)$ let $\langle \eta_{\nu,n} : n \in \mathbb{N}\rangle$ list $\mathrm{suc}_B(\nu)$ so that the function $(\nu, n) \mapsto \eta_{\nu,n}$ belongs to $\mathbf{V}$. In $\mathbf{V}^{\mathbb{Q}}$, by downward induction on $\nu \in B$, we choose $k_\nu = k(\nu)$, A_ν, $A_{\nu,\rho}$ and $\mathbf{t}_{\nu,\rho}$ so that the following requirements (a)–(d) are satisfied:

(a) $k_\nu \in \mathbb{N}$, $A_\nu \in D_*$,
(b) if $\nu \in \max(B)$, then $k_n = \mathbf{c}(\nu)$, so > 0,
(c) if $\nu \notin \max(B)$ then $(\alpha)_\nu$ or $(\beta)_\nu$ where:
  $(\alpha)_\nu$ $k_\nu = 0$ and $\langle k(\eta_{\nu,n}) : n \in A_\nu\rangle$ is with no repetitions, all non-zero,
  $(\beta)_\nu$ $\langle k(\eta_{\nu,n}) : n \in A_\nu\rangle$ is constantly k_ν,

(d) for $\nu, \rho \in B\backslash \max(B)$ we have $A_{\nu,\rho} \in D_*$ and $\mathbf{t}_{\nu,\rho} \in \{0,1\}$ and
either $\mathbf{t}_{\nu,\rho} = 1$ and $n \in A_{\nu,\rho} \Rightarrow k(\eta_{\rho,n}) = k(\eta_{\nu,n})$
or $\mathbf{t}_{\nu,\rho} = 0$ and $\{k(\eta_{\rho,n}) : n \in A_{\nu,\ell}\}$ is disjoint to $\{k(\eta_{\nu,n}) : n \in A_{\nu,\rho}\}$.

This is possible by assumption (C). By the same assumption, there is $A_* \in D_*$ such that:

if $\nu \in B\backslash \max(B)$ then $A_* \subseteq^* A_\nu$,
if $\nu, \rho \in B\backslash \max(B)$ then $A_* \subseteq^* A_{\nu,\rho}$.

Let $\langle \nu_n : n \in \mathbb{N}\rangle$ list $B\backslash \max(B)$ and let f_1 be the function with domain $B\backslash \max(B)$ such that

$$f_1(\nu) = \{\eta_{\nu,n} : n \in A_*\backslash A_\nu \text{ or for some } k < \ell \text{ we have } \nu = \nu_\ell \ \wedge \ n \in A_* \setminus A_{\nu_k,\nu_\ell}\}$$

(so $f_1(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}$).

As the forcing $\mathbb{Q}$ satisfies (C), it is bounding, so there is a function $f_2 \in \mathbf{V}$ with domain $B\backslash \max(B)$ such that $f_1(\nu) \subseteq f_2(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}$. Clearly, letting

$$\begin{aligned} B_1 := A_{B,f} := \big\{ \nu \in B : \textbf{if } & \rho \in B \text{ satisfies } \mathrm{rt}_{\mathbf{x}} \leq_B \rho <_B \nu \\ & \text{and } n \text{ is such that } \eta_{\rho,n} \leq_B \nu, \\ & \textbf{then } n \in A_* \text{ but } \eta_{\rho,n} \notin f_2(\nu) \big\} \end{aligned}$$

we have $B_1 \in \mathrm{psb}(B)^{\mathbf{V}}$.

Define

$$Y := \{ \nu \in B_1 : k_\nu \neq 0 \text{ and } \rho <_B \nu \Rightarrow k_\rho = 0 \}.$$

Plainly,

the set Y is a front of B_1,
and if $\nu \in Y$ then $\mathbf{c}{\restriction}(B_1)_{\geq \nu}$ is constantly k_ν.

Note that

if $\nu \in B_1$ and $k_\nu = 0$, then either $k_\eta = 0$ for all $\eta \in \mathrm{suc}_{B_1}(\nu)$,
or $k_\eta > 0$ for all $\eta \in \mathrm{suc}_{B_1}(\nu)$.

Hence:

if $\nu \in B_1\backslash \max(B_1)$ and $\mathrm{suc}_{B_1}(\nu)$ is not disjoint to Y,
then $\mathrm{suc}_{B_1}(\nu) \subseteq Y$.

If $Y = \{\mathrm{rt}_{\mathbf{x}}\}$ we are done, so assume not. Let $Z = \{\eta \in B_1 : \eta \notin \max(B_1)$ and $\mathrm{suc}_{B_1}(\eta) \subseteq Y\}$. So

both Z and Y are fronts of B_1,
both Z and Y belong to $\mathbf{V}$,
if $\nu \in Y$ then $\langle k_\rho : \rho \in \max\big((B_1)_{\geq \nu}\big)\rangle$ is constantly k_ν.

Also if $Z = \{\mathrm{rt}_{\mathbf{x}}\}$ we are done, so assume not. Let $\langle \nu_n : n \in \mathbb{N}\rangle$ list Z. As $\mathrm{fil}(D_*)$ is a Ramsey ultrafilter we can find $\bar{n}$ such that

- $\bar{n} = \langle n(i) : i \in \mathbb{N}\rangle$ is an increasing enumeration of a member of D_*, hence $\bar{n} \in \mathbf{V}$,
- if $\ell \leq i$ then $\eta_{\nu_\ell, n(i)} \in B_1$,
- if $\ell < i$, $\mathbf{t}_{\nu_\ell,\nu_i} = 0$ and $\nu_\ell, \nu_i \in B_1[\leq Z]$, then $\{k(\eta_{\nu_i,n(j)}) : i \leq j\}$ is disjoint from $\{k(\eta_{\nu_\ell,n(j)}) : i \leq j\}$, moreover it is disjoint from $\{k(\eta_{\nu_\ell,n(j)} : j \in \mathbb{N}\}$.

Lastly, as $\bar{n} \in \mathbf{V}$ we can find in $\mathbf{V}$ a partition $\langle C_\ell : \ell \in \mathbb{N}\rangle$ of $\mathbb{N}$ to (pairwise disjoint) infinite sets and let

$$\begin{aligned} B_2 = \{\varrho \in B_1 : \ & \text{if } \nu_\ell <_{B_1} \varrho \text{ and } \nu_\ell \in B_1[\leq Z], \\ & \text{then for some } i \in C_\ell \text{ we have } i > \ell \text{ and } \eta_{\nu_\ell, n(i)} \leq_{B_2} \varrho\}. \end{aligned}$$

Easily $B_2 \in \mathbf{V}$, $B_2 \in \mathrm{psb}(B_1)$ and it is as required. $\square$

Motivated by Definition 4.1 we introduce the following bounding games for a forcing notion $\mathbb{Q}$.

Definition 4.8. Let $\mathbb{Q}$ be a forcing notion and $p \in \mathbb{Q}$. We will define 3 games: $\Game^{\rm bd}_p = \Game^{\rm bd}_{\mathbb{Q},p}$, $\Game^{\rm ufbd}_p = \Game^{\rm ufbd}_{\mathbb{Q},p}$, and $\Game^{\rm vfbd}_p = \Game^{\rm vfbd}_{\mathbb{Q},p}$. Each of the games lasts ω rounds, and in each round player NU moves first, and player BND second.

The games $\Game^{\rm bd}, \Game^{\rm ufbd}, \Game^{\rm vfbd}$ are defined analogously, but here the condition p will be chosen by player NU in his first move.

(1) In the n-th round of the game $\Game^{\rm bd}_p$, first the NU player gives a $\mathbb{Q}$-name $\underset{\sim}{\tau}_n$ of a member of $\mathbf{V}$ and then the BND player gives a finite set $w_n \subseteq \mathbf{V}$. After ω rounds, the BND player wins the play iff there is $q \in \mathbb{Q}$ above p forcing "$\underset{\sim}{\tau}_n \in w_n$" for every n.

(2) In the n-th round of the game $\Game^{\rm ufbd}_p$, first the NU player chooses an ultrafilter E_n on some set I_n from $\mathbf{V}$ and a $\mathbb{Q}$-name $\underset{\sim}{E}^+_n$ of an ultrafilter on I_n extending E_n and a $\mathbb{Q}$-name $\underset{\sim}{X}_n$ of a member of $\underset{\sim}{E}^+_n$; then the BND player chooses $t_n \in I_n$.
In the end of the play the BND player wins the play iff there is $q \in \mathbb{Q}$ above p forcing "$t_n \in \underset{\sim}{X}_n$" for every n.

(3) The game $\Game^{\rm vfbd}_p$ is similar to $\Game^{\rm ufbd}_p$, but now we demand

$$\Vdash_{\mathbb{Q}} \text{" } \underset{\sim}{X}_n \in E_n \text{ or just includes a member of } E_n \text{ "},$$

so $\underset{\sim}{E}^+_n$ is redundant.

Basic relations between the games introduced above are given by the following result.

Proposition 4.9. *Let $\mathbb{Q}$ be a forcing notion.*

(1) If BND wins in $\Game^{\rm sb}_{\mathbb{Q},p}$ then *BND wins in $\Game^{\rm bd}_{\mathbb{Q},p}$ which implies that $\mathbb{Q}$ is a bounding forcing.*
(2) The player BND wins in $\Game^{\rm bd}_{\mathbb{Q},p}$ iff *BND wins in $\Game^{\rm vfbd}_{\mathbb{Q},p}$.*
(3) If the player BND wins in $\Game^{\rm ufbd}_{\mathbb{Q},p}$ then *BND wins in $\Game^{\rm vfbd}_{\mathbb{Q},p}$.*
(4) We can replace in (1)–(3) above "wins" by "does not lose".

Proof. (1) The second implication is obvious, so we concentrate on the first. For every $\underset{\sim}{\tau}$, a $\mathbb{Q}$-name of an ordinal we define a pair $(T_{\underset{\sim}{\tau}}, \underset{\sim}{F}_{\underset{\sim}{\tau}})$ as follows:

– let $u = \{\alpha : \nVdash_{\mathbb{Q}} \text{"}\underset{\sim}{\tau} \neq \alpha\text{"}\}$, it is a non-empty set of $\leq |\mathbb{Q}|$ ordinals,
– $T_{\underset{\sim}{\tau}}$ is the tree $\{\eta : \eta \in {}^{\omega >}u\}$, i.e., ordered by $\triangleleft$ (being an initial segment),
– $\underset{\sim}{F}_{\underset{\sim}{\tau}}(\eta) = \eta^\frown\langle\underset{\sim}{\tau}\rangle$ for $\eta \in T_{\underset{\sim}{\tau}}$.

Clearly,

– $T_{\underset{\sim}{\tau}}$ is in $\mathbf{V}$, a tree with ω levels,
– $\underset{\sim}{F}_{\underset{\sim}{\tau}}$ is a $\mathbb{Q}$-name of a function with domain $T_{\underset{\sim}{\tau}}$ such that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{F}_{\underset{\sim}{\tau}}(\eta) \in \mathrm{suc}_{T_{\underset{\sim}{\tau}}}(\eta)$".
– if $q \in \mathbb{Q}$ and $\eta \in T_{\underset{\sim}{\tau}}$ (so $\mathrm{Rang}(\eta)$ is a finite subset of u) then the following are equivalent:
  (i) $q \Vdash$ "$\underset{\sim}{\tau} \in \mathrm{Rang}(\eta)$",
  (ii) $q \Vdash$ "for some $\nu \triangleleft \eta$ we have $\nu^\frown\langle\underset{\sim}{F}_{\underset{\sim}{\tau}}(\nu)\rangle \trianglelefteq \eta$".

So playing the game $\partial^{\rm bd}_{\mathbb{Q},p}$ we can "translate" it to a play of $\partial^{\rm sb}_{\mathbb{Q},p}$ replacing the NU choice of $\underset{\sim}{\tau}_n$ by the choice of $(T_{\underset{\sim}{\tau}}, \underset{\sim}{F}_{\underset{\sim}{\tau}})$. Thus every strategy $\mathbf{st}_1$ of BND in $\partial^{\rm sb}_{\mathbb{Q},p}$ translates it to a strategy $\mathbf{st}_2$ of the player BND in $\partial^{\rm bd}_{\mathbb{Q},p}$.

(2) We now need two translations.

Translating $\partial^{\rm vfbd}_{\mathbb{Q},p}$ to $\partial^{\rm bd}_{\mathbb{Q},p}$:

So we are given a move $y = (I, E, \underset{\sim}{X})$ of NU in a play of $\partial^{\rm vfbd}_{\mathbb{Q},p}$ as in Definition 4.8, i.e.,

- $I \in \mathbf{V}$, E is an ultrafilter on I, in $\mathbf{V}$, and
- $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{X} \in E$ or just includes a member $\underset{\sim}{X}'$ of E".

Now we have:
if $q \Vdash$ "$\underset{\sim}{X}' \in \mathscr{W}$" where $\mathscr{W} \subseteq E$ is finite ($\mathscr{W}$ an object in $\mathbf{V}$ not a name),
then $\bigcap\{A : A \in \mathscr{W}\}$ is non-empty and $t \in \bigcap\{A : A \in \mathscr{W}\} \Rightarrow q \Vdash$ "$t \in \underset{\sim}{X}' \subseteq \underset{\sim}{X}$".

Translating $\partial^{\rm bd}_{\mathbb{Q},p}$ to $\partial^{\rm vfbd}_{\mathbb{Q},p}$:

Given $y = (I, \underset{\sim}{\tau}), \underset{\sim}{\tau}$ a $\mathbb{Q}$-name of a member I of $\mathbf{V}$ we define $I_y = [I]^{<\aleph_0} \in \mathbf{V}$ and choose $E_y \in \mathbf{V}$ an ultrafilter on I_y such that $u_* \in [I]^{<\aleph_0} \Rightarrow \{u \in [I]^{<\aleph_0} : u_* \subseteq u\} \in E$; lastly we choose

$$\underset{\sim}{X}_y = \{u \in [I]^{<\aleph_0} : \underset{\sim}{\tau} \in u\}.$$

So $(I_y, E_y, \underset{\sim}{X}_y)$ is a legal move in $\partial^{\rm vfbd}_{\mathbb{Q},p}$ and for a finite subset t of I:

if $q \Vdash$ "$t \in \underset{\sim}{X}_y$" then $q \Vdash$ "$\underset{\sim}{\tau} \in t$".

(3) Obvious.

(4) The same proof. □

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