

ON THE COFINALITY OF THE SPLITTING NUMBER

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ABSTRACT. The splitting number \mathfrak{s} can be singular. The key method is to construct a forcing poset with finite support matrix iterations of ccc posets introduced by Blass and the second author [*Ultrafilters with small generating sets*, Israel J. Math., **65**, (1989)]

1. INTRODUCTION

The cardinal invariants of the continuum discussed in this article are very well known (see [5, van Douwen, p111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. A set $S \subset \omega$ is *unsplit* by a family $\mathcal{Y} \subset [\omega]^{\aleph_0}$ if S is mod finite contained in one member of $\{Y, \omega \setminus Y\}$ for each $Y \in \mathcal{Y}$. The splitting number \mathfrak{s} is the minimum cardinal of a family \mathcal{Y} for which there is no infinite set unsplit by \mathcal{Y} (equivalently every $S \in [\omega]^{\aleph_0}$ is *split* by some member of \mathcal{Y}). It is mentioned in [2] that it is currently unknown if \mathfrak{s} can be a singular cardinal.

Proposition 1.1. *The cofinality of the splitting number is not countable.*

Proof. Assume that θ is the supremum of $\{\kappa_n : n \in \omega\}$ and that there is no splitting family of cardinality less than θ . Let $\mathcal{Y} = \{Y_\alpha : \alpha < \theta\}$ be a family of subsets of ω . Let $S_0 = \omega$ and by induction on n , choose an infinite subset S_{n+1} of S_n so that S_{n+1} is not split by the family $\{Y_\alpha : \alpha < \kappa_n\}$. If S is any pseudointersection of $\{S_n : n \in \omega\}$, then S is not split by any member of \mathcal{Y} . \square

Date: September 26, 2020.

1991 Mathematics Subject Classification. 03E15 .

Key words and phrases. splitting number, cardinal invariants of the continuum, matrix forcing.

The research of the first author was supported by the NSF grant No. NSF-DMS 1501506. The research of the second author was supported by the United States-Israel Binational Science Foundation (BSF Grant no. 2010405), and by the NSF grant No. NSF-DMS 1101597.

One can easily generalize the previous result and proof to show that the cofinality of the splitting number is at least \mathfrak{t} . In this paper we prove the following.

Theorem 1.2. *If κ is any uncountable regular cardinal, then there is a $\lambda > \kappa$ with $\text{cf}(\lambda) = \kappa$ and a ccc forcing \mathbb{P} satisfying that $\mathfrak{s} = \lambda$ in the forcing extension.*

To prove the theorem, we construct \mathbb{P} using matrix iterations.

2. A SPECIAL SPLITTING FAMILY

Definition 2.1. *Let us say that a family $\{x_i : i \in I\} \subset [\omega]^\omega$ is θ -Luzin (for an uncountable cardinal θ) if for each $J \in [I]^\theta$, $\bigcap\{x_i : i \in J\}$ is finite and $\bigcup\{x_i : i \in J\}$ is cofinite.*

Clearly a family is θ -Luzin if every θ -sized subfamily is θ -Luzin. We leave to the reader the easy verification that for a regular uncountable cardinal θ , each θ -Luzin family is a splitting family. A poset being θ -Luzin preserving will have the obvious meaning. For example, any poset of cardinality less than a regular cardinal θ is θ -Luzin preserving.

Lemma 2.2. *If θ is a regular uncountable cardinal then any ccc finite support iteration of θ -Luzin preserving posets is again θ -Luzin preserving.*

Proof. We prove this by induction on the length of the iteration. Fix any θ -Luzin family $\{x_i : i \in I\}$ and let $\langle\langle \mathbb{P}_\alpha : \alpha \leq \gamma \rangle\rangle, \langle\langle \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle\rangle$ be a finite support iteration of ccc posets satisfying that \mathbb{P}_α forces that $\dot{\mathbb{Q}}_\alpha$ is ccc and θ -Luzin preserving, for all $\alpha < \gamma$.

If γ is a successor ordinal $\beta + 1$, then for any \mathbb{P}_β -generic filter G_β , the family $\{x_i : i \in I\}$ is a θ -Luzin family in $V[G_\beta]$. By the hypothesis on $\dot{\mathbb{Q}}_\beta$, this family remains θ -Luzin after further forcing by $\dot{\mathbb{Q}}_\beta$.

Now we assume that α is a limit. Let \dot{J}_0 be any \mathbb{P}_α -name of a subset of I and assume that $p \in \mathbb{P}_\alpha$ forces that $|\dot{J}_0| = \theta$. We must produce a $q < p$ that forces that \dot{J}_0 is as in the definition of θ -Luzin. There is a set $J_1 \subset I$ of cardinality θ satisfying that, for each $i \in J_1$, there is a $p_i < p$ with $p_i \Vdash i \in \dot{J}_0$. The case when the cofinality of α not equal to θ is almost immediate. There is a $\beta < \alpha$ such that $J_2 = \{i \in J_1 : p_i \in \mathbb{P}_\beta\}$ has cardinality θ . There is a \mathbb{P}_β -generic filter G_β such that $J_3 = \{i \in J_2 : p_i \in G_\beta\}$ has cardinality θ . By the induction hypothesis, the family $\{x_i : i \in I\}$ is θ -Luzin in $V[G_\beta]$ and so we have that $\bigcap\{x_i : i \in J_3\}$ is finite and $\bigcup\{x_i : i \in J_3\}$ is co-finite. Choose any $q < p$ in G_β and a name \dot{J}_3 for J_3 so that q forces this

property for \dot{J}_3 . Since q forces that $\dot{J}_3 \subset \dot{J}_0$, we have that q forces the same property for \dot{J}_0 .

Finally we assume that α has cofinality θ . Naturally we may assume that the collection $\{\text{dom}(p_i) : i \in J_1\}$ forms a Δ -system with root contained in some $\beta < \alpha$. Again, we may choose a \mathbb{P}_β -generic filter G_β satisfying that $J_2 = \{i \in J_1 : p_i \upharpoonright \beta \in G_\beta\}$ has cardinality θ . In $V[G_\beta]$, let $\{J_{2,\xi} : \xi \in \omega_1\}$ be a partition of J_2 into pieces of size θ . For each $\xi \in \omega_1$, apply the induction hypothesis in the model $V[G_\beta]$, and so we have that $\bigcap\{x_i : i \in J_{2,\xi}\}$ is finite and $\bigcup\{x_i : i \in J_{2,\xi}\}$ is co-finite. For each $\xi \in \omega_1$ let m_ξ be an integer large enough so that $\bigcap\{x_i : i \in J_{2,\xi}\} \subset m_\xi$ and $\bigcup\{x_i : i \in J_{2,\xi}\} \supset \omega \setminus m_\xi$. Let m be any integer such that $m_\xi = m$ for uncountably many ξ . Choose any condition $\bar{p} \in \mathbb{P}_\alpha$ so that $\bar{p} \upharpoonright \beta \in G_\beta$. We prove that for each $n > m$ there is a $\bar{p}_n < \bar{p}$ so that $\bar{p}_n \Vdash n \notin \bigcap\{x_i : i \in \dot{I}\}$ and $\bar{p}_n \Vdash n \in \bigcup\{x_i : i \in \dot{I}\}$. Choose any $\xi \in \omega_1$ so that $m_\xi = m$ and $\text{dom}(p_i) \cap \text{dom}(\bar{p}) \subset \beta$ for all $i \in J_{2,\xi}$. Now choose any $i_0 \in J_{2,\xi}$ so that $n \notin x_{i_0}$. Next choose a distinct ξ' with $m_{\xi'} = m$ so that $\text{dom}(p_i) \cap (\text{dom}(\bar{p}) \cup \text{dom}(p_{i_0})) \subset \beta$ for all $i \in J_{2,\xi'}$. Now choose $i_1 \in J_{2,\xi'}$ so that $n \in x_{i_1}$. We now have that $\bar{p} \cup p_{i_0} \cup p_{i_1}$ is a condition that forces $\{i_0, i_1\} \subset \dot{I}$. \square

Next we introduce a σ -centered poset that will render a given family non-splitting.

Definition 2.3. For a filter \mathfrak{D} on ω , we define the Laver style poset $\mathbb{L}(\mathfrak{D})$ to be the set of trees $T \subset \omega^{<\omega}$ with the property that T has a minimal branching node $\text{stem}(T)$ and for all $\text{stem}(T) \subseteq t \in T$, the branching set $\{k : t \frown k \in T\}$ is an element of \mathfrak{D} . If \mathfrak{D} is a filter base for a filter \mathfrak{D}^* , then $\mathbb{L}(\mathfrak{D})$ will also denote $\mathbb{L}(\mathfrak{D}^*)$.

The name $\dot{L} = \{(k, T) : (\exists t) t \frown k \subset \text{stem}(T)\}$ will be referred to as the canonical name for the real added by $\mathbb{L}(\mathfrak{D})$.

If \mathfrak{D} is a principal (fixed) ultrafilter on ω , then $\mathbb{L}(\mathfrak{D})$ has a minimum element and so is forcing isomorphic to the trivial poset. If \mathfrak{D} is principal but not an ultrafilter, then $\mathbb{L}(\mathfrak{D})$ is isomorphic to Cohen forcing. If \mathfrak{D} is a free filter, then $\mathbb{L}(\mathfrak{D})$ adds a dominating real and has similarities to Hechler forcing. As usual, for a filter (or filter base) \mathfrak{D} of subsets of ω , we use \mathfrak{D}^+ to denote the set of all subsets of ω that meet every member of \mathfrak{D} .

Definition 2.4. If E is a dense subset of $\mathbb{L}(\mathfrak{D})$, then a function ρ_E from $\omega^{<\omega}$ into ω_1 is a rank function for E if $\rho_E(t) = 0$ if and only if $t = \text{stem}(T)$ for some $T \in E$, and for all $t \in \omega^{<\omega}$ and $0 < \alpha \in \omega_1$, $\rho_E(t) \leq \alpha$ providing the set $\{k \in \omega : \rho_E(t \frown k) < \alpha\}$ is in \mathfrak{D}^+ .

When \mathfrak{D} is a free filter, then $\mathbb{L}(\mathfrak{D})$ has cardinality \mathfrak{c} , but nevertheless, if \mathfrak{D} has a base of cardinality less than a regular cardinal θ , $\mathbb{L}(\mathfrak{D})$ is θ -Luzin preserving.

Lemma 2.5. *If \mathfrak{D} is a free filter on ω and if \mathfrak{D} has a base of cardinality less than a regular uncountable cardinal θ , then $\mathbb{L}(\mathfrak{D})$ is θ -Luzin preserving.*

Proof. Let $\{x_i : i \in \theta\}$ be a θ -Luzin family with θ as in the Lemma. Let \dot{J} be a $\mathbb{L}(\mathfrak{D})$ -name of a subset of θ . We prove that if $\bigcap\{x_i : i \in \dot{J}\}$ is not finite, then \dot{J} is bounded in θ . By symmetry, it will also prove that if $\bigcup\{x_i : i \in \dot{J}\}$ is not cofinite, then \dot{J} is bounded in θ . Let \dot{y} be the $\mathbb{L}(\mathfrak{D})$ -name of the intersection, and let T_0 be any member of $\mathbb{L}(\mathfrak{D})$ that forces that \dot{y} is infinite. Let M be any $< \theta$ -sized elementary submodel of $H((2^\omega)^+)$ such that $T_0, \mathfrak{D}, \dot{J}$, and $\{x_i : i \in \theta\}$ are all members of M and such that $M \cap \mathfrak{D}$ contains a base for \mathfrak{D} . Let $i_M = \sup(M \cap \theta)$. If $x \in M \cap [\omega]^\omega$, then $I_x = \{i \in \theta : x \subset x_i\}$ is an element of M and has cardinality less than θ . Therefore, if $i \in \theta \setminus i_M$, then x_i does not contain any infinite subset of ω that is an element of M . We prove that x_i is forced by T_0 to also not contain \dot{y} . This will prove that \dot{J} is bounded by i_M . Let $T_1 < T_0$ be any condition in $\mathbb{L}(\mathfrak{D})$ and let $t_1 = \text{stem}(T_1)$. We show that T_1 does not force that $x_i \supset \dot{y}$. We define the relation \Vdash_w on $T_0 \times \omega$ to be the set

$$\{(t, n) \in T_0 \times \omega : \text{there is no } T \leq T_0, \text{stem}(T) = t, \text{s.t. } T \Vdash_w n \notin \dot{y}\}.$$

For convenience we may write, for $T \leq T_0$, $T \Vdash_w n \in \dot{y}$ providing $(\text{stem}(T), n)$ is in \Vdash_w , and this is equivalent to the relation that T has no stem preserving extension forcing that n is not in \dot{y} . Let $T_2 \in M$ be any extension of T_0 with stem t_1 . Let L denote the set of $\ell \in \omega$ such that $T_2 \Vdash_w \ell \in \dot{y}$. If L is infinite, then, since $L \in M$, there is an $\ell \in L \setminus x_i$. This implies that T_1 does not force $x_i \supset \dot{y}$, since $T_2 \Vdash_w j \in \dot{y}$ implies that T_1 fails to force that $\ell \notin \dot{y}$.

Therefore we may assume that L is finite and let ℓ be the maximum of L . Define the set $E \subset \mathbb{L}(\mathfrak{D})$ according to $T \in E$ providing that either $t_1 \notin T$ or there is a $j > \ell$ such that $T \Vdash_w j \in \dot{y}$. Again this set E is in M and is easily seen to be a dense subset of $\mathbb{L}(\mathfrak{D})$. By the choice of ℓ , we note that $\rho_E(t_1) > 0$. If $\rho_E(t_1) > 1$, then the set $\{k \in \omega : 0 < \rho_E(t_1 \hat{\ } k) < \rho_E(t_1)\}$ is in \mathfrak{D}^+ and so there is a k_1 in this set such that $t_1 \hat{\ } k_1 \in T_1 \cap T_2$. By a finite induction, we can choose an extension $t_2 \supseteq t_1$ so that $t_2 \in T_1 \cap T_2$ and $\rho_E(t_2) = 1$. Now, there is a set $D \in \mathfrak{D} \cap M$ contained in $\{k : t_2 \hat{\ } k \in T_1 \cap T_2\}$ since M contains a base for \mathfrak{D} . Also, $D_E = \{k \in D : \rho_E(t_2 \hat{\ } k) = 0\}$ is in \mathfrak{D}^+ . For each $k \in D_E$, choose the minimal j_k so that $T_2 \hat{\ } k \Vdash_w j_k \in \dot{y}$. The set

$\{j_k : k \in D_E\}$ is an element of M . This set is not finite because if it were then there would be a single j such that $\{k \in D_E : j_k = j\} \in \mathcal{D}^+$, which would contradict that $\rho_E(t_2) > 0$. This means that there is a $k \in D_E^+$ with $j_k \notin x_i$, and again we have shown that T_1 fails to force that x_i contains \dot{y} . \square

3. MATRIX ITERATIONS

The terminology “matrix iterations” is used in [3], see also forthcoming preprint (F1222) from the second author. The paper [3] nicely expands on the method of matrix iterated forcing first introduced in [1].

Let us recall that a poset $(P, <_P)$ is a complete suborder of a poset $(Q, <_Q)$ providing $P \subset Q$, $<_P \subset <_Q$, and each maximal antichain of $(P, <_P)$ is also a maximal antichain of $(Q, <_Q)$. Note that it follows that incomparable members of $(P, <_P)$ are still incomparable in $(Q, <_Q)$, i.e. $p_1 \perp_P p_2$ implies $p_1 \perp_Q p_2$. We use the notation $(P, <_P) <_{\circ} (Q, <_Q)$ to abbreviate the complete suborder relation, and similarly use $P <_{\circ} Q$ if $<_P$ and $<_Q$ are clear from the context. An element p of P is a reduction of $q \in Q$ if $r \not\perp_Q q$ for each $r <_P p$. If $P \subset Q$, $<_P \subset <_Q$, $\perp_P \subset \perp_Q$, and each element of Q has a reduction in P , then $P <_{\circ} Q$. The reason is that if $A \subset P$ is a maximal antichain and $p \in P$ is a reduction of $q \in Q$, then there is an $a \in A$ and an r less than both p and a in P , such that $r \not\perp_Q q$.

Definition 3.1. *We will say that an object \mathbf{P} is a matrix iteration if there is an infinite cardinal κ and an ordinal γ (thence a (κ, γ) -matrix iteration) such that $\mathbf{P} = \langle \langle \mathbb{P}_{i,\alpha}^{\mathbf{P}} : i \leq \kappa, \alpha \leq \gamma \rangle, \langle \dot{\mathbb{Q}}_{i,\alpha}^{\mathbf{P}} : i \leq \kappa, \alpha < \gamma \rangle \rangle$ where, for each $(i, \alpha) \in \kappa + 1 \times \gamma$ and each $j < i$,*

- (1) $\mathbb{P}_{j,\alpha}^{\mathbf{P}}$ is a complete suborder of the poset $\mathbb{P}_{i,\alpha}^{\mathbf{P}}$ (i.e. $\mathbb{P}_{j,\alpha}^{\mathbf{P}} <_{\circ} \mathbb{P}_{i,\alpha}^{\mathbf{P}}$),
- (2) $\dot{\mathbb{Q}}_{i,\alpha}^{\mathbf{P}}$ is a $\mathbb{P}_{i,\alpha}^{\mathbf{P}}$ -name of a ccc poset, $\mathbb{P}_{i,\alpha+1}^{\mathbf{P}}$ is equal to $\mathbb{P}_{i,\alpha}^{\mathbf{P}} * \dot{\mathbb{Q}}_{i,\alpha}^{\mathbf{P}}$,
- (3) for limit $\delta \leq \gamma$, $\mathbb{P}_{i,\delta}^{\mathbf{P}}$ is equal to the union of the family $\{\mathbb{P}_{i,\beta}^{\mathbf{P}} : \beta < \delta\}$
- (4) $\mathbb{P}_{\kappa,\alpha}^{\mathbf{P}}$ is the union of the chain $\{\mathbb{P}_{j,\alpha}^{\mathbf{P}} : j < \kappa\}$.

When the context makes it clear, we omit the superscript \mathbf{P} when discussing a matrix iteration. Throughout the paper, κ will be a fixed uncountable regular cardinal

Definition 3.2. *A sequence $\vec{\lambda}$ is κ -tall if $\vec{\lambda} = \langle \mu_{\xi}, \lambda_{\xi} : \xi < \kappa \rangle$ is a sequence of pairs of regular cardinals satisfying that $\mu_0 = \omega < \kappa < \lambda_0$ and, for $0 < \eta < \kappa$, $\mu_{\eta} < \lambda_{\eta}$ where $\mu_{\eta} = (2^{\sup\{\lambda_{\xi} : \xi < \eta\}})^+$.*

Also for the remainder of the paper, we fix a κ -tall sequence $\vec{\lambda}$ and λ will denote the supremum of the set $\{\lambda_\xi : \xi \in \kappa\}$. For simpler notation, whenever we discuss a matrix iteration \mathbf{P} we shall henceforth assume that it is a (κ, γ) -matrix iteration for some ordinal γ . We may refer to a forcing extension by \mathbf{P} as an abbreviation for the forcing extension by $\mathbb{P}_{\kappa, \gamma}^{\mathbf{P}}$.

For any poset P , any P -name \dot{D} , and P -generic filter G , $\dot{D}[G]$ will denote the valuation of \dot{D} by G . For any ground model x , \check{x} denotes the canonical name so that $\check{x}[G] = x$. When x is an ordinal (or an integer) we will suppress the accent in \check{x} . A P -name \dot{D} of a subset of ω will be said to be *nice* or *canonical* if for each integer $j \in \omega$, there is an antichain A_j such that $\dot{D} = \bigcup \{\{j\} \times A_j : j \in \omega\}$. We will say that $\dot{\mathcal{D}}$ is a nice P -name of a family of subsets of ω just to mean that $\dot{\mathcal{D}}$ is a collection of nice P -names of subsets of ω . We will use $(\dot{\mathcal{D}})_P$ if we need to emphasize that we mean the P -name. Similarly if we say that $\dot{\mathcal{D}}$ is a nice P -name of a filter (base) we mean that $\dot{\mathcal{D}}$ is a nice P -name such that, for each P -generic filter, the collection $\{\dot{D}[G] : \dot{D} \in \dot{\mathcal{D}}\}$ is a filter (base) of infinite subsets of ω .

Following these conventions, the following notation will be helpful.

Definition 3.3. For a (κ, γ) -matrix \mathbf{P} and $i < \kappa$, we let $\mathbb{B}_{i, \gamma}^{\mathbf{P}}$ denote the set of all nice $\mathbb{P}_{i, \gamma}^{\mathbf{P}}$ -names of subsets of ω . We note that this then is the nice $\mathbb{P}_{i, \gamma}^{\mathbf{P}}$ -name for the power set of ω . As usual, when possible we suppress the \mathbf{P} superscript.

For a nice \mathbf{P} -name $\dot{\mathcal{D}}$ of a filter (or filter base) of subsets of ω , we let $(\dot{\mathcal{D}})^+$ denote the set of all nice \mathbf{P} -names that are forced to meet every member of $\dot{\mathcal{D}}$. It follows that $(\dot{\mathcal{D}})^+$ is the nice \mathbf{P} -name for the usual defined notion $(\dot{\mathcal{D}})^+$ in the forcing extension by \mathbf{P} . We let $\langle \dot{\mathcal{D}} \rangle$ denote the nice \mathbf{P} -name of the filter generated by $\dot{\mathcal{D}}$. We use the same notational conventions if, for some poset \mathbb{P} , $\dot{\mathcal{D}}$ is a nice \mathbb{P} -name of a filter (or filter base) of subsets of ω .

The main idea for controlling the splitting number in the extension by \mathbf{P} will involve having many of the subposets being θ -Luzin preserving for $\theta \in \{\lambda_\xi : \xi \in \kappa\}$. Motivated by the fact that posets of the form $\mathbb{L}(\dot{\mathcal{D}})$ (our proposed iterands) are θ -Luzin preserving when $\dot{\mathcal{D}}$ is sufficiently small we adopt the name $\vec{\lambda}$ -thin for this next notion.

Definition 3.4. For a κ -tall sequence $\vec{\lambda}$, we will say that a (κ, γ) -matrix-iteration \mathbf{P} is $\vec{\lambda}$ -thin providing that for each $\xi < \kappa$ and $\alpha \leq \gamma$, $\mathbb{P}_{\xi, \alpha}^{\mathbf{P}}$ is λ_ξ -Luzin preserving.

Now we combine the notion of $\vec{\lambda}$ -thin matrix-iteration with Lemma 2.2. We adopt Kunen's notation that for a set I , $\text{Fn}(I, 2)$ denotes the usual poset for adding Cohen reals (finite partial functions from I into 2 ordered by superset).

Lemma 3.5. *Suppose that \mathbf{P} is a $\vec{\lambda}$ -thin (κ, γ) -matrix iteration for some κ -tall sequence $\vec{\lambda}$. Further suppose that $\dot{Q}_{i,0}$ is the $\mathbb{P}_{i,0}$ -name of the poset $\text{Fn}(\lambda_\xi, 2)$ for each $\xi \in \kappa$, and therefore $\mathbb{P}_{\kappa,1}$ is isomorphic to $\text{Fn}(\lambda, 2)$. Let \dot{g} denote the generic function from λ onto 2 added by $\mathbb{P}_{\kappa,1}$ and, for $i < \lambda$, let \dot{x}_i be the canonical name of the set $\{n \in \omega : \dot{g}(i+n) = 1\}$. Then the family $\{\dot{x}_i : i < \lambda\}$ is forced by \mathbf{P} to be a splitting family.*

Proof. Let $G_{\kappa,\gamma}$ be a $\mathbb{P}_{\kappa,\gamma}$ -generic filter. For each $\xi \in \kappa$ and $\alpha \leq \gamma$, let $G_{\xi,\alpha} = G_{\kappa,\gamma} \cap \mathbb{P}_{\xi,\alpha}$. Let \dot{y} be any nice $\mathbb{P}_{\kappa,\gamma}$ -name for a subset of ω . Since \dot{y} is a countable name, we may choose a $\xi < \kappa$ so that \dot{y} is a $\mathbb{P}_{\xi,\gamma}$ -name. It is easily shown, and very well-known, that the family $\{\dot{x}_i : i < \lambda_\xi\}$ is forced by $\mathbb{P}_{\xi,1}$ (i.e. $\text{Fn}(\lambda_\xi, 2)$) to be a λ_ξ -Luzin family. By the hypothesis that \mathbf{P} is $\vec{\lambda}$ -thin, we have, by Lemma 2.2, that $\{\dot{x}_i : i < \lambda_\xi\}$ is still λ_ξ -Luzin in $V[G \cap \mathbb{P}_{\xi,\gamma}]$. Since \dot{y} is a $\mathbb{P}_{\xi,\gamma}$ -name, there is an $i < \lambda_\xi$ such that $\dot{y}[G_{\xi,\gamma}] \cap \dot{x}_i[G_{\xi,\gamma}]$ and $\dot{y}[G_{\xi,\gamma}] \setminus \dot{x}_i[G_{\xi,\gamma}]$ are infinite. \square

4. THE CONSTRUCTION OF \mathbf{P}

When constructing a matrix-iteration by recursion, we will need notation and language for extension. We will use, for an ordinal γ , \mathbf{P}^γ to indicate that \mathbf{P}^γ is a (κ, γ) -matrix iteration.

Definition 4.1. (1) *A matrix iteration \mathbf{P}^γ is an extension of \mathbf{P}^δ providing $\delta \leq \gamma$, and, for each $\alpha \leq \delta$ and $i \leq \kappa$, $\mathbb{P}_{i,\alpha}^{\mathbf{P}^\delta} = \mathbb{P}_{i,\alpha}^{\mathbf{P}^\gamma}$. We can use $\mathbf{P}^\gamma \upharpoonright \delta$ to denote the unique (κ, δ) -matrix iteration extended by \mathbf{P}^γ .*

(2) *If, for each $i < \kappa$, $\dot{Q}_{i,\gamma}$ is a $\mathbb{P}_{i,\gamma}^{\mathbf{P}}$ -name of a ccc poset satisfying that, for each $i < j < \kappa$, $\mathbb{P}_{i,\gamma} * \dot{Q}_{i,\gamma}$ is a complete subposet of $\mathbb{P}_{j,\gamma} * \dot{Q}_{j,\gamma}$, then we let $\mathbf{P} * \langle \dot{Q}_{i,\gamma} : i < \kappa \rangle$ denote the $(\kappa, \gamma + 1)$ -matrix $\langle \langle \mathbb{P}_{i,\alpha} : i \leq \kappa, \alpha \leq \gamma + 1 \rangle, \langle \dot{Q}_{i,\alpha} : i \leq \kappa, \alpha < \gamma + 1 \rangle \rangle$, where $\dot{Q}_{\kappa,\gamma}$ is the \mathbf{P} -name of the union of $\{\dot{Q}_{i,\gamma} : i < \kappa\}$ and, for $i \leq \kappa$, $\mathbb{P}_{i,\gamma} = \mathbb{P}_{i,\gamma}^{\mathbf{P}}$, $\mathbb{P}_{i,\gamma+1} = \mathbb{P}_{i,\gamma}^{\mathbf{P}} * \dot{Q}_{i,\gamma}$, and for $\alpha < \gamma$, $(\mathbb{P}_{i,\alpha}, \dot{Q}_{i,\alpha}) = (\mathbb{P}_{i,\alpha}^{\mathbf{P}}, \dot{Q}_{i,\alpha}^{\mathbf{P}})$.*

The following, from [3, Lemma 3.10], shows that extension at limit steps is canonical.

Lemma 4.2. *If γ is a limit and if $\{\mathbf{P}^\delta : \delta < \gamma\}$ is a sequence of matrix iterations satisfying that for $\beta < \delta < \gamma$, $\mathbf{P}^\delta \upharpoonright \beta = \mathbf{P}^\beta$, then there is a unique matrix iteration \mathbf{P}^γ such that $\mathbf{P}^\gamma \upharpoonright \delta = \mathbf{P}^\delta$ for all $\delta < \gamma$.*

Proof. For each $\delta < \gamma$ and $i < \kappa$, we define $\mathbb{P}_{i,\delta}^{\mathbf{P}^\gamma}$ to be $\mathbb{P}_{i,\delta}^{\mathbf{P}^\delta}$ and $\dot{\mathbb{Q}}_{i,\delta}^{\mathbf{P}^\gamma}$ to be $\dot{\mathbb{Q}}_{i,\delta}^{\mathbf{P}^{\delta+1}}$. It follows that $\dot{\mathbb{Q}}_{i,\delta}^{\mathbf{P}^\gamma}$ is a $\mathbb{P}_{i,\delta}^{\mathbf{P}^\gamma}$ -name. Since γ is a limit, the definition of $\mathbb{P}_{i,\gamma}^{\mathbf{P}^\gamma}$ is required to be $\bigcup\{\mathbb{P}_{i,\delta}^{\mathbf{P}^\gamma} : \delta < \gamma\}$ for $i < \kappa$. Similarly, the definition of $\mathbb{P}_{\kappa,\gamma}^{\mathbf{P}^\gamma}$ is required to be $\bigcup\{\mathbb{P}_{i,\gamma}^{\mathbf{P}^\gamma} : i < \kappa\}$. Let us note that $\mathbb{P}_{\kappa,\gamma}^{\mathbf{P}^\gamma}$ is also required to be the union of the chain $\bigcup\{\mathbb{P}_{\kappa,\delta}^{\mathbf{P}^\gamma} : \delta < \gamma\}$, and this holds by assumption on the sequence $\{\mathbf{P}^\delta : \delta < \gamma\}$.

To prove that \mathbf{P}^γ is a (κ, γ) -matrix it remains to prove that for $j < i \leq \kappa$, and each $q \in \mathbb{P}_{i,\gamma}^{\mathbf{P}^\gamma}$, there is a reduction p in $\mathbb{P}_{j,\gamma}^{\mathbf{P}^\gamma}$. Since γ is a limit, there is an $\alpha < \gamma$ such that $q \in \mathbb{P}_{i,\alpha}^{\mathbf{P}^\alpha}$ and, by assumption, there is a reduction, p , of q in $\mathbb{P}_{j,\alpha}^{\mathbf{P}^\alpha}$. By induction on β ($\alpha \leq \beta \leq \gamma$) we note that $q \in \mathbb{P}_{i,\beta}^{\mathbf{P}^\beta}$ and that p is a reduction of q in $\mathbb{P}_{j,\beta}^{\mathbf{P}^\beta}$. For limit β it is trivial, and for successor β it follows from condition (1) in the definition of matrix iteration. \square

We also will need the next result taken from [3, Lemma 13], which they describe as well known, for stepping diagonally in the array of posets.

Lemma 4.3. *Let \mathbb{P}, \mathbb{Q} be partial orders such that \mathbb{P} is a complete suborder of \mathbb{Q} . Let $\dot{\mathbb{A}}$ be a \mathbb{P} -name for a forcing notion and let $\dot{\mathbb{B}}$ be a \mathbb{Q} -name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subset \dot{\mathbb{B}}$, and every \mathbb{P} -name of a maximal antichain of $\dot{\mathbb{A}}$ is also forced by \mathbb{Q} to be a maximal antichain of $\dot{\mathbb{B}}$. Then $\mathbb{P} * \dot{\mathbb{A}} <_{\circ} \mathbb{Q} * \dot{\mathbb{B}}$*

Let us also note if $\dot{\mathbb{B}}$ is equal to $\dot{\mathbb{A}}$ in Lemma 4.3, then the hypothesis and the conclusion of the Lemma are immediate. On the other hand, if $\dot{\mathbb{A}}$ is the \mathbb{P} -name of $\mathbb{L}(\dot{\mathcal{D}})$ for some \mathbb{P} -name of a filter $\dot{\mathcal{D}}$, then the \mathbb{Q} -name of $\mathbb{L}(\dot{\mathcal{D}})$ is not necessarily equal to $\dot{\mathbb{A}}$.

Lemma 4.4 ([6, 1.9]). *Suppose that \mathbb{P}, \mathbb{Q} are posets with $\mathbb{P} <_{\circ} \mathbb{Q}$. Suppose also that $\dot{\mathcal{D}}_0$ is a \mathbb{P} -name of a filter on ω and $\dot{\mathcal{D}}_1$ is a \mathbb{Q} -name of a filter on ω . If $\Vdash_{\mathbb{Q}} \dot{\mathcal{D}}_0 \subseteq \dot{\mathcal{D}}_1$ then $\mathbb{P} * \mathbb{L}(\dot{\mathcal{D}}_0)$ is a complete subset of $\mathbb{Q} * \mathbb{L}(\dot{\mathcal{D}}_1)$ if either of the two equivalent conditions hold:*

- (1) $\Vdash_{\mathbb{Q}} ((\dot{\mathcal{D}}_0)^+)_{\mathbb{P}} \subseteq \dot{\mathcal{D}}_1^+$,
- (2) $\Vdash_{\mathbb{Q}} \dot{\mathcal{D}}_1 \cap V^{\mathbb{P}} \subseteq \langle \dot{\mathcal{D}}_0 \rangle$ (where $V^{\mathbb{P}}$ is the class of \mathbb{P} -names).

Proof. Let \dot{E} be any \mathbb{P} -name of a maximal antichain of $\mathbb{L}(\dot{\mathcal{D}}_0)$. By Lemma 4.3, it suffices to show that \mathbb{Q} forces that every member of

$\mathbb{L}(\dot{\mathcal{D}}_1)$ is compatible with some member of \dot{E} . Let G be any \mathbb{Q} -generic filter and let E denote the valuation of \dot{E} by $G \cap \mathbb{P}$. Working in the model $V[G \cap \mathbb{P}]$, we have the function ρ_E as in Lemma 2.4. Choose $\delta \in \omega_1$ satisfying that $\rho_E(t) < \delta$ for all $t \in \omega^{<\omega}$. Now, working in $V[G]$, we consider any $T \in \mathbb{L}(\dot{\mathcal{D}}_1)$ and we find an element of E that is compatible with T . In fact, by induction on $\alpha < \delta$, one easily proves that for each $T \in \mathbb{L}(\dot{\mathcal{D}}_1)$ with $\rho_E(\text{stem}(T)) \leq \alpha$, T is compatible with some member of E . \square

Definition 4.5. For a (κ, γ) -matrix-iteration $\underline{\mathbf{P}}$, and ordinal $i_\gamma < \kappa$, we say that an increasing sequence $\langle \dot{\mathcal{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases, if for each $i < j < \kappa$

- (1) $\dot{\mathcal{D}}_i$ is a subset of $\mathbb{B}_{i,\gamma}$ (hence a nice $\mathbb{P}_{i,\gamma}^{\underline{\mathbf{P}}}$ -name)
- (2) $\Vdash_{\mathbb{P}_{i,\gamma}} \dot{\mathcal{D}}_i$ is a filter with a base of cardinality at most μ_{i_γ} ,
- (3) $\Vdash_{\mathbb{P}_{j,\gamma}} \langle \dot{\mathcal{D}}_j \rangle \cap \mathbb{B}_{i,\gamma} \subseteq \langle \dot{\mathcal{D}}_i \rangle$.

Notice that a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases can be (essentially) eventually constant. Thus we will say that a sequence $\langle \dot{\mathcal{D}}_i : i \leq j \rangle$ (for some $j < \kappa$) is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases if the sequence $\langle \dot{\mathcal{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases where $\dot{\mathcal{D}}_i$ is the $\mathbb{P}_{i,\gamma}$ -name for $\mathbb{B}_{i,\gamma} \cap \langle \dot{\mathcal{D}}_j \rangle$ for $j < i \leq \kappa$. When $\underline{\mathbf{P}}$ is clear from the context, we will use $\vec{\lambda}(i_\gamma)$ -thin as an abbreviation for $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin.

Corollary 4.6. For a (κ, γ) -matrix-iteration $\underline{\mathbf{P}}$, ordinal $i_\gamma < \kappa$, and a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases $\langle \dot{\mathcal{D}}_\xi : \xi < \kappa \rangle$, $\underline{\mathbf{P}} * \langle \dot{\mathcal{Q}}_{i,\gamma} : i \leq \kappa \rangle$ is a $\gamma + 1$ -extension of $\underline{\mathbf{P}}$, where, for each $i \leq i_\gamma$, $\dot{\mathcal{Q}}_{i,\gamma}$ is the trivial poset, and for $i_\gamma \leq i < \kappa$, $\dot{\mathcal{Q}}_{i,\gamma}$ is $\mathbb{L}(\dot{\mathcal{D}}_i)$.

Definition 4.7. Whenever $\langle \dot{\mathcal{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases, let $\underline{\mathbf{P}} * \mathbb{L}(\langle \dot{\mathcal{D}}_i : i_\gamma \leq i < \kappa \rangle)$ denote the $\gamma + 1$ -extension described in Corollary 4.6.

This next corollary is immediate.

Corollary 4.8. If $\underline{\mathbf{P}}$ is a $\vec{\lambda}$ -thin (κ, γ) -matrix and if $\langle \dot{\mathcal{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases, then $\underline{\mathbf{P}} * \mathbb{L}(\langle \dot{\mathcal{D}}_i : i_\gamma \leq i < \kappa \rangle)$ is a $\vec{\lambda}$ -thin $(\kappa, \gamma + 1)$ -matrix.

We now describe a first approximation of the scheme, $\mathcal{K}(\vec{\lambda})$, of posets that we will be using to produce the model.

Definition 4.9. For an ordinal $\gamma > 0$ and a (κ, γ) -matrix iteration $\underline{\mathbf{P}}$, we will say that $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ providing for each $0 < \alpha < \gamma$,

- (1) for each $i \leq \kappa$, $\mathbb{P}_{i,1}^{\mathbf{P}}$ is $\text{Fn}(\lambda_i, 2)$, and
- (2) there is an $i_\alpha = i_\alpha^{\mathbf{P}} < \kappa$ and a $(\mathbf{P} \upharpoonright \alpha, \vec{\lambda}(i_\alpha))$ -thin sequence $\langle \dot{\mathcal{D}}_i^\alpha : i < \kappa \rangle$ of filter bases, such that $\mathbf{P} \upharpoonright \alpha + 1$ is equal to $\mathbf{P} \upharpoonright \alpha * \mathbb{L}(\langle \dot{\mathcal{D}}_i^\alpha : i_\alpha \leq i < \kappa \rangle)$.

For each $0 < \alpha < \gamma$, we let $\dot{\mathcal{D}}_\kappa^\alpha$ denote the $\mathbf{P} \upharpoonright \alpha$ -name of the union $\bigcup \{ \dot{\mathcal{D}}_i^\alpha : i_\alpha \leq i < \kappa \}$, and we let \dot{L}_α denote the canonical $\mathbf{P} \upharpoonright \alpha + 1$ -name of the subset of ω added by $\mathbb{L}(\dot{\mathcal{D}}_\kappa^\alpha)$.

Let us note that each $\mathbf{P} \in \mathcal{K}(\vec{\lambda})$ is $\vec{\lambda}$ -thin. Furthermore, by Lemma 3.5, this means that each $\mathbf{P} \in \mathcal{K}(\vec{\lambda})$ forces that $\mathfrak{s} \leq \lambda$. We begin a new section for the task of proving that there is a $\mathbf{P} \in \mathcal{K}(\vec{\lambda})$ that forces that $\mathfrak{s} \geq \lambda$.

It will be important to be able to construct $(\mathbf{P}, \vec{\lambda}(i_\gamma))$ -thin sequences of filter bases, and it seems we will need some help.

Definition 4.10. For an ordinal $\gamma > 0$ and a (κ, γ) -matrix iteration \mathbf{P} we will say that $\mathbf{P} \in \mathcal{H}(\vec{\lambda})$ if \mathbf{P} is in $\mathcal{K}(\vec{\lambda})$ and for each $0 < \alpha < \gamma$, if $i_\alpha = i_\alpha^{\mathbf{P}} > 0$ then $\omega_1 \leq \text{cf}(\alpha) \leq \mu_{i_\alpha}$ and there is a $\beta_\alpha < \alpha$ such that

- (1) for $\beta_\alpha \leq \xi < \alpha$, $i_\xi \in \{0, i_\alpha\}$,
- (2) if $\beta_\alpha \leq \eta < \alpha$, $i_\eta > 0$ and $\xi = \eta + \omega_1 \leq \alpha$, then $\dot{L}_\eta \in \dot{\mathcal{D}}_{i_\xi}^\xi$, and $\mathbb{P}_{i_\xi, \xi} \Vdash \dot{\mathcal{D}}_{i_\xi}^\alpha$ has a descending mod finite base of cardinality ω_1 ,
- (3) if $\beta_\alpha < \xi \leq \alpha$, $i_\xi > 0$, and $\eta + \omega_1 < \xi$ for $\eta < \xi$, then $\{\dot{L}_\eta : \beta_\alpha \leq \eta < \alpha, \text{cf}(\eta) \geq \omega_1\}$ is a base for $\dot{\mathcal{D}}_{i_\xi}^\xi$.

5. PRODUCING $\vec{\lambda}$ -THIN FILTER SEQUENCES

In this section we prove this main lemma.

Lemma 5.1. Suppose that $\mathbf{P}^\gamma \in \mathcal{H}(\vec{\lambda})$ and that \mathcal{Y} is a set of fewer than λ nice \mathbf{P}^γ -names of subsets of ω , then there is a $\delta < \gamma + \lambda$ and an extension \mathbf{P}^δ of \mathbf{P}^γ in $\mathcal{H}(\vec{\lambda})$ that forces that the family \mathcal{Y} is not a splitting family.

The main theorem follows easily.

Proof of Theorem 1.2. Let θ be any regular cardinal so that $\theta^{<\lambda} = \theta$ (for example, $\theta = (2^\lambda)^+$). Construct $\mathbf{P}^\theta \in \mathcal{H}(\vec{\lambda})$ so that for all $\mathcal{Y} \subset \mathbb{B}_{\kappa, \theta}$ with $|\mathcal{Y}| < \lambda$, there is a $\gamma < \delta < \theta$ so that $\mathcal{Y} \subset \mathbb{B}_{\kappa, \gamma}$ and, by applying Lemma 5.1, such that $\mathbf{P}^\theta \upharpoonright \delta$ forces that \mathcal{Y} is not a splitting family. \square

We begin by reducing our job to simply finding a $(\mathbf{P}, \vec{\lambda}(i_\gamma))$ -thin sequence.

Definition 5.2. For a (κ, γ) -matrix-iteration \mathbf{P}^γ , we say that a subset \mathcal{E} of $\mathbb{B}_{\kappa, \gamma}$ is $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter subbase if, $i_\gamma < \kappa$, $|\mathcal{E}| \leq \mu_{i_\gamma}$, and the sequence $\langle \langle \mathcal{E} \cap \mathbb{B}_{i_\gamma} \rangle : i < \kappa \rangle$ is a $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin sequence of filter bases.

Lemma 5.3. For any $\mathbf{P}^\gamma \in \mathcal{H}(\vec{\lambda})$, and any $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter base \mathcal{E} , there is an $\alpha \leq \gamma + \mu_{i_\gamma} + 1$ and extensions $\mathbf{P}^\alpha, \mathbf{P}^{\alpha+1}$ of \mathbf{P}^γ in $\mathcal{H}(\vec{\lambda})$, such that, $\mathbf{P}^{\alpha+1} = \mathbf{P}^\alpha * \mathbb{L}(\langle \dot{\mathcal{D}}_i^\alpha : i_\alpha \leq i < \kappa \rangle)$ and \mathbf{P}^α forces that $\mathcal{E} \cap \mathbb{B}_{i_\gamma}$ is a subset of $\dot{\mathcal{D}}_i^\alpha$ for all $i < \kappa$.

Proof. The case $i_\gamma = 0$ is trivial, so we assume $i_\gamma > 0$. There is no loss of generality to assume that $\mathcal{E} \cap \mathbb{B}_{i_\gamma, \gamma}$ has character μ_{i_γ} . Let $\{\dot{E}_\xi : \xi < \mu_{i_\gamma}\} \subset \mathcal{E} \cap \mathbb{B}_{i_\gamma, \gamma}$ enumerate a filter base for $\langle \mathcal{E} \rangle \cap \mathbb{B}_{i_\gamma, \gamma}$. We can assume that this enumeration satisfies that $\dot{E}_\xi \setminus \dot{E}_{\xi+1}$ is forced to be infinite for all $\xi < \mu_{i_\gamma}$. Let \mathcal{A} be any countably generated free filter on ω that is not principal mod finite. By induction on $\xi < \mu_{i_\gamma}$ we define $\mathbf{P}^{\gamma+\xi}$ by simply defining $i_{\gamma+\xi}$ and the sequence $\langle \dot{\mathcal{D}}_i^{\gamma+\xi} : i_{\gamma+\xi} \leq i \leq \kappa \rangle$. We will also recursively define, for each $\xi < \mu_{i_\gamma}$, a $\mathbf{P}^{\gamma+\xi}$ -name \dot{D}_ξ such that $\mathbf{P}^{\gamma+\xi}$ forces that $\dot{D}_\xi \subset \dot{E}_\xi$. An important induction hypothesis is that $\{\dot{D}_\eta : \eta < \xi\} \cup \{\dot{E}_\zeta : \zeta < \mu_{i_\gamma}\} \cup \mathcal{E}$ is forced to have the finite intersection property.

For each $\xi < \gamma + \omega_1$, let $i_\xi = 0$ and $\dot{\mathcal{D}}_i^\xi$ be the \mathbf{P}^ξ -name $\langle \mathcal{A} \rangle \cap \mathbb{B}_{i_\xi}$ for all $i \leq \kappa$. The definition of \dot{D}_0 is simply \dot{E}_0 . By recursion, for each $\eta < \omega_1$ and $\xi = \eta + 1$, we define \dot{D}_ξ to be the intersection of \dot{D}_η and \dot{E}_ξ . For limit $\xi < \omega_1$, we note that $\mathbb{P}_{i_\gamma, \xi}$ forces that $\mathbb{L}(\langle \mathcal{A} \rangle)$ is isomorphic to $\mathbb{L}(\langle \{\dot{D}_\eta \cap \dot{E}_\xi : \eta < \xi\} \rangle)$. Therefore, we can let \dot{D}_ξ be a $\mathbf{P}^{\xi+1}$ -name for the generic real added by $\mathbb{L}(\langle \{\dot{D}_\eta \cap \dot{E}_\xi : \eta < \xi\} \rangle)$. A routine density argument shows that this definition satisfies the induction hypothesis.

The definition of $i_{\gamma+\omega_1}$ is i_γ and the definition of $\dot{\mathcal{D}}_{i_\gamma}^{\gamma+\omega_1}$ is the filter generated by $\{\dot{D}_\xi : \xi < \omega_1\}$. The definition of \dot{D}_{ω_1} is $\dot{L}_{\gamma+\omega_1}$.

Let S denote the set of $\eta < \mu_{i_\gamma}$ with uncountable cofinality. We now add additional induction hypotheses:

- (1) if $\zeta = \sup(S \cap \xi) < \xi$ and $\xi = \nu + 1$, then $\dot{D}_\xi = \dot{D}_\nu \cap \dot{E}_\xi$, and $i_\xi = 0$ and $\dot{\mathcal{D}}_i^{\gamma+\xi} = \langle \mathcal{A} \rangle$ for all $i \leq \kappa$
- (2) if $\zeta = \sup(S \cap \xi) < \xi$ and ξ is a limit of countable cofinality, then $i_\xi = 0$ and $\dot{\mathcal{D}}_i^{\gamma+\xi} = \langle \mathcal{A} \rangle$ for all $i \leq \kappa$, and \dot{D}_ξ is forced by $\mathbf{P}^{\gamma+\xi+1}$ to be the generic real added by $\mathbb{L}(\langle \{\dot{D}_\eta \cap \dot{E}_\xi : \zeta \leq \eta < \xi\} \rangle)$,
- (3) if $\zeta = \sup(S \cap \xi)$ and $\xi = \zeta + \omega_1$, then $i_\xi = i_\gamma$, $\dot{\mathcal{D}}_{i_\xi}^{\gamma+\xi}$ is the filter generated by $\{\dot{E}_\xi \cap \dot{D}_\eta : \zeta \leq \eta < \xi\}$ and \dot{D}_ξ is $\dot{L}_{\gamma+\xi}$,

- (4) if $S \cap \xi$ is cofinal in ξ and $\text{cf}(\xi) > \omega$, then $i_\xi = i_\gamma$ and $\dot{\mathfrak{D}}_{i_\xi}^{\gamma+\xi}$ is the filter generated by $\{\dot{D}_{\gamma+\eta} : \eta \in S \cap \xi\}$ and $\dot{D}_\xi = \dot{L}_{\gamma+\xi}$,
- (5) if $S \cap \xi$ is cofinal in ξ and $\text{cf}(\xi) = \omega$, then $i_\xi = 0$ and $\dot{\mathfrak{D}}_i^{\gamma+\xi} = \langle \mathcal{A} \rangle$ for all $i \leq \kappa$, and \dot{D}_ξ is forced by $\mathbf{P}^{\gamma+\xi+1}$ to be the generic real added by $\mathbb{L}(\{\dot{D}_{\eta_n} \cap \dot{E}_\xi : n \in \omega\})$, where $\{\eta_n : n \in \omega\}$ is some increasing cofinal subset of $S \cap (\gamma, \xi)$.

It should be clear that the induction continues to stage μ_{i_γ} and that $\mathbf{P}^{\gamma+\xi} \in \mathcal{H}(\vec{\lambda}(i_\gamma))$ for all $\xi \leq \mu_{i_\gamma}$, with $\beta_{\gamma\xi} = \gamma$ being the witness to Definition 4.10 for all ξ with $\text{cf}(\xi) > \omega$.

The final definition of the sequence $\langle \dot{\mathfrak{D}}_i^\delta : i_\delta = i_\gamma \leq i \leq \kappa \rangle$, where $\delta = \gamma + \mu_{i_\gamma}$ is that $\dot{\mathfrak{D}}_{i_\gamma}^\delta$ is the filter generated by $\{\dot{L}_{\gamma+\xi} : \text{cf}(\xi) > \omega\}$, and for $i_\gamma < i \leq \kappa$, $\dot{\mathfrak{D}}_i^\delta$ is the filter generated by $\dot{\mathfrak{D}}_{i_\gamma}^\delta \cup (\mathcal{E} \cap \mathbb{B}_{i,\gamma})$. \square

Lemma 5.4. *Suppose that \mathcal{E} is a $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter base. Also assume that $i < \kappa$ and $\alpha \leq \gamma$ and $\mathcal{E}_1 \subset \mathbb{B}_{i,\alpha}$ is a $(\mathbf{P}^\alpha, \vec{\lambda}(i_\gamma))$ -thin filter base satisfying that $\langle \mathcal{E} \rangle \cap \mathbb{B}_{i,\alpha} \subset \langle \mathcal{E}_1 \rangle$, then $\mathcal{E} \cup \mathcal{E}_1$ is a $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter subbase.*

Proof. Let \mathcal{E}_2 be equal to $\mathcal{E} \cup \mathcal{E}_1$. The fact that each member of the sequence $\langle \dot{\mathcal{D}}_j = \langle \mathcal{E}_2 \cap \mathbb{B}_{j,\gamma} \rangle : j < \kappa \rangle$ is a name of a filter base with character at most μ_{i_γ} is immediate. Now we verify that if $j_1 < j_2 < \kappa$, then $\Vdash_{\mathbb{P}_{j_2,\gamma}} \dot{\mathcal{D}}_{j_2} \cap \mathbb{B}_{j_1,\gamma} \subset \dot{\mathcal{D}}_{j_1}$. Let $\dot{b} \in \mathbb{B}_{j_2,\gamma}$ and suppose there are $p \in \mathbb{P}_{j_2,\gamma}$, $\dot{E}_0 \in \mathcal{E} \cap \mathbb{B}_{j_2,\gamma}$, and $\dot{E}_1 \in \mathcal{E}_1$ such that $p \Vdash \dot{b} \cap \dot{E}_0 \cap \dot{E}_1$. It suffices to produce an $\dot{E} \in \langle \mathcal{E}_2 \rangle \cap \mathbb{B}_{j_1,\gamma}$ satisfying that $p \Vdash \dot{b} \cap \dot{E} = \emptyset$. First, using that \mathcal{E} is $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin, choose $\dot{E}_2 \in \langle \mathcal{E} \rangle \cap \mathbb{B}_{j_1,\gamma}$ such that $p \Vdash (\dot{b} \setminus \dot{E}_0) \cap \dot{E}_2 = \emptyset$. Equivalently, we have that $p \Vdash (\dot{b} \cap \dot{E}_2) \subset \dot{E}_0$, and therefore $p \Vdash (\dot{b} \cap \dot{E}_2) \cap \dot{E}_1 = \emptyset$. Since \dot{E}_1 is a $\mathbb{P}_{j_2,\alpha}$ -name, there is a $\mathbb{P}_{j_1,\alpha}$ -name (which we can denote as) $(\dot{b} \cap \dot{E}_2) \upharpoonright \alpha$ satisfying that $p \Vdash \dot{E}_2 \cap (\dot{b} \cap \dot{E}_2) \upharpoonright \alpha$ is empty and that $p \Vdash (\dot{b} \cap \dot{E}_2) \subset (\dot{b} \cap \dot{E}_2) \upharpoonright \alpha$. Now using that \mathcal{E}_1 is $(\mathbf{P}^\alpha, \vec{\lambda}(i_\gamma))$ -thin, choose $\dot{E}_3 \in \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{j_1,\alpha}$ so that $p \Vdash \dot{E}_3 \cap (\dot{b} \cap \dot{E}_2) \upharpoonright \alpha$ is empty. Naturally we have that $p \Vdash \dot{E}_3 \cap (\dot{b} \cap \dot{E}_2)$ is also empty. This completes the proof since $\dot{E}_2 \cap \dot{E}_3$ is in $\langle \mathcal{E}_2 \rangle \cap \mathbb{B}_{j_1,\gamma}$. \square

Let $\mathbf{P}^\gamma \in \mathcal{H}(\vec{\lambda})$ and let $\dot{y} \in \mathbb{B}_{\kappa,\gamma}$. For a family $\mathcal{E} \subset \mathbb{B}_{\kappa,\gamma}$ and condition $p \in \mathbf{P}^\gamma$ say that p forces that \mathcal{E} measures \dot{y} if $p \Vdash_{\mathbf{P}^\gamma} \{\dot{y}, \omega \setminus \dot{y}\} \cap \langle \mathcal{E} \rangle \neq \emptyset$. Naturally we will just say that \mathcal{E} measures \dot{y} if 1 forces that \mathcal{E} measures \dot{y} .

Given Lemma 5.3, it will now suffice to prove.

Lemma 5.5. *If $\mathcal{Y} \subset \mathbb{B}_{\kappa,\gamma}$ for some $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$ and $|\mathcal{Y}| \leq \mu_{i_\gamma}$ for some $i_\gamma < \kappa$, then there is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter $\mathcal{E} \subset \mathbb{B}_{\kappa,\gamma}$ that measures every element of \mathcal{Y} .*

In fact, to prove Lemma 5.5, it is evidently sufficient to prove:

Lemma 5.6. *If $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$, $\dot{y} \in \mathbb{B}_{\kappa,\gamma}$, and if \mathcal{E} is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter, then there is a family $\mathcal{E}_1 \supset \mathcal{E}$ measuring \dot{y} that is also a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter.*

Proof. Throughout the proof we suppress mention of $\underline{\mathbf{P}}^\gamma$ and refer instead to component member posets $\mathbb{P}_{i,\alpha}, \dot{\mathbb{Q}}_{i,\alpha}$ of $\underline{\mathbf{P}}^\gamma$. Let $i_{\dot{y}}$ be minimal such that \dot{y} is in $\mathbb{B}_{i_{\dot{y}},\gamma}$. Proceeding by induction, we can assume that the lemma holds for all $\dot{x} \in \mathbb{B}_{j,\gamma}$ and all $j < i_{\dot{y}}$.

We can replace \dot{y} by any $\dot{x} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ that has the property that $1 \Vdash \dot{x} \in \{\dot{y}, \omega \setminus \dot{y}\}$ since if we measure \dot{x} then we also measure \dot{y} . With this reduction then we can assume that no condition forces that $\omega \setminus \dot{y}$ is in the filter generated by \mathcal{E} .

Fact 1. If $i_{\dot{y}} \leq i_\gamma$, then there is a $\dot{E} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ such that $\mathcal{E} \cup \{\dot{E}\}$ is contained a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter that measures \dot{y} .

Proof of Fact 1. It is immediate that $\langle \{\dot{y}\} \cup (\mathbb{B}_{i_{\dot{y}},\gamma} \cap \mathcal{E}) \rangle$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter. Therefore, by Lemma 5.4, $\mathcal{E} \cup \{\dot{y}\}$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter subbase. \square

We may thus assume that $0 < i_{\dot{y}}$ and that the Lemma has been proven for all members of $\mathbb{B}_{i,\gamma}$ for all $i < i_{\dot{y}}$. Similarly, let $\alpha_{\dot{y}}$ be minimal so that $\dot{y} \in \mathbb{B}_{i_{\dot{y}},\alpha_{\dot{y}}}$, and assume that the Lemma has been proven for all members of $\mathbb{B}_{i_{\dot{y}},\beta}$ for all $\beta < \alpha_{\dot{y}}$. We skip proving the easy case when $\alpha_{\dot{y}} = 1$ and henceforth assume that $1 < \alpha_{\dot{y}}$. Notice also that $\alpha_{\dot{y}}$ has countable cofinality since $\mathbb{P}_{i_{\dot{y}},\gamma}$ is ccc.

Now choose an elementary submodel M of $H((2^{\lambda_\gamma})^+)$ containing $\vec{\lambda}, \underline{\mathbf{P}}^\gamma, \mathcal{E}, \dot{y}$ and so that M has cardinality equal to μ_{i_γ} and, by our cardinal assumptions, $M^{\lambda_j} \subset M$ for all $j < i_\gamma$. Naturally this implies that $M^\omega \subset M$.

By the inductive assumption we may assume that there is an $\mathcal{E}_1 \supset \mathcal{E}$ that is $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin and measures every element of $M \cap \mathbb{B}_{j,\gamma}$ for $j < i_{\dot{y}}$ as well as every element of $M \cap \mathbb{B}_{i_{\dot{y}},\beta}$ for all $\beta \in M \cap \alpha_{\dot{y}}$. Moreover, it is easily checked that we can assume that \mathcal{E}_1 is a subset of M . Furthermore, we may assume that \mathcal{E}_1 contains a maximal family of subsets of $M \cap \mathbb{B}_{i_{\dot{y}},\alpha_{\dot{y}}}$ that forms a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$ -thin filter subbase.

Fact 2. There is a maximal antichain $A \subset \mathbb{P}_{i_{\dot{y}},\gamma}$ and a subset $A_1 \subset A$ such that

- (1) each $p \in A_1$ forces that \mathcal{E}_1 measures \dot{y} ,
- (2) for each $p \in A \setminus A_1$, p forces that there is an $i_p < i_{\dot{y}}$ such that $\mathbb{B}_{i_p, \gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle$ is not generated by the elements in M ,
- (3) for each $p \in A \setminus A_1$, p forces that there is a $j_p < i_{\dot{y}}$ such that $i_p \leq j_p$ and $\mathbb{B}_{j_p, \gamma} \cap \langle \mathcal{E}_1 \cup \{\omega \setminus \dot{y}\} \rangle$ is not generated by the elements in M .

Proof of Fact 2. Suppose that $p \in \mathbb{P}_{i_{\dot{y}}, \gamma}$ forces that the conclusion (2) fails. We have already arranged that $p \Vdash_{\mathbb{P}_{i_{\dot{y}}, \gamma}} \dot{y} \in \langle \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}}, \gamma} \rangle^+$. Define $\dot{E} \in \mathbb{B}_{i_{\dot{y}}, \gamma}$ so that p forces $\dot{E} = \dot{y}$ and each $q \in \mathbb{P}_{i_{\dot{y}}, \gamma} \cap p^\perp$ forces that $\dot{E} = \omega$. It is easily checked that $\mathbb{B}_{i_{\dot{y}}, \gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{E}\} \rangle$ is then $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin and that p forces that it measures \dot{y} . This condition ensures that p is compatible with an element of A_1 .

If (2) holds but (3) fails, then by a symmetric argument as in the previous paragraph we can again define \dot{E} so that $\mathbb{B}_{i_{\dot{y}}, \gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{E}\} \rangle$ is then $(\mathbf{P}^\gamma, \vec{\lambda}(i_\gamma))$ -thin and that p forces that it measures $\omega \setminus \dot{y}$. \square

If by increasing M we can enlarge A_1 we simply do so. Since \mathbf{P}^γ is ccc we may assume that this is no longer possible, and therefore we may also assume that A is a subset of M . Now we choose any $p \in A \setminus A_1$. It suffices to produce an $\dot{E}_p \in \mathbb{B}_{i_{\dot{y}}, \gamma}$ that can be added to \mathcal{E}_1 that measures \dot{y} and satisfies that $q \Vdash \dot{E}_p = \omega$ for all $q \in p^\perp$. This is because we then have that $\mathcal{E}_1 \cup \{\dot{E}_p : p \in A \setminus A_1\}$ is contained in a $\vec{\lambda}(i_\gamma)$ -thin filter that measures \dot{y} .

Fact 3. There is an α such that $\alpha_{\dot{y}} = \alpha + 1$.

Proof of Fact 3. Otherwise, let $j = i_p$ and for each $r < p$ in $\mathbb{P}_{i_{\dot{y}}, \alpha_{\dot{y}}}$, choose $\beta \in M \cap \alpha_{\dot{y}}$ such that $r \in \mathbb{P}_{i_{\dot{y}}, \beta}$, and define a name $\dot{y}[r]$ in $M \cap \mathbb{B}_{j, \gamma}$ according to $(\ell, q) \in \dot{y}[r]$ providing there is a pair $(\ell, p_\ell) \in \dot{y}$ such that $q <_j p_\ell$ and $q \upharpoonright \beta$ is in the set $M \cap \mathbb{P}_{j, \beta} \setminus (r \wedge p_\ell \upharpoonright \beta)^\perp$. This set, namely $\dot{y}[r]$, is in M because $\mathbb{P}_{j, \beta}$ is ccc and $M^\omega \subset M$.

We prove that r forces that $\dot{y}[r]$ contains \dot{y} . Suppose that $r_1 < r$ and there is a pair $(\ell, p_\ell) \in \dot{y}$ with $r_1 < p_\ell$. Choose an $r_2 \in \mathbb{P}_{j, \gamma}$ so that $r_2 <_j r_1$. It suffices to show $r_2 \Vdash \ell \in \dot{y}[r]$. Let $q <_j p_\ell$ with $q \in M$. Then $r_2 \not\leq_j p_\ell$ implies $r_2 \not\leq_j q$. Since r_2 was any $<_j$ -projection of r_1 we can assume that $r_2 < q$. Since $r_2 \upharpoonright \beta$ is in $(\mathbb{P}_{j, \beta} \cap (r \wedge p_\ell \upharpoonright \beta)^\perp)^\perp$, it follows that $q \upharpoonright \beta \notin (r \wedge p_\ell \upharpoonright \beta)^\perp$. This implies that $(\ell, q) \in \dot{y}[r]$ and completes the proof that $r_2 \Vdash \ell \in \dot{y}[r]$.

Now assume that $\beta < \alpha_{\dot{y}}$ and $r \Vdash \dot{b} \cap \dot{E} \cap \dot{y}$ is empty for some $r < p$ in $\mathbb{P}_{i_{\dot{y}}, \beta}$, $\dot{b} \in \mathbb{B}_{j, \gamma}$, and $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}}, \gamma}$. Let $\dot{x} = (\dot{E} \cap \dot{y})[r]$ (defined as above for $\dot{y}[r]$). We complete the proof of Fact 3 by proving that $r \Vdash \dot{b} \cap \dot{x}$ is

empty. Since each are in $\mathbb{B}_{j,\gamma}$, we may choose any $r_1 <_j r$, and assume that $r_1 \Vdash \ell \in \dot{b} \cap \dot{x}$. In addition we can suppose that there is a pair $(\ell, q) \in \dot{x}$ such that $r_1 < q$. The fact that $(\ell, q) \in \dot{x}$ means there is a p_ℓ with (ℓ, p_ℓ) in the name $\dot{E} \cap \dot{y}$ such that $q <_j p_\ell$. Since $r_1 \in \mathbb{P}_{j,\gamma}$ and $r_1 < q$, it follows that $r_1 \not\leq p_\ell$. Now it follows that r_1 has an extension forcing that $\ell \in \dot{b} \cap (\dot{E} \cap \dot{y})$ which is a contradiction. \square

Fact 4. $i_{\dot{y}} = i_\alpha$ and so also $i_p < i_\alpha$.

Proof of Fact 4. Since $\mathbb{P}_{i,\alpha+1} = \mathbb{P}_{i,\alpha}$ for $i < i_\alpha$, we have that $i_\alpha \leq i_{\dot{y}}$. Now assume that $i_\alpha < i_{\dot{y}}$ and we proceed much as we did in Fact 3 to prove that i_p does not exist. Assume that $r < p$ (in $\mathbb{P}_{i_{\dot{y}},\alpha+1}$) and $r \Vdash \dot{b} \cap (\dot{E} \cap \dot{y})$ is empty for some $\dot{E} \in M \cap \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_{\dot{y}},\gamma}$ and $\dot{b} \in \mathbb{B}_{i_p,\gamma}$. It follows from Lemma 5.4 that we can simply assume that $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}},\alpha+1}$, and similarly that $\dot{b} \in \mathbb{B}_{i_p,\alpha+1}$.

Let \dot{T}_α be the $\mathbb{P}_{i_{\dot{y}},\alpha}$ -name such that $r \upharpoonright \alpha \Vdash r(\alpha) = \dot{T}_\alpha \in \mathbb{L}(\mathfrak{D}_{i_{\dot{y}}}^\alpha)$. We may assume that there is a $t_\alpha \in \omega^{<\omega}$ such that $r \upharpoonright \alpha \Vdash t_\alpha = \text{stem}(\dot{T}_\alpha)$.

Choose any $M \cap \mathbb{P}_{i_\alpha,\alpha}$ -generic filter \bar{G} such that $r \upharpoonright \alpha \in \bar{G}^+$. Since $\mathbb{P}_{i_\alpha,\alpha}$ is ccc and $M^\omega \subset M$, it follows that $M[\bar{G}]$ is closed under ω -sequences in the model $V[\bar{G}]$.

In this model, define an $\mathbb{L}(\mathfrak{D}_{i_\alpha}^\alpha)$ -name \dot{x} . A pair $(\ell, T_\ell) \in \dot{x}$ if $t_\alpha \leq \text{stem}(T_\ell) \in T_\ell \in \mathbb{L}(\mathfrak{D}_{i_\alpha}^\alpha)$ and for each $\text{stem}(T_\ell) \leq t \in T_\ell$, there is a pair $(\ell, q_{\ell,t}) \in M$ in the name $(\dot{y} \cap \dot{E})$ such that $q_{\ell,t} \upharpoonright \alpha \in \bar{G}^+$, $q_{\ell,t} \upharpoonright \alpha \Vdash t = \text{stem}(q_{\ell,t}(\alpha))$, and $(q_{\ell,t} \upharpoonright \alpha \wedge r \upharpoonright \alpha)$ does not force (over the poset \bar{G}^+) that $t \notin \dot{T}_\alpha$. We will show that r forces over the poset \bar{G}^+ that \dot{x} contains $\dot{E} \cap \dot{y}$ and that $\dot{x} \cap \dot{b}$ is empty. This proves that p forces that $\langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_p,\alpha+1}$ generates $\langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle \cap \mathbb{B}_{i_p,\alpha+1}$ since \dot{x} must be forced to be in $\langle \mathcal{E}_1 \rangle$. It then follows from Lemma 5.4 that $\mathcal{E}_1 \cap \mathbb{B}_{i_p,\gamma}$ generates $\langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle \cap \mathbb{B}_{i_p,\gamma}$, contradicting the assumption on i_p .

To prove that r forces that \dot{x} contains $\dot{y} \cap \dot{E}$, we consider any $r_\ell < r$ that forces over \bar{G}^+ that $\ell \in \dot{y} \cap \dot{E}$. We may choose $(\ell, p_\ell) \in M$ in the name $(\dot{E} \cap \dot{y})$ such that (wlog) $r_\ell < p_\ell$. We may assume that $r_\ell \upharpoonright \alpha$ forces a value t on $\text{stem}(r_\ell(\alpha))$ and that this equals $\text{stem}(p_\ell(\alpha))$. Now show there is a $T_\ell \in \mathbb{L}(\mathfrak{D}_{i_\alpha}^\alpha)$. In fact, assume $t \in T_\ell$ with $q_{\ell,t}$ as the witness. Let $L^- = \{k : t \frown k \notin T_\ell\}$; it suffices to show that $L^- \notin (\mathfrak{D}_{i_\alpha}^\alpha)^+$.

By assumption that $q_{\ell,t}$ is the witness, there is an $r_t < (q_{\ell,t} \upharpoonright \alpha \wedge r \upharpoonright \alpha)$ such that $r_t \Vdash t \in \dot{T}_\alpha$ and $r_t \Vdash t = \text{stem}(q_{\ell,t}(\alpha))$. By strengthening r_t we can assume that r_t forces a value $\dot{D} \in \mathfrak{D}_{i_{\dot{y}}}^\alpha$ on $\{k : t \frown k \in \dot{T}_\alpha \cap q_{\ell,t}(\alpha)\}$. But now, it follows that r_t forces that \dot{D} is disjoint from L^- since if $r_{t,k} \Vdash k \in \dot{D}$ for some $r_{t,k} < r_t$, $r_{t,k}$ is the witness to $(\ell, q_{\ell,t \frown k})$ is in $(\dot{y} \cap \dot{E})$ etc., where $q_{\ell,t \frown k} \upharpoonright \alpha = q_{\ell,t} \upharpoonright \alpha$ and $q_{\ell,t \frown k}(\alpha) = (q_{\ell,t}(\alpha))_{t \frown k}$.

Since some condition forces that L^- is not in $(\dot{\mathfrak{D}}_{i\dot{y}}^\alpha)^+$ it follows that L^- is not in $(\dot{\mathfrak{D}}_{i\alpha}^\alpha)^+$

Finally we must show that r forces over \bar{G}^+ that \dot{b} is disjoint from \dot{x} . Since each are $\mathbb{P}_{i_p, \alpha+1}$ -names, it suffices to assume that $\bar{r} \in \bar{G}^+$ is some $\mathbb{P}_{i_p, \alpha+1}$ -reduct of r that forces some ℓ is in $\dot{b} \cap \dot{x}$, and to then show that r fails to force that $\ell \notin \dot{b} \cap (\dot{E} \cap \dot{y})$. Choose $(\ell, q_{\ell, t}) \in (\dot{y} \cap \dot{E})$ witnessing that $\bar{r} \Vdash \ell \in \dot{x}$. That is, we may assume that $\bar{r} \upharpoonright \alpha \Vdash t = \text{stem}(\bar{r}(\alpha))$, that $q_{\ell, t} \upharpoonright \alpha \in \bar{G}^+$, and $(q_{\ell, t} \wedge r \upharpoonright \alpha)$ does not force over \bar{G}^+ that $t \notin \dot{T}_\alpha$. Of course this means that the condition $\bar{r} \wedge r \wedge [[t \in \dot{T}_\alpha]] \wedge q_{\ell, t}$ is not 0. This condition forces that ℓ is in $\dot{b} \cap (\dot{E} \cap \dot{y})$ as required. \square

Fact 5. The character of $\mathfrak{D}_{i\alpha}^\alpha$ is greater than μ_{i_γ} .

Proof of Fact 5. We know that $\mathfrak{D}_{i\alpha}^\alpha$ is forced to have an ω -closed base (in fact, descending mod finite with uncountable cofinality). Even more, $\mathbb{P}_{i_\alpha, \alpha}$ forces that for all $T \in \mathbb{L}(\mathfrak{D}_{i\alpha}^\alpha)$, there is a $D \in \mathfrak{D}_{i\alpha}^\alpha$ such that the condition $([D]^{<\omega})_{\text{stem}(T)}$ is below T . Let χ_α be the cofinality of α and fix a list $\{\dot{D}_\beta : \beta < \chi_\alpha\} \in M$ (closed under mod finite changes) of $\mathbb{P}_{i_\alpha, \alpha}$ -names of elements of $\mathfrak{D}_{i\alpha}^\alpha$ that is forced to be a base.

Now, suppose that $\dot{b} \in \mathbb{B}_{i_p, \alpha+1} = \mathbb{B}_{i_p, \alpha}$ and there is an $\dot{E} \in \mathcal{E}_1$ and an $r < p$ forcing that $\dot{b} \cap (\dot{E} \cap \dot{y})$ is empty. We prove there is an $\dot{x} \in \mathcal{E}_1$ and an $r_2 < r \upharpoonright \alpha$ in $\mathbb{P}_{i_\alpha, \alpha}$ such that $r_2 \Vdash \dot{b} \cap \dot{x}$ is empty. We may assume that r_2 forces a value t on $\text{stem}(r(\alpha))$ and that, for some $\beta < \chi_\alpha$, $r_2 \Vdash (\dot{D}_\beta^{<\omega})_t < r(\alpha)$. Let

$$\dot{x} = \{(\ell, q_\ell \upharpoonright \alpha) : (\ell, q_\ell) \in (\dot{E} \cap \dot{y}) \text{ and } q_\ell \upharpoonright \alpha \Vdash q_\ell(\alpha) \leq (\dot{D}_\beta^{<\omega})_t\}.$$

It is immediate that $\dot{x} \in M$ and that $(r_2 \wedge r) \Vdash_{\mathbb{P}_{i_\alpha, \alpha+1}} \dot{x} \supseteq (\dot{E} \cap \dot{y})$. Since $\dot{E} \cap \dot{y}$ is forced to be in \mathcal{E}_1^+ , it follows that \dot{x} is forced by r_2 to be in $\langle \mathcal{E}_1 \rangle$. Now we verify that $r_2 \Vdash \dot{b} \cap \dot{x}$ is empty. Assume that $r_3 < r_2$ in $\mathbb{P}_{i_\alpha, \alpha}$ and that $r_3 \Vdash \ell \in \dot{b} \cap \dot{x}$. We may assume there is $(\ell, q_\ell \upharpoonright \alpha) \in \dot{x}$ such that $r_3 < q_\ell \upharpoonright \alpha$. But now $r_2 \Vdash q_\ell(\alpha) \leq r(\alpha)$ and so $r_2 \wedge r \Vdash \ell \in \dot{b} \cap (\dot{E} \cap \dot{y})$ – a contradiction.

The conclusion now follows from Lemma 5.4. \square

Definition 5.7. For each $t \in \omega^{<\omega}$, define that $\mathbb{P}_{i_\alpha, \alpha}$ -name \dot{E}_t according to the rule that $r \Vdash \ell \in \dot{E}_t$ providing $r \in \mathbb{P}_{i_\alpha, \alpha}$ forces that there is a \dot{T} with $r \Vdash \dot{T} \in \mathbb{L}(\mathfrak{D}_{i\alpha}^\alpha)$, $r \Vdash t = \text{stem}(\dot{T})$, and $r \cup \{(\alpha, \dot{T})\} \Vdash \ell \notin \dot{y}$.

Fact 6. There is a $\dot{T} \in \mathbb{L}(\mathfrak{D}_{i\alpha}^\alpha) \cap M$ such that $p \upharpoonright \alpha$ forces the statement: $\dot{E}_t \in \mathcal{E}_1$ for all t such that $\text{stem}(\dot{T}) \leq t \in \dot{T}$.

Proof of Fact 6. By elementarity, there is a maximal antichain of $\mathbb{P}_{i_\alpha, \alpha}$ each element of which decides if there is a \dot{T} with $\dot{E}_t \in \mathcal{E}_1$ for all $t \in \dot{T}$ above $\text{stem}(\dot{T})$. Since $p \in A \setminus A_1$ it follows that there is an $i_p < i_\alpha$ as in condition (2) of Fact 2. Let $t_0 \in \omega^{<\omega}$ so that $p \upharpoonright \alpha \Vdash t_0 = \text{stem}(p(\alpha))$. By the maximum principle, there is a $\dot{b} \in \mathbb{B}_{i_p, \gamma}$ and a $\dot{E}_0 \in \mathcal{E}_1$ satisfying that $p \Vdash \dot{b} \cap \dot{E}_0 \cap \dot{y}$ is empty, while $p \Vdash \dot{b} \cap \dot{E}$ is infinite for all $\dot{E} \in \langle \mathcal{E}_1 \rangle$. This means that p forces that $\dot{b} \cap \dot{E}_0$ is an element of $\langle \mathcal{E}_1 \rangle^+$ that is contained in $\omega \setminus \dot{y}$. As in the proof of Lemma 5.4, there is an $\dot{E}_2 \in \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_p, \gamma}$ such that p forces that $\dot{b} \cap \dot{E}_2$ is contained in \dot{E}_0 . We also have that $(\dot{b} \cap \dot{E}_2) \upharpoonright \alpha$ is forced to be contained in $\omega \setminus \dot{y}$. It now follows that $p \upharpoonright \alpha$ forces that for all $t_0 \leq t \in p(\alpha)$, $p \upharpoonright \alpha$ forces that \dot{E}_t contains $(\dot{b} \cap \dot{E}_2) \upharpoonright \alpha$ and so is in $\langle \mathcal{E}_1 \rangle^+$. Since \dot{E}_t is also measured by \mathcal{E}_1 , we have that $p \upharpoonright \alpha$ forces that such \dot{E}_t are in \mathcal{E}_1 . This completes the proof. \square

Now we show how to extend $\mathcal{E}_1 \cap \mathbb{B}_{i_\alpha, \gamma}$ so as to measure \dot{y} . Let $\beta = \text{sup}(M \cap \alpha)$. By Fact 5, $\beta < \alpha$ and by the definition of $\mathcal{H}(\vec{\lambda})$, $M \cap \dot{\mathcal{D}}_{i_\alpha}^\alpha$ is a subset of $\langle \dot{\mathcal{D}}_{i_\beta}^\beta \rangle$, $\dot{L}_\beta \in \dot{\mathcal{D}}_{i_\alpha}^\alpha$, and $i_\beta = i_\alpha$. We also have that the family $\{\dot{L}_\xi : \text{cf}(\xi) \geq \omega_1 \text{ and } \beta_\alpha \leq \xi \in M \cap \beta\}$ is a base for $\dot{\mathcal{D}}_{i_\beta}^\beta$. For convenience let $q <_M p$ denote the relation that q is an $M \cap \mathbb{P}_{i_\alpha, \alpha+1}$ -reduct of p . Let \bar{p} be any condition in $\mathbb{P}_{i_\beta, \beta+1}$ satisfying that $\bar{p} \upharpoonright \beta = p \upharpoonright \alpha$ and $\bar{p} \upharpoonright \beta \Vdash \text{stem}(\bar{p}(\beta)) = t_\alpha$ (recall that $p \upharpoonright \alpha \Vdash t_\alpha = \text{stem}(p(\alpha))$).

Let us note that for each $q \in M \cap \mathbb{P}_{i_\alpha, \alpha+1}$, $q \upharpoonright \alpha = q \upharpoonright \beta$ and $q \upharpoonright \beta \Vdash q(\alpha)$ is also a $\mathbb{P}_{\beta, i_\beta}$ -name of an element of $\mathbb{L}(\dot{\mathcal{D}}_{i_\beta}^\beta)$. Let \dot{x} be the following $\mathbb{P}_{i_\beta, \beta+1}$ -name

$$\dot{x} = \{(\ell, q \upharpoonright \beta \cup \{(\beta, q(\beta))\}) : (\ell, q) \in \dot{y} \cap M \text{ and } q <_M p\}.$$

We will complete the proof by showing that there is an extension of p that forces that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ measures \dot{y} and that 1 forces that $\langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i_\beta, \beta+1}$ is $\vec{\lambda}(i_\gamma)$ -thin. Here $\dot{x}[\dot{L}_\beta]$ abbreviates the $\mathbb{P}_{i_\beta, \beta+1}$ -name

$$\{(\ell, r) : (\exists q) (\ell, q) \in \dot{x}, q \upharpoonright \beta = r \upharpoonright \beta, \text{ and } r \Vdash \text{stem}(q(\beta)) \in \dot{L}_\beta^{<\omega}\}.$$

The way to think of $\dot{x}[\dot{L}_\beta]$ is that if \bar{p} is in some $\mathbb{P}_{i_\alpha, \alpha}$ -generic filter G , then $\dot{y}[G]$ is now an $\mathbb{L}(\dot{\mathcal{D}}_{i_\alpha}^\alpha)$ -name, $L_\beta^{<\omega} = (\dot{L}_\beta[G])^{<\omega}$ is in $\mathbb{L}(\dot{\mathcal{D}}_{i_\alpha}^\alpha)$, and $(\dot{x}[\dot{L}_\beta])[G]$ is equal to $\{\ell : L_\beta^{<\omega} \not\Vdash \ell \notin \dot{y}\}$. We will use the properties of \dot{x} to help show that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ is $\vec{\lambda}(i_\gamma)$ -thin. This semantic description of $\dot{x}[\dot{L}_\beta]$ makes clear that $\bar{p} \cup \{(\alpha, (L_\beta)^{<\omega})\} \in \mathbb{P}_{i_\alpha, \alpha+1}$ forces that $\dot{x}[\dot{L}_\beta]$ contains \dot{y} . This implies that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ measures \dot{y} .

Claim: It is forced by \bar{p} that $\omega \setminus \dot{x}$ is not measured by \mathcal{E}_1 .

Each element of \mathcal{E}_1 is in M and simple elementarity will show that for any condition in q in M that forces $\dot{E} \cap (\omega \setminus \dot{y})$ is infinite, the corresponding $\bar{q} = q \upharpoonright \alpha \cup \{(\beta, q(\alpha))\}$ will also force that $\dot{E} \cap (\omega \setminus \dot{x})$ is infinite.

It follows from Fact 5, with $\omega \setminus \dot{x}$ playing the role of \dot{y} , that $\mathcal{E}_1 \cup \{\omega \setminus \dot{x}\}$ is $\vec{\lambda}(i_\gamma)$ -thin. Recall that $q \Vdash \dot{x} = \emptyset$ for all $q \perp \bar{p}$. Now to prove that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ is also $\vec{\lambda}(i_\gamma)$ -thin, we prove that

$$\langle \mathcal{E}_1 \cup \{\omega \setminus \dot{x}\} \rangle \cap \mathbb{B}_{i,\alpha} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i,\alpha}$$

for all $i < i_\alpha$. In fact, first we prove

$$\langle \mathcal{E}_1 \cup \{\omega \setminus \dot{x}\} \rangle \cap \mathbb{B}_{i,\beta} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i,\beta}$$

for all $i < i_\alpha$.

We begin with this main Claim.

Claim 1. If $\dot{b} \in \mathbb{B}_{i,\beta}$ ($i < i_\beta$) and there is an $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_\alpha,\beta}$ and a $\bar{p} \geq q \in \mathbb{P}_{i_\beta,\beta+1}$ such that $q \Vdash \dot{b} \cap (\dot{E} \setminus \dot{x}) = \emptyset$ then $q \upharpoonright \beta \Vdash (\exists \dot{E} \in \mathcal{E}_1) \dot{b} \cap \dot{E} = \emptyset$.

Proof of Claim: We may assume that $q \upharpoonright \beta$ forces a value t on $\text{stem}(q(\beta))$. Recall that $q \upharpoonright \beta$ forces the statement: there is a $\dot{D} \in M \cap \dot{\mathfrak{D}}_{i_\alpha}^\alpha$ such that $(\dot{D}^{<\omega})_t \leq q(\beta)$. The definition of \dot{x} ensures that $q \upharpoonright \beta \cup \{(\alpha, (\dot{D}^{<\omega})_t)\} \Vdash \dot{b} \cap (\dot{E} \setminus \dot{y})$ is empty. There is a $\mathbb{P}_{i_\alpha,\alpha}$ -name $\dot{E}_1 \in M$ such that $q \upharpoonright \alpha \Vdash \dot{E}_1 = \{\ell : (\dot{D}^{<\omega})_t \Vdash \ell \notin (\dot{E} \setminus \dot{y})\}$. By assumption $q \upharpoonright \alpha \Vdash \dot{E}_1 \in \langle \mathcal{E}_1 \rangle$. Since \dot{b} is also a $\mathbb{P}_{i_\alpha,\alpha}$ -name, we have that $q \upharpoonright \alpha \Vdash \dot{b} \cap \dot{E}_1 = \emptyset$. \square

Now assume that $\dot{b} \in \mathbb{B}_{i_\beta,\beta}$ and $q \Vdash \dot{b} \cap (\dot{E} \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])))$ is empty for some $q < \bar{p}$ in $\mathbb{P}_{i_\beta,\beta+1}$. By Lemma 5.4 it suffices to assume that $\dot{E} \in \mathbb{B}_{i_\beta,\beta}$. To prove that q forces that $\dot{b} \notin \langle \mathcal{E}_1 \rangle^+$, it suffices to prove that there is some $\dot{E}_1 \in \mathcal{E}_1$ such that $q \Vdash \dot{b} \cap (\dot{E}_1 \cap (\omega \setminus \dot{x}))$ is finite. We proceed by contradiction.

We may again assume that $q \upharpoonright \beta$ forces that $q(\beta)$ is $(\dot{D}^{<\omega})_t$ for some $t \supset t_\alpha$ and some $\dot{D} \in \dot{\mathfrak{D}}_{i_\alpha}^\alpha \cap M$. Let H be the range of t . Let, for the moment, G be a $\mathbb{P}_{i_\alpha,\alpha}$ -generic filter with $q \in G$. Now in $M[G]$ we have the value L_β of \dot{L}_β and $H \subset L_\beta$. We can also let E denote the value of $\dot{E}[G]$. Recall that for each $s \in H^{<\omega}$, E_s denotes the set of $\ell \in E$ such that there is some $T \in \mathbb{L}(\dot{\mathfrak{D}}_{i_\alpha}^\alpha)$ with $s = \text{stem}(T)$ and $T \Vdash \ell \notin \dot{y}$. We have shown in Fact 6 that there is a $T \in \mathbb{L}(\dot{\mathfrak{D}}_{i_\alpha}^\alpha) \cap M$ such that $E_s \in \mathcal{E}_1$ for all $s \in T$ above $\text{stem}(T)$. This means that there is an $\ell \in \dot{b} \cap E$ such that $\ell \in E_s$ for each of the finitely many suitable s . For each s , choose $T_s \subset T$ witnessing $\ell \in E_s$. As before, and since there are only finitely many s involved, we can assume that $\dot{T}_s = (\dot{D}^{<\omega})_s$ for

some $H \subset \dot{D} \in \dot{\mathfrak{D}}_{i_\alpha}^\alpha \cap M$ and we then define an extension q of q' so that $q'(\beta) = (\dot{D}^{<\omega})_{t_\alpha}$ ensures that $(\dot{L}_\beta^{<\omega})_s < T_s$ for each s . Note that such a condition q' we have that $q' \cup \{(\alpha, (\dot{L}_\beta)^{<\omega})\}$ forces that $\ell \notin \dot{y}$. But then it should be clear that q' that forces $\ell \notin \dot{x}[\dot{L}_\beta]$. This contradicts that q forces $\ell \notin \dot{b} \cap (\dot{E} \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])))$. \square

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