EXISTENTIALLY CLOSED LOCALLY FINITE GROUPS SH312

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Abstract. We investigate this class of groups originally called ulf (universal locally finite groups) of uncountable cardinality. We prove that for every locally finite group G there is a canonical existentially closed extension of the same cardinality, unique up to isomorphism and increasing with G. Also we get, e.g. existence of complete members (i.e. with no non-inner automorphisms) in many cardinals (provably in ZFC). The main point here is having a parallel to stability theory in the sense of investigating definable types though the class is very unstable.

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§ 0. Introduction

§ 0(A). Background.

On lf (locally finite) groups and exlf (existentially closed locally finite) groups, see the book by Kegel-Wehrfritz [?]; exlf groups were originally called ulf (= universal locally finite) groups, we change as the word "universal" has been used in this context with a different meaning, see Definition 0.21 and Claim 0.14.
Recall

Definition 0.1. 1) G is a lf (locally finite) group if G is a group and every finitely generated subgroup is finite.
2) G is an exlf (existentially closed lf) group (in [?] it is called ulf, universal locally finite group) when G is a locally finite group and for any finite groups $K \subseteq L$ and embedding of K into G, the embedding can be extended to an embedding of L into G.
3) Let $\mathbf{K}_{\rm lf}$ be the class of lf (locally finite) groups (partially ordered by $\subseteq$, being a subgroup) and let $\mathbf{K}_{\rm exlf}$ be the class of existentially closed $G \in \mathbf{K}_{\rm lf}$.

In particular there is one and only one exlf group of cardinality $\aleph_0$. Hall proved that every lf group can be extended to an exlf group, as follows. It suffices for a given lf group G to find $H \supseteq G$ such that if $K \subseteq L$ are finite and f embeds K into G, then some $g \supseteq f$ embed L into H. To get such H, for finite $K \subseteq G$ let $E_{G,K} = \{(a,b) : a,b \in G \text{ and } aK = bK\}$ and let $G^\oplus$ be the group of permutations f of G such that for some finite $K \subseteq G$ we have $a \in G \Rightarrow aE_{G,K}f(a)$; now $b \in G$ should be identified with $f_b \in G^\oplus$ where f_b is defined by $f_b(x) = xb$ hence $f_b \in G^\oplus$ because if $b \in K \subseteq G$ then $a \in G \Rightarrow f_b(a) = ab \in abK = aK$ and $f_{b_2} \circ f_{b_1}(x) = (xb_1)b_2 = x(b_1b_2) = f_{b_1b_2}(x)$. Now $H = G^\oplus$ is essentially as required.

The proof gives a canonical extension. This means for example that every automorphism of G can be extended to an automorphism of $G^\oplus$ and, moreover, we can do it uniformly so preserving isomorphisms. Still we may like to have more; (for a given lf infinite group G) the extension $G^\oplus$ defined above is of cardinality $2^{|G|}$ rather than the minimal value - $|G| + \aleph_0$ (not to mention having to repeat this ω times in order to get an exlf extension). Also if $G_1 \subseteq G_2$ then the connection between $G_1^\oplus$ and $G_2^\oplus$ is not clear, i.e. failure of "naturality". A major point of the present work is a construction of a canonical existentially closed extension of G which has those two additional desirable properties, see e.g. 3.15.

Note that in model theoretic terminology the exlf groups are the $(\mathbf{D}, \aleph_0)$-homogeneous groups, with $\mathbf{D}$ the set of isomorphism types of finite groups or more exactly complete qf (= quantifier free) types of finite tuples generating a finite group, see e.g. [?, §2]. We use quantifier free types as we use embeddings (rather than, e.g. elementary embeddings). Let $\mathbf{D}(G)$ be the set of qf-complete types of finite sequences from the group G.

Let $\mathbf{K}_{\rm exlf}$ be the class of exlf groups. By Grossberg-Shelah [?], if $\lambda = \lambda^{\aleph_0}$ then no $G \in \mathbf{K}^{\rm exlf}_\lambda := \{H \in K_{\rm exf} : |H| = \lambda\}$ is universal in it, i.e., such that every other member is embeddable into it. But if κ is a compact cardinal and $\lambda > \kappa$ is strong limit of cofinality $\aleph_0$ then there is a universal exlf in cardinality λ, (this is a special case of a general theorem).

Wehrfritz asked about the categoricity of the class of exlf groups in any $\lambda > \aleph_0$. This was answered by Macintyre-Shelah [?] which proved that in every $\lambda > \aleph_0$ there are 2^λ non-isomorphic members of $\mathbf{K}^{\rm exlf}_\lambda$. This was disappointing in some sense: in $\aleph_0$ the class is categorical, so the question was perhaps motivated by the hope that also general structures in the class can be understood to some extent.

A natural and frequent question on a class of structures is the existence of rigid members, i.e. ones with no non-trivial automorphism. Now any exlf group $G \in \mathbf{K}_{\rm exlf}$ has non-trivial automorphisms - the inner automorphisms (recalling it has a trivial center). So the natural question is about complete members where a group is called complete iff it has no non-inner automorphism.

Concerning the existence of a complete, locally finite group of cardinality λ: Hickin [?] proved one exists in $\aleph_1$ (and more, e.g. he finds a family of $2^{\aleph_1}$ such groups pairwise far apart, i.e. no uncountable group is embeddable in two of them). Thomas [?] assumed G.C.H. and built one in every successor cardinal (and more, e.g. it has no Abelian or just solvable subgroup of the same cardinality). Related are Giorgetta-Shelah [?], Shelah-Zigler [?], which investigate $\mathbf{K}_{G_*}$ getting similar results where

- $(*)$ assume G_* an existentially closed countable group we let
  - (a) $\mathbf{K}_{G_*}$ is the class of groups G such that every finitely generated subgroup of G is embeddable into G_*
  - (b) $\mathbf{K}^{\rm excl}_{G_*}$ is the class of groups G which are $\mathbb{L}_{\infty,\aleph_0}$-equivalent to G_* (excl stands for existentially closed); equivalently $G \in \mathbf{K}_{G_*}$, every finitely generated subgroup of G_* is embeddable into G and if $\bar a, \bar b \in {}^n G$ realize the same qf type in G then some inner automorphism of G maps $\bar a$ to $\bar b$
- $(**)$ we can replace "group G_*" by any other structure.

Giorgetta-Shelah [?] build in cardinality continuum $G \in \mathbf{K}_{\rm exlf}$ with no uncountable Abelian subgroup and similarly for $\mathbf{K}^{\rm excl}_{G_*}, G_*$ as in $(*)$ and also for the similarly defined $\mathbf{K}^{\rm excl}_{F_*}, F_*$ an existentially closed countable fixed division ring. Shelah-Zigler [?] build, for G_* as in $(*)$ and $\lambda > \aleph_0$; $N^\ell_\lambda \in \mathbf{K}^{\rm exlf}_{G_*}$ of cardinality λ for $\ell = 1,2$ such that N^1_λ has no Abelian group of cardinality λ and every subgroup of cardinality λ has a free subgroup of the same cardinality; moreover, there are 2^λ pairwise non-isomorphic N like N^ℓ_λ.

In 1985 the author wrote notes (in Hebrew) for proving that there are anti-prime constructions and complete exlf groups when, e.g., $\lambda = \mu^+, \mu^{\aleph_0} = \mu$; using black boxes and "anti-prime" construction, i.e. using definable types as below; here we exclusively use qf (quantifier free) types; this was announced in [?, pg.418], but the work was not properly finished. To do so is our aim here.

Meanwhile Dugas-Göbel [?, Th.2] prove that for $\lambda = \lambda^{\aleph_0}$ and $G_0 \in \mathbf{K}^{\rm lf}_{\le\lambda}$ there is a complete $G \in \mathbf{K}^{\rm exlf}_{\lambda^+}$ extending G_0; moreover 2^{λ^+} pairwise non-isomorphic ones. Then Braun-Göbel [?] got better results for complete locally finite p-groups. Those constructions build an increasing continuous chain $\langle G_\alpha : \alpha < \lambda^+\rangle$, each G_α of cardinality λ, such that $G_{\alpha+1}$ is the wreath product of G_α and suitable Abelian locally finite groups, $G = \{G_\alpha : \alpha < \lambda^+\}$ is the desired group. This gives a tight control over the group and implies, e.g. that only few (i.e. $\le \lambda$) members commute with G_0. Here we are interested in groups G' which are "more existentially closed", e.g. "for every $G' \subseteq G$ of cardinality $< |G|$, there are $|G|$ elements commuting with it"; such properties are called "being full", note that fullness implies that a restriction on the cardinal is necessary and not so without it, see 5.5.

We show that though the class $\mathbf{K}_{\rm exlf}$ is very "unstable" there is a large enough set of definable types so we can imitate stability theory and have reasonable control in building exlf groups, using quantifier free types. This may be considered a "correction" to the non-structure results discussed above.

In §1 we present somewhat abstractly our results relying on the existence of a dense and closed so called $\mathfrak{S}$, a set of schemes of definitions of the relevant types. So before we turn to explaining our results we deal with the so called schemes, needed for explaining them.

§ 0(B). **Schemes.**

We deal with a class $\mathbf{K}$ of structures, usually it is the class of locally finite groups, but some of the results holds for suitable universal classes, see §6.

Central here are so-called schemes. For <u>models theorists</u> they are for a given $G \in \mathbf{K}_{\rm lf}$ and finite sequence $\bar a \subseteq G$ (realizing a suitable quantifier free type) a definition of a complete (quantifier free) type over G so realized in some extension of G from $\mathbf{K}_{\rm lf}$, which does not split over $\bar a$; alternatively you may say that they are definitions of a complete-free type quantifier over G which does not split over $\bar a$ and its restriction.

For <u>algebraists</u> they are our replacement of free products $G_1 *_{G_0} G_2$, but $\mathbf{K}_{\rm lf}$ is not closed under free product, in fact, fail amalgamation. So we are interested in replacements in the cases G_0 is finite, also we waive symmetry.

Convention 0.2. 1) $\mathbf{K}$ a universal class of structures (i.e. all of the same vocabulary, closed under isomorphisms and $M \in \mathbf{K}$ iff every finite generated substructure belongs to $\mathbf{K}$; usually $\mathbf{K} = \mathbf{K}_{\rm lf}$).
2) $G, H, \ldots \in \mathbf{K}$.

Definition 0.3. For $H \in \mathbf{K}, n < \omega$, a set $A \subseteq H$ and $\bar a \in {}^n H$ let $\mathrm{tp}(\bar a, A, H) = \mathrm{tp}_{\rm bs}(\bar a, A, H)$ be the basic type of $\bar a$ in H over A, that is:

$$\begin{aligned}\{\varphi(\bar x, \bar b) : \quad & \varphi \text{ is a basic (atomic or negation of atomic) formula in the variables} \\ & \bar x = \langle x_\ell : \ell < n\rangle \text{ and the parameters } \bar b, \text{ a finite sequence from } A, \\ & \text{which is satisfied by } \bar a \text{ in } H\}.\end{aligned}$$

So if $\mathbf{K}$ is a class of groups withough loss of generality φ is $\sigma(\bar x, \bar b) = e$ or $\sigma(\bar x, \bar b) \neq e$ for some group-term σ, a so called "word", (for $\mathbf{K}_{\rm olf}$ we also have $\sigma_1(\bar x, \bar b) < \sigma_2(\bar x, \bar b)$) <u>but</u> we may write $p(\bar y) = \mathrm{tp}_{\rm bs}(\bar b, A, H)$ or $p(\bar z) = \mathrm{tp}_{\rm bs}(\bar c, A, H)$ or just p when the sequence of variables is clear from the context.
2) We say $p(\bar x)$ is an $n - bs$-<u>type</u> over G <u>when</u> it is a set of basic formulas in the variables $\bar x = \langle x_\ell : \ell < n\rangle$ and parameters from G, such that $p(\bar x)$ is consistent, which means: if $K \subseteq G$ is f.g. and $q(\bar x)$ is a finite subset of $p(\bar x)$ and $q(\bar x)$ is over K (i.e. all the parameters appearing in $q(\bar x)$ are from K) <u>then</u> $q(\bar x)$ is realized in some $L \in \mathbf{K}$ extending K. We say $\bar a$ realizes p in H if $G \subseteq H$ and $\varphi(\bar x, \bar b) \in p \Rightarrow H \models \varphi[\bar a, \bar b]$.
3) $\mathbf{S}^n_{\rm bs}(G) = \{\mathrm{tp}_{\rm bs}(\bar a, G, H) : G \subseteq H, H \text{ is from } \mathbf{K} \text{ and } \bar a \in {}^n H\}$ and $\mathbf{S}_{\rm bs}(G) = \bigcup\limits_n \mathbf{S}_{\rm bs}(G)$; if $\mathbf{K}$ is not clear from the context we should write $\mathbf{S}^n_{\rm bs}(G, \mathbf{K}), \mathbf{S}_{\rm bs}(G, \mathbf{K})$.

Observation 0.4. For every $p \in \mathbf{S}^n_{\rm bs}(M)$ and $M \in \mathbf{K}$ there are $N, \bar a$ such that $M \subseteq N \in \mathbf{K}, \bar a \in {}^n N$ realizes $p, G_N = c\ell(G_M + \bar a, N)$ <u>and</u> if $M \subseteq N' \in \mathbf{K}$ and $\bar a'$ realizes p in N' then there is $N'' \subseteq N'$ and an isomorphism f from N onto N'' extending id_M such that $f(\bar a) = \bar a'$.

Remark 0.5. 0) In 0.4 we shall later use the convention of 0.15(1),(3).
1) We are particularly interested in types which are definable in some sense over small sets.
2) We can define "$p \in \mathbf{S}^n_{\rm bs}(M)$" syntactically, because for a set p of basic formulas $\varphi(\bar x, \bar a), \bar a$ from M which is complete (i.e. if $\varphi(\bar x, \bar a)$ is an atomic formula over M then $\varphi(\bar x, \bar a) \in p$ or $\neg\varphi(\bar x, \bar a) \in p$), we have $p \in \mathbf{S}^n_{\rm bs}(M)$ iff for every f.g. $N \subseteq M$ we have $p{\restriction}N := \{\varphi(\bar x, \bar a) \in p : \bar a \subseteq N\} \in \mathbf{S}^n_{\rm bs}(N)$.
3) Why do we use below types which do not split over a finite subgroup and the related set of schemes? As we like to get a canonical extension of $M \in \mathbf{K}$ it is natural to use a set of types closed under automorphisms of M, and as their number is preferably $\le \|M\|$, it is natural to demand that any such type is, in some sense, definable over some finite subset of M.

As in [**?**]:

Definition 0.6. We say that $p = \mathrm{tp}_{\rm bs}(\bar a, G, H) \in \mathbf{S}^n_{\rm bs}(G)$ <u>does not split</u> over $K \subseteq G$ <u>when</u> for every $m < \omega$ and $\bar b_1, \bar b_2 \in {}^m G$ satisfying $\mathrm{tp}_{\rm bs}(\bar b_1, K, G) = \mathrm{tp}_{\rm bs}(\bar b_2, K, G)$ we have $\mathrm{tp}_{\rm bs}(\bar b_1 \hat{}\, \bar a, K, H) = \mathrm{tp}_{\rm bs}(\bar b_2 \hat{}\, \bar a, K, H)$.

Definition 0.7. 1) Let $\mathbf{D}(\mathbf{K}) = \bigcup\limits_n \mathbf{D}_n(\mathbf{K})$, where $\mathbf{D}_n(\mathbf{K}) = \{\mathrm{tp}_{\rm bs}(\bar a, \emptyset, M) : \bar a \in {}^n M$ and $M \in \mathbf{K}\}$.
2) Assume[1] $p(\bar x)$ is a k-type, that is, $\bar x = \langle x_\ell : \ell < k\rangle$ and for some $p'(\bar x)$ we have $p(\bar x) \subseteq p'(\bar x) \in \mathbf{D}_k(\mathbf{K})$ and $m < \omega$. We let $\mathbf{D}_{p(\bar x),m}(\mathbf{K}) = \mathbf{D}_m(p(\bar x), \mathbf{K})$ be the set of $q(\bar x, \bar y) \in \mathbf{D}_{k+m}(\mathbf{K})$ such that $q(\bar x, \bar y) \supseteq p(\bar x)$, which means that there is $M \in \mathbf{K}$ and $\bar a \in {}^k M$ realizing $p(\bar x)$ and $(\bar a, \bar b)$ realizing $q(\bar x, \bar y)$ in M, i.e. $\ell g(\bar a) = k, \ell g(\bar b) = m$ and $\bar a \hat{}\, \bar b$ realizes $q(\bar x, \bar y)$.
3) In part (2) let $\mathbf{D}_{p(\bar x)}(\mathbf{K}) = \cup\{\mathbf{D}_m(p(\bar x), \mathbf{K}) : m < \omega\}$.

Remark 0.8. Below $\mathfrak{s} \in \Omega_{n,k}[\mathbf{K}]$ is a scheme to fully define a type $q(\bar z) \in \mathbf{S}^n_{\rm bs}(M)$ for a given parameter $\bar a \in {}^k M$ such that $q(\bar z)$ does not split over $\bar a$. Sometimes $\mathfrak{s}$ is not unique but if, e.g., $M \in \mathbf{K}_{\rm exlf}$ it is.

Definition 0.9. 1) Let $\Omega[\mathbf{K}]$ be the set of schemes, i.e. $\cup\{\Omega_{n,k}[\mathbf{K}] : k, n < \omega\}$ where $\Omega_{n,k}[\mathbf{K}]$ is the set of (k,n)-schemes $\mathfrak{s}$ which means, see below.
1A) We say $\mathfrak{s}$ is a (k,n)-<u>scheme</u> when for some $p(\bar x) = p_{\mathfrak{s}}(\bar x_{\mathfrak{s}})$ with $\ell g(\bar x_{\mathfrak{s}}) = k$, (and $k_{\mathfrak{s}} = k(\mathfrak{s}) = k, n_{\mathfrak{s}} = n(\mathfrak{s}) = n$) we have:

- (a) $\mathfrak{s}$ is a function with domain $\mathbf{D}_{p(\bar x)}(\mathbf{K})$ such that for each m it maps $\mathbf{D}_{p(\bar x),m}(\mathbf{K})$ into $\mathbf{D}_{k+m+n}(\mathbf{K})$
- (b) if $s(\bar x, \bar y) \in \mathbf{D}_{p(\bar x),m}(\mathbf{K})$ and $r(\bar x, \bar y, \bar z) = \mathfrak{s}(s(\bar x, \bar y))$ <u>then</u> $r(\bar x, \bar y, \bar z){\restriction}(k+m) = s(\bar x, \bar y)$; that is, if $(\bar a, \bar b, \bar c)$, i.e. $\bar a \hat{}\, \bar b \hat{}\, \bar c$, realizes $r(\bar x, \bar y, \bar z)$ in $M \in \mathbf{K}$ so $k = \ell g(\bar a), m = \ell g(\bar b), n = \ell g(\bar c)$, <u>then</u> $\bar a \hat{}\, \bar b$ realizes $s(\bar x, \bar y)$ in M; see 1.2(1)

[1] This is used to define the set $\mathfrak{S}$ of schemes; for this section the case $p(\bar x) = p'(\bar x)$ is enough as we can consider all the completions but the general version is more natural in counting a set $\mathfrak{S}$ of schemes and in considering actual examples.

(c) in clause (b), moreover if $\bar b' \in {}^{\omega>}M$, $\mathrm{Rang}(\bar b') \subseteq \mathrm{Rang}(\bar a\char`^\bar b)$ then $\bar a\char`^\bar b'\char`^\bar c$ realizes the type $\mathfrak{s}(\mathrm{tp}_{\mathrm{bs}}(\bar a\char`^\bar b', \emptyset, M))$; this is to avoid $\mathfrak{s}$'s which define contradictory types[2].

2) Assume $\mathfrak{s} \in \Omega_{n,k}[\mathbf{K}]$ and $M \in \mathbf{K}$ and $\bar a \in {}^kM$ realizes $p_{\mathfrak{s}}(\bar x_{\mathfrak{s}})$; we let $q_{\mathfrak{s}}(\bar a, M)$ be the unique $r(\bar z) = r(z_{\mathfrak{s}}) \in \mathbf{S}^n_{\mathrm{bs}}(M)$ such that for any $\bar b \in {}^{\omega>}M$ letting $r_{\bar b}(\bar x, \bar y, \bar z) := \mathfrak{s}(\mathrm{tp}_{\mathrm{bs}}(\bar a\char`^\bar b, \emptyset, M))$ we have $r_{\bar b}(\bar a, \bar b, \bar z) \subseteq r(\bar z)$.
3) We call $\mathfrak{s}$ full when $p_{\mathfrak{s}}(\bar x) \in \mathbf{D}_{k(\mathfrak{s})}(\mathbf{K})$.
4) For technical reasons we allow $\bar x_{\mathfrak{s}} = \langle x_{\mathfrak{s},\ell} : \ell \in u\rangle, u \subseteq \mathbb{N}, |u| = k_{\mathfrak{s}}$ and in this case ${}^{k(\mathfrak{s})}M$ will mean ${}^uM = \{\langle a_\ell : \ell \in u\rangle : a_\ell \in M \text{ for } \ell \in u\}$ and we do not pedantically distinguish between u and $k_{\mathfrak{s}}$. Similarly for $n_{\mathfrak{s}}$ and $\bar z$, the reason is 1.1, 1.6(4).

Convention 0.10. $\mathfrak{S}$ will denote a subset of $\Omega[\mathbf{K}]$.

§ 0(C). The Results.

In particular (in the so-called first avenue, see below):

Theorem 0.11. *Let λ be any cardinal $\geq |\mathfrak{S}|$.*
1) For every $G \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ there is $H_G \in K^{\mathrm{exlf}}_\lambda$ which is λ-full over G (hence over any $G' \subseteq G$; see Definition 1.15) and $\mathfrak{S}$-constructible over it (see 1.19).
2) If $H \in \mathbf{K}^{\mathrm{lf}}_{<\lambda}$ is λ-full over $G (\in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda})$ then H_G from above can be embedded into H over G, see 1.23(4).

This is proved by 1.23 + §2. So in some sense H_G is prime over G, that is, it is prime but not among the members of $\mathbf{K}^{\mathrm{exlf}}_\lambda$, i.e. for a different class. Still we would like to have canonicity so uniqueness. There are some additional avenues helpful toward this.

The second avenue tries to get results which are nicer by assuming $\mathfrak{S}$ is so called symmetric which is the parallel of being stable in this context. Under this assumption we prove the existence of a canonical closure of a locally finite group to an exlf one. This is done in 1.12, 1.13.

The third avenue is without assuming "$\mathfrak{S}$ is symmetric" but using a more complicated construction, for which we have similar, somewhat weaker results using special linear orders. The failure of symmetry seems to draw you to order the relevant pairs $(\mathfrak{s}, \bar a)$ for G. That is, trying to repeat the construction in 1.12(2), without symmetry we have to well order or at least linearly order $\mathrm{def}(G) = \mathrm{def}_{\mathfrak{S}}(G)$ which is essentially the set of relevant complete quantifier types over G over a finite set of parameters, see Definition 1.1; this suffices by 1.8(9). At first glance we have to linearly order $\mathrm{def}(G)$, but we take a list of $\mathrm{def}(G)$, with each appearing λ times and linearly order it such that it does not induce a linear order of $\mathrm{def}(G)$. See below. So we prove (in 1.30, 1.31, 1.33)

Theorem 0.12. *1) We can for every* lf *group G, define $G^{c\ell}$ such that:*

(a) *if $G \in K^{\mathrm{lf}}_{\leq\lambda}$ then $G \subseteq G^{c\ell} \in K^{\mathrm{exlf}}_\lambda$*
(b) *$G^{c\ell}$ is unique up to isomorphism over G.*

[2]But some $\mathfrak{s}$'s satisfying clauses (a),(b) of 0.9(1A) but failing clause (c) this may give a consistent type in an interesting class of cases.

2) Also[3] *essentially it commutes with extensions, i.e.* $G_1 \subseteq G_2 \Rightarrow G_1^{c\ell} \subseteq G_1^{c\ell}$, *pedantically*

- (c) *if* $G_1 \subseteq G_2$ *and* $G_\ell^{c\ell}$ *is as above* then *there is an embedding* h *of* $G_1^{c\ell}$ *into* $G_2^{c\ell}$ *such that* $h(G_1^{c\ell}) \cap G_2 = G_1$
- $(c)'$ *restricting ourselves to* $\{G \in \mathbf{K}_{\rm lf}$*: every* $x \in G$ *is a singleton*$\}$ *we have:*
- $(b)''$ $G^{c\ell}$ *is really unique*
- $(c)''$ $G_1 \subseteq G_2 \Rightarrow G_1^{c\ell} \subseteq G_2^{c\ell}$.

To stress the generality in addition to the class $\mathbf{K}_{\rm lf}$ of lf-groups we use $\mathbf{K}_{\rm olf}$, the class of ordered locally finite groups (see 0.15); for them the proof of the existence of a suitable $\mathfrak{S}$ is easier. Naturally for $\mathbf{K}_{\rm olf}$ we certainly do not have a symmetric $\mathfrak{S}$.

In §2 we show that $\mathfrak{S}$ as needed in §1 exists, but not necessarily symmetric and define and investigate some specific schemes used later; also we define and investigate NF, a relative of free amalgamation. In §3 we find a fourth avenue which is more specific to the class of lf groups. We show that we can induce symmetry, i.e. define symmetric constructions even for non-symmetric $\mathfrak{S}$ hence get somewhat better results, see 3.15. In particular we construct reasonable closures.

In §4(A), we show that we can find amalgamation preserving commuting and so can get a new relative NF^3 of NF. In §4(B) we deal with some related schemes (of types). In §4(C) we deal with types with infinitely many variables.

In §5 we prove the existence of a complete group $G_* \in \mathbf{K}_\lambda^{\rm exlf}$ when $\lambda = \mu^+, \mu = \mu^{\aleph_0}$. Moreover, we prove the existence of a complete extension $G_* \in \mathbf{K}_\lambda^{\rm exlf}$ of an arbitrary $G \in \mathbf{K}_{\leq \mu}^{\rm lf}$.

Some of the definitions and claims work also in quite a general framework, but it is not clear at present how interesting this is. Still we consider some expansions of $\mathbf{K}_{\rm lf}$, and comment on them in §6.

We here also consider the partial order $\leq_{\mathfrak{S}}$ on $\mathbf{K}$, where $G_1 \leq_{\mathfrak{S}} G_2$ means that every finite $\bar a \subseteq G_2$ realizes over G_2 a type from $\mathrm{def}_{\mathfrak{S}}(G_1)$. Note that on $(\mathbf{K}, \leq_{\mathfrak{S}})$ we may generalize stability theory, in particular when $\mathfrak{S}$ is symmetric (see §1) or when we use the symmetrized version (see §3). In particular, we can investigate orthogonality, parallelism, super-stability, and indiscernible sets which Δ-converge ([**?**] or [**?**]). A class somewhat similar to $\mathbf{K}_{\rm lf}$, for an existentially closed countable group L is $\mathbf{K}_L$, the class of groups G such that every f.g. subgroup is embeddable into L. We further investigate $\mathbf{K}_{\rm lf}$ in [**?**] and in more general direction in a work in preparation with G. Paolini.

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§ 0(D). **Preliminaries.**

Definition 0.13. 1) Let $\mathbf{K}_\lambda^{\rm lf}$ be the class of $G \in \mathbf{K}_{\rm lf}$ of cardinality λ, let $\mathbf{K}_\lambda^{\rm exlf}$ be the class of $G \in \mathbf{K}_{\rm exlf}$ of cardinality λ; see Definition 0.1.
2) Let $\mathrm{fsb}(M)$ be the set of f.g. (finitely generated) sub-structures of M.

Note that $\mathbf{K}_{\rm exlf}$ is the same $\mathbf{K}_{\rm ulf}$ as defined by Hall as proved in Macintyre-Shelah [**?**], Wood [**?**]; that is:

[3]See on this in 3.14.

Claim 0.14. *The following conditions on a locally finite group G are equivalent:*

- (A) *G is* ulf *which means:*
  - (a) *every finite group is embeddable into G*
  - (b) *if H_1, H_2 are isomorphic finite subgroups of G, then for some $x \in G$, conjugation by x maps H_2 onto (here equivalently into) H_2, i.e. $x^{-1}H_1x = H_2$*
- (B) $G \in \mathbf{K}_{\rm exlf}$.

Proof. $(B) \Rightarrow (A)$

Clause (A)(a): let H be a finite group, let $H_1 = \{e_H\} \subseteq H$ so a sub-group of H and let $H_2 = H$ and let $h_1 : H_1 \to G$ be defined by $h_2(e_H) = e_G$. So by clause (B) there is an extension h_2 of h_1 embedding $H_2 = H$ into G, so $h_2(H)$ is as required.

Clause (A)(b): let $H_1, H_2 \subseteq G$ be finite sub-groups and let $H_3 \subseteq G$ be the finite subgroup which $H_1 \cup H_2$ generates. There is a finite group H_4 extending H_3 such that: any partial automorphism of H_3 is included in some conjugation in H_4. Let $h_3 : H_3 \to H_3 \subseteq G$ be the identity, hence by Clause (B) recalling $G \in \mathbf{K}_{\rm exlf}$, there is an embedding h_4 of H_4 into G extending h_3.

So in $h_4(H_4) \subseteq G$ there is a conjugation as required.

$(A) \Rightarrow (B)$:

Let $H_1 \subseteq H_2$ be finite groups and h_1 be an embedding of H_1 into G. Let $H_4 \supseteq H_2$ be a finite group such that any automorphism of H_1 is included in an inner automorphism of H_1. By Clause (A)(a) there is an embedding h_4 of H_4 into G. By Clause (A)(b) there is $x \in G$ such that $H'_4 := x^{-1}h_1(H_1)x \subseteq G$ is equal to $h_4(H_1)$.

Recalling 0.23(7) $h'_4 = (\Box_x {\restriction} h_4(H_4)) \circ h_4$ embeds H_4 into G and maps H_1 onto $h_1(H_1)$; but the embedding h'_4 does not necessarily extend h_1. However, by clause (A)(b), for some $y \in h'_4(H_4), \Box_y h'_4$ embeds H_4 (hence H_2) and extends h_1 as required. $\Box_{0.14}$

We may use the class $\mathbf{K}_{\rm olf}$ of linearly ordered lf groups, it is closely related and some issues are more transparent for it; $\mathbf{K}_{\rm olf}$ is defined as follows.

Definition 0.15. 1) Let $\mathbf{K}_{\rm olf}$ be the class of structures M which are an expansion of a lf group $G = G_M$ by a linear order $<_M$, also this class is partially ordered by $M_1 \subseteq M_2, M_1$ a sub-structure of M_2.
2) We say that $M \in \mathbf{K}_{\rm olf}$ is existentially closed as in 0.13(2) and define $\mathbf{K}^{\rm olf}_\lambda$ as in 0.1(2).
3) If $M \in \mathbf{K}_{\rm lf}$ then we let $G_M = M$.

Remark 0.16. For $\mathbf{K}_{\rm lf}$ conceivably there is a symmetric dense $\mathfrak{S}$, hence a very natural canonical exlf-closure. Without it we can either use a somewhat less natural one (using linear orders, see end of §1) or "make it symmetric by brute force" (see §3). But for the class $\mathbf{K}_{\rm olf}$ we can use only the linear orders, so every M has a canonical existentially closed extension, but it is more difficult to make it unique up to isomorphism. We shall in 6.2 introduce another class, $\mathbf{K}_{\rm clf}$, locally finite groups with choice.

Convention 0.17. 1) Except in §6, $\mathbf{K}$ is the class $\mathbf{K}_{\rm lf}$ of locally finite groups or $\mathbf{K}_{\rm olf}$ of ordered locally finite groups (we may use $\le_{\mathbf{K}}$ but here $\mathbf{K}$ is partially ordered by $\subseteq$, being a substructure) and see 0.16.
2) Let xlf-group mean a member of $\mathbf{K}$. Let $\mathbf{K}_{\rm ec}$ be the class of existentially closed members of $\mathbf{K}$.
3) In §2, §3, §4, §5 we use only $\mathbf{K}_{\rm lf}$; in §1 you can restrict yourself to $\mathbf{K} = \mathbf{K}_{\rm lf}$ but in §6 we have further cases on which we comment.

The following definition is for the more general framework.

Definition 0.18. 1) For $M, N \in \mathbf{K}$ let $M \le_{\rm fsb} N$ mean that if $K \subseteq L$ are f.g., $K \subseteq M, L \subseteq N$, <u>then</u> there is an embedding of L into M over K.
2) For $M, N \in \mathbf{K}$ let $M \le_{\Sigma_1} N$ <u>means</u> that $M \subseteq N$ and if $\bar a \in {}^{\ell g(\bar y)}M, \bar b \in {}^{\ell g(\bar x)}N$ and $\varphi(\bar x, \bar y) \in \mathbb{L}(\tau_{\mathbf{K}})$ is quantifier free and $N \models \varphi[\bar b, \bar a]$ then for some $\bar b' \in {}^{\ell g(\bar x)}M$ we have $M \models \varphi[\bar b', \bar a]$.
3) Let $M_\ell \in \mathbf{K}, \bar a_\ell \in {}^{n(\ell)}(M_\ell)$ for $\ell = 1, 2$. We say that a relation on $M_1 \times M_2$ is quantifier-free definable in $(M_1, \bar a_1, M_2, \bar a_2)$ <u>when</u> it is a Boolean combination of finitely many simple ones, where R is a simple n-place relation on $M_1 \times M_2$ <u>when</u> R is the set of n-tuples $((b_0, c_0), \ldots, (b_{n-1}, c_{n-1}))$ such that $b_i \in M_1, c_i \in M_2$ for $i < n$ and

$$M_1 \models \varphi_1[b_0, \ldots, b_{n-1}, \bar a_1]$$

$$M_2 \models \varphi_2[c_0, \ldots, c_{n-1}, \bar a_2]$$

for some quantifier-free formulas φ_1, φ_2 in $\mathbb{L}(\tau_{\mathbf{K}})$ and finite sequences $\bar a_1, \bar a_2$ from M_1, M_2 respectively.

Remark 0.19. 1) Note that 0.18(3) is not actually used, but just indicate the form of definability used.
2) Note that $\le_{\Sigma_1}$ for $\mathbf{K}_{\rm lf}$ and $\mathbf{K}_{\rm olf}$ is the same as $\le_{\rm fsb}$. For other classes, see §6, if the vocabulary is finite and we deal with locally finite structures they are still the same. Otherwise, by our choice of "does not split" we have to use $\le_{\rm fsb}$. But if we prefer to use $\le_{\Sigma_1}$ we have to strengthen the definition of "does not split" to make the proof of 1.10(1) work.

Convention 0.20. Let $M_1, M_2 \in \mathbf{K}, M_1 \subseteq M_2$ and $\bar a \in {}^n(M_2)$, so $\bar a = (a_0, a_1, a_2, \ldots, a_{n-1})$.
1) Denote by $cl(M_1 + \bar a, M_2)$ the sub-structure generated by $M_1 \cup \bar a = M_1 \cup \{a_0, a_1, \ldots, a_{n-1}\}$ in M_2.
2) For a group G and $A \subseteq G$ let

- $\mathbf{C}_G(A) = \{g \in G : G \models \text{"}ag = ga\text{"} \text{ for every } a \in A\}$
- $\mathbf{Z}(G) = \mathbf{C}_G(G)$
- $\mathbf{N}_G(A) = \{c \in G : c^{-1}Ac = A\}$.

4) For a group G, aut(G) is the group of automorphisms of G and inner(G) is the normal subgroup of aut(G) consisting of the inner automorphisms of G.

A side issue here is:

Definition 0.21. 1) For a class $\mathbf{K}$ of structures (of a fixed vocabulary) we say $M \in \mathbf{K}$ is λ-universal in $\mathbf{K}$ when every $N \in \mathbf{K}$ of cardinality λ can be embedded into it.
2) We say $M \in \mathbf{K}$ is $(\leq \lambda)$-universal in $\mathbf{K}$ when every $N \in \mathbf{K}$ of cardinality $\leq \lambda$ can be embedded into M.
3) We say $M \in \mathbf{K}$ is universal when it is λ-universal for $\lambda = \|M\|$.
4) Assume $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}}), K_{\mathfrak{k}}$ as a class of τ-structures (for some vocabulary $\tau = \tau_{\mathfrak{k}}$), closed under isomorphism, and $\leq_{\mathfrak{k}}$ a partial order on $K_{\mathfrak{k}}$ preserved under isomorphisms. Above "$M \in K_{\mathfrak{k}}$ is λ-universal in $\mathfrak{k}$" means that if $N \in K_{\mathfrak{k}}$ has cardinality λ then there is a $\leq_{\mathfrak{k}}$-embedding f of N into M, i.e. f is an isomorphism from N onto some $N' \leq_{\mathfrak{k}} M$. Similarly in the other variants.

The problem of the existence of universal members of $\mathbf{K}^{\mathrm{lf}}_\lambda$ is connected to

Question 0.22. Fixing κ and an ideal J on κ, what is $\lambda_{\mu,\kappa}(J, \mathbf{K})$, which is the minimal cardinal (or ∞) λ which is $> \mu$ and there is no sequence $\langle (G_\alpha, \bar{a}_\alpha) : \alpha < \lambda \rangle$ such that $G_\alpha \in \mathbf{K}_{\leq \mu}, \bar{a}_\alpha \in {}^\kappa(G_\alpha)$ and there are no $H \in \mathbf{K}$ and $\alpha < \beta < \lambda$ and embeddings f_1, f_2 of G_α, G_β respectively into H such that $\{i < \kappa : f(a_{\alpha,i}) \neq a_{\beta,i}\} \in J$.

Notation 0.23. 1) Let G, H, K denote members of $\mathbf{K}$.
2) Let p, q, r and s denote types.
3) $\mathfrak{s}$ denotes a scheme of defining types, here qf.
4) t denotes a member of some $\mathrm{def}(G)$, i.e. a pair $(\mathfrak{s}, \bar{a})$ which defines a type in $\mathbf{S}^{n(\mathfrak{s})}_{\mathrm{bs}}(G)$.
5) For $A \subseteq M$ let $c\ell(A, M) = \langle A \rangle_M$ be the closure of the set A under the functions of M, i.e. the sub-structure of M which A generates when M is, as usual, a group.
6) We may write, e.g. $A+B$, $A+\bar{a}$, $\sum\limits_{i<\alpha} \bar{a}_i$ instead of $A \cup B$, $A \cup \mathrm{Rang}(\bar{a})$, $\bigcup\limits_{i<\alpha} \mathrm{Rang}(\bar{a}_i)$.
7) For a group G and $x \in G$ let $\Box_x$ be conjugation by x, that is, the mapping $y \mapsto x^{-1} y x$ for $y \in G$.

§ 1. Definable types

What is accomplished in §1 and under what assumptions? We have to assume that there are dense $\mathfrak{S} \in \Omega[\mathbf{K}]$ to get existentially closed H (see §2). Still there are $\mathfrak{S}$'s and any $\mathfrak{S}$ can be extended to a closed one, preserving density. For any $\mathfrak{S}$, the partial order $\leq_{\mathfrak{S}}$ on $\mathbf{K}$ is quite reasonable: not fully so called a.e.c. still close enough. In $(\mathbf{K}, \leq_{\mathfrak{S}})$ for regular λ we can find over any $G \in K_{\leq \lambda}$ a prime H among the $H \in \mathbf{K}_\lambda$ extending G which are so-called $(\lambda, \mathfrak{S})$-full over it, see 1.23. Also we can find such H quite definable in three ways. First avenue is to allow a well order. Second avenue is to assume $\mathfrak{S}$ is symmetric, then H is canonical and commutes with extensions (1.13, 1.16, 1.23, 1.17). Third avenue relies on linear order. We still get uniqueness, but rely on linear ordering of $\mathrm{def}(G)$ and the commutation with extension is problematic. However, we may use pair $(I, E), I$ a linear order, E an equivalence relation on I and "dedicate" each equivalence class to some $t \in \mathrm{def}(G)$, so can avoid linearly ordering $\mathrm{def}(G)$, see 1.30, 1.33; see more in §3.

§ 1(A). **The Framework.**

Definition 1.1. 1) For $G \in \mathbf{K}$ let $\mathrm{def}(G)$ be the set of pairs $t = (\mathfrak{s}, \bar{a}) = (\mathfrak{s}_t, \bar{a}_t)$ such that $\mathfrak{s} \in \Omega[\mathbf{K}]$ and $\bar{a} \in {}^{\omega >}G$ realizes $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$ and let $q_t(G) = q_{\mathfrak{s}_t}(\bar{a}_t, G)$ and $p_t(\bar{x}_t) = p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}), k(t) = k(\mathfrak{s}), n(t) = n(\mathfrak{s})$.
2) We say $\mathfrak{s}_1, \mathfrak{s}_2$ are disjoint when $\bar{x}_{\mathfrak{s}_1}, \bar{x}_{\mathfrak{s}_2}$ are <u>disjoint</u> as well as $\bar{z}_{\mathfrak{s}_1}, \bar{z}_{\mathfrak{s}_2}$ recalling 0.9(4). Similarly for $t_1, t_2 \in \mathrm{def}(G)$.
3) We say $\mathfrak{s}_1, \mathfrak{s}_2$ are <u>congruent</u>, written $\mathfrak{s}_1 \equiv \mathfrak{s}_2$ <u>when</u> we get $\mathfrak{s}_2$ from $\mathfrak{s}_1$ by replacing $\bar{x}_{\mathfrak{s}_1}, \bar{z}_{\mathfrak{s}_1}$ by other sequences of variables, $\bar{x}_{\mathfrak{s}_2}, \bar{z}_{\mathfrak{s}_2}$ (again with no repetitions, of the same length respectively, of course). Similarly for $t_1, t_2 \in \mathrm{def}(G)$ (the aim is to be able to get <u>disjoint</u> congruent copies; we do not always remember to replace a scheme by some congruent copy).
4) We say $\mathfrak{S}$ is <u>invariant</u> <u>when</u>: if $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$ are congruent then $\mathfrak{s}_1 \in \mathfrak{S} \Leftrightarrow \mathfrak{s}_2 \in \mathfrak{S}$.
5) The invariant closure of $\mathfrak{S}$ is defined naturally. Let $|\mathfrak{S}|$ mean its cardinality up to congruency, that is, $|\mathfrak{S}/\equiv|$; if not said otherwise we use invariant $\mathfrak{S}$.
6) We define the (equivalence) relation $\approx_G$ on $\mathrm{def}(G)$ by $t_1 \approx_G t_2$ <u>iff</u> $t_1, t_2 \in \mathrm{def}(G)$ and $q_{t_1}(G) = q_{t_2}(G)$.

Claim 1.2. *1) If $\mathfrak{s} \in \Omega_{n,k}[\mathbf{K}]$ and $G \in \mathbf{K}, \bar{a} \in {}^k M$ <u>then</u> indeed $q_{\mathfrak{s}}(\bar{a}, G) \in \mathbf{S}^n_{\mathrm{bs}}(G)$ so exist and is unique and does not split over $\bar{a}$, see Definition 0.9(2); if $\bar{a}$ is empty, i.e. $k_{\mathfrak{s}} = 0$ we may write $q_{\mathfrak{s}}(G)$.*
1A) If $G_1 \subseteq G_2 \subseteq \mathbf{K}$ and $t \in \mathrm{def}(G_1)$ <u>then</u> $t \in \mathrm{def}(G_2)$ and $q_t(G_1) \subseteq q_t(G_2)$.
2) Assume $G \subseteq H \in \mathbf{K}$ and G is existentially closed or just $G \leq_{\Sigma_1} H \in \mathbf{K}$. If $t_1, t_2 \in \mathrm{def}(G)$ <u>then</u> $q_{t_1}(G) = q_{t_2}(G)$ iff $q_{t_1}(H) = q_{t_2}(H)$.
3) Let $K \subseteq G \in \mathbf{K}, G$ be existentially closed or just every $r \in \mathbf{S}^{<\omega}_{\mathrm{bs}}(K)$ is realized in G, K is finite, and $p \in \mathbf{S}^n_{\mathrm{bs}}(G)$.

The type p does not split over K <u>iff</u> there are $\mathfrak{s} \in \Omega[\mathbf{K}]$ and a finite sequence $\bar{a}$ from K (even listing K) realizing $p_{\mathfrak{s}}(\bar{x})$ such that $p = q_{\mathfrak{s}}(\bar{a}, M)$.
4) If $G \subseteq H, \mathfrak{s} \in \Omega[\mathbf{K}], \bar{a} \in {}^{k(\mathfrak{s})}G$ realizes $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$ and $\bar{c} \in {}^{n(\mathfrak{s})}H$ realizes $q_{\mathfrak{s}}(\bar{a}, G)$ in H and $\sigma(\bar{z}_{\mathfrak{s}}, \bar{x}_{\mathfrak{s}})$ is a group-term <u>then</u> $\sigma^H(\bar{c}, \bar{a}) \in G \Rightarrow \sigma^H(\bar{c}, \bar{a}) \in c\ell(\bar{a}, G)$.
4A) In (4), if $\bar{a}' = \bar{a}{}^\frown\bar{a}''$ then $\sigma^H(\bar{c}, \bar{a}'') \in G \Rightarrow \sigma^H(\bar{c}, \bar{a}'') \in c\ell(\bar{a}'', G)$ because p also does not split over $\bar{a}^*$ if $\bar{a}^* \subseteq G, \bar{a} \subseteq c\ell(\bar{a}^*, H)$.

Proof. 1) Let $K_* \subseteq G$ be the subgroup of G generated by $\bar{a}$.

First, there are H and $\bar{c}$ such that:

$(*)^1_{H,\bar{c}}$ $G \subseteq H \in \mathbf{K}$ and $\bar{c} \in {}^n H$ such that $\mathrm{tp}(\bar{c}, G, H) = q_{\mathfrak{s}}(\bar{a}, G)$.

Why? For every $K \in \mathbf{K}^* := \{K \subseteq G : K$ finite extending $K_*\}$ we can choose a pair $(H_K, \bar{c}_K)$ such that: $K \subseteq H_K \in \mathbf{K}, H_K$ is finite, $\bar{c}_K \in {}^n(H_K), H_K$ is generated by $K \cup \bar{c}_K$ and for some $\bar{b}$ listing $K, \mathrm{tp}_{\mathrm{bs}}(\bar{a}\hat{\ }\bar{b}\hat{\ }\bar{c}_K, \emptyset, H_K) = \mathfrak{s}(\mathrm{tp}_{\mathrm{bs}}(\bar{a}\hat{\ }\bar{b}, \emptyset, G))$.

[Why? By Definition 0.9(1A)(b).] Now for every $K_1 \subseteq K_2$ from $\mathbf{K}_*$ we can choose an embedding f_{K_2,K_1} from H_{K_1} into H_{K_2} extending id_{K_1} and mapping $\bar{c}_{K_1}$ to $\bar{c}_{K_2}$. [Why? By Definition 0.9(1A)(c).]

As H_{K_1} is generated by $K_1 \cup \bar{c}$, this mapping is unique. Now if $K_1 \subseteq K_2 \subseteq K_3$ are from $\mathbf{K}_*$ then $f_{K_3,K_2} \circ f_{K_2,K_1}$ is an embedding of H_{K_1} into H_{K_3} extending id_{K_1} and mapping $\bar{c}_{K_1}$ to $\bar{c}_{K_3}$; hence by the previous sentence $f_{K_3,K_2} \circ f_{K_2,K_1} = f_{K_3,K_1}$. Hence $\langle H_{K_1}, f_{K_2,K_1} : K_1 \subseteq K_2$ are from $\mathbf{K}_*\rangle$ has a direct limit, i.e. we can find a group H and $\bar{f} = \langle f_K : K \in \mathbf{K}_*\rangle$ such that f_K embed H_K into H and for every $K_1 \subseteq K_2$ from $\mathbf{K}_*$ we have $f_{K_1} = f_{K_2} \circ f_{K_2,K_1}$. Without loss of generality $H = \cup\{f_K(H_K) : K \in \mathbf{K}_*\}$ hence H is a locally finite group and $\{f_K : K \in \mathbf{K}_*\}$ embeds G into H, so without loss of generality $G \subseteq H$ and $f_K{\restriction}K = \mathrm{id}_K$ for $K \in \mathbf{K}_*$. Letting $\bar{c} = f_K(\bar{c}_K)$ for any $K \in \mathbf{K}_*$, clearly $(H, \bar{c})$ is as required in $(*)^1_{H,\bar{c}}$.

$(*)_2$ $\mathrm{tp}_{\mathrm{bs}}(\bar{c}, G, H)$ belongs to $\mathbf{S}^n_{\mathrm{bs}}(G)$.

[Why? By the definitions of $\mathbf{S}^n_{\mathrm{bs}}(G)$ because $G \subseteq H \in \mathbf{K}$ and $\bar{c} \in {}^n H$.]

$(*)_3$ $q_{\mathfrak{s}}(\bar{a}, G)$ is unique and does not split over $\bar{a}$.

[Why? See Definition 0.9(1A)(c).]

1A) See Definition 0.9(2).

2) For $\ell = 1, 2$ we have $q_{t_\ell}(G) \subseteq q_{t_\ell}(H)$, moreover, $q_{t_\ell}(G) = \{\varphi(\bar{z}_{n(t)}, \bar{b}) \in q_{t_\ell}(H) : \bar{b} \subseteq G\}$. For the other direction, note that $\bar{a}_{t_1}, \bar{a}_{t_2} \subseteq G$ and assume $q_{t_1}(H) \neq q_{t_2}(H)$, hence there are m and $\bar{b} \in {}^m H$ and a basic formula $\varphi(\bar{y}_m, \bar{z}_n)$ such that $\varphi(\bar{b}, \bar{z}_n) \in q_{t_1}(H), \neg\varphi(\bar{b}, \bar{z}_n) \in q_{t_2}(H)$. Now there is $\bar{b}' \in {}^m G$ such that $\mathrm{tp}_{\mathrm{bs}}(\bar{b}', \bar{a}_{t_1}\hat{\ }\bar{a}_{t_2}, G) = \mathrm{tp}_{\mathrm{bs}}(\bar{b}, \bar{a}_{t_1}\hat{\ }\bar{a}_{t_2}, H)$ because $G \leq_{\Sigma_1} H$ and our choice of $\mathbf{K}$. As $q_{t_\ell}(H)$ does not split over $\bar{a}_{t_\ell}$, clearly $\varphi(\bar{b}', \bar{z}_n) \in q_{t_\ell}(H) \Leftrightarrow \varphi(\bar{b}, \bar{z}_n) \in q_{t_\ell}(H)$ for $\ell = 1, 2$.

Together with an earlier sentence, $\varphi(\bar{b}', \bar{z}_n) \in q_{t_1}(H), \neg\varphi(\bar{b}', \bar{z}_n) \in q_{t_2}(H)$ hence by the first sentence in the proof of 1.2(2) we have $\varphi(\bar{b}', \bar{z}_n) \in q_{t_1}(G)$ and $\neg\varphi(\bar{b}', \bar{z}_n) \in q_{t_2}(G)$ hence $q_{t_2}(G) \neq q_{t_2}(G)$ so we are also done with the "other" direction.

3) The implication "if" holds by 1.2(1). For the other direction assume p does not split over K. As K is finite, let $k = |K|$ and let $\bar{a} \in {}^n K \subseteq {}^n G$ list K.

We now define $\mathfrak{s}$ by:

(a) $p_{\mathfrak{s}} = \mathrm{tp}_{\mathrm{bs}}(\bar{a}, \emptyset, K)$ so $k_{\mathfrak{s}} = k$
(b) $q = \mathfrak{s}(s(\bar{x}, \bar{y}))$ <u>iff</u> for some $\bar{b} \in {}^m G$ we have:
   - $s(\bar{x}, \bar{y}) = \mathrm{tp}(\bar{a}\hat{\ }\bar{b}, \emptyset, G)$
   - $q = \mathrm{tp}_{\mathrm{bs}}(\bar{a}\hat{\ }\bar{b}\hat{\ }\bar{c}, \emptyset, G)$ for some $\bar{c} \in {}^n G$ realizing $p{\restriction}(\bar{a}\hat{\ }\bar{b})$.

Now $\mathfrak{s}$ is well defined because on the one hand p does not split over $\bar{a}$, and on the other hand G is existentially closed or just every $r \in \mathbf{S}^{<\omega}_{\mathrm{bs}}(K)$ is realized in G.

4) By 1.2(2) without loss of generality G is existentially closed, assume $\sigma^H(\bar{c},\bar{a}) \in G$ and let $b = \sigma^H(\bar{c},\bar{a})$. If $b \notin c\ell(\bar{a},G)$ there is $b' \in G\backslash\{b\}$ realizing $\mathrm{tp}_{\mathrm{bs}}(b,K,G)$ because $\mathbf{K}$ has disjoint amalgamation for finite members. As $q_{\mathfrak{s}}(\bar{a},G)$ does not split over $\bar{a}$ and $b',b \in G$ realize the same type over $\bar{a}$ it follows that $H \models$ "$(\sigma(\bar{c},\bar{a}) = b) \equiv (\sigma(\bar{c},\bar{a}) = b')$", an obvious contradiction.
4A) Should be clear. $\square_{1.2}$

Example 1.3. There is $\mathfrak{s} \in \Omega[\mathbf{K}]$ such that:

(a) $k_{\mathfrak{s}} = 0$ and $n_{\mathfrak{s}} = 1$;
(b) if $G \subseteq H \in \mathbf{K}$ and $a \in H$, <u>then</u>: a realizes $q_{\mathfrak{s}}(<>,G)$ <u>iff</u> $a \in H\backslash G$ has order 2 and commute with every member of G.

Definition 1.4. 1) For $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ we define the two place relation $\leq_{\mathfrak{S}}$ on $\mathbf{K}$ as follows: $M \leq_{\mathfrak{S}} N$ iff $M \subseteq N$ (and they belong to $\mathbf{K}$) and for every $n < \omega$ and $\bar{c} \in {}^n N$ we can find $k < \omega$ and $\bar{a} \in {}^k M$ and $\mathfrak{s} \in \mathfrak{S}$ such that $p_{\mathfrak{s}}(\bar{x}) \subseteq \mathrm{tp}_{\mathrm{bs}}(\bar{a},\emptyset,M) \in \mathbf{D}_k(\mathbf{K})$ and $\mathrm{tp}_{\mathrm{bs}}(\bar{c},M,N) = q_{\mathfrak{s}}(\bar{a},M)$ recalling $q_{\mathfrak{s}}(\bar{a},M) \in \mathbf{S}^n_{\mathrm{bs}}(M)$.
2) For $M \in \mathbf{K}$ and $\mathfrak{S} \subseteq \mathfrak{S}[\mathbf{K}]$ let

(a) $\mathbf{S}^n_{\mathfrak{S}}(M) = \{q_{\mathfrak{s}}(\bar{a},M) : \mathfrak{s} \in \mathfrak{S}$ satisfies $n_{\mathfrak{s}} = n$ and $\bar{a} \in {}^{k(\mathfrak{s})}M$ realizes $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})\}$
(b) $\mathrm{def}_{\mathfrak{S}}(M) = \{t \in \mathrm{def}(M) : \mathfrak{s}_t \in \mathfrak{S}\}$
(c) $\mathbf{S}_{\mathfrak{S}}(M) = \cup\{\mathbf{S}^n_{\mathfrak{S}}(M) : n < \omega\}$.

3) We say $M \in \mathbf{K}$ is <u>$\mathfrak{S}$-existentially closed</u> <u>when</u> for every $\mathfrak{s} \in \mathfrak{S}$, finite[4] $G \subseteq M$ and $\bar{a} \in {}^{\omega>}G$ realizing $p_{\mathfrak{s}}(\bar{x})$ the type $q_{\mathfrak{s}}(\bar{a},G)$ is realized in M; (this is equivalent to being existentially closed if $\mathfrak{S}$ is dense, see Definition 1.6(2) below).

Definition 1.5. We say $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is <u>symmetric</u> <u>when</u> : if $\mathfrak{s}_1,\mathfrak{s}_2 \in \mathfrak{S}, M \subseteq N$ are from $\mathbf{K}$ and $\bar{c}_\ell \in {}^{n(\mathfrak{s}_\ell)}N$ realizes $q_{\mathfrak{s}_\ell}(\bar{a}_\ell,M)$ in N (so $\bar{a}_\ell \in {}^{k(\mathfrak{s}_\ell)}M$ realizes $p_{\mathfrak{s}_\ell}(\bar{x}_{\mathfrak{s}_\ell})$) and $M_\ell = c\ell(M+\bar{c}_\ell,N) \subseteq N$ for $\ell = 1,2$ <u>then</u> $\bar{c}_1$ realizes $q_{\mathfrak{s}_1}(\bar{a}_1,M_2)$ in N iff $\bar{c}_2$ realizes $q_{\mathfrak{s}_2}(\bar{a}_2,M_1)$ in N.

Definition 1.6. 1) We say $\mathfrak{S}$ is closed <u>when</u> it is dominating-<u>closed</u> and composition-closed, see below and invariant of course.
1A) $\mathfrak{S}$ is <u>composition-closed</u> <u>when</u> if $H_0 \subseteq H_1 \subseteq H_2 \in \mathbf{K}, \bar{a}_\ell \in {}^{n(\ell)}(H_\ell)$ for $\ell = 0,1,2$ and $\mathrm{tp}_{\mathrm{bs}}(\bar{a}_{\ell+1},H_\ell,H_{\ell+1}) = q_{\mathfrak{s}_\ell}(\bar{a}_\ell,H_\ell) \in \mathbf{S}^{n(\ell+1)}_{\mathfrak{S}}(H_\ell)$ and $H_{\ell+1} = c\ell(H_\ell + \bar{a}_\ell,H_{\ell+1}), \mathfrak{s}_\ell \in \mathfrak{S}$ for $\ell = 0,1$ <u>then</u> $\mathrm{tp}_{\mathrm{bs}}(\bar{a}_1{}^\frown\bar{a}_2,H_0,H_2) = q_{\mathfrak{s}}(\bar{a}_0,H_0)$ for some $\mathfrak{s} \in \mathfrak{S} \cap \Omega_{n(1)+n(2),n(0)}[\mathbf{K}]$.
1B) $\mathfrak{S}$ is <u>dominating-closed</u> <u>when</u>: if $H_0 \subseteq H_1 \in \mathbf{K}, \bar{a}_1 \in {}^{k(1)}(H_0), \bar{c}_1 \in {}^{n(1)}(H_1)$, $\mathrm{tp}_{\mathrm{bs}}(\bar{c}_1,H_0,H_1) = q_{\mathfrak{s}}(\bar{a}_1,H_0) \in \mathbf{S}^{n(1)}_{\mathfrak{S}}(H_0)$ and $\bar{c}_2 \in {}^{n(2)}(H_1)$ and $\bar{a}_2 \in {}^{k(2)}(H_0)$, $\mathrm{Rang}(\bar{a}_2) \supseteq \mathrm{Rang}(\bar{a}_1)$ and $\bar{c}_2 \subseteq c\ell(\bar{a}_2+\bar{c}_1,H_1)$ <u>then</u> $\mathrm{tp}(\bar{c}_2,H_0,H_1) = q_{\mathfrak{s}}(\bar{a}_2,H_0)$ for some $\mathfrak{s} \in \mathfrak{S}$.
2) We say $\mathfrak{S}$ is weakly dense <u>when</u>: every $\mathfrak{S}$-existentially closed $G \in K$ is existentially closed.
3) We say $\mathfrak{S}$ is <u>dense</u> when: for every $G_0 \subseteq H \in \mathbf{K}, G_0 \subseteq G_1 \in \mathbf{K}, G_0,G_1$ are finite and $\bar{c} \in {}^n(G_1)$ <u>there is</u> $p(\bar{z}) \in \mathbf{S}^n_{\mathfrak{S}}(H)$ which extends $\mathrm{tp}_{\mathrm{bs}}(\bar{c},G_0,G_1)$. Moreover $p(\bar{z}) = q_{\mathfrak{s}}(\bar{a},H)$ for some $\mathfrak{s} \in \mathfrak{S}$ and $\bar{a}$ from G_0.

[4]For general $\mathbf{K}$: we use finitely generated $G \subseteq M$; generally this change is needed.

4) For disjoint $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathfrak{S}$ define $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ with $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = p_{\mathfrak{s}_1}(\bar{x}_{\mathfrak{s}_1}) \cup p_{\mathfrak{s}_2}(\bar{x}_{\mathfrak{s}_2})$, recalling $\bar{x}_{\mathfrak{s}_1}, \bar{x}_{\mathfrak{s}_2}$ are disjoint, as follow: if $G_0 \subseteq G_1 \subseteq G_2$ are from $\mathbf{K}$ and $\bar{a}_\ell \in {}^{k(\mathfrak{s}_\ell)}G_0, \bar{a}_\ell$ realizes $p_{\mathfrak{s}_\ell}(\bar{x}_{\mathfrak{s}_\ell})$ in $G_0 \in \mathbf{K}$ and $\bar{c}_\ell \in {}^{n(\mathfrak{s}_\ell)}(G_{\ell+1})$ realizes $q_{\mathfrak{s}_\ell}(\bar{a}_\ell, G_\ell)$ for $\ell = 1, 2$ <u>then</u> $\bar{c}_1{}^\frown\bar{c}_2$ realizes $q_{\mathfrak{s}}(\bar{a}_1{}^\frown\bar{a}_2, G_0)$ in G_2.
4A) For (disjoint) $t_1, t_2 \in$ def(G) we define $t_1 \oplus t_2 = t_1 \oplus_G t_2$ similarly.
5) We define $\bigoplus\limits_{k<m} \mathfrak{s}_k, \bigoplus\limits_{k<m} t_k$ similarly using associativity, see 1.8(5).
6) Let $\mathfrak{s}_1 \le \mathfrak{s}_2$ means: if $G \in \mathbf{K}, \bar{a}_2 \in {}^{u(2)}G$ realizes $p_{\mathfrak{s}_2}(\bar{x}_{\mathfrak{s}_2}), G \subseteq H, \bar{c}_2 \in {}^{n(\mathfrak{s}_2)}H$ realizes $q_{\mathfrak{s}_2}(\bar{a}_2, G)$ <u>then</u> $\mathrm{dom}(\bar{x}_{\mathfrak{s}_1}) \subseteq u(2)$ and $\bar{c}_2 \restriction \mathrm{dom}(\bar{z}_{\mathfrak{s}_2})$ realizes $q_{\mathfrak{s}_1}(\bar{a}_2 \restriction k(\mathfrak{s}_1), G)$ and $p_{\mathfrak{s}_2}(\bar{x}_{\mathfrak{s}_2}) \restriction \bar{x}_{\mathfrak{s}_1} = p_{\mathfrak{s}_1}(\bar{x}_{\mathfrak{s}_1})$.
7) Let $\mathfrak{s}_1 \le_{\bar{h}} \mathfrak{s}_2$ means that $\bar{h} = (h', h''), h'$ is a one-to-one function from $\mathrm{dom}(\bar{x}_{\mathfrak{s}_1})$ into $\mathrm{dom}(\bar{x}_{\mathfrak{s}_2})$ and h'' is a one-to-one function from $\mathrm{dom}(\bar{z}_{\mathfrak{s}_1})$ into $\mathrm{dom}(\bar{z}_{\mathfrak{s}_2})$ such that: if $\mathrm{tp}_{\mathrm{bs}}(\bar{c}_2, G, H) = q_{\mathfrak{s}_2}(\bar{a}_2, G)$ and $\bar{a}_1 = \langle a_{2,h''(\ell)} : \ell \in \mathrm{dom}(\bar{a}_1)\rangle$ and $\bar{c}_1 = \langle c_{2,h(\ell)} : \ell \in \mathrm{dom}(\bar{c}_2)\rangle$ then $\mathrm{tp}_{\mathrm{bs}}(\bar{c}_1, G, H) = q_{\mathfrak{s}_1}(\bar{a}_1, G, H)$. Similarly $t_1 \le_{\bar{h}} t_2$ for $t_1, t_2 \in$ def(G). If $h' \cup h''$ is well defined we may write $h' \cup h''$ instead of $\bar{h}$.

Remark 1.7. 0) Concerning 1.6(7) the point of disjoint $\mathfrak{s}_1, \mathfrak{s}_2$ and congruency is to avoid using it. So we may ignore it as well as 1.9(2),(3), 3.4(3), 3.5(4), 3.6(5).
1) Note that the operation $\mathfrak{s}_1 \oplus \mathfrak{s}_2$ is not necessarily commutative, e.g. for $\mathbf{K}_{\mathrm{olf}}$ it cannot be.
2) In e.g. Definition 1.6(1A), in general $\mathfrak{s}$ is not uniquely determined by the relevant information $\mathrm{tp}_{\mathrm{bs}}(\bar{a}_1{}^\frown\bar{a}_2{}^\frown\bar{c}_1{}^\frown\bar{c}_2, H_0, H_2)$ and the lengths of $\bar{a}_1, \bar{a}_2, \bar{c}_1, \bar{c}_2$ <u>but</u> if H_1 is existentially closed, it is. We could have written the definition in a computational form.
3) So $\mathfrak{s}_1 \le \mathfrak{s}_1$ means $\mathfrak{s}_1 \le_{\bar{h}} \mathfrak{s}_2$ with h_ℓ the identity for $\ell = 1, 2$.

Definition/Claim 1.8. 1) For any $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ we can define its closure as the minimal closed (and <u>invariant</u>, of course) $\mathfrak{S}_1 \subseteq \Omega[\mathbf{K}]$ which includes it, see 1.6(1); we denote it by $c\ell(\mathfrak{S}) = c\ell(\mathfrak{S}; \mathbf{K})$.
2) Similarly for <u>dominating-closure</u> docl$(\mathfrak{S})$ and <u>composition-closure</u> cocl$(\mathfrak{S})$.
3) Those closures preserve density and countability (and being invariant), and have the obvious closure properties.
4) Also dominating-closure preserve being composition closed.
5) The operation $\oplus$ on $\Omega[\mathbf{K}]$ is well defined and associative. If $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is closed under $\oplus$, for transparency, <u>then</u> $\mathfrak{S}$ is symmetric (see 1.5) <u>iff</u> the operation $\oplus$ on $\mathfrak{S}$ is commutative (when defined). Similarly for $\mathrm{def}_{\mathfrak{S}}(G)$.
6) $\Omega[\mathbf{K}]$ has cardinality $\le 2^{\aleph_0}$; generally $\le 2^{|\tau(\mathbf{K})|+\aleph_0}$.
7) $\le_{\mathfrak{S}}$ is a transitive relation on $\mathbf{K}$, if $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is closed.
8) If $H_0 \subseteq H_1 \subseteq H_2, \mathfrak{s} \in \Omega[\mathbf{K}]$ and $\mathrm{tp}_{\mathrm{bs}}(\bar{c}, H_1, H_2) = q_{\mathfrak{s}}(\bar{a}, H_1)$ and $\bar{a} \in {}^{k(\mathfrak{s})}H_0$ <u>then</u> $\mathrm{Rang}(\bar{c}) \cap H_1 = \mathrm{Rang}(\bar{c}) \cap H_0$.
9) Assume $\mathfrak{S}$ is dense and closed. If $G \subseteq H \in \mathbf{K}$ and G is finite then $G \le_{\mathfrak{S}} H$.
10) If $\mathfrak{s} = \mathfrak{s}_0 \oplus \ldots \oplus \mathfrak{s}_{n-1}$ and $i(0) < \ldots i(k-1) < n$ and $\mathfrak{s}' = \mathfrak{s}_{i(0)} \oplus \ldots \oplus \mathfrak{s}_{i(k-1)}$ <u>then</u> $\mathfrak{s}' \le \mathfrak{s}$.

Proof. Natural, noting that (8) is specific for our present $\mathbf{K}$, see 1.2(4). $\square_{1.8}$

Claim 1.9. *0) The operation $\oplus$ is well defined, that is:*

(a) *if $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$ are disjoint <u>then</u> $\mathfrak{s}_1 \oplus \mathfrak{s} \in \Omega[\mathbf{K}]$ is well defined;*
(b) *if $t_1, t_2 \in \mathrm{def}(G)$ are disjoint <u>then</u> $t_1 \oplus t_2 \in \mathrm{def}(G)$.*

1) The operation $\oplus$ on disjoint pairs from $\mathrm{def}(G)$ *respects congruency, see Definition 1.1(3). If $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$ then $(\mathfrak{s}_1/\equiv) \oplus (\mathfrak{s}_2/\equiv)$ is well defined, i.e. if $\mathfrak{s}'_\ell, \mathfrak{s}''_\ell$ are congruent to $\mathfrak{s}_\ell$ for $\ell = 1,2$ and $\mathfrak{s}' = \mathfrak{s}'_1 \oplus \mathfrak{s}'_2, \mathfrak{s}'' = \mathfrak{s}''_1 \oplus \mathfrak{s}''_2$ are well defined (equivalently for $\ell = 1,2$ the two schemes $\mathfrak{s}'_\ell, \mathfrak{s}''_\ell$ are disjoint) then $\mathfrak{s}', \mathfrak{s}''$ are congruent. (So we may forget to be pedantic about this.)*
2) If $(\mathfrak{s}, \bar a) = (\mathfrak{s}_1, \bar a_1) \oplus_G (\mathfrak{s}_2, \bar a_2)$ then $(\mathfrak{s}_\ell, \bar a_\ell) \le (\mathfrak{s}, \bar a)$.
3) If in $\mathrm{def}(G)$ *we have $(\mathfrak{s}_\ell, \bar a_\ell) \le_{h_\ell} (\mathfrak{s}'_\ell, \bar a'_\ell)$ for $\ell = 1,2$ and* $\mathrm{Dom}(h_1) \cap \mathrm{Dom}(h_2) = \emptyset$, $\mathrm{Rang}(h_1) \cap \mathrm{Rang}(h_1) = \emptyset$ *then $(\mathfrak{s}_1, \bar a_1) \oplus (\mathfrak{s}_2, \bar a_2) \le_{h_1 \cup h_2} (\mathfrak{s}'_1, \bar a_1) \oplus (\mathfrak{s}'_2, \bar a_2)$. Similarly for $\bar h_1, \bar h_2$.*

Proof. Straightforward. $\square_{1.9}$

Claim 1.10. *Assume $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is dominating-closed and $G_0 \subseteq G_1 \in \mathbf{K}$ and $G_0 \le_{\mathfrak{S}} G_2$ and, for transparency, $G_1 \cap G_2 = G_0$ and*[5] *$G_0 \le_{\Sigma_1} G_2$ (holds if G_0 is existentially closed in $\mathbf{K}$).*
1) There is $G_3 \in \mathbf{K}$ such that $G_1 \le_{\mathfrak{S}} G_3$ and $G_2 \subseteq G_3$ and $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$ and $G_1 \le_{\Sigma_1} G_3$.
2) Above G_3 above is unique up to isomorphism over $G_1 \cup G_2$.
3) If $\mathfrak{S}$ is symmetric and $G_0 \le_{\mathfrak{S}} G_1$ in part (1) then also $G_2 \le_{\mathfrak{S}} G_3$.

Proof. Straightforward, e.g.
1) Let $\bar c = \langle c_\alpha : \alpha < \alpha(*) \rangle$ list the elements of G_2, and for every finite $u \subseteq \alpha(*)$ let $\bar x_u = \langle x_\alpha : \alpha \in u \rangle$ and $p^0_u(\bar x_u) = \mathrm{tp}_{\mathrm{bs}}(\bar c \restriction u, G_0, G_2)$ hence by assumption, there is $\mathfrak{s}_u \in \mathfrak{S}$ (up to congruency) and $\bar a_u \in {}^{k(\mathfrak{s}_u)}(G_0)$ such that $p^0_u(\bar x) = q_{\mathfrak{s}_u}(\bar a_u, G_0)$ so $\mathrm{dom}(\bar x_{\mathfrak{s}_u}) = u$. We define $p^1_u(\bar x_u) \in \mathbf{S}(G_1)$ as $q_{\mathfrak{s}_u}(\bar a_u, G_1)$. We define G_3 as a group extending G_1 generated by $G_1 \cup \{c_\alpha : \alpha < \alpha(*)\}$ such that $\bar c \restriction u$ realizes $p^1_u(\bar x_u)$ for every finite $u \subseteq \alpha(*)$. But for this to work we have to prove that for finite $u \subseteq v \subseteq \alpha(*)$ we have $p^1_u(\bar x_u) \subseteq p^1_v(\bar x_v)$. This is straightforward recalling 1.2(1A).
Lastly, $G_1 \le_{\Sigma_1} G_3$ is easy, too. $\square_{1.10}$

Remark 1.11. 1) We may consider an alternative definition of $\le_{\mathfrak{S}}$:

- $\bullet_1$ $G \le_{\mathfrak{S}} H$ <u>iff</u> for every finite $A \subseteq H$ there are $\bar c \in {}^{\omega>}H, \bar a \in {}^{\omega>}G$ and $\mathfrak{s} \in \mathfrak{S}$ such that: $\bar a$ realizes $p_{\mathfrak{s}}(\bar x_{\mathfrak{s}}), \bar c$ realizes $q_{\mathfrak{s}}(\bar a, G)$ in H and $A \subseteq \mathrm{Rang}(\bar c)$.

An even weaker version is:

- $\bullet_2$ as in $\bullet_1$ but "$A \subseteq \mathrm{Rang}(\bar c)$" is replaced by $A \subseteq c\ell(G \cup \bar c, H)$.

2) But, e.g. for $\bullet_1$, to prove $\le_{\mathfrak{S}}$ is transitive we need a stronger version of composition-closed: if $G_0 \subseteq G_1 \subseteq G_2$ and for $\ell = 0,1, \bar c_\ell \in {}^{n(\ell)}(G_{\ell+1})$ realizes $q_{\mathfrak{s}_\ell}(\bar a_\ell, G_\ell)$ and $\mathrm{Rang}(\bar b_0) \subseteq \mathrm{Rang}(\bar a_1)$ <u>then</u> for some $\mathfrak{s} \in \mathfrak{S}, p_{\mathfrak{s}}(\bar x_{\mathfrak{s}}) = p_{\mathfrak{s}_0}(\bar x_{\mathfrak{s}})$ and $\bar a_1 {}^\frown \bar a_2$ realizes $q_{\mathfrak{s}}(\bar a_0, G_0)$.
3) In any case for closed $\mathfrak{S}$ the three definitions are equivalent, i.e. those in $\bullet_1$, in $\bullet_2$ and in 1.4(1).
4) Does the operation $\oplus_G$ respect $\approx_G$, see Definition 1.1, i.e. if $t_1 \approx_G t'_1$ and $t_2 \approx_G t'_2$ <u>then</u> $t_1 \oplus_G t_2 \approx_G t'_1 \oplus_G t'_2$?; all this assuming the operations are well defined, i.e. the disjointness demands from 1.6(4) are satisfied. We do not see a reason for this to hold.

[5] If $G_2 = \langle G_0 \cup A \rangle$, A finite then for part (1) this is not necessary.

§ 1(B). **Constructions.**

Before we present the more systematic construction from [**?**, Ch.IV], we give a self-contained direct definition and proof for the existence of a canonical existentially closed extension of $G \in \mathbf{K}$ when $\mathfrak{S}$ is symmetric, i.e. the "second avenue" in §0B. We shall deal with the non-symmetric case later.

Definition 1.12. Assume $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is symmetric.
1) We say H is a $\mathfrak{S}$-<u>closure</u> of G <u>when</u> there is a sequence $\langle G_n : n < \omega \rangle$ such that $G_0 = G, H = \cup\{G_n : n < \omega\}$ and G_{n+1} is a one-step $\mathfrak{S}$-closure of G_n, see below.
2) We say that H is a <u>one-step</u> $\mathfrak{S}$-<u>closure</u> of G <u>when</u>:

- (a) $G \subseteq H$ are from $\mathbf{K}$;
- (b) $S := \operatorname{def}(G) = \{(\mathfrak{s}, \bar{a}) : \mathfrak{s} \in \mathfrak{S}$ and $\bar{a} \in {}^{\omega >}G$ realizes $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})\}$ and let $t = (\mathfrak{s}_t, \bar{a}_t) = (\mathfrak{s}(t), \bar{a}(t))$ for $t \in S$;
- (c) $\bar{c}_t \in {}^{n(\mathfrak{s}(t))}H$ realizes $q_{\mathfrak{s}_t}(\bar{a}_t, G)$ for $t \in S$;
- (d) H is generated by $\{\bar{c}_t : t \in S\} \cup G$;
- (e) $\bar{c}_t$ realizes $q_{\mathfrak{s}_t}(\bar{c}_t, c\ell(\cup\{\bar{c}_s : s \in S \backslash \{t\}\} \cup G, H)$ inside H for every $t \in S$.

Claim 1.13. *Let $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ be symmetric.*
1) For every $G \in \mathbf{K}$ there is a one-step $\mathfrak{S}$-closure H of G.
2) For every $G \in \mathbf{K}$ there is an $\mathfrak{S}$-closure H of G.
3) In both parts (1) and (2) we have $|G| \leq |H| \leq |G| + |\mathfrak{S}| + \aleph_0$.
4) In both parts (1) and (2), H is unique up to isomorphism over G.
5) If the pair (G_ℓ, H_ℓ) is as in part (1), or as in part (2) for $\ell = 1, 2$ and $G_1 \subseteq G_2$ <u>then</u> H_1 can be embedded into H_2 over G_1.
6) In both parts (1) and (2) there is a set theoretic class function $\mathbf{F}$ computing H from G, pedantically for every $G \in \mathbf{K}$ and ordinal α not in the transitive closure $\mathrm{tr} - c\ell(G)$ of $G, \mathbf{F}_\alpha(G)$ is well defined such that:

- (A)
  - (a) $\mathbf{F}_\alpha(G) \in \mathbf{K}_{\mathrm{lf}}$ *is of cardinality* $\leq |G| + \aleph_0 + |\alpha|$
  - (b) *if* $\alpha = 0$ *then* $\mathbf{F}_\alpha(G) = G$
  - (c) *the sequence* $\langle \mathbf{F}_\beta(G) : \beta \leq \alpha \rangle$ *is increasing continuous*
  - (d) $\mathbf{F}_{\alpha+1}(G)$ *is a one step closure of* $\mathbf{F}_\alpha(G)$
- (B) *if* $G_1 \subseteq G_2 \wedge G_2 \cap \mathbf{F}_\alpha(G_1) = G_1 \wedge \bigwedge\limits_{\ell=1}^{2} \emptyset = (\alpha + 1) \cap \mathrm{tr} - c\ell(G_\ell)$ *then* $\mathbf{F}_\alpha(G_1) \subseteq \mathbf{F}_\alpha(G_2)$*; this is "naturality"; an alternative is 0.12(2).*

7) In fact we do not have to use the axiom of choice.

Proof. Should be clear (alternatively, below we do more). $\square_{1.13}$

Remark 1.14. Similarly in §3.

Definition 1.15. 1) We say N is $(\lambda, \mathfrak{S})$-<u>full</u> over M when: $M \subseteq N$ and <u>if</u> $M \subseteq M_1 \subseteq N$ and $M_1 = c\ell(M + A, N)$ for some $A \subseteq M_1$ of cardinality $< \lambda$ and $\mathfrak{s} \in \mathfrak{S}$ and $\bar{a} \in {}^{k(\mathfrak{s})}M_1$ realizes $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$ in M_1 then $q_{\mathfrak{s}}(\bar{a}, M_1)$ is realized in N.

2) We may write "N is $\mathfrak{S}$-full over M" when $\lambda = \|N\|$ is regular or, in general, when[6] there is a list $\langle a_\alpha : \alpha < \|N\|\rangle$ of N such that for every $\alpha < \|N\|$ and $\mathfrak{s} \in \mathfrak{S}$ we have: if $M_\alpha = c\ell(M + \{a_\beta : \beta < \alpha\}, N)$ and $\bar{a} \in {}^{k(\mathfrak{s})}M_\alpha$ realizes $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$ then the type $q_{\mathfrak{s}}(\bar{a}, M_\alpha)$ is realized in N by $\|N\|$ elements.
3) We may omit $\mathfrak{S}$ when $\mathfrak{S} = \Omega[\mathbf{K}]$.

Claim 1.16. *Let $\mathfrak{S}$ be symmetric.*
1) If $\mathfrak{S} \subseteq \mathfrak{S}(\mathbf{K})$ is closed (see 1.6(1)) then $(\mathbf{K}, \leq_{\mathfrak{S}})$ is a weak a.e.c. with amalgamation[7] (even canonical), see [?, 1.2] or [?, Ch.I], i.e. in the Definition of a.e.c. we have Ax 0,(I),(II),(III),(V) but $\mathrm{LST}(\mathbf{K}, \leq_{\mathfrak{S}})$ may be ∞ and we omit Ax(IV), see 1.18 below.
2) If $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is dense and closed (see 1.6) then for every $M \in \mathbf{K}_\lambda$ there is an existentially closed $N \in \mathbf{K}_\lambda$ which $\leq_{\mathfrak{S}}$-extends it, in fact any $\mathfrak{S}$-closure of M can serve.
3) If N is $(\lambda, \mathfrak{S})$-full over M_1 and $M_0 \subseteq M_1$, then N is $(\lambda, \mathfrak{S})$-full over M_0; also in Definition 1.15 without loss of generality *$\bar{a}$ is from $M \cup A$, i.e. $\bar{a} \in {}^{k(\mathfrak{s})}(M \cup A)$.*
4) If $M \in \mathbf{K}_{\leq\lambda}$ then there is a model N, $(\lambda, \mathfrak{S})$-full over M of cardinality $\leq \lambda + \|M\| = \lambda$; moreover if $\mathfrak{S}$ is dense, then M is existentially closed.
5) In (4), we can add: if $N' \in \mathbf{K}$ is $(\lambda, \mathfrak{S})$-full over M then we can find an embedding of N into N' over M.

Proof. 1) Easy.
2) Easy by 1.13 and see more below.
3) Easy.
4) We choose $G_n \in \mathbf{K}$ by induction on n such that:

- (a) $G_0 = M$;
- (b) $G_{n+1} \supseteq G_n$ is as in Definition 1.12 but each t appears λ times, i.e.
  - $G_{n+1} = c\ell(\cup\{\bar{c}^n_{t,\alpha} : t \in \mathrm{def}_{\mathfrak{S}}(G_n) \text{ and } \alpha < \lambda\} \cup G_n, G_{n+1})$ where
  - $\mathrm{tp}_{\mathrm{bs}}(\bar{c}^n_{t,\alpha}, G_{n,t,\alpha}, G_{n+1}) = q_t(\bar{a}_t, G_{n,t,\alpha})$ where
  - $G_{n,t,\alpha} = c\ell(\cup\{\bar{c}^n_{t_1,\alpha_1} : t_1 \in \mathrm{def}_{\mathfrak{S}}(G_n), \alpha_1 < \lambda \text{ but } (t_1, \alpha_1) \neq (t, \alpha)\} \cup G_n, G_{n+1})$.

Let $\hat{G} = \bigcup\limits_n G_n$ and we shall show that $\hat{G}$ is $(\lambda, \mathfrak{S})$-full over G. We can ignore the case $\lambda = \aleph_0$ being obvious. Assume $A \subseteq \hat{G}, |A| < \lambda$ and $t_* \in \mathrm{def}_{\mathfrak{S}}(\hat{G})$ and $\bar{a}_{t_*} \subseteq c\ell(G_0 + A, \hat{G})$, hence we can find $\bar{S}$ such that:

$(*)$
- (a) $\bar{S} = \langle S_n : n < \omega\rangle$;
- (b) $S_n \subseteq \mathrm{def}_{\mathfrak{S}}(G_n) \times \lambda$ and $\bigcup\limits_m S_m$ has cardinality $< |A|^+ + \aleph_0$;
- (c) if $(t, \alpha) \in S_n$ then $\bar{a}_t \subseteq c\ell(\cup\{\bar{c}^m_{t_1,\alpha_1} : m < n \text{ and } (t_1, \alpha_1) \in S_m\} \cup G_0, G_n)$;
- (d) $A \subseteq \bigcup\limits_n A_n \cup G_0$ where $A_n = \cup\{\bar{c}^m_{t,\alpha} : (t, \alpha) \in S_m \text{ and } m < n\}$;

[6] For the case $\mathfrak{S}$ is not symmetric and λ is singular, if we like to have "prime", (as in 1.16(5)) we should add: for every pair $t = (\mathfrak{s}, \bar{a})$ as in 1.15(2), for every large enough $\mu < \lambda$, for every $\alpha < \mu^+$ for some $\bar{c} \subseteq M_{\mu^+}$ requires $q_{\mathfrak{s}}(\bar{a}, M_\alpha)$ is realized in M_{μ^+}; also we can in 1.23(1) have such $\mathscr{A}$, i.e. strengthen (d) there as here so weakens the assumption in 1.23(4).

[7] Not enough for quoting results.

(e) for some $n_*, (t_*, 0) \in S_{n_*}$.

We have to prove that $q_{t_*}(c\ell(A \cup G, \hat{G}))$ is realized in M. Choose α_* such that $(t_*, \alpha_*) \notin S_{n_*}$ and prove by induction on $n \geq n_*$ that $\bar{c}^{n_*}_{t_*,\alpha_*}$ realizes $q_{t_*}(c\ell(A_n \cup G_0, \hat{G}))$.

For $n = n_*$ this is obvious, so assume this holds for n and we shall prove for $n+1$.

For this it suffices to prove, for every finite $u \subseteq S_n$ that $\bar{c}^{n(*)}_{t_*,\alpha_*}$ realizes $q_{t_*}(c\ell(A_n \cup G_0 \cup \{\bar{c}^n_{t,\alpha} : (t,\alpha) \in u\}, \hat{G})$; we prove this by induction on $|u|$. Now if $|u| = 0$ this holds by the induction hypothesis on n and if $|u| > 0$, let $\beta \in u$ and use the induction for $u' = u \backslash \{\beta\}$ and $\mathfrak{S}$ being symmetric.

5) We can find a list $\langle (n_\zeta, t_\zeta, \alpha_\zeta) : \zeta < \lambda \rangle$ of $\{(n,t,\alpha) : n < \omega$ and $(t,\alpha) \in S_n\}$ such that $\bar{a}_{t_\zeta} \subseteq c\ell(\cup\{(\bar{c}^{n_\xi}_{t_\xi,\alpha_\xi} : \xi < \zeta\} \cup M, N)$.

Now choose $f(\bar{c}^{n_\zeta}_{t_\zeta,\alpha_\zeta}) \subseteq N'$ by induction on ζ. $\square_{1.16}$

Discussion 1.17. 1) So by 1.13(2), 1.16(2) if there is a symmetric closed dense $\mathfrak{S}$ <u>then</u> for every lf group G there is a "nice" extension of G to an existentially closed one $\hat{G}$, that is we have:

(a) uniqueness (by 1.13(4))
(b) cardinality $\leq |\theta| + |\mathfrak{S}|$ (by 1.13(3))
(c) extending G (see 1.12(1))
(d) being existentially closed (see 1.16(2)).

2) Fixing λ and demanding $G \in \mathbf{K}_{\leq \lambda}$ we can add

(e) $\hat{G}$ is $(\lambda, \mathfrak{S})$-full over M
(f) if $H \supseteq G$ is $(\lambda, \mathfrak{S})$-full <u>then</u> there is an embedding of $\hat{G}$ into H over G.

Discussion 1.18. Concerning 1.16(1), if we assume $\langle G_\alpha : \alpha \leq \delta{+}1 \rangle$ is $\subseteq$-increasing continuous and $\alpha < \delta \Rightarrow G_\alpha \leq_{\mathfrak{S}} G_{\delta+1}$, does it follow that $G_\delta \leq_{\mathfrak{S}} G_{\delta+1}$? This is Ax(IV) of the definition of a.e.c. Well, if δ has uncountable cofinality and each G_α is existentially closed then yes. The point is that the relevant types do not split over <u>finite</u> sets. If we deal with "not split over countable sets" we need cf$(\delta) \geq \aleph_2$, etc.

So $(\mathbf{K}, \leq_{\mathfrak{S}})$ is not an a.e.c. in general failing Ax(IV); in fact, e.g. we may prove for the maximal $\mathfrak{S}$ that this axiom fails, see the proof of 5.1.

Now we turn to constructions not necessarily assuming "$\mathfrak{S}$ is symmetric" presenting the "first avenue" in §0(B).

Definition 1.19. 1) We say that $\mathscr{A} = \langle G_i, \bar{a}_j, w_j, K_j : i \leq \alpha, j < \alpha \rangle$ is an $\mathbf{F}^{\rm sch}_{\aleph_0} - \mathfrak{S}$-<u>construction</u> (for $\mathbf{K}$) <u>when</u> :

(a) G_i for $i \leq \alpha$ is an $\leq_{\mathfrak{S}}$-increasing continuous sequence of members of $\mathbf{K}$;
(b) G_{i+1} is generated by $G_i \cup \bar{a}_i, \bar{a}_i$ a finite sequence;
(c) w_i is a finite subset of i;
(d) $K_i \subseteq G_i$ is finite;
(d)$^+$ moreover $K_i \subseteq \langle G_0 + \sum\limits_{j \in w_i} \bar{a}_j \rangle_{G_i}$; we may add "$K_j$ generated by $\cup\{\bar{a}_j : j \in w_i\} \cup (K_i \cap G_i)$;

(e) $\mathrm{tp}_{\mathrm{bs}}(\bar a_i, G_i, G_{i+1}) \in \mathbf{S}^{\ell g(\bar a_i)}_{\mathfrak{S}}(G_i)$ as witnessed by K_i, i.e. it is $q_{\mathfrak{s}}(\bar a, G_i)$ for some $\bar a \in {}^{\omega>}K_i$ realizing $p_{\mathfrak{s}}$ for some $\mathfrak{s} \in \mathfrak{S}$.

2) We may say above that G_α is $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-constructible over G_0; and may also say that $\mathscr{A}$ is an $\mathfrak{S}$-construction over G_0. We let $\alpha = \ell g(\mathscr{A}), G_i = G^{\mathscr{A}}_i, \bar a_i = \bar a^{\mathscr{A}}_i, w_j = w^{\mathscr{A}}_j, K_j = K^{\mathscr{A}}_j$.
3) We say above that $\mathscr{A}$ is a <u>definite</u> $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-<u>construction</u> <u>when</u> for every $j < \alpha$ we have also $t_j = t^{\mathscr{A}}_j \in \mathrm{def}(G^{\mathscr{A}}_j)$ such that $\bar a_{t_j} \in {}^{\omega>}(K_j)$ and $\bar a^{\mathscr{A}}_j$ realizes $q_{t_j}(G_j)$ (note that in 1.19(1)(e) we have "for some $\mathfrak{s}_j$", so $\mathscr{A}$ does not determine the $\mathfrak{s}$'s (or here the t_j; so every $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-construction can be expanded to a definite one, but not necessarily uniquely).
4) We say $\mathscr{A}$ is a λ-<u>full definite</u> $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-<u>construction</u> <u>when</u> α is divisible by λ and for every $i < \alpha$ and $t \in \mathrm{def}(G_i)$, the set $\{j : j \in (i, \alpha) \text{ and } t^{\mathscr{A}}_j = t\}$ is an unbounded subset of $\alpha(*)$ of order type divisible by λ.

Discussion 1.20. We may replace 1.19(1)(e) by "$\mathrm{tp}_{\mathrm{bs}}(\bar a_i, G_i, G_{i+1})$ does not split over K_i", this is like the case $\mathbf{F}^p_{\aleph_0}$ in [**?**, Ch.IV,Def.2.6,pg.168] and [**?**, Ch.IV,Lemma 2.20,pg.168] and is equal to $\mathbf{F}^{\mathrm{nsp}}_{\aleph_0}$ in [**?**, §1,1.1-1.12], both for first order theories, but we seemingly lose the following:

Observation 1.21. 1) If $\mathscr{A}$ is a $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-construction and $G^{\mathscr{A}}_0 \subseteq G$ and $G \cap G^{\mathscr{A}}_{\ell g(\mathscr{A})} = G^{\mathscr{A}}_0$ <u>then</u> there is an $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-construction $\mathscr{B}$ with $G^{\mathscr{B}}_0 = G, \ell g(\mathscr{B}) = \ell g(\mathscr{A})$ and $G^{\mathscr{B}}_{\ell g(\mathscr{B})} = \langle G^{\mathscr{A}}_{\ell g(\mathscr{A})} \cup G\rangle_{G^{\mathscr{A}}_{\ell g(\mathscr{A})}}$.
2) Like (1) but with definite $F^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-constructions and then add in the end $t^{\mathscr{B}}_j = t^{\mathscr{A}}_j$ for $j < \ell g(\mathscr{A})$.
3) For the definite version, see 1.19(3), we get even uniqueness in (2).

Discussion 1.22. In 1.24 below, we may consider (see [**?**, Ch.IV,§1]):

<u>Ax(V.1)</u>: If $(q, G, L) \in \mathbf{F}, G \subseteq H \in \mathbf{K}; \bar a, \bar b \in {}^{\omega>}H$; $q = \mathrm{tp}_{\mathrm{bs}}(\bar a\char`^\bar b, G, H)$ and $p = \mathrm{tp}_{\mathrm{bs}}(\bar a, \langle G + \bar b\rangle_H, H)$ <u>then</u> $(p, \langle G + \bar b\rangle_H, L) \in \mathbf{F}$.

<u>Ax(V.2)</u>: A notational variant of (V1) so we ignore it.

The following claim (together with §2, the existence of countable dense $\mathfrak{S}$) proves Theorem 0.11.

Claim 1.23. *1) If $G \in \mathbf{K}$ is of cardinality $\leq \lambda$ and $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is closed and dense and of cardinality $\leq \lambda$ (if $\lambda \geq 2^{\aleph_0}$ this follows) <u>then</u> there is an $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-construction $\mathscr{A}$ such that:*

(a) *$\alpha^{\mathscr{A}} = \lambda$;*
(b) *$G^{\mathscr{A}}_0 = G$;*
(c) *$G^{\mathscr{A}}_\lambda \in \mathbf{K}$ is existentially closed of cardinality λ;*
(d) *$\mathscr{A}$ is λ-full, that is for every $\mathfrak{s} \in \mathfrak{S}$ and $\bar a \in {}^{k(\mathfrak{s})}(G^{\mathscr{A}}_\lambda)$ realizing $p_{\mathfrak{s}}(\bar x)$, for λ ordinals $\alpha < \lambda$ we have: $\mathrm{tp}_{\mathrm{bs}}(\bar a_\alpha, G^{\mathscr{A}}_\alpha, G^{\mathscr{A}}_{\alpha+1}) = q_{\mathfrak{s}}(\bar a, G^{\mathscr{A}}_\alpha)$.*

2) Assume $\lambda \geq \|G\| + |\mathfrak{S}|$ is regular. <u>Then</u> we can find $H \in \mathbf{K}_\lambda$ which is $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-constructible over G, is $(\lambda, \mathfrak{S})$-full over M and is embeddable over M into any N' which is $(\lambda, \mathfrak{S})$-full over G, in fact G_λ from part (1) is as required.
3) If $\mathfrak{S}$ is symmetric and is closed and H_1, H_2 are $\mathbf{F}^{\mathrm{sch}}_{\aleph_0} - \mathfrak{S}$-constructible over G and $(\lambda, \mathfrak{S})$-full over G and of cardinality λ <u>then</u> H_1, H_2 are isomorphic over G.

4) If $\lambda \geq \|G\|$ and $\mathscr{A}$ is an $\mathbf{F}^{\rm sch}_{\aleph_0} - \mathfrak{S}$-construction of H over G and $\ell g(\mathscr{A}) = \lambda$ then for every $H' \in \mathbf{K}$ which is $(\lambda, \mathfrak{S})$-full over G, we have H is embeddable into H' over G.

Proof. By [?, Ch.VI,§3] as all the relevant axioms there apply (see below or [?, Ch.IV,§1,pg.153]) or just check directly. Of course, we can use a monster $\mathfrak{C}$ for groups, but use only sets A such that $c\ell(A, \mathfrak{C}) = \langle A \rangle_{\mathfrak{C}}$ is locally finite, and we use quantifier-free types. $\square_{1.23}$

Now we make the connection to [?, Ch.IV].

Definition/Claim 1.24. 1) Let $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ be closed and below let $\lambda = \lambda(\mathbf{F}_{\mathfrak{S}})$ be $\aleph_0$. Then $\mathbf{F} = \mathbf{F}_{\mathfrak{S}}$ is defined as the set of triples (p, G, A) such that A is finite, for some $B \subseteq G \in \mathbf{K}$ we have $A \subseteq B, c\ell(B) = c\ell(B, G) = G \in \mathbf{K}$, $p \in \mathbf{S}^{<\omega}_{\mathfrak{S}}(c\ell(B))$ is $q_{\mathfrak{s}}(\bar{b}, c\ell(B))$ for some $\mathfrak{s} \in \mathfrak{S}, \bar{b} \subseteq c\ell(A)$ over A; we may restrict ourselves to the case $B = c\ell(B, G) = G$. Note that: as here we do not have a monster model $\mathfrak{C}$ we can either demand $B \in \mathbf{K}$ or demand $B \subseteq G \in \mathbf{K}$ but then it is more natural to write (p, G, A) instead of (p, A).
2) $\mathbf{F}$ satisfies the axioms (from [?, Ch.IV,§1] written below in the present notation) except possibly V, VI, VIII, X.1, X.2, XI.1, XI.2.
3) If $\mathfrak{S}$ is symmetric then $\mathbf{F}$ satisfies also Ax(VI).
4) If $\mathfrak{S}$ is dense then $\mathbf{F}$ satisfies also Ax(X.1).

Remark 1.25. If $\mathfrak{S}$ is compact (see 1.6(5)), then $\mathbf{F}$ satisfies Ax(VIII), i.e.

Ax(VIII) when $\mathfrak{S}$ is compact: If $\langle G_i : i \leq \delta + 1\rangle$ is $\subseteq$-increasing continuous in $\mathbf{K}, L \subseteq G_0$ finite, $p \in \mathbf{S}_{\mathfrak{S}}(G_\lambda)$ and $i < \delta \Rightarrow (p{\restriction}G_i, G_i, L) \in \mathbf{F}$ then $(p, G_\delta, L) \in \mathbf{F}$.

[Why? By the Definition; also holds when $\mathrm{cf}(\delta) > \aleph_0$.]

Proof. Isomorphism - Ax(I): preservation under isomorphism.

Obvious.

Concerning trivial $\mathbf{F}$-types:
Ax(II1): If $K \subseteq L \subseteq G \in \mathbf{K}, |L| < \lambda, K$ is finite, $\bar{a} \in {}^{\omega>}K$ and $p = \mathrm{tp}_{\rm bs}(\bar{a}, L, G)$ then $(p, G, K) \in \mathbf{F}$.

[Why? Trivially; recall $\lambda = \aleph_0$.]

Axiom(II2)-(II3)-(II4): irrelevant here.

Concerning monotonicity:
Ax(III1): If $L \subseteq G_1 \subseteq G_2$ and $(p, G_2, L) \in \mathbf{F}$ then $(p{\restriction}G_1, G_1, L) \in \mathbf{F}$.

[Why? Because if $\bar{a} \in {}^{\omega>}L, L \subseteq G_1 \subseteq G_2 \in \mathbf{K}$ and $q_{\mathfrak{s}}(\bar{a}, G_2)$ is well defined and equal to p, then $q_{\mathfrak{s}}(\bar{a}, G_1) = q_{\mathfrak{s}}(\bar{a}, G_2){\restriction}G_1)$, see Claim 1.2(1A).]

Ax(III2): If $L \subseteq L_1 \subseteq G, |L_1| < \lambda$, i.e. L_1 is finite and $(p, G, L) \in \mathbf{F}$ then $(p, G, L_1) \in \mathbf{F}$.

[Why? By the definition.]

Ax(IV): If $\bar{a}, \bar{b} \in {}^{\omega>}H, L \subseteq G \subseteq H, (\mathrm{tp}_{\rm bs}(\bar{b}, G, H), G, L) \in \mathbf{F}$ and $\mathrm{Rang}(\bar{a}) \subseteq \mathrm{Rang}(\bar{b})$ then $(\mathrm{tp}_{\rm bs}(\bar{a}, G, H), G, L) \in \mathbf{F}$.

[Why? Straightforward as $\mathfrak{S}$ is domination closed, see Definition 1.6(1B).]

Concerning transitivity and symmetry:

Ax(VI): ($\mathfrak{S}$ is symmetric). If $G \subseteq H \in \mathbf{K}, \bar{a}, \bar{b} \in {}^{\omega >}H$ and $L_1, L_2 \subseteq G$ are finite and $(\mathrm{tp}_{\mathrm{bs}}(\bar{b}, \langle G + \bar{a}\rangle_H, H), \langle G + \bar{a}\rangle_H, L_1) \in \mathbf{F}$ and $(\mathrm{tp}_{\mathrm{bs}}(\bar{a}, G, H), G, L_2) \in \mathbf{F}$ then $(\mathrm{tp}_{\mathrm{bs}}(\bar{a}, \langle G + \bar{b}\rangle_H, H), \langle G + \bar{b}\rangle_H, L_1) \in \mathbf{F}$.

[Why? By $\mathfrak{S}$ being symmetric when we claim this axiom, i.e. in 1.24(3).]

Ax(VII): If $G \subseteq H \in \mathbf{K}, \bar{a}, \bar{b} \in {}^{\omega >}H, (\mathrm{tp}_{\mathrm{bs}}(\bar{a}, \langle G + \bar{b}\rangle_H, H), \langle G + \bar{b}\rangle_H, L) \in \mathbf{F}$ and $(\mathrm{tp}_{\mathrm{bs}}(\bar{b}, G, H), G, L) \in \mathbf{F}$ hence $L \subseteq G$ is finite, then $(\mathrm{tp}_{\mathrm{bs}}(\bar{a}\hat{\ }\bar{b}, G, H), G, L) \in \mathbf{F}$.

[Why? By $\mathfrak{S}$ being composition-closed, see Definition 1.6(1A).]

Concerning continuity:

Ax(IX): irrelevant as $\lambda = \aleph_0$.

Concerning existence:
Ax(X.1): If $L_1 \subseteq G \in \mathbf{K}, L_1 \subseteq L_2$ finite, $\bar{a} \in {}^{\omega >}(L_2)$ then for some p extending $\mathrm{tp}_{\mathrm{bs}}(\bar{a}, L_1, L_2)$ and finite $L \subseteq G$ we have $(p, G, L) \in \mathbf{F}$, moreover without loss of generality $L = L_1$.

[Why? By $\mathfrak{S}$ being dense.]

Ax(X.2): irrelevant and follows by the moreover in Ax(X.1).

Ax(XI.1): If $p \in \mathbf{S}_{\mathrm{bs}}(G_1), (p, G_1, L) \in \mathbf{F}$ hence $p \in \mathbf{S}^n_{\mathfrak{S}}(G_1)$ for some n and $G_1 \subseteq G_2$ then there is $q \in \mathbf{S}^n_{\mathrm{bs}}(G_2)$ extending p such that $(q, G_1, L_2) \in \mathbf{F}$ for L_2, so $\bar{q} \in \mathbf{S}^n_{\mathfrak{S}}(G_2)$; moreover, in fact, $L_2 = L$ is O.K.

[Why? Use the same $\mathfrak{s} \in \mathfrak{S}$.]

Ax(XI.2): irrelevant and really follows by the moreover in (XI.1). $\square_{1.24}$

Definition 1.26. A sequence $\mathbf{I} = \langle \bar{a}_s : s \in I\rangle$ in $G \in \mathbf{K}$ is κ-convergent when for some $m, s \in I \Rightarrow \bar{a}_s \in {}^mG$ and for every finite $K \subseteq G$ and some $q \in \mathbf{S}^m(K)$ for all but $< \kappa$ members s of $\mathbf{I}, q = \mathrm{tp}_{\mathrm{bs}}(\bar{a}_s, K, G)$.

Remark 1.27. 1) So $\mathbf{F}_{\mathfrak{S}}$-constructions preserve "$\mathbf{I}$ is κ-convergent". Moreover, if $\mathbf{I}$ is κ-convergent in $G \in \mathbf{K}$ and $G \le_{\mathfrak{S}} H$, where $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ then I is κ-convergent in H.
2) We can assume I is a linear order with no last member and of cofinality $\ge \kappa$ and replace "all but $< \kappa$ of the $s \in I$" by "every large enough $s \in I$". See more in [**?**, §(1C)].

§ 1(C). Using Order.

We now turn to the third avenue of §(0B) to deal with the general and not necessarily symmetric case. Can we get uniqueness for non-symmetric $\mathfrak{S}$? Can we get every automorphism extendable, etc.? The answer is that at some price, yes. A major point in the construction was the use of linear well ordered index set (λ in 1.23(1) or $\alpha^{\mathscr{A}}$ in general). But actually we can use linear non-well ordered index sets, so those index sets can have automorphisms which help us toward uniqueness. The solution here is not peculiar to locally finite groups.

Definition 1.28. We say (I, E) is λ-suitable when (we may omit λ when $\lambda = |I|$, we may write $(I, P_i)_{i<\lambda}$ with $\langle P_i : i < \lambda\rangle$ listing the E-equivalence classes (with no repetitions)):

(a) I is a linear order;
(b) E is an equivalence relation on I with λ equivalence classes;
(c) every permutation of I/E is induced by some automorphism of the linear order which preserves equivalence and non-equivalence by E;
(d) each E-equivalence class has cardinality $|I|$.

Claim 1.29. *Let $T = \mathrm{Th}(\mathbb{R}, <, E)$ where Th stands for "the first order theory of", $E := \{(a,b) : a, b \in \mathbb{R}$ and $a - b \in \mathbb{Q}\}$; so $(A, <, E) \models T$ iff $(A, <)$ is a dense linear order with neither first nor last element, E an equivalence relation with each equivalence class a dense subset of A and with infinitely many equivalence classes.*
1) If $\lambda = \lambda^{<\lambda}$ and (I, E) is a saturated model of T of cardinality λ, then (I, E) is suitable[8]
2) For every λ the $(I, P_i)_{i<\lambda}$ from [?, §2] (see history there) is λ-suitable and $|I| = \lambda$.
3) There is a definable sequence $\langle (I_\lambda, P_i^\lambda)_{i<\lambda} : \lambda$ an infinite cardinal$\rangle$ such that $(I_\lambda, P_i^\lambda)_{i<\lambda}$ is λ-suitable and is increasing with λ and this definition is absolute.

Proof. 1) Obvious.
2),3) See there. $\square_{1.29}$

Claim 1.30. *Assume*

(A) *$G \in \mathbf{K}$ is of cardinality λ;*
(B) *$\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is closed and dense;*
(C) (a) *$i(*) \leq \lambda$ and $\mathscr{S} = \{t_i = (\mathfrak{s}_i, \bar{a}_i) : i < i(*)\}$ lists $\mathrm{def}_{\mathfrak{S}}(G)$, i.e. the pairs $(\mathfrak{s}, \bar{a})$, as in clause (d) of 1.23(1) or 1.23(3);*
  (b) *each such pair appears exactly once;*
  (c) *let $t_i = (\mathfrak{s}_*, <>)$ for $i \in [i(*), \lambda)$ so $\mathfrak{s}_* \in \mathfrak{S}, k_{\mathfrak{s}_*} = 0, n_{\mathfrak{s}} = 1, i(*) = \|\mathrm{def}_{\mathfrak{S}}(G)\|, \mathfrak{s}_*$ is from 1.3; so $\bar{a}_i = \langle\rangle$;*
(D) *$(I, P_i)_{i<\lambda}$ is λ-suitable, see Definition 1.28.*

Then we can find $H, \mathbf{c} = \langle \bar{c}_r : r \in I\rangle$ (the ordered one step $(\lambda, \mathfrak{S})$-closure), such that:

(a) *$H \in \mathbf{K}$ is a $\leq_{\mathfrak{S}}$-extension of G;*
(b) *$\bar{c}_r \in {}^{n(t_i)}H$ if $r \in P_i, i < \lambda$;*
(c) *H is generated by $G \cup \{\bar{c}_r : r \in I\}$;*
(d) *if $i < \lambda$ and $r \in P_i$ then $\bar{c}_r$ realizes in H over $c\ell(G \cup \{\bar{c}_s : s <_I r\}, H)$ the type defined by $(\mathfrak{s}_i, \bar{a}_i)$;*
(e) *every automorphism of G can be extended to an automorphism of H.*

Proof. Straightforward; e.g. to define H we should choose $q_{r_0,\ldots,r_{n-1}}$ for every $r_0 <_I \ldots <_I r_{n-1}$ by induction on n such that in the end $q_{r_0,\ldots,r_{n-1}} = \mathrm{tp}_{\mathrm{bs}}(\bar{c}_{r_0}{}^\frown \ldots {}^\frown \bar{c}_{r_{n-1}}, G, H)$, by clause (d), and prove that:

(*) if $m \leq n$ and $h : \{0, \ldots, m-1\} \to \{0, \ldots, n-1\}$ is increasing then $q_{r_{h(0)},\ldots,r_{h(m-1)}} \leq_h q_{r_0,\ldots,r_{n-1}}$.

[8]By similar arguments, if $\lambda \geq 2^\mu$ then there is a μ-suitable $(I, P_i)_{i<\mu}$ but $|I/E| = \mu < \lambda$. We can use any model of cardinality λ which is strongly μ^+-sequence homogeneous; this means that every partial automorphism of cardinality $\leq \mu$ can be extended to an automorphism.

Note that clause (e) follows by clauses (a)-(d) above recalling clause (c) of Definition 1.28.

Why? Let π be an automorphism of G, for each $i < \lambda$ we have $(\mathfrak{s}_i, \bar{a}_i) \in \mathscr{S}$ and also $(\mathfrak{s}_i, \pi(\bar{a}_i)) \in \mathscr{S}$, so by the choice of $\langle (\mathfrak{s}_i, \bar{a}_i) : i < \lambda \rangle$ there is a unique $j < \lambda$ such that $i \geq i(*) \Rightarrow j = i$ and $(\pi(\bar{a}_i), \mathfrak{s}_i) = (\bar{a}_j, \mathfrak{s}_j)$, so let $j = \hat{\pi}(i)$. So $\hat{\pi}$ is a permutation of λ. By "$(I, P_i)_{i<\lambda}$ is λ-suitable" there is an automorphism $\breve{\pi}$ of the linear order I such that $i < \lambda \Rightarrow \breve{\pi}(P_i) = P_{\hat{\pi}(j)}$. Clearly there is a unique automorphism $\dot{\pi}$ of H such that $\pi = \dot{\pi} \restriction G$ and $\dot{\pi}(\bar{c}_i) = \bar{c}_{\hat{\pi}(i)}$. $\square_{1.30}$

Definition 1.31. 1) We say H is an ordered one-step $(\lambda, \mathfrak{S})$-closure of G, pedantically the ordered one step $(I, E) - \mathfrak{S}$-closure of G, <u>when</u> $G, H, \mathbf{c}$ are as in 1.30.
2) We say H is an ordered $(\lambda, \mathfrak{S})$-closure of G, pedantically the ordered $(I, E) - \mathfrak{S}$-closure of G when:

- (a) $H = \bigcup\limits_n H_n$
- (b) $H_0 = G$
- (c) H_{n+1} is the one step $(I, E) - \mathfrak{S}$-closure of H_n.

Remark 1.32. 1) In what way is 1.30 weaker? We have to choose the listing of $\mathrm{def}(G)$ in clause (C). Also for $G_1 \subseteq G_2$ it is not clear why $H_1 \subseteq H_2$, where (G_ℓ, H_ℓ) is as above. But see 1.29(3).
2) On naturality see Paolini-Shelah [**?**].

Conclusion 1.33. *The parallel of parts (2)-(6) of 1.13 holds.*

Proof. Straightforward, for part (6) of 1.13 use 1.29(3). $\square_{1.33}$

§ 2. There are enough reasonable schemes

§ 2(A). **There is a Dense Set of Schemes.**

We like to find $\mathfrak{S}$'s as in §1 for $\mathbf{K}_{\rm lf}$, in particular to prove that there are dense $\mathfrak{S}$, so we have to look in details at amalgamations of lf-groups under special assumptions.

Recall the well known: for finite groups $G_0 \subseteq G_\ell \in \mathbf{K}$ for $\ell = 1,2$ we can amalgamate G_1, G_2 over G_0 by embedding into suitable finite permutations group; see the proof of the theorem of Hall, explained in the second paragraph of §(0A).

Concerning the $\mathbf{K}_{\rm olf}$ versions of 2.2, see later in 6.7.

Convention 2.1. $\mathbf{K}$ is $\mathbf{K}_{\rm lf}$.

Definition 2.2. 1) Let $\mathbf{X}_{\mathbf{K}} = \mathbf{X}(\mathbf{K})$, the set of amalgamation tries, be the set of $\mathbf{x}$ such that: $\mathbf{x}$ is a quintuple $(G_0, G_1, G_2, \mathbf{I}_1, \mathbf{I}_2) = (G_{\mathbf{x},0}, G_{\mathbf{x},1}, \ldots)$ satisfying:

- (a) $G_0 \subseteq G_\ell \in \mathbf{K}$ for $\ell = 1,2$;
- (b) $\mathbf{I}_\ell$ is a set of representatives of the left G_0-cosets in G_ℓ, i.e. $\langle gG_0 : g \in \mathbf{I}_\ell \rangle$ is a partition of G_ℓ (so without repetitions) for $\ell = 1,2$;
- (c) $e_{G_{\mathbf{x},0}} \in \mathbf{I}_{\mathbf{x},1} \cap \mathbf{I}_{\mathbf{x},2}$.

2) For $\mathbf{x}$ as above let

- (a) $\mathscr{U} = \mathscr{U}_{\mathbf{x}} = \{(g_0, g_1, g_2) : g_\ell \in G_\ell$ for $\ell = 0,1,2$ and $g_1 \in \mathbf{I}_1, g_2 \in \mathbf{I}_2\}$;
- (b) for $\ell = 1,2$ and $g \in \mathbf{I}_\ell$ let $\mathscr{U}^\ell_g = \mathscr{U}^\ell_{\mathbf{x},g} := \{(g_0, g_1, g_2) \in \mathscr{U}_{\mathbf{x}} : g_\ell = g\}$;
- (c) if $G_1 \cap G_2 = G_0$ then we let $\mathbf{j}_{\mathbf{x}} = \mathbf{j}_{\mathbf{x},1} \cup \mathbf{j}_{\mathbf{x},2}$, see below;
- (d) for $\ell = 0,1,2$ let $\mathbf{j}_\ell = \mathbf{j}_{\mathbf{x},\ell}$ be the following embedding of G_ℓ into per$(\mathscr{U}_{\mathbf{x}})$, the group of permutations of $\mathscr{U}_{\mathbf{x}}$, so let $g \in G_\ell$ and we should define $\mathbf{j}_\ell(g)$, so let $(g_0, g_1, g_2) \in \mathscr{U}_{\mathbf{x}}$ and we define $(g'_0, g'_1, g'_2) = (\mathbf{j}_\ell(g))(g_0, g_1, g_2)$ from $\mathscr{U}_{\mathbf{x}}$ as follows:

<u>$\ell = 0$</u>: $g'_0 = g_0 g$ in G_0 and $g'_1 = g_1, g'_2 = g_2$;

<u>$\ell = 1$</u>: $g'_1 g'_0 = g_1 g_0 g$ in G_1 and $g'_2 = g_2$;

<u>$\ell = 2$</u>: $g'_2 g'_0 = g_2 g_0 g$ in G_2 and $g'_1 = g_1$.

3) Let $G_{\mathbf{x}} = G_{\mathbf{x},3}$ be the subgroup of Sym$(\mathscr{U}_{\mathbf{x}})$ which Rang$(\mathbf{j}_{\mathbf{x},1}) \cup$ Rang$(\mathbf{j}_{\mathbf{x},2})$ generates where $G_{\mathbf{x}} \models$ "$f_1 f_2 = f_3$" means that for every $u \in \mathscr{U}_{\mathbf{x}}$, $f_3(u) = f_2(f_1(u))$, i.e. we look at the permutation as acting from the right.
4) Let $\le_{\mathbf{X}(\mathbf{K})}$ be the following partial order on $\mathbf{X}_{\mathbf{K}} : \mathbf{x} \le_{\mathbf{X}(\mathbf{K})} \mathbf{y}$ <u>iff</u>:

- (a) $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathbf{K}}$;
- (b) $G_{\mathbf{x},0} = G_{\mathbf{y},0}$;
- (c) $G_{\mathbf{x},\ell} \subseteq G_{\mathbf{y},\ell}$ for $\ell = 1,2$;
- (d) $\mathbf{I}_{\mathbf{x},\ell} = \mathbf{I}_{\mathbf{y},\ell} \cap G_{\mathbf{x},\ell}$ for $\ell = 1,2$.

5) We say (f_1, f_2) embeds $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ into $\mathbf{y} \in \mathbf{X}_{\mathbf{K}}$ <u>when</u>:

- (a) f_ℓ embeds $G_{\mathbf{x},\ell}$ into $G_{\mathbf{y},\ell}$ for $\ell = 1,2$;
- (b) $f_1 \restriction G_{\mathbf{x},0} = f_2 \restriction G_{\mathbf{y},0}$ maps $G_{\mathbf{x},0}$ onto $G_{\mathbf{y},0}$.

6) We say (f_1, f_2) is an isomorphism from $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ onto $\mathbf{y} \in \mathbf{X}_{\mathbf{K}}$ <u>when</u> above f_ℓ is onto $G_{\mathbf{y},\ell}$ for $\ell = 1, 2$.

Observation 2.3. Let $\mathbf{x}$ be as in Definition 2.2, i.e. it is an amalgamation try.
0) If $G_0 \subseteq G_\ell \in \mathbf{K}$ for $\ell = 1, 2$ <u>then</u> for some $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ we have $G_{\mathbf{x},\ell} = G_\ell$ for $\ell = 0, 1, 2$.
1) In Definition 2.2(2), for $\ell = 0, 1, 2$ if $g \in G_{\mathbf{x},\ell}$ <u>then</u> $\mathbf{j}_{\mathbf{x},\ell}(g)$ is a permutation of $\mathscr{U}_{\mathbf{x}}$, in fact, its restriction to $\mathscr{U}^{3-\ell}_{g_1}$ is a permutation for each $g_1 \in G_{3-\ell}$.
2) Moreover in part (1) the mapping $\mathbf{j}_{\mathbf{x},\ell}$ embeds the group $G_{\mathbf{x},\ell}$ into the group of permutation of $\mathscr{U}_{\mathbf{x}}$ hence into $G_{\mathbf{x}}$.
3) The mapping $\mathbf{j}_{\mathbf{x},0}$ is equal to $\mathbf{j}_{\mathbf{x},1} \restriction G_{\mathbf{x},0}$ and also to $\mathbf{j}_{\mathbf{x},2} \restriction G_{\mathbf{x},0}$.
4) If $G_{\mathbf{x},\ell}$ is finite for $\ell = 0, 1, 2$ <u>then</u> $|G_{\mathbf{x}}| \le (|G_{\mathbf{x},1}| \times |G_{\mathbf{x},2}|/|G_{\mathbf{x},0}|)!$
5) If $\mathbf{x}$ is an amalgamation try and $G_{\mathbf{x},0} \subseteq G'_\ell \subseteq G_{\mathbf{x},\ell}$ so G'_ℓ is a subgroup of $G_{\mathbf{x},\ell}$, for $\ell = 1, 2$ <u>then</u> for one and only one amalgamation try $\mathbf{y}$ we have $G_{\mathbf{y},0} = G_{\mathbf{x},0}, G_{\mathbf{y},\ell} = G'_\ell$ for $\ell = 1, 2$ and $\mathbf{I}_{\mathbf{y},\ell} = \mathbf{I}_{\mathbf{x},\ell} \cap G'_\ell$ so $\mathbf{y} \le_{\mathbf{X}(\mathbf{K})} \mathbf{x}$.
6) Moreover in part (5), if $\mathbf{z}$ is an amalgamation try with $(G_{\mathbf{z},0}, G_{\mathbf{z},1}, G_{\mathbf{z},2}) = (G_{\mathbf{x},0}, G'_1, G'_2)$ <u>then</u> for some $\mathbf{x}'$, the pair $(\mathbf{x}', \mathbf{z})$ is like $(\mathbf{x}, \mathbf{y})$ in (5) and $(G_{\mathbf{x}',0}, G_{\mathbf{x}',1}, G_{\mathbf{x}',2}) = (G_{\mathbf{x},0}, G_{\mathbf{x},1}, G_{\mathbf{x},2})$.
7) In part (5) there is a unique homomorphism f from $\langle \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)\rangle_{\mathrm{Sym}(\mathscr{U}_{\mathbf{x}})}$ onto $G_{\mathbf{y}}$ such that $\ell \in \{1, 2\} \wedge g \in G'_\ell \Rightarrow \mathbf{j}_{\mathbf{y},\ell}(g) = f(\mathbf{j}_{\mathbf{x},\ell}(g))$.
8) In part (5), if G'_1, G'_2 are finite then $\langle \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)\rangle_{G_{\mathbf{x}}}$ has at most $(n_*!)^{m_*}$ members where $n_* = |G'_1| \times |G'_2| \times |G_{\mathbf{x},0}|^3$ and $m_* = (n_*!)^{|G'_1|+|G'_2|}$.

Proof. Straightforward. E.g.:
2) E.g. let $\ell = 1$ and $f, h \in G_1$. For $(g_0, g_1, g_2) \in \mathscr{U}_{\mathbf{x}}$ let $(\mathbf{j}_1(f))(g_0, g_1, g_2) = (g'_0, g'_1, g'_2)$ and $(\mathbf{j}_1(h))(g'_0, g'_1, g'_2) = (g''_0, g''_1, g''_2)$ hence

$(*)_1$ $(\mathbf{j}_1(h))(\mathbf{j}_1(f))(g_0; g_1, g_2) = (g''_0, g''_1, g''_2)$.

Then $g_2 = g'_2$ and $g'_2 = g''_2$ and in G_1 we have $g_1 g_0 f = g'_1 g'_0$ and $g'_1 g'_0 h = g''_1 g''_0$, hence $g_2 = g''_2$ and $g''_1 g''_0 = g'_1 g'_0 h = (g_1 g_0 f) h = (g_1 g_0)(fh)$, so by the definition of $\mathbf{j}_1(f_h)$ we have

$(*)_2$ $\mathbf{j}_1(fh)(g_0, g_1, g_2) = (g''_0, g''_1, g''_2)$.

By $(*)_1 + (*)_2$ we have

$(*)_3$ $\mathbf{j}_1(f_h)(g_0, g_1, g_2) = (\mathbf{j}_1(h))(\mathbf{j}_1(f))(g_0, g_1, g_2)$.

As this holds for every $(g_0, g_1, g_2) \in \mathscr{U}_{\mathbf{x}}$ we have $G_{\mathbf{x}} \models$ "$\mathbf{j}_1(fh) = \mathbf{j}_1(f)\mathbf{j}_1(h)$".
4) Clearly $|G_{\mathbf{x},\ell}| = |\mathbf{I}_{\mathbf{x},\ell}| \times |G_{\mathbf{x},0}|$ for $\ell = 1, 2$ hence $|\mathscr{U}_{\mathbf{x}}| = |\mathbf{I}_{\mathbf{x},1}| \times |\mathbf{I}_{\mathbf{x},2}| \times |G_{\mathbf{x},0}| = (|G_{\mathbf{x},1}|/|G_{\mathbf{x},0}) \times (|G_{\mathbf{x},2}|/|G_{\mathbf{x},0}| \times |G_{\mathbf{x},0}| = |G_{\mathbf{x},1}| \times |G_{\mathbf{x},2}|/|G_{\mathbf{x},0}|$.

Hence $|G_{\mathbf{x}}| \le |\mathrm{Sym}(\mathscr{U}_{\mathbf{x}})| = (|\mathscr{U}_{\mathbf{x}}|)! = (|G_{\mathbf{x},1}| \times |G_{\mathbf{x},2}|/|G_{\mathbf{x},0}|)!$ as stated).
7) First, why there is such a homomorphism? If $b \in \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$ then b is a permutation of $\mathscr{U}_{\mathbf{x}}$ which maps the set $\mathscr{U}_{\mathbf{y}} = G_0 \times \mathbf{I}_{\mathbf{y},1} \times \mathbf{I}_{\mathbf{y},2}$ onto itself. It follows that every $b \in G' := \langle \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)\rangle_{\mathrm{Sym}(\mathscr{U}_{\mathbf{x}})}$ maps the set $\mathbf{U}_{\mathbf{y}} = G_0 \times \mathbf{I}_{\mathbf{y},1} \times \mathbf{I}_{\mathbf{y},2}$ onto itself. Hence the mapping with domain G' defined by $f(b) = b \restriction \mathscr{U}_{\mathbf{y}}$ is a homomomorphism from G' into $\mathrm{Sym}(\mathscr{U}_{\mathbf{y}})$. However, for each $b \in \mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$ we have $b \restriction \mathscr{U}_{\mathbf{y}}$ belongs to $G_{\mathbf{y},3}$ so $b \in G' \Rightarrow f(b) \in G_{\mathbf{y},3}$, hence f is as required.

Second, why f is unique? Because $\mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$ generates G' and on it f is determined.

8) Let $G_0 = G_{\mathbf{x},0}$. We define $E = \{((g'_0, g'_1, g'_2), (g''_0, g''_1, g''_2))) \in \mathscr{U}_{\mathbf{x}} \times \mathscr{U}_{\mathbf{x}} : G_0 g'_1 G'_1 = G_0 g''_1 G'_1$ and $G_0 g'_2 G'_2 = G_0 g''_2 G'_2\}$, this is an equivalence relation on $\mathscr{U}_{\mathbf{x}}$, each equivalence class has $\leq (|G'_1| \times |G'_2| \times |G_{\mathbf{x},0}|^3) = n_*$ members.

[Why? As if $(g'_0, g'_1, g'_2) \in (g_0, g_1, g_2)/E$ then $g'_0 \in G_0, g'_1 \in G_0 g_1 G'_1, g'_2 \in G_0 g_2 G'_2$ and $|G_0 g_\ell G'_\ell| \leq |G_0| \times |G'_\ell|$.]

Also each of the permutations of $\mathscr{U}_{\mathbf{x}}$ from $\mathbf{j}_{\mathbf{x},1}(G'_1) \cup \mathbf{j}_{\mathbf{x},2}(G'_2)$ maps each E-equivalence class onto itself. Hence for $n \in [1, n_*]$ there are $\leq m^*_n := n!^{|G'_1|+|G'_2|-1}$ isomorphism types of structures of the form: $N = (|N|, F^N_f)_{f \in G'_1 \cup G'_2}$, where $|N|$, the universe, has exactly n elements and is an E-equivalence class, and for each $f \in G'_1 \cup G'_2$ we have: F^N_f is a permutation of this equivalence class and $F^N_{e(G_0)}$ is the identity. Clearly as $\sum\limits_{n \leq n_*} (n!)^{|G'_1|+|G'_2|-1} \leq (n_*!)^{|G'_1|+|G'_2|} = m_*$, the subgroup $\langle \mathbf{j}_{1,\mathbf{x}}(G'_1) \cup \mathbf{j}_{2,\mathbf{x}}(G'_2)\rangle_{G_{\mathbf{x}}}$ of $G_{\mathbf{x}}$ has at most $(n_*!)^{m_*}$ members. Of course[9], the argument gives better bounds, e.g. the number of relevant N's is much smaller and using a finer E. $\square_{2.3}$

Claim 2.4. *In Definition 2.2,* $\mathbf{j}_{\mathbf{x},1}(G_1) \cap \mathbf{j}_{\mathbf{x},2}(G_2) = \mathbf{j}_{\mathbf{x},\ell}(G_0)$.

Proof. Assume that $a_\ell \in G_\ell$ and $b_\ell = \mathbf{j}_{\mathbf{x},\ell}(a_\ell)$ for $\ell = 1, 2$. It suffices to show that: if $b_1 = b_2$ <u>then</u> $a_1, a_2 \in G_{\mathbf{x},0}$ and $a_1 = a_2$. We check to what b_ℓ maps the triple $(e, e, e) \in \mathscr{U}_{\mathbf{x}}$: by the definition of $\mathbf{j}_{\mathbf{x},1}, \mathbf{j}_{\mathbf{x},2}$ we have:

- $b_1((e, e, e)) = (g'_0, g_1, e) \in \mathscr{U}_{\mathbf{x}}$ where $G_1 \models g_1 g'_0 = b_1$;
- $b_2((e, e, e)) = (g''_0, e, g_2) \in \mathscr{U}_{\mathbf{x}}$ where $G_2 \models g_2 g''_0 = b_2$.

So if $b_1 = b_2$ then $(g'_0, g_1, e) = b_1((e, e, e) = b_2((e, e, e,)) = (g''_0, e, g_2)$, hence $g'_0 = g''_0 \wedge g_1 = e \wedge e = g_2$; this implies that $g'_0 = b_1, g''_0 = b_2$ hence $g'_0 = g''_0$, also $g''_0 \in G_0$ together $a_1 = a_2$ so we are done. $\square_{2.4}$

Definition 2.5. 1) Let[10] $\mathrm{NF}_{\mathrm{rfin}}(G_0, G_1, G_2, G_3)$ means that $G_\ell \subseteq G_3 (\in \mathbf{K})$ for $\ell < 3$ and $\mathrm{NF}_{\mathrm{fin}}(G_0, G_1, G_2, \langle G_1 \cup G_2\rangle_{G_3})$, see below.
2) Let $\mathrm{NF}_{\mathrm{fin}}(G_0, G_1, G_2, G_3)$ mean that:

(a) $G_0 \subseteq G_\ell \subseteq G_3 \in \mathbf{K}$ are finite groups for $\ell = 1, 2$;
(b) $G_3 = \langle G_1 \cup G_2\rangle_{G_3}$;
(c) if $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ and $G_0 = G_{\mathbf{x},0}, G_1 \subseteq G_{\mathbf{x},1}, G_2 \subseteq G_{\mathbf{x},2}$ <u>then</u> there is a homomorphism $\mathbf{f}$ from G_3 into $G_{\mathbf{x}}$ such that $\mathbf{f} \restriction G_\ell = \mathbf{j}_{\mathbf{x},\ell} \restriction G_\ell$ for $\ell = 1, 2$;
(d) if $a \in G_3 \backslash \{e_{G_3}\}$ <u>then</u> for some $\mathbf{x}, \mathbf{f}$ as above we have $\mathbf{f}(a) \neq e_{G_3}$.

Remark 2.6. Note the choice "$G_\ell \subseteq G_{\mathbf{x},\ell}$" rather than $G_\ell = G_{\mathbf{x},\ell}$ in clause (c) of 2.5.

Now the amalgamation in Definition 2.5 is very nice but do we have existence, in $\mathbf{K}_{\mathrm{lf}}$ of course? The following Claim 2.7(3) answers positively.

Claim 2.7. *1) In clause (c) of Definition 2.5(2), the homomorphism* $\mathbf{f}$ *is unique.*
1A) If $\mathrm{NF}_{\mathrm{fin}}(G^\iota_0, G^\iota_1, G^\iota_2, G^\iota_3)$ *for* $\iota = 1, 2$ *and* $\mathbf{f}_\ell$ *is an isomorphism from* G^1_ℓ *onto* G^2_ℓ *such that* $\mathbf{f}_0 \subseteq \mathbf{f}_\ell$ *for* $\ell = 0, 1, 2$ <u>*then*</u> *there is one and only one isomorphism* $\mathbf{f}_3$ *from* G^1_3 *onto* G^2_3 *extending* $\mathbf{f}_1 \cup \mathbf{f}_2$.
2) In Definition 2.5, necessarily $G_1 \cap G_2 = G_0$.

[9] See more in [?].
[10] NF stands for non-forking.

3) If $G_0 \subseteq G_\ell \in \mathbf{K}$ are finite for $\ell = 1,2$ then we can find $\bar{f}, \bar{H}$ such that

(a) $\bar{f} = \langle f_0, f_1, f_2 \rangle$;
(b) $\bar{H} = \langle H_\ell : \ell \le 3 \rangle$;
(c) $\mathrm{NF}_{\rm fin}(H_0, H_1, H_2, H_3)$;
(d) f_ℓ *is an isomorphism from* G_ℓ *onto* H_ℓ *for* $\ell = 0,1,2$;
(e) $f_0 \subseteq f_1$ *and* $f_0 \subseteq f_2$.

Proof. 1), 1A) Obvious.
2) By Claim 2.4 recalling clause (c) of 2.5(2).
3) Follows by 2.3(8) but we elaborate. Let $\bar{G} = \langle G_\ell : \ell = 0,1,2 \rangle$ and

$(*)_1$ let $\mathbf{X}_{\bar{G}} := \{\mathbf{x} \in \mathbf{X}_{\mathbf{x}} : G_{\mathbf{x},0} = G_0$ and $G_{\mathbf{x},\ell}$ is a lf group extending G_ℓ for $\ell = 1,2\}$;
$(*)_2$ for $\mathbf{x} \in \mathbf{X}_{\bar{G}}$ let: $\mathbf{n}_{\bar{g}}(\mathbf{x}) =$ the number of elements of $\langle \mathbf{j}_{\mathbf{x},1}(G_1) \cup \mathbf{j}_{\mathbf{x},2}(G_2) \rangle_{G_{\mathbf{x},3}}$.

We define $\mathbf{X}^{\rm mx}_{\bar{G}}$ as the set of $\mathbf{x}$'s such that:

$(*)_3$ (a) $\mathbf{x} \in \mathbf{X}_{\bar{G}}$;
(b) if $\mathbf{y} \in \mathbf{X}_{\bar{G}}$ and $\mathbf{x} \le \mathbf{y}$ then $n_{\bar{G}}(\mathbf{x}) = n_{\bar{G}}(\mathbf{y})$;
$(*)_4$ if $\mathbf{x}, \mathbf{z} \in \mathbf{X}^{\rm mx}_{\bar{G}}$ and $\mathbf{x} \le_{\mathbf{X}(\mathbf{K})} \mathbf{z}$, then $n_{\bar{G}}(\mathbf{x}) \le n_{\bar{G}}(\mathbf{z})$.

[Why? Because by 2.3(7) there is a homomorphism from $G_{\mathbf{z}} = \langle \mathbf{j}_{\mathbf{z},1}(G_1) \cup \mathbf{j}_{\mathbf{z},2}(G_2) \rangle$ onto $G_{\mathbf{x}} = \langle \mathbf{j}_{\mathbf{x},1}(G_1) \cup \mathbf{j}_{\mathbf{x},2}(G_2) \rangle$.]

$(*)_5$ for every $\mathbf{x} \in \mathbf{X}_{\bar{G}}$ there is $\mathbf{y} \in \mathbf{X}^{\rm mx}_{\bar{G}}$ such that $x \le \mathbf{y}$; hence $\mathbf{X}^{\rm mx}_{\bar{G}} \ne \emptyset$.

[Why? By $(*)_4$ and 2.3(8).]

$(*)_6$ $(\mathbf{X}_{\bar{G}}, \le_{\mathbf{X}[\mathbf{K}]})$ has amalgamation, that is
- if $\mathbf{x}_0 \le_{\mathbf{X}[\mathbf{K}]} \mathbf{x}_\iota$ for $\iota = 1,2$ then we can find $\mathbf{x}_3$ and (f^ι_1, f^ι_2) for $\iota = 1,2$ such that:
  (a) $\mathbf{x}_3 \in \mathbf{X}_{\mathbf{K}}$
  (b) $\mathbf{x}_0 \le_{\mathbf{X}[\mathbf{K}]} \mathbf{x}_3$
  (c) (f^ι_1, f^ι_2) embeds $\mathbf{x}_\iota$ into $\mathbf{x}_3$ over $\mathbf{x}_0$
  (over $\mathbf{x}_0$ means: $f^\iota_1 \restriction G_{\mathbf{x}_0,1} = \mathrm{id}_{G_{\mathbf{x}_0,1}}, f^\iota_2 \restriction G_{\mathbf{x}_0,2} = \mathrm{id}_{G_{\mathbf{x}_0,2}}$).

[Why? For $\ell = 1,2$, we use the disjoint amalgamation for finite groups, i.e. find $(G_\ell, f^1_\ell, f^2_\ell)$ such that:

$\bullet_1$ G_ℓ is a finite group extending $G_{\mathbf{x}_0,0}$
$\bullet_2$ f^1_ℓ embeds $G_{\mathbf{x}_1,\ell}$ into G_ℓ over $G_{\mathbf{x}_0,0}$
$\bullet_3$ f^2_ℓ embeds $G_{\mathbf{x}_2,\ell}$ into G_ℓ over $G_{\mathbf{x}_0,0}$
$\bullet_4$ $f^1_\ell(G_{\mathbf{x}_1,\ell}) \cap f^2_\ell(G_{\mathbf{x},\ell}) = G_{\mathbf{x}_0,0}$.

Note that $f^1_\ell(\mathbf{I}_{\mathbf{x}_1,\ell}) \cap f^2_\ell(\mathbf{I}_{\mathbf{x}_2,\ell}) = \{e_{G_{\mathbf{x}_0,0}}\}$, moreover, working inside G_ℓ, $\langle gG_{\mathbf{x}_0,0} : g \in f^1_\ell(\mathbf{I}_{\mathbf{x}_1,\ell}) \cup f^2_\ell(\mathbf{I}_{\mathbf{x}_2,\ell}) \rangle$ is a sequence of pairwise disjoint sets. Hence there is $\mathbf{I}_\ell \subseteq G_\ell$ extending $f^1_\ell(\mathbf{I}_{\mathbf{x}_1,\ell}) \cup f^2_\ell(\mathbf{I}_{\mathbf{x}_2,\ell})$ such that $\langle gG_{\mathbf{x}_0,0} : g \in \mathbf{I}_\ell \rangle$ is a partition of G_ℓ.

Define $\mathbf{x}_3$ by:

$\bullet'_1$ $G_{\mathbf{x}_3,0} = G_{\mathbf{x}_0,0}$

- $\bullet_2'$ $G_{\mathbf{x}_3,\ell} = G_\ell$ for $\ell = 1,2$
- $\bullet_3'$ $\mathbf{I}_{\mathbf{x}_3,\ell} = \mathbf{I}_\ell$ for $\ell = 1,2$.

Now check that $\mathbf{x}_3, (f_1^\iota, f_2^\iota)$ for $\iota = 1,2$ are as required.]

$(*)_7$ if $\mathbf{y} \in \mathbf{x}^{\rm mx}_{\bar G}$ then $\mathrm{NF}_{\rm fin}(G_{\mathbf{y},0}, G_{\mathbf{y},1}, G_{\mathbf{y},2}, G_{\mathbf{y},3})$.

[Should be clear now.]

Alternatively[11], use 2.15 below and 2.3(8). $\square_{2.7}$

We give now further basic properties, mainly connecting it to non-splitting (in 2.8(4)).

Claim 2.8. *Assume* $\mathrm{NF}_{\rm rfin}(G_0, G_1, G_2, G_3)$ *hence* $\mathrm{NF}_{\rm fin}(G_0, G_1, G_2, G_3) \Leftrightarrow G_3 = \langle G_1 \cup G_2\rangle_{G_3}$.
1) Symmetry: Also $\mathrm{NF}_{\rm rfin}(G_0, G_2, G_1, G_3)$ *holds.*
2) Monotonicity: If $G_0 \subseteq G'_\ell \subseteq G_\ell$ *for* $\ell = 1,2$ *and* $G'_1 \cup G'_2 \subseteq G'_3 \subseteq G_3$ *<u>then</u>* $\mathrm{NF}_{\rm rfin}(G_0, G'_1, G'_2, G'_3)$.
3) Uniqueness: if $\mathrm{NF}_{\rm fin}(G'_0, G'_1, G'_2, G'_3)$ *hence* $G'_3 = \langle G'_1 \cup G'_2\rangle_{G'_3}$, f_ℓ *is an isomorphism from* G'_ℓ *into* G_ℓ *for* $\ell = 0,1,2$ *such that* $f_1 \restriction G'_0 = f_0 = f_2 \restriction G'_0$ *and* f_0 *is onto* G_0, *<u>then</u> there is an embedding* f_3 *of* G'_3 *into* G_3 *extending* $f_1 \cup f_2$ *(unique, of course; it is onto if and only if* $G_3 = \langle G_1 \cup G_2\rangle_{G_3}$ *and* f_ℓ *is onto* G_ℓ *for* $\ell = 1,2$*).*
4) Definability: If $\bar a \in {}^{\omega>}(G_2)$ *<u>then</u>* $\mathrm{tp}_{\rm bs}(\bar a, G_1, G_3)$ *does not split over* G_0.

Proof. Straightforward but we elaborate.
1) Use the symmetry in the definition (recall that in §2 we have $\mathbf{K} = \mathbf{K}_{\rm lf}$ not $\mathbf{K}_{\rm olf}$!)
2) By 2.3(7) and use the uniqueness in 2.7(1). Alternatively use 2.15 below and 2.3(8).
3) Easily, too.
4) Obvious by parts (2) and (3). $\square_{2.8}$

Now above the restriction of G_1, G_2 to be finite is undesirable.

Definition 2.9. Let $\mathrm{NF}_f(G_0, G_1, G_2, G_3)$ or "G_1, G_2 are NF_f-stably amalgamated over G_0 inside G_3" mean that:

- (a) $G_\ell \in \mathbf{K}$ for $\ell \le 3$
- (b) G_0 is finite
- (c) $G_0 \subseteq G_\ell \subseteq G_3$ for $\ell = 1,2$ and $G_1 \cap G_2 = G_0$
- (d) if G'_1, G'_2 are finite groups and $G_0 \subseteq G'_\ell \subseteq G_\ell$ for $\ell = 1,2$ and $G'_3 = \langle G'_1 \cup G'_2\rangle_{G_3}$ then $\mathrm{NF}_{\rm fin}(G_0, G'_1, G'_2, G'_3)$.

Claim 2.10. <u>Stable Amalgamation over Finite Claim</u> *1) Existence: If* $G_0 \in \mathbf{K}$ *is finite and* $G_0 \subseteq G_\ell \in \mathbf{K}$ *for* $\ell = 1,2$ *and for transparency* $G_1 \cap G_2 = G_0$ *<u>then</u> for some* G_3 *we have* $\mathrm{NF}_f(G_0, G_1, G_2, G_3)$ *and* $G_3 = \langle G_1 \cup G_2\rangle_{G_3}$.
2) Uniqueness: In part (1), G_3 *is unique up to isomorphism over* $G_1 \cup G_2$.
3) Monotonicity: If $G_0 \subseteq G'_\ell \subseteq G_\ell$ *for* $\ell = 1,2$ *and* $\mathrm{NF}_\ell(G_0, G_1, G_1, G_2)$ *<u>then</u>* $\mathrm{NF}_f(G_0, G'_1, G'_2, G_3)$.
4) Symmetry: $\mathrm{NF}_f(G_0, G_1, G_2, G_3)$ *holds <u>iff</u>* $\mathrm{NF}_f(G_0, G_2, G_1, G_3)$ *holds.*
5) Definability: If $\mathrm{NF}_f(G_0, G_1, G_2, G_3)$, *<u>then</u>* $G_1 \le_{\Omega[\mathbf{K}]} G_3$.

[11] Or see [?].

Proof. 1) Straightforward by 2.7(1) and 2.10(2),(3), i.e. existence follows from the existence for finite G_1, G_2 using uniqueness and monotonicity.
2) - 5) easy too; holds by 2.8(3), 2.8(1), 2.8(1), 2.8(3), i.e. uniqueness, monotonicity, symmetry and definability respectively. $\square_{2.10}$

Now we go back to the major problem left in §1.

Claim 2.11. *There is one and only one full $\mathfrak{s} \in \Omega[\mathbf{K}]$ such that $q = q_{\mathfrak{s}}(\bar{a}, G_1)$ <u>when</u>:*

- (a) $G_1 \in \mathbf{K}$ *is existentially closed*
- (b) $q(\bar{x}) \in \mathbf{S}^n_{\mathfrak{S}_{\rm atdf}}(G)$ *is* $\mathrm{tp}_{\rm bs}(\bar{c}, G_1, G_3)$, *see below*
- (c) $\mathrm{NF}_f(G_0, G_1, G_2, G_3)$
- (d) $\bar{c} \in {}^{n(\mathfrak{s})}(G_2)$ *and* $\bar{a} \in {}^{k(\mathfrak{s})}(G_0)$ *generate* G_0.

Proof. By 2.10 and 1.2(3). $\square_{2.11}$

Definition 2.12. Let $\mathfrak{S}_{\rm df} \subseteq \Omega[\mathbf{K}]$ be the closure of $\mathfrak{S}_{\rm atdf}$, see Definition 1.6 where $\mathfrak{S}_{\rm atdf} \subseteq \mathfrak{S}(\mathbf{K})$ is the set of $\mathfrak{s} \in \Omega[\mathbf{K}]$ as in 2.11.

Claim 2.13. *1) $\mathfrak{S}_{\rm df}$ is well defined, see Definition 2.12, 2.8(3).*
2) $\mathfrak{S}_{\rm df}$ is dense (see Definition 1.6(2)), closed and countable.

Proof. 1) Obvious.
2) <u>$\mathfrak{S}_{\rm df}$ is dense</u>: holds by 2.10 and 2.11 recalling Definition 2.9, 2.12.
<u>$\mathfrak{S}_{\rm df}$ is closed</u>: by its definition.
<u>$\mathfrak{S}_{\rm df}$ is countable</u>: as $\mathfrak{S}_{\rm atdf}$ is by 2.8(3), 2.10(2) recalling 1.8(3). $\square_{2.13}$

Discussion 2.14. Is $\mathfrak{S}_{\rm df}$ symmetric? Not clear, however, in the end of §1 we have circumvented this and we shall in §3 circumvent this in another way.

Claim 2.15. *1) Assume $G_0 \subseteq G_\ell \in \mathbf{K}$ and G_ℓ is existentially closed for $\ell = 1, 2$ and G_0 finite.*

<u>Then</u> we can find $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ such that $G_\ell = G_{\mathbf{x},\ell}$ for $\ell = 0, 1, 2$ and $(\mathbf{j}_{\mathbf{x},0}(G_0) \subseteq \mathbf{j}_{\mathbf{x},\ell}(G_\ell) \le_{\mathfrak{S}_{\rm df}} G_{\mathbf{x}})$ and $\mathrm{NF}_f(\mathbf{j}_{\mathbf{x},0}(G_0), \mathbf{j}_{\mathbf{x},1}(G_1), \mathbf{j}_{\mathbf{x},2}(G_2), G_{\mathbf{x}})$.
2) Assume $G_0 \subseteq G'_\ell \subseteq G_\ell$ and G'_ℓ finite (or just $(G_\ell : G'_\ell) = |G_\ell|$) and $\mathbf{y} \in \mathbf{X}_{\mathbf{K}}, G_{\mathbf{y},0} = G_0$ and $G_{\mathbf{y},\ell} = G'_\ell$ for $\ell = 1, 2$. <u>Then</u> in part (1) we can demand that $\mathbf{x}$ extends $\mathbf{y}$.

Remark 2.16. If $G_0 \subseteq G_\ell \in \mathbf{K}$ for $\ell = 1, 2$ then we can find infinite $G'_1, G'_2 \in \mathbf{K}$ extending G_1, G_2 respectively as $\mathbf{K}$ is closed under (finite) product (for $\mathbf{K}_{\rm olf}$ use lexicographic order).

Proof. 1) By the definitions it is easy. That is, for $\ell = 1, 2$ we can choose $\mathbf{I}_\ell$ as in 2.2(1)(b) satisfying:

- $(*)$ if $G'_\ell \subseteq G_\ell$ is finite and extends G_0 and $\mathbf{I}' \subseteq G'_\ell$ is such that $e_{G_0} \in \mathbf{I}'$ and $\langle gG_0 : g \in \mathbf{I}' \rangle$ is a partition of G'_ℓ <u>then</u> we can find $g^* \in G_\ell$ such that $\{g^* g : g \in \mathbf{I}'\} \subseteq \mathbf{I}_\ell$.

Now think.
2) Similarly. $\square_{2.15}$

§ 2(B). Constructing Reasonable Schemes.

We now give some examples of $\mathfrak{s} \in \Omega[\mathbf{K}]$.

Definition 2.17. 1) Let $\mathfrak{s}_{\rm cg}$ be the $\mathfrak{s}$ from 2.18(2) below.
2) Let $\mathfrak{s}_{\rm gl}$ be the $\mathfrak{s}$ from 2.18(3) below.

Claim 2.18. *1) For every $G \in \mathbf{K}_{\rm lf}$ there are G^+ and a such that $G \subseteq G^+ \in \mathbf{K}_{\rm lf}, G^+ = \langle G \cup \{a\}\rangle_{G^+}$; in G^+ the element a does not commute with any $b \in G\backslash\{e_G\}, a$ has order 2 and the sets $G, a^{-1}Ga$ commute in G^+ and their intersection is $\{e_G\}$.*
2) There is unique $\mathfrak{s} \in \Omega[\mathbf{K}]$ such that $k_{\mathfrak{s}} = 0, n_{\mathfrak{s}} = 1, p_{\mathfrak{s}}$ is empty and in part (1) above $\mathrm{tp}_{\rm bs}(a, G, G^+)$ is $q_{\mathfrak{s}}(<>, G)$.
3) There is $\mathfrak{s} \in \Omega[\mathbf{K}]$ with $k_{\mathfrak{s}} = 1, n_{\mathfrak{s}} = 4, p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = \{x_0 = x_0^{-1} \wedge x_0 \neq e\}$ such that: if $G \in \mathbf{K}_{\rm lf}$ and $a \in G$ realizes $p_{\mathfrak{s}}(x_0)$ then there are $G^+, \bar{c}$ such that $G \subseteq G^+ = \langle G \cup \bar{c}\rangle_{G^+}, \mathrm{tp}_{\rm bs}(\bar{c}, G, G^+) = q_{\mathfrak{s}}(\langle a\rangle, G)$ and c_ℓ realizes $q_{\mathfrak{s}_{\rm cg}}(<>, G)$ in G^+ for $\ell < n_{\mathfrak{s}}$ and $a \in \langle\bar{c}\rangle_{G^+}$.

Proof. We first make a less specific construction for any $G \in \mathbf{K}$.

For $n \geq 2$ let $\mathscr{U}_n = G \times n = \{(g, \iota) : g \in G, \iota < n\}$. For finite $K \subseteq G$ let $E_K := \{((g_1, \iota_1), (g_2, \iota_2)) : g_1, g_2 \in G$ and $\iota_1, \iota_2 < n$ and $g_1 K = g_2 K\}$, this is an equivalence relation on $\mathscr{U}_n$, each equivalence class has $\leq n \times |K|$ elements. For $\bar{a} \in {}^{\omega>}G$ let $E_{\bar{a}} = E_K$ when $K = \langle \mathrm{Rang}(\bar{a})\rangle_G$ which is finite.

For $\bar{a} \in {}^n G$ and π a permutation of $\{0, \ldots, n-1\}$ let $h_{\bar{a},\pi}$ be the following function from $\mathscr{U}_n$ into $\mathscr{U}_n$:

$(*)_1$ $h_{\bar{a},\pi}((g, \iota)) = (g a_\iota, \pi(\iota))$.

Clearly

$(*)_2$ $h_{\bar{a},\pi}$ is a permutation of $\mathscr{U}_n$ which maps every $E_{\bar{a}}$-equivalence class onto itself.

Let H be the group of permutations of $\mathscr{U}_n$ generated by $\{h_{\bar{a},\pi} : \bar{a} \in {}^n G$ and π is a permutation of $\{0, \ldots, n-1\}\}$, now by $(*)_2$ it is easy to see that $H \in \mathbf{K}_{\rm lf}$ where, as in earlier cases,

- $H \models$ "$h = h_1 h_2$" iff $x \in \mathscr{U}_n \Rightarrow h(x) = h_2(h_1(x))$.

Now for $\iota < n$ let $\mathbf{j}_\iota$ be the following function from G into H:

$(*)_3$ $\mathbf{j}_\iota(a) = h_{\bar{b},\pi}$ when $\pi =$ the identity and b_k is a if $k = \iota$ and is e_G otherwise.

Now

$(*)_4$ for $\iota < n, \mathbf{j}_\iota$ is an embedding of G into H.

[Why? Check.]

We let $G^*, \mathbf{j}_*$ be such that $G^* \supseteq G$ and $\mathbf{j}_*$ is an isomorphism from G^* onto H extending $\mathbf{j}_0$.

For later use note:

$(*)_5$ for transparency we can use existentially closed G.

Also

$(*)_6$ (a) $G \le_{\Omega[\mathbf{K}]} G^+$ equivalently $\mathbf{j}_0(G) = \mathbf{j}_*(G) \le_{\Omega[\mathbf{K}]} H$;

(b) if $A \subseteq G$ and for $\ell < m$ we have $\bar a_\ell \in {}^n A$ and π_ℓ is a permutation of $\{0, \dots, n-1\}$ then $p = \mathrm{tp}_{\mathrm{bs}}(\langle h_{\bar a_\ell, \pi_\ell} : \ell < m\rangle, \mathbf{j}_*(G), H)$ does not split over A;

(c) if above A is finite and $\bar a$ lists A <u>then</u> for some $\mathfrak{s} \in \Omega[\mathbf{K}], p = q_s(\bar a, \mathbf{j}_*(G)$.

[E.g. why clause (c) holds? By 2.10(2) recalling $(*)_5$.]

Now we prove each part.

1) Let $n = 2$ and π be the permutation of $\{0,1\}$ such that $\pi(0) = 1, \pi(1) = 0$, and let $a = \mathbf{j}_*^{-1}(h_{<e_G, e_G>, \pi})$.

2) Should be clear by $(*)_6(c)$.

3) First note that

$\oplus_1$ $\mathbf{j}_*^{-1}(h_{\bar a, \pi})$ realizes $q_{\mathfrak{s}_{\mathrm{cg}}}(\langle\rangle, G)$ in G^* <u>when</u> for some $k \in \{1, \dots, n-1\}$ we have

- $\bullet_1$ π is a permutation of $\{0, \dots, n-1\}$ and has order two
- $\bullet_2$ $\pi(0) = k$
- $\bullet_3$ $\pi(k) = 0$
- $\bullet_4$ $\bar a \in {}^n G$ satisfies $a_{\pi(\iota)} = a_\iota^{-1}$ for $\iota < n$
- $\bullet_5$ if $\pi(\iota) \neq \iota, \iota < n$ then $a_\iota = e_G$, (or just a_0 belongs to the center of G).

[Why? By $(*)_2$ and the choice of H clearly $h_{\bar a, \pi} \in H$ and inspecting $(*)_1$, easily $h_{\bar a, \pi}$ has order two. By the choice of $\mathbf{j}_*, \pi$ as $\pi(0) = k, \pi(k) = 0$ and $a_k = e_G = a_0$, for $g \in G$ we get $H \models$ "$h_{\bar a, \pi}^{-1} \mathbf{j}_0(g) h_{\bar a, \pi} = \mathbf{j}_k(g)$". However, for every $g_1, g_2 \in G$ the elements $\mathbf{j}_0(g_1), \mathbf{j}_k(g_2)$ of H commute as $h_{\bar a_1, \pi_1}, h_{\bar a_0, \pi_2}$ commute in H, e.g. when $\pi_1 = \mathrm{id}_n = \pi_2$ and $\bigwedge\limits_{\ell < n} (a_{1,\ell} = e \vee a_{2,\ell} = e)$. Lastly, $g_1, g_2 \in G \wedge \mathbf{j}_0(g_1) = \mathbf{j}_0(g_2) \Rightarrow g_1 = e_G = g_2$. Together we are done.]

Let $n = 3$ and for $\ell < 4$ let $g_\ell \in H$ be $h_{\bar a_\ell, \pi_\ell}$ where $\pi_\ell, \bar a_\ell$ are defined by (recall $a \in G$ is given and has order 2):

$\oplus_2$ for $\ell < 4$ let π_ℓ be such that:

<u>$\ell = 0, 3$</u>: the orbits are $\{0,1\}, \{2\}$;

<u>$\ell = 1, 2$</u>: the orbits are $\{0,2\}, \{1\}$.

$\oplus_3$ let $\bar a_\ell = \langle a_{\ell, i} : i < 3\rangle$ be $\langle e, e, e\rangle, \langle e, a, e\rangle, \langle e, e, e\rangle, (e, e, e)$ for $\ell = 0, 1, 2, 3$.

Now

$\oplus_4$ $c_\ell := \mathbf{j}_*^{-1}(h_{\bar a_\ell, \pi_\ell})$ realizes $q_{\mathfrak{s}_{\mathrm{cg}}}(\langle\rangle, G)$ for $\ell < 4$.

[Why? We apply $\oplus_1$ with k being 1 for $\ell = 0, 3$ and 2 for $\ell = 1, 2$. So we have to check $\bullet_1 - \bullet_4$ for each ℓ; now $\bullet_1 + \bullet_2 + \bullet_3$ holds by inspecting $\oplus_2$ and the choice of k and of π_ℓ.

Lastly, for $\bullet_4 + \bullet_5$ note that a, e has order 2 and $a_{\ell, 0} = e_G = a_{\ell, k}$ by inspecting $\oplus_3$.]

$\oplus_5$ $\mathrm{tp}_{\mathrm{bs}}(\langle c_0, c_1, c_2, c_3), G, G^*)$ does not split over $\langle a\rangle$, moreover is $q_{\mathfrak{t}}(\langle a\rangle, G)$ for some $\mathfrak{t} \in \Omega[\mathbf{K}]$.

[Why? Just think recalling $(*)_6$.]

Lastly,

$\oplus_6$ $G^+ \models$ "$c_0c_1c_2c_3 = a$".

[Why? This is equivalent to $H \models h_{\bar{a}_0,\pi_0}h_{\bar{a}_1,\pi_1}h_{\bar{a}_2,\pi_2}h_{\bar{a}_3,\pi_3} = \mathbf{j}_0(a)$. By the definition of the product we check how each $(g,\ell) \in \mathscr{U}_n$ is mapped (see above, so $h_{\bar{a}_0,\pi_0}$ is applied first) applying $h_{\bar{a}_\ell,\pi_\ell}$ in turn:

$$(g,0) \mapsto (ge,1) \mapsto (gea,1) \mapsto (geae,1) \mapsto (geaee,0) = (ga,0) = \mathbf{j}_0(a)(g,0)$$

and

$$(g,1) \mapsto (ge,0) \mapsto (gee,2) \mapsto (geee,0) \mapsto (geeee,1) = (g,1) = \mathbf{j}_0(a)(g,1)$$

$$(g,2) \mapsto (ge,2) \mapsto (gee,0) \mapsto (geee,2) \mapsto (geeee,2) = (g,2) = \mathbf{j}_0(a)(g,2).$$

So we are done.] $\square_{2.18}$

The following will be used in the proof of existence of complete existentially closed G.

Claim 2.19. *1) If (A) then (B) <u>where</u>:*

(A) (a) *$G_n \subseteq G_{n+1} \in \mathbf{K}$ for $n < \omega$ and I a set;*
(b) *$a^t_n \in G_{n+1}$ and let $b^t_n = a^t_0 \ldots a^t_n$ in G_{n+1} for $n < \omega, t \in I$;*
(c) *$\bar{a}_n = \langle a^t_n : t \in I\rangle, \bar{b}_n = \langle b^t_n : t \in I\rangle$;*
(d) (α) *$\mathrm{tp}_{\mathrm{bs}}(\bar{a}_n, G_n, G_{n+1})$ is increasing*[12] *with n;*
(β) *$c\ell(\bar{a}_n, G_{n+1}) \cap G_n = \{e_{G_n}\}$; if $I = \{t\}$ and a^t_n has order $k(t)$ this means that for every $i \in \{1,\ldots,k(t)\}$ we have:*
$G_{n+1} \models$ "$(a^t_n)^i = e_{G_1}$" *iff* $(a^t_n)^i \in G_n$ *iff* $i = k(t)$;
(e) *a^t_n commutes with every $c \in G_n$;*
(f) *$G_\omega = \cup\{G_n : n < \omega\}$ hence $\in \mathbf{K}$;*

(B) *for some $\bar{b}_\omega, G_{\omega+1}$ we have:*
(a) *$G_{\omega+1} \supseteq G_\omega$ belongs to $\mathbf{K}$;*
(b) *$\bar{b}_\omega = \langle b^t_\omega : t \in I\rangle$ and $b^t_\omega \in G_{\omega+1}$;*
(c) *$G_{\omega+1} = c\ell(G_\omega \cup \{b^t_\omega : t \in I\}, G_{\omega+1})$;*
(d) *if $n < \omega$ then $p_n = \mathrm{tp}_{\mathrm{bs}}(\bar{b}_\omega, G_n, G_{\omega+1}) = \mathrm{tp}_{\mathrm{bs}}(\bar{b}_n, G_n, G_{n+1})$.*

2) If we have (A) except omitting (A)(d)(β), still we have:

(B)$'$ *(a) − (c) as above;*
(d) $\bar{b}_\omega{\restriction}u$ realizes $\mathrm{tp}_{\mathrm{bs}}(\bar{b}_{n!}, G_{n!}{\restriction}u, G_{n+1})$ in $G_{\omega+1}$ when $u \subseteq I$ is finite and n is large enough.

Proof. 1) Letting $p_n(\bar{x}) = \mathrm{tp}_{\mathrm{bs}}(\bar{b}_n, G_n, G_{n+1})$, it is enough to prove:

$(*)_1$ $p_n \subseteq p_{n+1}$.

For this it is enough to prove, letting $\bar{y} = \langle y_t : t \in I\rangle$,

[12] So by (A)(e) this is equivalent to "$\mathrm{tp}(\bar{a}_n, \emptyset, G_{n+1})$ is constant".

$(*)_2$ if $\sigma(\bar{y},\bar{z})$ is a group-term and $\bar{c} \in {}^{(\ell g(\bar{z}))}(G_n)$ <u>then</u> $G_{n+2} \models$ "$\sigma(\bar{b}_{n+1},\bar{c}) = e$" iff $G_{n+1} \models$ "$\sigma(\bar{b}_n,\bar{c}) = e$".

Towards proving $(*)_2$ note:

- $\bullet_{2.1}$ $\bar{c}$ and $\bar{b}_n$, and hence $\sigma(\bar{b}_n,\bar{c})$ are from G_{n+1},
- $\bullet_{2.2}$ a^t_{n+1} commutes with every $c_i (i < \ell g(\bar{c}))$ and with b^s_n for $s \in I$.

By clause (A)(b) of the assumption of the claim,

- $\bullet_{2.3}$ $b^t_{n+1} = b^t_n a^t_{n+1}$ and $b^t_n = b^t_{n-1} a^t_n$ stipulating $b^t_{-1} = e$.

Similarly,

- $\bullet_{2.4}$ $\bar{c}$ and $\bar{b}_{n-1}$ are from G_n;
- $\bullet_{2.5}$ a^t_n commute with every $c_i (i < \ell g(\bar{c}))$ and with b^s_{n-1} for $s \in I$.

Hence for some group term $\sigma_*(\bar{x})$:

- $\bullet_{2.6}$ $G_{n+2} \models$ "$\sigma(\bar{b}_{n+1},\bar{c}) = \sigma(\bar{b}_n,\bar{c})\sigma_*(\bar{a}_{n+1})$";
- $\bullet_{2.7}$ $G_{n+1} \models$ "$\sigma(\bar{b}_n,\bar{c}) = \sigma(\bar{b}_{n-1},\bar{c})\sigma_*(\bar{a}_n)$".

Hence by clauses (A)(d)$(\alpha),(\beta)$:

- $\bullet_{2.8}$ $\sigma_*(\bar{a}_n) \in G_n$ iff $\sigma_*(\bar{a}_n) = e_{G_n}$ iff $\sigma_*(\bar{a}_{n+1}) = e_{G_n}$ iff $\sigma_*(\bar{a}_{n+1}) \in G_{n+1}$;
- $\bullet_{2.9}$ if $\sigma_*(\bar{a}_n) \notin G_n$, hence $\sigma_*(\bar{a}_{n+1}) \notin G_{n+1}$, then both statements in $(*)_2$ fail because:
  - (α) $\sigma(\bar{b}_n,\bar{c})$ is from G_{n+1} and $\sigma_*(\bar{a}_{n+1}) \notin G_{n+1}$ so $\sigma(\bar{b}_{n+1},\bar{c}) \notin G_{n+1}$ and thus $\sigma(\bar{b}_{n+1},\bar{c}) \neq e_{G_n}$;
  - (β) similarly $\sigma(\bar{b}_n,\bar{c}) \notin G_n$ and thus $\sigma(\bar{b}_n,\bar{c}) \neq e_{G_n}$;
- $\bullet_{2.10}$ if $\sigma_*(\bar{a}_n) \in G_n$ hence $\sigma_*(\bar{a}_n) = e = \sigma_*(\bar{a}_{n+1})$, then $\sigma(\bar{b}_{n+1},\bar{c}) = \sigma(\bar{b}_n,\bar{c})$ and again we are done.

Together $(*)_2$ holds.
2) Similarly (and the same as part (1) when G_n is existentially closed for every n) but we elaborate. Without loss of generality I is finite; letting $p_n(\bar{y}) = \mathrm{tp}_{\mathrm{bs}}(\bar{a}_n, G_n)$, we need:

$(*)_1$ if $\bar{c}$ is a finite sequence from G_ω <u>then</u> the sequence $\langle \mathrm{tp}_{\mathrm{bs}}(\bar{b}_{n!}\hat{\ }\bar{c},\emptyset,G_{n!+1}) : n < \omega\rangle$ is eventually constant.

Let $K_n = c\ell(\bar{a}_n, G_{n+1})$, so by clause $(A)(d)(\alpha)$ of the assumption $|K_n|$ is constant, finite and $K_n \cap G_n$ is $\subseteq$-increasing with n. Hence for some $K_*, n(*)$ we have $n \geq n(*) \Rightarrow K_n \cap G_n = K_*$ and let $k(*) = |K_*|$. Without loss of generality $n(*) \geq k(*)$; so it is enough to prove

$(*)_2$ if $\bar{y} = \langle y_t : t \in I\rangle$ and $\sigma(\bar{y},\bar{z})$ is a group term, $\bar{c} \in {}^{\ell g(\bar{z})}(G_n)$ and $n \geq n(*)$, <u>then</u> $G_{n+1} \models$ "$\sigma(\bar{b}_n,\bar{c}) = e$" <u>iff</u> $G_{n+k(*)+1} \models \sigma(\bar{b}_{n+k(*)},\bar{c}) = e$".

As in part (1) we can prove that for some group term $\sigma_*(\bar{y})$ we have

$\boxplus$ if $n \geq n(*)$ then $G_{n+2} \models \sigma(\bar{b}_{n+1},\bar{c}) = \sigma(\bar{b}_n,\bar{c})\sigma_*(\bar{a}_{n+1})$.

Case 1: "$\sigma_*(\bar{a}_n) \notin G_n$ for some, equivalently every, $n \geq n(*)$.

In this case $G_{n+1} \models$ "$\sigma(\bar{b}_n, \bar{c}) \neq e$" for every $n \geq n(*)$.

Case 2: "$\sigma_*(\bar{a}_n) \in G_n$ for some, equivalently every, $n \geq n(*)$.

In this case there is b such that $\sigma_*(\bar{a}_n, \bar{c}) = b$ for every $n \geq n(*)$. So for every $n \geq n(*)$ by induction on m we can prove $\sigma(\bar{b}_{n+m}, \bar{c}) = \sigma(\bar{b}_n, \bar{c}) \cdot b^m$. But necessarily $b \in K_*$ hence b has order dividing $|K_*| = k(*)$. Hence $n \geq n(*) \Rightarrow \sigma(\bar{b}_{n+k(*)}, \bar{c}) = \sigma(\bar{b}_n, \bar{c})$ and thus $n_2 > n_1 \geq n(*) \wedge k(*)|(n_2 - n_1) \Rightarrow \sigma(\bar{b}_{n-2}, \bar{c}) = \sigma(\bar{b}_{n_1}, \bar{c})$, and so we can finish easily. $\square_{2.19}$

Definition/Claim 2.20. 1) For $k = 2, 3...$ let $\mathfrak{s}_{\mathrm{ab}(k)}$ be the unique $\mathfrak{s} \in \Omega[\mathbf{K}_{\mathrm{lf}}]$ such that:

- (a) $n(\mathfrak{s}) = 1, k(\mathfrak{s}) = 0$;
- (b) if $G \subseteq H$ and $c \in H$ realizes $q_{\mathfrak{s}}(<>, G) = \mathrm{tp}_{\mathrm{bs}}(c, G, H)$ then c commutes with every $a \in G$;
- (c) also for every $m < \omega, a^m = e_H$ iff $a^m \in G$ iff $k|m$.

2) Assume $K \in \mathbf{K}_{\mathrm{lf}}$ is finite and $\bar{c} \in {}^{|K|}K$ list it. Then let $\mathfrak{s} = \mathfrak{s}_{\mathrm{ab}}(\bar{c}, K)$ be the unique $\mathfrak{s} \in \Omega[\mathbf{K}_{\mathrm{lf}}]$ such that:

- (a) $n(\mathfrak{s}) = \ell g(\bar{c}), k(\mathfrak{s}) = 0$ so $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = \emptyset$;
- (b) if $G \subseteq H \in \mathbf{K}_{\mathrm{lf}}$ and $\bar{c}' \in {}^{n(\mathfrak{s})}H$ then the following are equivalent:
  - (α) $\mathrm{tp}(\bar{c}', G, H) = q_{\mathfrak{s}}(<>, G)$,
  - (β) $\bar{c}'$ commutes with G, realizes $\mathrm{tp}(\bar{c}, \emptyset, K)$ and $\langle \bar{c}' \rangle_H \cap G = \{e\}$.

Claim 2.21. *Assume* $\mathrm{NF}_f(G_0, G_1, G_2, G_3)$ *and* $a \in G_1 \backslash G_0, b \in G_2 \backslash G_0$. *Then* a, b *commute in* G_3 *iff* $a \in \mathbf{C}_{G_1}(G_0), b \in \mathbf{C}_{G_2}(G_0)$ *and* G_0 *is commutative.*

Remark 2.22. 1) NF_f is from Definition 2.9.
2) Recall $g^{[a]} = a^{-1}ga$.

Proof. Without loss of generality G_1, G_2 are existentially closed (by monotonicity of NF_f, see 2.10(3) and existence of existentially closed extensions).

First assume

$\oplus$ $a \in \mathbf{N}_{G_1}(G_0)$ and $b \in \mathbf{N}_{G_2}(G_0)$.

By 2.15, without loss of generality we can find $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ such that $G_3 = G_{\mathbf{x}}, G_{\mathbf{x},\ell} = G_\ell$ for $\ell < 3$ and let $f_a = \mathbf{j}_{\mathbf{x},1}(a), f_b = \mathbf{j}_{\mathbf{x},2}(b)$; we shall use the fact that: we have some freedom in the choice of $\mathbf{x}$, see 2.15.

Let $(g_0, g_1, g_2) \in \mathscr{U}_{\mathbf{x}}$ and we should see whether $f_b \circ f_a((g_0, g_1, g_2)) = f_a \circ f_b((g_0, g_1, g_2))$; there are unique a', h_a, b', h_b such that:

$(*)_0$ (a) $g_1 a = a' h_a$ with $h_a \in G_0, a' \in \mathbf{I}_{\mathbf{x},1}$;
(b) $g_2 b = b' h_b$ with $h_b \in G_0, b' \in \mathbf{I}_{\mathbf{x},2}$.

Now

$(*)_1$ $f_a((g_0, g_1, g_2)) = (h_a g_0^{[a]}, a', g_2)$.

[Why? As $g_1 g_0 a = g_1 a g_0^{[a]} = a'(h_a g_0^{[a]})$, noting that $g_0^{[a]} \in G_0$ because we are assuming that a normalize G_0 inside G_1.]

$(*)_2$ $f_b((h_a g_0^{[a]}, a', g_2)) = (h_b h_a^{[b]} g_0^{[a][b]}, a', b')$.

[Why? As $g_2(h_a g_0^{[a]})b = g_2 b(h_a^{[b]} g_0^{[a][b]}) = b'(h_b h_a^{[b]} g_0^{[a][b]})$.]

So,

$(*)_3$ $(f_b \circ f_a)((g_0, g_1, g_2)) = (h_b h_a^{[b]} g_0^{[a][b]}, a', b')$.

Now,

$(*)_4$ $f_b((g_0, g_1, g_2)) = (h_b g_0^{[b]}, g_1, b')$.

[Why? As $g_2 g_0 b = g_2 b g_0^{[b]} = b' h_b g_0^{[b]}$).]

$(*)_5$ $f_a((h_b g_0^{[b]}, g_1, b')) = (h_a h_b^{[a]} g_0^{[b][a]}, a', b')$.

[Why? As $g_1(h_b g_0^{[b]})a = g_1 a(h_b^{[a]} g_0^{[b][a]}) = a'(h_a h_b^{[a]} g_0^{[b][a]})$.]

Hence,

$(*)_6$ $(f_a \circ f_b)((g_0, g_1, g_2)) = (h_a h_b^{[a]} g_0^{[b][a]}, a', b')$.

Together we can deduce:

$(*)_7$ $(f_b \circ f_a)(g_0, g_1, g_2)) = (f_a \circ f_b)(g_0, g_1, g_2)$ iff $h_b h_a^{[b]} g_0^{[a][b]} = h_a h_b^{[a]} g_0^{[b][a]}$ in G_0.

Now, <u>not</u> assuming $\oplus$ we shall prove the claim by cases (using $(*)_7$ when $\oplus$ holds).

$\oplus_1$ a, b commute in G_3 <u>when</u>:
- $\bullet_1$ a commutes with G_0 in G_1,
- $\bullet_2$ b commutes with G_0 in G_2,
- $\bullet_3$ G_0 is commutative.

[Why? Note that the assumption $\oplus$ holds (by $\bullet_1 + \bullet_2$), and so let $\mathbf{x} \in \mathbf{X}_\mathbf{K}$ be as above. For any $(g_0, g_1, g_2) \in \mathscr{U}_\mathbf{x}$, we can apply $(*)_7$ thus $h_a, h_b \in G_0$ are well defined, by $(*)_0$. Now as $h_b, h_a, g_0 \in G_0$ and $a \in \mathbf{C}_{G_1}(G_0), b \in \mathbf{C}_{G_1}(G_0)$ and G_0 is commutative, by the present assumptions, clearly $h_b h_a^{[b]} g_0^{[a][b]} = h_b h_a g_0 = h_a h_b g_0 = h_a h_b^{[a]} g_0^{[b][a]}$. As $G_3 = G_\mathbf{x}, G_\mathbf{x}$ is a group of permutations of $\mathscr{U}_\mathbf{x}$ and $(*)_7$ holds for any $(g_0, g_1, g_3) \in \mathscr{U}_\mathbf{x}$, clearly $f_a, f_b \in G_\mathbf{x}$ commute, so we are done.]

$\oplus_2$ a, b do not commute in G_3 <u>when</u>:
- a commutes with G_0,
- b commutes with G_0,
- G_0 is not commutative.

[Why? Choose $h_1, h_2 \in G_0$ which do not commute and let $(g_0, g_1, g_2) = (e_{G_0}, e_{G_1}, e_{G_2}) = (e, e, e)$; note that $a h_\ell^{-1} \notin G_0, b h_\ell^{-1} \notin G_0$ for $\ell = 1, 2$.

Above we could have chosen $\mathbf{x} \in \mathbf{X}_\mathbf{K}$ such that ($G_{\mathbf{x},\ell} = G_\ell$ for $\ell < 3$ and) $a h_1^{-1} \in \mathbf{I}_{\mathbf{x},1}, b h_2^{-1} \in \mathbf{I}_{\mathbf{x},2}$. Again $\oplus$ holds hence $(*)_7$ holds for any relevant $\mathbf{x}, g_0, g_1, g_2$. Recall $(g_0, g_1, g_2) := (e, e, e)$, so $g_1 a = ea = a = (a h_1^{-1}) h_1$. So in $(*)_0(a)$, we get $a' = a h_1^{-1}$ and $h_a = h_1$. Similarly in $(*)_0(b)$ we get $b' = b h_2^{-1}, h_b = h_2$. So $f_a, f_b \in G_\mathbf{x}$ do not commute by $(*)_7$, because we get $h_b h_a^{[b]} g_0^{[a][b]} = h_b h_a g_0 =$

$h_2h_1g_0 \neq h_1h_2g_0 = h_ah_bg_0 = h_ah_b^{[a]}g_0^{[b][a]}$, the inequality as $G_0 \models h_1h_2 \neq h_2h_1$ and we are done by $(*)_7$.]

$\oplus_3$ a, b does not commute in G_3 <u>when</u>:
- a normalizes G_0 in G_1,
- b normalizes G_0 in G_2,
- b does not commute with G_0 in G_2.

[Why? Again $\oplus$ holds hence we can apply $(*)_7$ for any relevant $\mathbf{x}, g_0, g_1, g_2$. Let $h_1 \in G_0$ be such that it does not commute with b in G_2 and let $h_2 = e_{G_0}$. Choose above $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ such that $ah_1^{-1} \in \mathbf{I}_{\mathbf{x},1}$ and $b = bh_2^{-1} \in \mathbf{I}_{\mathbf{x},2}$ and let $(g_0, g_1, g_2) = (e, e, e)$. Again in $(*)_0$ we get $a' = ah_1^{-1}, h_a = h_1$ and $b' = bh_2^{-1}, h_b = h_2 = e$. Now $h_bh_a^{[b]}g_0^{[a][b]} = eh_a^{[b]}e = h_a^{[b]} \neq h_a = h_aee = h_ah_b^{[a]}g_0^{[b][a]}$, the inequality by the choice of $h_a = h_1$.]

$\oplus_4$ a, b do not commute in G_3 <u>when</u>:
- a normalizes G_0 in G_1,
- b normalizes G_0 in G_2,
- a does not commute with G_0 in G_2.

[Why? Like $\oplus_3$.]

Next

$\oplus_5$ a, b does not commute in G_3 <u>when</u>:
- $a \in G_1 \backslash G_0$ does not normalize G_0.

Why? Choose $h \in G_0$ such that $a^{-1}ha \notin G_0$ hence $ha \notin aG_0$ and, of course, $ha \notin G_0$ as $a \notin G_0, h \in G_0$ and similarly $bh^{-1} \in G_2 \backslash G_0$. Let $a' = ha$ so $a' \neq a$ because $h \neq e$.

Choose above $\mathbf{x} \in \mathbf{X}_{\mathbf{K}_{\mathrm{lf}}}$ such that $bh^{-1} \in \mathbf{I}_{\mathbf{x},2}$ and $a, a' \in \mathbf{I}_{\mathbf{x},1}$. Why can we choose such $\mathbf{I}_{\mathbf{x},1}$? Because $a' = ha \in G_1 \backslash G_0, a \in G_1 \backslash G_0$ and $aG_0 \neq a'G_0$, as otherwise for some $h_1 \in G_0$ we have $a' = ah_1$, and so $a^{-1}ha = a^{-1}a' = h_1 \in G_0$, contradicting the choice of h.

Let f_a, f_b be as above for this choice of $\mathbf{x}$.

Now consider $(e, e, e) \in \mathscr{U}_{\mathbf{x}}$ so

$(*)'_1$ $f_a((e, e, e)) = (e, a, e)$.

[Why? As $a \in \mathbf{I}_{\mathbf{x},1}$.]

$(*)'_2$ $f_b((e, a, e)) = (h, a, bh^{-1})$.

[Why? Because $bh^{-1} \in \mathbf{I}_{\mathbf{x},2}, h \in G_0$.]

$(*)'_3$ $(f_b \circ f_a)(e, e, e) = (h, a, bh^{-1})$.

[Why? By $(*)'_1 + (*)'_2$.]

$(*)'_4$ $f_b((e, e, e)) = (h, e, bh^{-1})$.

[Why? Because $bh^{-1} \in \mathbf{I}_{\mathbf{x},2}$ and $h \in G_0$.]

$(*)'_5$ $f_a((h, e, bh^{-1}) = (e, a', bh^{-1})$.

[Why? As $eha = ha = a' = a'e$ and $a' \in \mathbf{I}_{\mathbf{x},1}$.]

$(*)'_6$ $(f_a \circ f_b)((e,e,e) = (e, a', bh^{-1}))$.

[Why? By $(*)'_4 + (*)'_5$.]

By $(*)'_3 + (*)'_6$, as $a' \neq a$ the triple (e,e,e) exemplifies $\mathbf{j}_{\mathbf{x},1}(a), \mathbf{j}_{\mathbf{x},2}(b)$ do not commute in $G_{\mathbf{x}}$.

Lastly,

$\oplus_6$ a, b do not commute in G_3 <u>when</u>:

- $b \in G_2 \backslash G_0$ does not normalize G_0.

[Why? As in $\oplus_5$.]

As we have covered all the cases we are done. $\square_{2.21}$

Claim 2.23. *Assume $\mathfrak{S} \subseteq \Omega[\mathbf{K}_{\rm lf}]$ and $G_1 \leq_{\mathfrak{S}} G_2, G_1$ is existentially closed and $d \in G_2$. If conjugation by d (in G_2) maps G_1 onto itself <u>then</u> for some $c \in G_1$ we have $a \in G_1 \Rightarrow c^{-1}ac = d^{-1}ad$, i.e. $dc^{-1}a = adc^{-1}$, i.e. dc^{-1}, a commute in G_1 so dc^{-1} commute with G_1.*

Proof. Easy. Clearly there is $(\mathfrak{s}, \bar{a}) \in \mathrm{def}(G_1)$ such that $\mathrm{tp}_{\mathrm{bs}}(d, G_1, G_2) = q_{\mathfrak{s}}(\bar{a}, G_1))$, hence if $b, b_1, c_1 \in G_1$ and $\mathrm{tp}_{\mathrm{bs}}(\langle b_1, c_1\rangle, \bar{a}, G_1) = \mathrm{tp}_{\mathrm{bs}}(\langle b, d^{-1}bd\rangle, \bar{a}, G_1)$ then $d^{-1}b_1 d = c_1$. Having disjoint amalgamation we have $x \in G_1 \Rightarrow d^{-1}xd \in c\ell(\bar{a}\hat{\ }\langle x\rangle, G_1)$. We can continue or note that if there is no $c \in G_1$ as desired, then every existentially closed G has a non-inner automorphism, contradiction. $\square_{2.23}$

§ 3. Symmetryzing

Our intention is to start with $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ which may contain $\mathfrak{s}_1, \mathfrak{s}_2$ failing symmetry but have the nice conclusion as for symmetric $\mathfrak{S}$. Towards this we define the operation $\otimes$, related to $\oplus$ defined in Definition 1.6(4),(4A), and $\mathfrak{S}-\otimes$-constructions (close but not the same as the constructions in Definition 1.12, 1.19, 1.24) and $\mathfrak{S} - \oplus$-constructions.

Note that $\mathfrak{S}_{\rm atdf}$ has "quasi symmetry", i.e. when the parameter (= base of amalgamation) is the same, but when we allow increasing the base this is not clear. Now $\otimes$ is like $\oplus$ when we insist on it being symmetric. We use the construction here in §4,§5 where we sometimes give more details. Recall def(G) for $G \in \mathbf{K}$ is from Definition 1.1(1).
Recall

Definition 3.1. For $t \in \text{def}(G)$ let $q_t(G) = q_{\mathfrak{s}_t}(\bar a_t, G)$ and $n_t = n_{\mathfrak{s}_t}, k_t = k_{\mathfrak{s}_t}$ and see Definition 1.1(6).

Definition 3.2. 1) On def(G) we define a (partial) operation $\otimes$ by $t_1 \otimes t_2 = (\mathfrak{s}_{t_1} \otimes \mathfrak{s}_{t_2}, \bar a_{t_1}{}^\frown \bar a_{t_2})$, see below.
2) $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2$ means that $\mathfrak{s}_1, \mathfrak{s}_2$ are disjoint[13], $\bar x_{\mathfrak{s}} = \bar x_{\mathfrak{s}_1}{}^\frown \bar x_{\mathfrak{s}_2}, \bar z_{\mathfrak{s}} = \bar z_{\mathfrak{s}_1}{}^\frown \bar z_{\mathfrak{s}_2}$, so $k(\mathfrak{s}) = k(\mathfrak{s}_1) + k(\mathfrak{s}_n), n(\mathfrak{s}) = n(\mathfrak{s}_1) + n(\mathfrak{s}_2)$ and:

> $\boxplus$ if $H \subseteq H^+ \in \mathbf{K}, \bar a_\ell \in {}^{k(\mathfrak{s}_\ell)}H$ realizes $p_{\mathfrak{s}_\ell}(\bar x_{\mathfrak{s}_\ell})$ in H and $\bar c_\ell \in {}^{n(\mathfrak{s}_\ell)}(H^+)$ for $\ell = 1, 2$, then $\bar c_1{}^\frown \bar c_2$ realizes $q_{\mathfrak{s}}(\bar a_1{}^\frown \bar a_2, H)$ <u>iff</u>:
>
> (a) $\bar c_\ell$ realizes $q_{\mathfrak{s}_\ell}(\bar a_\ell, H)$ in H^+, for $\ell = 1, 2$;
>
> (b) if $\sigma(\bar z_1, \bar z_2, \bar y)$ is a group-term, $\ell g(\bar z_1) = n(\mathfrak{s}_1), \ell g(\bar z_2) = n(\mathfrak{s}_2)$ and $\bar b \in {}^{\ell g(\bar y)}(H)$, then $(\alpha) \Leftrightarrow (\beta)$ where:
>
> (α) $H^+ \models$ "$\sigma(\bar c_1, \bar c_2, \bar b) = e_H$",
>
> (β) $(\sigma(\bar z_1, \bar c_2, \bar b) = e) \in q_{\mathfrak{s}_1}(\bar a_1, H^+)$ and $(\sigma(\bar c_1, \bar z_2, \bar b) = e) \in q_{\mathfrak{s}_2}(\bar a_2, H^+)$.

Claim 3.3. *1) If $\mathfrak{s}_1, \mathfrak{s}_2 \in \Omega[\mathbf{K}]$ <u>then</u> $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2$ belongs to $\Omega[\mathbf{K}]$.*
2) If $G \in \mathbf{K}$ and $t_1, t_2 \in \text{def}(G)$ <u>then</u> $t = t_1 \otimes t_2 \in \text{def}(G)$.

Proof. Straightforward. $\square_{3.3}$

Definition 3.4. 1) Let $\approx^*_G$ be the following two-place relation on def(G) : $(\mathfrak{s}_1, \bar a_1) \approx^*_G (\mathfrak{s}_2, \bar a_2)$ if both are in def(G) and $G \subseteq G^+ \in \mathbf{K} \Rightarrow q_{\mathfrak{s}_1}(\bar a_1, G^+) = q_{\mathfrak{s}_2}(\bar a_2, G^+)$, (compare with $\approx_G$ from 1.1(6)).
2) For $t_1, t_2 \in \text{def}(G)$ let $t_1 \le t_2$ means $\text{dom}(\bar x_{t_1}) \subseteq \text{dom}(\bar x_{t_2}), \text{dom}(\bar z_{t_1}) \subseteq \text{dom}(z_{\bar t_2})$ and $\bar a_{t_1} = \bar a_{t_2} \restriction \text{dom}(\bar x_{t_1})$, and if $G \subseteq G_1 \subseteq G_2$ and $\bar c_2$ realizes $q_{t_2}(G_1)$ in G_2 then $\bar c_2 \restriction \text{dom}(\bar z_{t_1})$ realizes $q_{t_1}(G)$ in G_2.
3) $t_1 \le_{\bar h} t_2$ is defined similarly as in 1.6(7).

Claim 3.5. *0) $\approx^*_G$ is an equivalence relation on def(G).*
1) If $(\mathfrak{s}, \bar a) \in \text{def}(G_1)$ and $G_1 \subseteq G_2 \in \mathbf{K}$ <u>then</u> $q_{\mathfrak{s}}(\bar a, G_1) \subseteq q_{\mathfrak{s}}(\bar a, G_2)$ and $(\mathfrak{s}, \bar a) \in \text{def}(G_2)$.
2) If $G \in \mathbf{K}$ and $(\mathfrak{s}_\ell, \bar a_\ell) \in \text{def}(G)$ for $\ell = 1, 2$, <u>then</u> the satisfaction of $(\mathfrak{s}_1, \bar a_1) \approx^*_G (\mathfrak{s}_2, \bar a_2)$ depends just on $\mathfrak{s}_1, \mathfrak{s}_2$ and $\text{tp}_{\rm bs}(\bar a_1{}^\frown \bar a_2, \emptyset, G)$.
3) Transitivity: in Definition 3.4(2), $\le$ is indeed a partial order.
4) Moreover if $(\mathfrak{s}_1, \bar a_1) \le_{\bar h_1} (\mathfrak{s}_2, \bar a_2) \le_{\bar h_2} (\mathfrak{s}_3, \bar a_3)$ <u>then</u> $(\mathfrak{s}_1, \bar a_1) \le_{\bar h_2 \circ \bar h_1} (\mathfrak{s}_3, \bar a_3)$.

[13]As we use only invariant $\mathfrak{S}$, this is not a real restriction.

Proof. Easy. $\square_{3.5}$

Claim 3.6. *0) The operation $\otimes$ on disjoint pairs respects congruency (see Definition 1.1(3), Claim 1.9(1)).*
1) The operation $\otimes$ respects $\approx^*_G$, i.e. if $t_1 \approx^*_G t'_1$ and $t_2 \approx^*_G t'_2$ <u>then</u> $t_1 \otimes t_2 \approx^*_G t'_1 \otimes t'_2$ assuming the operations are well defined, of course.
2) If $(\mathfrak{s}, \bar{a}) = (\mathfrak{s}_1, \bar{a}_1) \otimes (\mathfrak{s}_2, \bar{a}_2)$, <u>then</u> $(\bar{\mathfrak{s}}_\ell, \bar{a}_\ell) \le (\mathfrak{s}, \bar{a})$.
3) If in $\mathrm{def}(G)$ we have $t_\ell \le t'_\ell$ for $\ell = 1, 2$ and $t'_1 \otimes t'_2$ is well defined (i.e. t'_1, t'_2 are disjoint) <u>then</u> $t_1 \otimes t_2 \le t'_1 \otimes t'_2$.
4) The operation $\otimes$ is associative and is symmetric, e.g. symmetry means: if $G \subseteq G^+$ and $(\mathfrak{s}_\ell, \bar{a}_\ell) \in \mathrm{def}(G)$ and $\bar{c}^\ell_\ell {}^\frown \bar{c}^\ell_{3-\ell}$ realizes $q_{t_\ell}(G)$ in G^+, where $t_\ell = (\mathfrak{t}_\ell, \bar{b}_\ell) = (\mathfrak{s}_\ell, \bar{a}_\ell) \otimes (\mathfrak{s}_{3-\ell}, \bar{a}_{3-\ell})$, (so assuming disjointness for transparency), for $\ell = 1, 2$, <u>then</u> $\mathrm{tp}_{\mathrm{bs}}(\bar{c}^1_1 {}^\frown \bar{c}^1_2, G, G^+) = \mathrm{tp}_{\mathrm{bs}}(\bar{c}^2_1 {}^\frown \bar{c}^2_2, G, G^+)$.
5) If in $\mathrm{def}(G)$ we have $(\mathfrak{s}_\ell, \bar{a}_\ell) \le_{h_\ell} (\mathfrak{s}'_\ell, \bar{a}'_\ell)$ for $\ell = 1, 2$ and $\mathrm{Dom}(h_1) \cap \mathrm{Dom}(h_2) = \emptyset, \mathrm{Rang}(h_1) \cap \mathrm{Rang}(h_2) = \emptyset$ <u>then</u> $(\mathfrak{s}_1, \bar{a}_1) \otimes (\mathfrak{s}_2, \bar{a}_2) \le_{h_1 \cup h_2} (\mathfrak{s}'_1, \bar{a}_1) \otimes (\mathfrak{s}'_2, \bar{a}_2)$.

Proof. Straightforward. $\square_{3.6}$

Remark 3.7. 1) Also the operation $\oplus$ satisfies the parallels of 3.6(1),(2),(3) and the first demand in (4).
2) We may phrase 3.6(5) as in 3.6(3) and vice versa.

Definition 3.8. Assume $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is closed.
1) We say $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is $\otimes$-closed <u>when</u> (recalling it is invariant) if $\mathfrak{s}_\ell \in \mathfrak{S}$ for $\ell = 1, 2$ are disjoint <u>then</u> $\mathfrak{s} = \mathfrak{s}_1 \otimes \mathfrak{s}_2 \in \mathfrak{S}$.
2) The $\otimes$-closure of $\mathfrak{S}$ is the $\subseteq$-minimal $\otimes$-closed $\mathfrak{S}' \subseteq \Omega[\mathbf{K}]$ such that $\mathfrak{S} \subseteq \mathfrak{S}'$.
3) Let $G_3 = G_1 \bigotimes^{\mathfrak{S}}_{G_0} G_2$ or $G_3 = \otimes_{\mathfrak{S}}(G_0, G_1, G_2)$ mean:

- $(*)$ (a) $G_0 \le_{\mathfrak{S}} G_2 \subseteq G_3 \in \mathbf{K}$ and $G_0 \le_{\mathfrak{S}} G_1 \subseteq G_3$ and $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$
  - (b) if $\mathrm{tp}_{\mathrm{bs}}(\bar{c}_\ell, G_0, G_\ell) = q_{\mathfrak{s}_\ell}(\bar{a}_\ell, G_0)$ so $\bar{c}_\ell \in {}^{\omega>}(G_\ell), \bar{a}_\ell \in {}^{\omega>}(G_0)$ for $\ell = 1, 2$, <u>then</u> $\mathrm{tp}_{\mathrm{bs}}(\bar{c}_1 {}^\frown \bar{c}_2, G_0, G_3) = q_{\mathfrak{s}}(\bar{a}_1 {}^\frown \bar{a}_2, G_0)$ when $(\mathfrak{s}, \bar{a}_1 {}^\frown \bar{a}_2) = (\mathfrak{s}, \bar{a}_1) \otimes (\mathfrak{s}_2, \bar{a}_2)$; note that without loss of generality $\mathfrak{s}_1, \mathfrak{s}_2$ are disjoint, (i.e. as in the proof of 1.10).

4) $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G_1, G_2, G_3)$ means that $G_0 \le_{\mathfrak{S}} G_\ell \le_{\mathfrak{S}} G_3$ for $\ell = 1, 2$ and the demands in (3) hold except that possibly $G_3 \ne \langle G_1 \cup G_2 \rangle_{G_3}$.

Claim 3.9. *Assume $\mathfrak{S}$ is closed and moreover $\otimes$-closed.*
1) $G_3 = \otimes_{\mathfrak{S}}(G_0, G_1, G_2)$ iff $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G_1, G_2, G_3)$ and $G_3 = \langle G_1 \cup G_2 \rangle_{G_3}$.
2) (disjointness): $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G_1, G_2, G_3)$ implies $G_1 \cap G_2 = G_0$.
3) (uniqueness): If $G^\iota_3 = \otimes_{\mathfrak{S}}(G^\iota_0, G^\iota_1, G^\iota_2)$ for $\iota = 1, 2$ and f_ℓ is an isomorphism from G^1_ℓ onto G^2_ℓ for $\ell = 1, 2$ and $G^1_0 = G^2_0, f_1 \restriction G^1_0 = f_2 \restriction G^2_0$ and G_0 is existentially closed[14] <u>then</u> there is one and only one isomorphism from G^1_3 onto G^2_3 extending $f_1 \cup f_2$ (which is well defined by (2)).

[14]Why? The problem is that $G \le_{\mathfrak{S}} H \in \mathbf{K}$ does not imply the existence of $\bar{t} = \langle t_{\bar{c}} : \bar{c} \in {}^{\omega>}H \rangle$ such that $t_{\bar{c}} \in \mathrm{def}(G), \mathrm{tp}_{\mathrm{bs}}(\bar{c}, G, H) = q_t(G)$ and if $\bar{c}^1, \bar{c}^2 \in {}^{\omega>}H, h : \ell g(\bar{c}^1) \to \ell g(\bar{c}^2)$ and $\bar{c}^2 = \langle c^2_{h(i)} : i < \ell g(\bar{c}^1) \rangle$ then $t_{\bar{c}^1} \le_h t_{\bar{c}^2}$. Moreover, even if there is such $\bar{t}$ we can "amalgamate for it" but this is not enough as $\bar{t}$ is not necessarily unique, which may give different results. Why 3.9(3) is O.K.? As in Definition 3.8(3) we ask "for every $\mathfrak{s}_1, \mathfrak{s}_2$". In other words if $G_0 \subseteq G_1, G_0 \subseteq G_2$ and $t_1, t_2 \in \mathrm{def}(G_0), \mathrm{tp}_{\mathrm{bs}}(\bar{c}_\ell, G_0, G_2) = q_{t_\ell}(G_0)$ for $\ell = 1, 2$ but $q_{t_1}(G_1) \ne q_{t_2}(G_1)$ we can amalgamate as in 3.8(3).

4) (symmetry): $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G_1, G_2, G_3)$ *iff* $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G_2, G_1, G_3)$.
5) (monotonicity): If $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G_1, G_2, G_3)$ *and* $G_0 \subseteq G'_\ell \subseteq G_\ell$ *for* $\ell = 1, 2$ *then* $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G'_1, G'_2, G_3)$.
6) (existence): If $G_0 \leq_{\mathfrak{S}} G_\ell$ *for* $\ell = 1, 2$ *and* G_0 *is existentially closed and* $G_1 \cap G_2 = G_0$ *then for some* $G_3 \in \mathbf{K}$ *we have* $\mathrm{NF}^2_{\mathfrak{S}}(G_0, G_1, G_2, G_3)$.

Remark 3.10. For parts (3) and (6) of 3.9 recall: for such G, if $t_1, t_2 \in \mathrm{def}(G)$, $q_{t_1}(G) = q_{t_2}(G)$ and $G \subseteq G^+ \in \mathbf{K}$ then $q_{t_1}(G^+) = q_{t_2}(G^+)$.

Proof. Straightforward, e.g. for disjointness (= part (2)) use Claim 1.2(4). $\square_{3.9}$

Alternative to §1 from 1.12 on is: we repeat it with changes being that we use $\otimes$ instead of $\oplus$ and we incorporated the λ-fullness, also in 3.12(3) we choose another version. We have not sorted out whether we can generalize 1.16(5) based on 1.15 and 1.23(2).

Definition 3.11. 1) We say that $\mathscr{A}$ is a one step $(\lambda, \mathfrak{S}) - \otimes$-construction when $\mathscr{A} = (G, H, \langle \bar{c}_\alpha, t_\alpha : \alpha < \alpha(\mathscr{A}) = \alpha_{\mathscr{A}} \rangle)$ satisfies:

- (a) $G \subseteq H \in \mathbf{K}$
- (b) $t_\alpha \in \mathrm{def}_{\mathfrak{S}}(G)$ for $\alpha < \alpha(\mathscr{A})$;
- (c) if $\alpha_0, \dots, \alpha_{n-1} < \alpha(\mathscr{A})$ with no repetitions then $\bar{c}_{\alpha_0}\hat{\ } \dots \hat{\ }\bar{c}_{\alpha_{n-1}}$ realizes $q_t(G_0)$ in H where $t = t_{\alpha_0} \otimes \dots \otimes t_{\alpha-1} \in \mathrm{def}(G)$;
- (d) $H = \langle \cup \{\bar{c}_\alpha : \alpha < \alpha(\mathscr{A})\} \cup G \rangle_H$;
- (e) $\langle t_\alpha : \alpha < \alpha(\mathscr{A}) \rangle$ lists $\mathrm{def}_{\mathfrak{S}}(G)$ each appearing exactly λ times.

2) In (1) we may use any index set instead of $\alpha(\mathscr{A})$, e.g. $\mathrm{def}_{\mathfrak{S}}(G)$ itself when $\lambda = 1$, $\mathrm{def}_{\mathfrak{S}}(G) \times \lambda$ in general.
3) We say $\mathscr{A}$ is an $\alpha(\mathscr{A})$-step-$(\lambda, \mathfrak{S}) - \otimes$-construction or $(\alpha(\mathscr{A}), \lambda, \mathfrak{S}) - \otimes$-construction when :

- (a) $\mathscr{A} = \langle G_\alpha, \langle \bar{c}_{\beta,s}, t_{\beta,s} : s \in S_\beta \rangle : \alpha \leq \alpha(\mathscr{A}), \beta < \alpha(\mathscr{A}) \rangle$
- (b) $(G_\alpha : \alpha \leq \alpha(\mathscr{A}))$ is increasing continuous (in $\mathbf{K}$)
- (c) $(G_\alpha, G_{\alpha+1}, \langle \bar{c}_{\alpha,s}, t_{\alpha,s} : s \in S_\alpha \rangle)$ is a one step $(\lambda, \mathfrak{S}) - \otimes$-construction.

4) In part (3), let $G^{\mathscr{A}}_\alpha = G_\alpha[\mathscr{A}]$ be G_α, etc., and in part (1) let $G^{\mathscr{A}} = G[\mathscr{A}]$ be G, etc.
5) In part (3) if $\alpha(\mathscr{A}) = \omega$ then we may omit it; also for every $\alpha < \alpha(\mathscr{A})$ the sequence $(G^{\mathscr{A}}_\alpha, G^{\mathscr{A}}_{\alpha+1}, \langle \bar{c}_{\alpha,s}, t_{\alpha,s} : s \in S^{\mathscr{A}}_\alpha \rangle)$ is called the α-th step of $\mathscr{A}$.

Definition 3.12. 1) We say H is a λ-full one step $\mathfrak{S} - \otimes$-closure of G when there is a one step $(\lambda, \mathfrak{S}) - \otimes$-construction $\mathscr{A}$ such that $G[\mathscr{A}] = G$, $H[\mathscr{A}] = H$. We may say H is λ-full one step $\mathfrak{S} - \otimes$-constructible over G; similarly in part (2).
2) We say H is λ-full α-step $\mathfrak{S}$-closure over G or H is $(\alpha, \lambda, \mathfrak{S})$-closure of G when there is a $(\alpha, \lambda, \mathfrak{S}) - \otimes$-construction $\mathscr{A}$ with $G = G^{\mathscr{A}}_0$, $H = G^{\mathscr{A}}_{\ell g(\mathscr{A})}$.
3) We say G_* is $(\delta, \lambda, \mathfrak{S}) - \otimes$-full over G when for some $\bar{G} = \langle G_i : i \leq \delta \rangle$ increasing continuous sequence in $\mathbf{K}$, $G_0 = G$, $G_\delta = G_*$ and G_{i+1} is $(1, \lambda, \mathfrak{S}) - \otimes$-full over G_i which means some $G' \subseteq G_{i+1}$ is a one step $(\lambda, \mathfrak{S}) - \otimes$-construction over G_i. If $\delta = \omega$ one may omit it writing $(\lambda, \mathfrak{S})$ instead of $(\delta, \lambda, \mathfrak{S})$.
4) We may in part (3) replace $\otimes$ by $\oplus$.

Claim 3.13. *Assume $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ is $\otimes$-closed, α an ordinal, λ a cardinal.*
1) If $G \in \mathbf{K}$ then there is a one step $(\lambda, \mathfrak{S}) - \otimes$-construction $\mathscr{A}$ over G (i.e. $G_0^{\mathscr{A}} = G$) of cardinality $\leq \lambda + |G| + |\mathfrak{S}|$ and $\geq \lambda$.
2) If in part (1), $\mathscr{A}_1, \mathscr{A}_2$ are one step-$(\lambda, \mathfrak{S}){-}\otimes$-constructions over G then $H[\mathscr{A}_1], H[\mathscr{A}_2]$ are isomorphic over G.
3) For any $G \in \mathbf{K}$ there is an $(\alpha, \lambda, \mathfrak{S}) - \otimes$-construction $\mathscr{A}$ over G and $G_\alpha[\mathscr{A}]$ is unique up to isomorphism over G.
4) If $\mathfrak{S}$ is dense, H is an $(\alpha, \lambda, \mathfrak{S}) - \otimes$-closure of G and α is a limit ordinal then H is existentially closed and is $(\alpha, \lambda, \mathfrak{S})$-full over G.

Proof. Straightforward, as in 1.23(3). $\square_{3.13}$

Discussion 3.14. Essentially we know that if "$G_1 \subseteq G_2$" implies the $(\alpha, \lambda, \mathfrak{S})$-closure of G_1 is a subgroup of the $(\alpha, \lambda, \mathfrak{S})$-closure of G_2.

But we have a delicate problem: what if the $(\alpha, \lambda, \mathfrak{S})$-closure of G_1 is not disjoint to $G_2 \backslash G_1$?

We have similar problems with "the algebraic closure of a field" or "the field of quotients of a field", but there if $G_1 \subseteq G_2$ then the closure G_1^+ of G_1 inside G_1 is definable (from G_2, G_1 and G_2^+). Here this is not true, but clearly this is not a serious problem. Ways to circumvent this appear in 0.12(2), 1.13(2) and below.

Claim 3.15. *1) We can choose $\hat{G} \in \mathbf{K}_{\mathrm{exlf}}$ such that $\hat{G}$ extends $G \in \mathbf{K}_{\mathrm{lf}}, G_1 \cong G_2 \Rightarrow \hat{G}_1 \cong \hat{G}_2$ and every embedding $f : G_1 \to G_2 \in \mathbf{K}_{\mathrm{lf}}$ can be extended to $\hat{f} : \hat{G}_1 \to \hat{G}_2$ canonically.*
1A) Moreover $G_1 \subseteq G_2 \Rightarrow \hat{G}_1 \subseteq \hat{G}_2$ but pedantically see (2).
2) There is a set theoretic class function $\mathbf{F}$, that computes from $G \in \mathbf{K}, \alpha \in \mathrm{Ord}, \lambda \in \mathrm{Card}, \gamma \in \mathrm{Ord}$ and $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ a group $H = \mathbf{F}(G, \alpha, \mathfrak{S}, \gamma)$ such that:

- (a) *$\mathbf{F}(G, \alpha, \mathfrak{S}, \gamma) \in \mathbf{K}$ extends G, moreover;*
- (b) *$\mathbf{F}(G, \alpha, \gamma, \mathfrak{S})$ is an $(\alpha, \lambda, \mathfrak{S})$-closure of G;*
- (c) *[uniqueness]: if $G_1, G_2 \in \mathbf{K}$ and g is an isomorphism from G_1 onto G_2 and $H_\ell = \mathbf{F}(G, \alpha, \mathfrak{S}, \gamma)$ for $\ell = 1, 2$ then there is an isomorphism g from H_1 onto H_2 extending g;*
- (d) *we have $H_1 \subseteq H_2$ and $G_1 = H_1 \cap G_2$ when $G_1 \subseteq G_1 \in \mathbf{K}, \gamma > \alpha$ and*[15] *$\gamma > \sup(\mathrm{Ord} \cap \mathrm{tr} - c\ell(G_\ell))$ for $\ell = 1, 2$ and $H_\ell = \mathbf{F}(G_\ell, \alpha, \mathfrak{S}, \gamma)$;*
- (e) *if we restrict ourselves to $G \in \mathbf{K}' = \{G \in \mathbf{K}$: if $x \in G$ then x is a singleton$\}$ then $G_1 \subseteq G_2 \Rightarrow \mathbf{F}(G, \alpha, \mathfrak{S}) \subseteq \mathbf{F}(G, \alpha, S, 0)$.*

* * *

In §4,§5 we intend to use also some relative of those constructions, including:

Definition 3.16. Assume $\bar{H} = \langle H_i : i < \delta \rangle$ is $\subseteq$-increasing in $\mathbf{K}$ and $H_\delta = \cup \{H_i : i < \delta\}$, (we shall use $\delta = \omega$). We say $\mathscr{A}$ is a one step atomic $\mathfrak{S} - \otimes$-construction above $\bar{H}$, when (and we may say H is weakly atomically $\mathfrak{S} - \otimes$-constructible over $\bar{H}$, omitting $\bar{H}$ means for some $\bar{H}$ of length ω and we may replace $\alpha_{\mathscr{A}} = \alpha(\mathscr{A})$ by any index set) $\mathscr{A}$ has the following objects satisfying the following additional conditions:

[15] Recalling tr-cl is the (set-theoretic) transitive closure.

(A) $(\bar H, H_\delta, H, \langle \bar c_\alpha, t_{\alpha,i}, \alpha < \alpha_{\mathscr A}, i < \delta\rangle)$;
(B) $H_\delta \subseteq H \in \mathbf{K}$;
(C) $t_{\alpha,i} \in \mathrm{def}_{\mathfrak S}(H_i)$;
(D) $H = \langle \cup\{\bar c_\alpha : \alpha < \alpha_{\mathscr A}\} \cup H_\delta\rangle_H$;
(E) $\bar c_\alpha$ realizes $q_{t_{\alpha,i}}(H_i)$ in H for $\alpha < \alpha_{\mathscr A}, i < \delta$;
(F) $\bar c_{\alpha,i} \subseteq H_{i+1}$ realizes $q_{t_{\alpha,i}}(H_i)$ for $i < \delta, \alpha < \alpha_{\mathscr A}$ and moreover;
$(F)^+$ assuming $\alpha(0) < \ldots < \alpha(n-1) < \alpha_{\mathscr A}$ and $\ell g(\bar x_\alpha) = \ell g(\bar c_\alpha)$ and $\varphi = \varphi(\bar x_{\alpha(0)}, \ldots, x_{\alpha(n-1)}, \bar y)$ we have[16] $\varphi(\bar x_{\alpha(0)}, \ldots, \bar x_{\alpha(n-1)}, \bar b) \in \mathrm{tp}_{\mathrm{at}}(\bar c_{\alpha(0)}\hat{}\ldots\hat{}\bar c_{\alpha(n-1)}, G_\delta, H)$ <u>iff</u> $\bar b \subseteq {}^{\ell g(\bar y)}G_\delta$ and for every permutation π of n, $(\forall^\infty i(0) < \delta)(\forall^\infty i(1) < \delta), \ldots, (\forall^\infty i(n-1) < \delta)$ $\varphi[\bar c_{\alpha(0),i(\pi(0)}, \bar c_{\alpha(1),i(\pi(1))}, \ldots, \bar c_{d(n-1),\pi(n-1)}, \bar b]$ (used in the proof of $(*)_{5.2}$ stage C in the proof of 5.1); note that φ is not necessarily atomic.

Remark 3.17. 1) We may consider replacing clause $(F)^+$ by:

$(F)^*$ $\bar c_{\alpha(0)}\hat{}\ldots\hat{}\bar c_{\alpha(n-1)}$ realizes $q_{t_{\alpha(0)}\otimes\ldots\otimes t_{\alpha(n-1)}}$ for $\alpha(0) < \ldots < \alpha(n-1) < \alpha(\mathscr A)$.

2) In this alternative version we do not need the existence of $\bar c_{\alpha,i} \subseteq H_{i+1}$, so it is easier to prove existence but the version above is the one we actually use. In particular the version in (1) would create problems in $(*)_{5.7}$ in the proof of 5.1; we may try to take care of this by changing the definition of L^*_β there.
3) A sufficient condition for having the assumptions of 3.16 appear in 2.19.

Observation 3.18. Let $\mathfrak S$ be closed and $\otimes$-closed. Assume $\langle G_i : i \le \alpha\rangle$ is $\subseteq$-increasing continuous in $\mathbf{K}$.
1) In 3.11(1) we can prove $G^{\mathscr A} \le_{\mathfrak S} H^{\mathscr A}$ and in 3.11(2), we can prove $\langle G^{\mathscr A}_\alpha : \alpha \le \alpha_{\mathscr A}\rangle$ is $\le_{\mathfrak S}$-increasing continuous.
2) In 3.16, if $\bar H$ is $\le_{\mathfrak S}$-increasing <u>then</u> we have $i < \delta \Rightarrow H_i \subseteq_{\mathfrak S} H$.
3) Assume S is a set of limit ordinals $< \delta, \langle G_i : i \le \delta\rangle$ is a $\subseteq$-increasing continuous sequence of members of $\mathbf{K}$ and G_{i+1} is a one step $\mathfrak S - \otimes$-constructible over G_i for $i \in \delta\backslash S$ and G_{i+1} is weakly one step $\mathfrak S - \otimes$-constructible over $\bar G\restriction C_i$ for some unbounded $C_i \subseteq i\backslash S$ for each $i \in S$, (hence i is a limit ordinal). <u>Then</u> $i < j \le \delta \wedge i \notin S \Rightarrow G_i \le_{\mathfrak S} G_j$.

Remark 3.19. The idea of $\mathfrak s_1 \otimes \mathfrak s_2$ can be applied to one $\mathfrak s$ (and is used in the end of the proof of $\boxplus_1$ in stage B the proof of Theorem 5.1).

Toward this in §4(B) we shall deal with finding such amalgamations and $\mathfrak s$'s.

Definition/Claim 3.20. Assume $\mathfrak s \in \Omega[\mathbf{K}_{\mathrm{lf}}]$ and $H_1 \subseteq H_2 \in \mathbf{K}$ are finite, $\bar a \in {}^{k(\mathfrak s)}(H_1), \bar c \in {}^{n(\mathfrak s)}(H_2)$ and $\bar a, \bar c$ generate H_1, H_2 respectively, and $\bar a$ realizes $p_{\mathfrak s}(\bar x_{\mathfrak s})$ in H_1 and $\bar c$ realizes $q_{\mathfrak s}(\bar a, H_1)$ in H_2. Assume further K is a group of automorphisms of H_2 mapping H_1 onto itself. <u>Then</u> there is a one and only one $\mathfrak t$ such that:

(a) $\mathfrak t \in \Omega[\mathbf{K}_{\mathrm{lf}}]$
(b) $k(\mathfrak t) = k(\mathfrak s)$ and $p_{\mathfrak t}(\bar x_{\mathfrak t}) = \mathrm{tp}_{\mathrm{qf}}(\bar a, \emptyset, H_1)$

[16]Yes! $\mathrm{tp}_{\mathrm{at}}$ and not $\mathrm{tp}_{\mathrm{bs}}$.

(c) if $H_1 \subseteq G_1 \subseteq G_2, H_2 \subseteq G_2$ and $\bar{c}$ realizes $q_{\mathfrak{s}}(\bar{a}, G_1)$ in G_2 and $\bar{c}' \in {}^n(G_2)$ realizes $q_{\mathfrak{t}}(\bar{a}, G_2)$ <u>then</u> $\mathrm{tp}_{\mathrm{at}}(\bar{c}', G_1, G_2) = \cap\{\mathrm{tp}_{\mathrm{at}}(\pi(\bar{c}), G_1, G_2) : \pi \in K\}$.

Remark 3.21. Toward this in §(4B) we deal with finding such amalgamations and $\mathfrak{s}$'s.

Proof. Straightforward. $\square_{3.20}$

§ 4. For fixing a distinguished subgroup

In the construction of complete members of $\mathbf{K}_{\rm exlf}$ (and related aims) we fix large enough $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$ and build a $\subseteq$-increasing continuous sequence $\langle G_\alpha : \alpha < \lambda\rangle, |G_\alpha| < \lambda$; normally we demand for $\alpha < \beta < \lambda$ that "usually" $G_\alpha \le_{\mathfrak{S}} G_\beta$ (i.e. except for $\delta \in S$, where $S \subseteq S^\lambda_{\aleph_0}$). But at some moment for $\alpha = \delta + n$, we like to use $p = {\rm tp}_{\rm bs}(c, G_\alpha, G_{\alpha+1})$ which extends some $r \in \mathbf{S}_{\rm bs}(K), K \subseteq G_\alpha$ finite but such that c commutes with G_δ. Also toward this in §(4A) we deal with a relative ${\rm NF}^3$ of ${\rm NF}_f$, in which we demand $\mathbf{C}_{G_1}(G_3)$ is large, this continues §2 concentrating on the case G_0 is with trivial center. In §(4B) we use this to define some schemes from $\Omega[\mathbf{K}]$, see e.g. 4.10.

Another problem is that given G_1 instead of extending G_1 to G_2 such that $q_t(G_1)$ is realized by $\bar c \in {}^{\omega >}(G_2)$ for some $t \in {\rm def}_{\mathfrak{S}}(G_1)$, we like to have an infinite $\bar c = (\ldots \hat{}\, \bar c_i \,\hat{}\, \ldots)_{i \in I}$, with ${\rm tp}(\bar c {\restriction} u, G_1, G_2) \in q_{t_u}(G_1)$ for every finite $u \subseteq I$; used in stage D of the proof of Theorem 5.1. This is done in §4(C).

§ 4(A). **Preserving Commutation.**

Claim 4.1. *The subgroups H'_1, H'_2 of G_3 commute* <u>*when*</u>*:*

$(*)$ (a) $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$;
- (b) $G_\ell = G_{\mathbf{x},\ell}, G'_\ell = \mathbf{j}_{\mathbf{x},\ell}(G_\ell)$ *for* $\ell = 0, 1, 2$;
- (c) $G_3 = G_{\mathbf{x}}$;
- (d) $H_1 \subseteq G_1$ *and* $H'_1 = \mathbf{j}_{\mathbf{x},1}(H_\ell)$;
- (e) $H_1 = \cup\{b(H_1 \cap G_0) : b \in \mathbf{I}_1\}$ *where* $\mathbf{I}_1 = \mathbf{I}_{\mathbf{x},1} \cap H_1$;
- (f) *if* $g \in \mathbf{I}_{\mathbf{x},1}$ *and*[17] $b \in \mathbf{I}_1$ *then* $gb \in \mathbf{I}_{\mathbf{x},1}$;
- (g) *the subgroups* G_0, H_1 *of* G_1 *commute;*
- (h) $H_2 \subseteq G_2$ *commutes with* $G_0 \cap H_1$ *and* $H'_2 = \mathbf{j}_{\mathbf{x},2}(H_2)$;
- (i) $H_2 = \cup\{a(G_0 \cap H_2) : a \in \mathbf{I}_2\}$ *where* $\mathbf{I}_2 = \mathbf{I}_{\mathbf{x},2} \cap H_2$.

Remark 4.2. 1) Really here it suffices to deal with the case $G_0 \cap H_1 = \{e\}$.
2) A natural case is $\mathbf{Z}(G_0) = \{e_{G_0}\}, H_1 = \mathbf{C}_{G_1}(G_0), H_2 = G_2$.
3) See the proof of 5.1.

Notation 4.3. Let $\mathbf{X}^3_{\rm lf} = \mathbf{X}^3_{\mathbf{K}_{\rm lf}}$ be the class of tuple $(\mathbf{x}, H_1, H_2)$ which satisfies $(*)$ of Claim 4.1.

Proof. Let $a \in H_2, b \in H_1, f_a = \mathbf{j}_{\mathbf{x},2}(a), f_b = \mathbf{j}_{\mathbf{x},1}(b)$, so by $(*)(d), (h)$ we just have to prove that $f_b f_a((g_0, g_1, g_2)) = f_a f_b((g_0, g_1, g_2))$ for any $(g_0, g_1, g_2) \in \mathscr{U}_{\mathbf{x}}$.

Clearly

- if $a \in G_0$ or $b \in G_0$ this holds.

[Why? First, if $a \in G_0$ then $f_a = \mathbf{j}_{\mathbf{x},2}(a) = \mathbf{j}_{\mathbf{x},0}(a) = \mathbf{j}_{\mathbf{x},1}(a) \in \mathbf{j}_{\mathbf{x},1}(G_1) = G'_1 \subseteq G_{\mathbf{x}}$ and as $b \in H_1 \subseteq G_{\mathbf{x}}$, by $(*)(g)$ we have $G_1 \models$ "a, b commute" hence $G_{\mathbf{x}} \models$ "$\mathbf{j}_{\mathbf{x},2}(a), \mathbf{j}_{\mathbf{x},1}(b)$ commute" and so $G_{\mathbf{x}} \models$ "f_a, f_b commute". Second, if $b \in G_0$ then $b \in G_0 \cap H_1 \subseteq G_0 \subseteq G_2$ and $a \in H_2 \subseteq G_2$, so by clause $(*)(h)$ clearly $G_2 \models$ "a, b commute" and we finish as above.]

[17] As G_1 is locally finite, necessarily $\mathbf{I}_1$ is a subgroup of H_1.

Moreover, as $H_1 = \langle (G_0 \cap H_1) \cup \mathbf{I}_1 \rangle_{G_1}$ by clause $(*)(e)$, recalling $\bullet$ above, without loss of generality

$\boxplus_1$ $b \in \mathbf{I}_1 \subseteq \mathbf{I}_{\mathbf{x},1}$.

Similarly as $H_2 = \langle (G_0 \cap H_2) \cup \mathbf{I}_1 \rangle$. By clause $(*)(i)$, recalling $\bullet$ above without loss of generality:

$\boxplus_2$ $a \in \mathbf{I}_2 \subseteq \mathbf{I}_{\mathbf{x},2}$.

Let[18] $f_x((g_0, g_1, g_2)) = (g_0^x, g_1^x, g_2^x)$ and $f_y f_x((g_0, g_1, g_2)) = (g_0^{x,y}, g_1^{x,y}, g_2^{x,y})$ for $x \in \{a, b\}$ and $y \in \{a, b\} \backslash \{x\}$.

We shall prove that $g_\ell^{a,b} = g_\ell^{b,a}$ for $\ell = 0, 1, 2$; this suffices.

Clearly,

- $\bullet_1$ $g_1^a = g_1$ and $g_2 g_0 a = g_2^a g_0^a$;
- $\bullet_2$ $g_2^{a,b} = g_2^a$ and $g_1^a g_0^a b = g_1^{a,b} g_0^{a,b}$;
- $\bullet_3$ $g_2^b = g_2$ and $g_1 g_0 b = g_1^b g_0^b$;
- $\bullet_4$ $g_1^{b,a} = g_1^b$ and $g_2^b g_0^b a = g_2^{b,a} g_0^{b,a}$.

Now

$\boxplus_3$ $g_1^{a,b} G_0 = g_1^{a,b} g_0^{a,b} G_0 = g_1^a g_0^a b G_0 = (g_1^a b)(g_0^a G_0) = (g_1^a b) G_0$.

[Why? As $g_0^{a,b} \in G_0$, by the second statement of $\bullet_2$, noting that b, g_0^a commute by $(*)(g)$, and as $g_0^a \in G_0$, respectively.]

But $g_1^a \in \mathbf{I}_{\mathbf{x},1}$ (as $(g_0^a, g_1^a, g_2^a) \in \mathscr{U}_{\mathbf{x}}$), and $b \in \mathbf{I}_1 \subseteq \mathbf{I}_{\mathbf{x},1}$ by $\boxplus_1$, hence by $(*)(f)$ we have $g_1^a b \in \mathbf{I}_{\mathbf{x},1}$ and also $g_1^{a,b} \in \mathbf{I}_{\mathbf{x},1}$ (as $(g_0^{a,b}, g_1^{a,b}, g_2^{a,b}) \in \mathscr{U}_{\mathbf{x}}$). Now by $\boxplus_3$, $g_1^{a,b} G_0 = (g_1^a b) G_0$ and by the last sentence $g_1^{a,b}, g_1^a \in \mathbf{I}_{\mathbf{x},1}$ and thus

$\bullet_5$ $g_1^{a,b} = g_1^a b$.

So by $\bullet_5$ and the second equation in $\bullet_2$ we have $g_1^a b g_0^{a,b} = g_1^{a,b} g_0^{ab} = g_1^a g_0^a b = g_1^a b g_0^a$, the last equality by recalling b, g_0^a commute by $(*)(g)$, hence we have:

$\bullet_6$ $g_0^{a,b} = g_0^a$.

Similarly to $\boxplus_3$ we have

$\boxplus_4$ $g_1^b G_0 = g_1^b g_0^b G_0 = g_1 g_0 b G_0 = (g_1 b)(g_0 G_0) = (g_1 b) G_0$.

[Why? As $g_0^b \in G_0$, by $\bullet_3$ second statement, as b, g_0 commute by $(*)(g)$, and as $g_0 \in G_0$ respectively.]

Also $g_1 \in \mathbf{I}_{\mathbf{x},1}$ as $(g_0, g_1, g_2) \in \mathscr{U}_{\mathbf{x}}$ and $b \in \mathbf{I}_1$ by $\boxplus_1$ so recalling $(*)(f)$ we deduce $g_1, g_1 b \in \mathbf{I}_{\mathbf{x},1}$ thus from $\boxplus_4$ we deduce:

$\bullet_7$ $g_1^b = g_1 b$.

Hence by $\bullet_7$ and $\bullet_3$ second statement we have $g_1 b g_0^b = g_1^b g_0^b = g_1 g_0 b = g_1 b g_0$, the last equation recalling b, g_0 commute (by $(*)(g)$), hence we have:

$\bullet_8$ $g_0^b = g_0$.

[18] Note that g_ℓ^x is <u>not</u> conjugation by x.

So by $\bullet_4, \bullet_7, \bullet_1, \bullet_6$, b commuting with G_0 and $\bullet_2$ second statement respectively, we have

$$\boxplus_5 \; g_1^{b,a} = g_1^b = (g_1 b) = (g_1^a b) = (g_1^a b)(g_0^a (g_0^{a,b})^{-1}) = (g_1^a g_0^a b)(g_0^{a,b})^{-1} = g_1^{a,b},$$

and thus

$$\bullet_9 \; g_1^{b,a} = g_1^{a,b}.$$

Also by $\bullet_4, \bullet_3, \bullet_8, \bullet_1, \bullet_6, \bullet_2$ we have

$$\boxplus_6 \; g_2^{b,a} g_0^{b,a} = g_2^b g_0^b a = g_2 g_0^b a = g_2 g_0 a = g_2^a g_0^a = g_2^a g_0^{a,b} = g_2^{a,b} g_0^{a,b}.$$

So

$$\bullet_{10} \; g_2^{b,a} g_0^{b,a} = g_2^{a,b} g_0^{a,b}$$

but $g_0^{b,a}, g_0^{a,b} \in G_0$ and $g_2^{b,a}, g_2^{a,b} \in \mathbf{I}_{\mathbf{x},2}$ hence recalling $(g_0^{a,b}, g_1^{a,b}, g_2^{a,b}), (g_0^{b,a}, g_1^{b,a}, g_2^{b,a}) \in \mathscr{U}_{\mathbf{x}}$ we have:

$$\bullet_{11} \; g_2^{b,a} = g_2^{a,b} \text{ and } g_0^{b,a} = g_0^{a,b}.$$

But $\bullet_{11} + \bullet_9$ imply that we are done. $\square_{4.1}$

The following claim is like Definition 2.5, but now we preserve a large $\mathbf{C}_{G_1}(G_0)$ using 4.1.

Definition 4.4. Let $\mathrm{NF}^3(\bar{G}, H_1, L, H_2)$ mean:

- (A)
  - (a) $\bar{G} = \langle G_\ell : \ell \leq 3\rangle$ are from $\mathbf{K}_{\mathrm{lf}}$;
  - (b) $G_0 \subseteq G_\ell$ for $\ell = 1, 2$;
  - (c) G_0 is finite;
  - (d) $H_1 \subseteq \mathbf{C}_{G_1}(G_0), L \subseteq H_1, L \cap G_0 = \{e_{G_0}\}, H_1 = \langle L, G_0 \cap H_1\rangle_{G_1}$;
  - (e) $G_1 \cap G_2 = G_0$;
  - (f) $H_2 \subseteq \mathbf{C}_{G_2}(H_1 \cap G_0)$;
- (B)
  - (a) $G_\ell \subseteq G_3$ for $\ell = 1, 2$;
  - (b) for $\sigma(\bar{x}, \bar{y})$ a group-term, $\bar{a} \in {}^{\ell g(\bar{x})}(G_1)$ and $\bar{b} \in {}^{\ell g(\bar{y})}(G_2)$ the following conditions are equivalent:
    - $G_3 \models$ “$\sigma(\bar{a}, \bar{b}) = e_{G_3}$”,
    - if $(\mathbf{x}, H_1, H_2) \in \mathbf{X}^3_{\mathrm{lf}}$, see 4.3, $G_\ell = G_{\mathbf{x},\ell}$ for $\ell = 0, 1, 2$ and $\bar{a}' = \mathbf{j}_{\mathbf{x},1}(\bar{a})$ and[19] $\bar{b}' = \mathbf{j}_{\mathbf{x},2}(\bar{b})$ <u>then</u> $G_{\mathbf{x}} \models$ “$\sigma(\bar{a}', \bar{b}') = e_{G_{\mathbf{x}}}$”.

Convention 4.5. In 4.4, if $H_1 = L$ we may in addition omit L. We may omit L, H_2 when $L = H_1, H_2 = \mathbf{C}_{G_2}(H_1 \cap G_0)$. Lastly, if $\mathbf{Z}(G_0) = \{e_{G_0}\}, L = \mathbf{C}_{G_1}(G_0)$ and $H_1 = L$ and $H_2 = G_2$, then we may omit H_1, L and H_2; see 4.6(3) below.

Claim 4.6. *Assume $\bar{G} = \langle G_\ell : \ell < 3\rangle, H_1, L, H_2$ are as in 4.4(A).*
1) We can find $\mathbf{x}$ such that $(\mathbf{x}, H_1, H_2) \in \mathbf{X}^3_{\mathrm{lf}}$.
2) There is $G_3 \in \mathbf{K}$ such that $\mathrm{NF}^3(\langle G_0, G_1, G_2, G_3\rangle, H_1, L, H_2)$ and G_3 is unique up to isomorphism over $G_1 \cup G_2$.
3) If $\bar{G}$ satisfies (A)(a),(b),(c) of Definition 4.4, $\mathbf{Z}(G_0) = \{e_{G_0}\}, H_1 = L = \mathbf{C}_{G_1}(G_0)$ and $H_2 = G_2$, <u>then</u> $(\bar{G}, H_1, L, H_2)$ satisfies 4.4(A).

[19]We may add $\mathbf{I}_1 = L$.

4) The relation $\mathrm{NF}^3(\bar{G}), \bar{G} = \langle G_\ell : \ell \le 3 \rangle$ *satisfies the parallel of 2.10 omitting symmetry, so having uniqueness, monotonicity and both sides definability, i.e.* $G_1 \le_{\Omega[\mathbf{K}]} G_3, G_2 \le_{\Omega[\mathbf{K}]} G_3$.

Proof. 1) It suffices to prove we can choose $\mathbf{I}^*_1, \mathbf{I}^*_2$ satisfying the demands on $\mathbf{I}_{\mathbf{x},1}, \mathbf{I}_{\mathbf{x},2}$ in 4.1.
Why can we do it? For $\mathbf{I}^*_2$ the demands are just clauses (b),(c) from 2.2(1) and $(*)(i)$ of 4.1 so just choose $\mathbf{I}_2 \subseteq H_2$ such that $e_{G_0} \in \mathbf{I}_2$ and $\langle g(G_0 \cap H_2) : g \in \mathbf{I}_2 \rangle$ is a partition of H_2 and then let $\mathbf{I}^*_2$ be such that $\mathbf{I}_2 \subseteq \mathbf{I}^*_2 \subseteq G_2$ and $\langle gG_0 : g \in \mathbf{I}^*_2 \rangle$ is a partition of G_2.

For $\mathbf{I}^*_1$ we have to take care of clauses (b),(c) from 2.2(1), of $(*)(e)$ (the parallel of $(*)(i)$) and of $(*)(f)$ from 4.1. For this let $H^+_1 := \langle G_0, H_1 \rangle_{G_1}$. First, choose $\mathbf{I}'_1 = L$ so clearly $e_{G_0} \in \mathbf{I}'_1$ and thus $\langle gG_0 : g \in \mathbf{I}'_1 \rangle$ is a partition of H^+_1. Why? Recalling that $L \subseteq H_1 \subseteq G_1, L \cap G_0 = \{e_{G_0}\}$ and $H_1 = \langle L, G_0 \cap H_1 \rangle_{G_1}$ and H_1 commute with G_0 in G_1; by clause (A)(d) we know that this is satisfied. Also let $\mathbf{J}_1 \subseteq G_1$ be such that $e_{G_0} = e_{G_1} \in \mathbf{J}_1$ and $\langle gH^+_1 : g \in \mathbf{J}_1 \rangle$ is a partition of G_1. Now let $\mathbf{I}^*_1 = \{gb : g \in \mathbf{J}_1 \text{ and } b \in \mathbf{I}'_1\}$.

Clearly $\langle gG_0 : g \in \mathbf{I}^*_1 \rangle = \langle g(bG_0) : b \in \mathbf{I}'_1, g \in \mathbf{J}_1 \rangle$ is a partition of G_1 (refining $\langle gH^+_1 : g \in \mathbf{J}_1 \rangle$), so clause 2.2(1)(b) holds. Furthermore, $\mathbf{I}^*_1 \cap H^+_2 = L = \mathbf{I}'_1$ so clause 4.1(e) holds.

Next as $e_{G_0} \in \mathbf{J}_1$ and $e_{G_0} \in \mathbf{I}'_1$ clearly $e_{G_0} \in \mathbf{I}^*_1$, so $\mathbf{I}^*_1$ satisfies clause 2.2(1)(c). Also if $g \in \mathbf{I}^*_1 \wedge b \in \mathbf{I}'_1$ then for some $g_1 \in \mathbf{J}_1, b_1 \in \mathbf{I}'_1$ we have $G_1 \models$ "$g = g_1 b_1$" hence $G_1 \models$ "$gb = (g_1 b_1)b = g_1(b_1 b)$" and recall $g_1 \in \mathbf{J}_1$ and $b_1 b \in \mathbf{I}'_1$ as $\mathbf{I}'_1 = L$ is closed under products. Thus together $gb \in \mathbf{I}^*_1$, hence clause 4.1(1)(f) is satisfied. So $\mathbf{I}^*_1, \mathbf{I}^*_2$ are as required in 2.2(1) and 4.1. Hence there is $\mathbf{x} \in \mathbf{X}_{\mathbf{K}}$ such that $G_{\mathbf{x},\ell} = G_\ell$ for $\ell = 0,1,2$ and $\mathbf{I}_{\mathbf{x},\ell} = \mathbf{I}^*_\ell$ for $\ell = 1,2$.
2) Consider clause (B) of 4.4, the "if $\mathbf{x} \in \ldots$" is not empty so G_3 is a well defined group. Easily $G_1 \subseteq G_3$ and $G_2 \subseteq G_3$ but is G_3 locally finite? This follows from the results in §2, in particular 2.10. That is, as there if G'_ℓ is finite, $G_0 \subseteq G'_\ell \subseteq G_\ell$ for $\ell = 1,2$ <u>then</u> we have finitely many possible choices of $(\mathbf{I}_{\mathbf{x},1} \cap x_1 G'_1, \mathbf{I}_{\mathbf{x},2} \cap x_2 G'_2)$ for $x_1 \in G_1, x_2 \in G_2$ hence the group G_3 that we get is locally finite. Probably better this is G'_3 such that $\mathrm{NF}_f(G_0, G_1, G_2, G'_3)$, by the definition there is a homomorphism from G'_3 onto G_3 over $G_1 \cup G_2$. Now as G'_3 is lf so is G_3.
3),4) Should be clear. $\square_{4.6}$

§ 4(B). Schemes and derived sets.

Definition 4.7. 1) Let $\mathbf{X}_0$ be the set of $\mathbf{x}$ such that:

- (a) $\mathbf{x}$ has the form $(K_1, K_2, \bar{a}_2, \bar{a}_1) = (K_1[\mathbf{x}], K_2[\mathbf{x}], \bar{a}_2[\mathbf{x}], \bar{a}_1[\mathbf{x}])$;
- (b) $K_1 \subseteq K_2$ are finite groups;
- (c) $\bar{a}_1$ is a finite sequence generating K_1;
- (d) $\bar{a}_2$ is a finite sequence from K_2 such that $\bar{a}_2{}^\frown\bar{a}_1$ generates K_2 (if $\bar{a}_2 = \langle a_2 \rangle$ we may write just a_2);
- (e) K_1 has trivial center.

2) Let $\mathbf{X}_1$ be the set of $\mathbf{x}$ such that:

(a) $\mathbf{x} = (K, \bar{a}) = (K[\mathbf{x}], \bar{a}[\mathbf{x}])$;
(b) $K \in \mathbf{K}$ is finite;
(c) $\bar{a}$ is a finite sequence from K generating $K, \ell g(\bar{a}) \geq 1$; let $a_* = a_*[\mathbf{x}] = a_0$, the first element of $\bar{a}$.

3) Let $\mathbf{X}_2$ be the set of $\mathbf{x} \in \mathbf{X}_1$ such that:

($*$) K has trivial center.

4) Let $\mathbf{X}_3$ be the set of $\mathbf{x} \in \mathbf{X}_1$ such that[20]:

($*$) if f is a non-trivial automorphism of K <u>then</u> for some conjugate b of $a_* = a_*[\mathbf{x}] = a_0[\mathbf{x}]$ we have $f(b) \notin \langle a_* \rangle_K$; equivalently, for some conjugate b of $a_*, \langle b \rangle_K \neq \langle a \rangle_K$.

Observation 4.8. If $m \in \{2, 3, \ldots\}$ then for some $\mathbf{x} \in \mathbf{X}_3$ the element $a_*[\mathbf{x}] \in K[\mathbf{x}]$ has order m.

Claim 4.9. *If $\mathbf{x} \in \mathbf{X}_0$, <u>then</u> there is one and only one $\mathfrak{s}$, call it $\mathfrak{s}_{\rm cm} = \mathfrak{s}_{\rm cm}[\mathbf{x}]$ such that:*

(a) *$\mathfrak{s} \in \Omega[\mathbf{K}_{\rm lf}]$;*
(b) *$k_{\mathfrak{s}} = \ell g(\bar{a}_1[\mathbf{x}])$ and $n_{\mathfrak{s}} = \ell g(\bar{a}_2[\mathbf{x}])$;*
(c) *$p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = {\rm tp}_{\rm bs}(\bar{a}_1[\mathbf{x}], \emptyset, K[\mathbf{x}])$;*
(d) *if $G_1 \subseteq G_3 \in \mathbf{K}_{\rm lf}$ and ${\rm tp}_{\rm bs}(\bar{a}, \emptyset, G_1) = {\rm tp}_{\rm bs}(\bar{a}_1[\mathbf{x}], \emptyset, K[\mathbf{x}])$ and $\bar{c}$ realizes $q_{\mathfrak{s}}(\bar{a}, G_1)$ in G_3 <u>then</u> ${\rm NF}^3(\langle \bar{a} \rangle_{G_1}, G_1, \langle \bar{a} \hat{\ } \bar{c} \rangle_{G_3}, G_3)$.*

Proof. As in §2 using §(4A). Let $K_\ell = K_\ell[\mathbf{x}]$ for $\ell = 1, 2$; and let $G_0 = K_1$ and $G_1 \in \mathbf{K}$ be existentially closed, extend K_1 and be such that $K_2 \cap G_1 = K_0$. Let $L = \mathbf{C}_{G_0}(G_1)$, so as $G_0 = K_1$ has trivial center (by 4.7(1)(e)), clearly we have $L \cap G_0 = \{e_{G_0}\}$ and let $H_1 = c\ell(G_0 \cup L, G_1), H_0 = \{e_{K_1}\}$ and let $H_2 = G_2 := K_2$. Now we apply Claim 4.6(2), so there is G_3 such that ${\rm NF}^3(G_0, G_1, G_2, G_3)$ see Definition 4.4. By it, the type ${\rm tp}_{\rm bs}(\bar{a}_2[\mathbf{x}], G_2, G_{\mathbf{x}})$ does not split over $G_0 = K_1$. From this it is easy to define $\mathfrak{s}$ and to prove it is as required. $\square_{4.9}$

Definition/Claim 4.10. For $\mathbf{x} \in \mathbf{X}_1$ let $\mathfrak{s} = \mathfrak{s}_{\rm ab}[\mathbf{x}]$ be such that:

(a) $\mathfrak{s} \in \Omega[\mathbf{K}_{\rm lf}]$;
(b) $k_{\mathfrak{s}} = 0$;
(c) if $\bar{c}$ realizes $q_2(<>, G_1)$ in G_2 so $G_1 \subseteq G_2$ <u>then</u> $\bar{c}$ realizes ${\rm tp}_{\rm bs}(\bar{a}[\mathbf{x}], \emptyset, K[\mathbf{x}])$ and commutes with G_1, and $\langle \bar{c} \rangle_{G_2} \cap G_1 = \{e\}$.

Proof. Easy. $\square_{4.11}$

Definition/Claim 4.11. For $\mathbf{x} \in \mathbf{X}_2$ we define $\mathfrak{s} = \mathfrak{s}_{\rm gm}[\mathbf{x}]$ such that:

(a) $\mathfrak{s} \in \Omega[\mathbf{K}_{\rm lf}]$;
(b) $k_{\mathfrak{s}} = 2\ell g(\bar{a}[\mathbf{x}])$ and $n_{\mathfrak{s}} = 1$;
(c) if $G_1 \subseteq G_2 \in \mathbf{K}_{\rm lf}$ and ${\rm tp}_{\rm bs}(\bar{a}_\ell, \emptyset, G_1) = {\rm tp}_{\rm bs}(\bar{a}[\mathbf{x}], \emptyset, K[\mathbf{x}])$ for $\ell = 1, 2$ and $\langle \bar{a}_1 \rangle_{G_1}, \langle \bar{a}_2 \rangle_{G_1}$ commute in G_1 and[21] have intersection $\{e_G\}$ then $p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}}) = {\rm tp}_{\rm bs}(\bar{a}_1 \hat{\ } \bar{a}_2, \emptyset, G_1)$;

[20] So $\mathbf{x}_3 \supseteq \mathbf{x}_2$.
[21] In fact this follows.

(d) moreover, in clause (c), if $c \in G_2$ realizes $q_{\mathfrak{s}}(\bar{a}_1 \hat{} \bar{a}_2, G_1)$ in G_2 <u>then</u> conjugation by c interchanges $\bar{a}_1, \bar{a}_2$ and is the identity on $\mathbf{C}_{G_1}(\bar{a}_1 \hat{} \bar{a}_2)$.

Proof. Let $G_2 \in \mathbf{K}_{\rm exlf}$ be an extension of $K[\mathbf{x}]$ in which some c realizes $q_{\mathfrak{s}_{\rm cg}}(K_{\mathbf{x}})$; let $\bar{a}_1 = \bar{a}[\mathbf{x}], \bar{a}_2 = c^{-1}\bar{a}_1 c := \langle c^{-1}a_{1,\ell}c : \ell < \ell g(\bar{a}_1)\rangle$ in G_2.

Note that, by inspection, $G_0 = \langle \bar{a}_1 \hat{} \bar{a}_2\rangle_{G_1}$ is finite with trivial center and let $G_0 \subseteq G_1 \in \mathbf{K}_{\rm lf}$. Now use 4.1 with $G_0, G_1, c\ell(\bar{a}_1 \hat{} \bar{a}_2 \hat{} \langle c\rangle, G_2), \mathbf{C}_{G_1}(G_0), \mathbf{C}_{G_1}(G_0), c\ell(\bar{a}_1 \hat{} \bar{a}_2 \hat{} \langle c\rangle, G_2)$ here standing for $G_0, G_1, G_2, G_1, H_1, L, H_2$ there. $\square_{4.10}$

Definition 4.12. 1) For $\mathfrak{s} \in \Omega[\mathbf{K}]$ and $G_1 \subseteq G_2$ let $\mathrm{cp}_{\mathfrak{s}}(G_1, G_2) = \{c_0 : \bar{c} \in {}^{n(\mathfrak{s})}(G_2)$ realizes $q_t(G_1)$ where $t \in \mathrm{def}(G_1)$ satisfies $\mathfrak{s}_t = \mathfrak{s}\}$.
2) For $\mathbf{x} \in \mathbf{X}_1$ and $G_1 \subseteq G_2$ let $\mathrm{cp}_{\mathbf{x}}(G_1, G_2) = \mathrm{cp}_{\mathfrak{s}_{\rm ab}[\mathbf{x}]}(G_1, G_2)$.
3) For $G_1 \subseteq G_2 \in \mathbf{K}_{\rm lf}$ and $\ell \in \{1, 2, 3\}$ let $\mathrm{cp}_\ell(G_1, G_2) = \cup\{\mathrm{cp}_{\mathfrak{s}_{ab}[\mathbf{x}]}(G_1, G_2) : \mathbf{x} \in \mathbf{X}_\ell\}$; if $\ell = 2$ we may omit it.

§ 4(C). Larger Definable Types.

Definition 4.13. 1) For $G \in \mathbf{K}, \mathfrak{S} \subseteq \Omega[\mathbf{K}]$ and set I let $\mathrm{Def}_{I,<\kappa}(G, \mathfrak{S})$ be the set of t such that:

(a) $t = \langle t_u : u \subseteq I \text{ finite}\rangle$;
(b) $t_u \in \mathrm{def}_{\mathfrak{S}}(G)$ with $\bar{x}_{t_u} = \langle x_i : i \in u\rangle$ and $\bar{a}_{t_u} = \bar{a}_t$ or pedantically $\bar{a}_{t_u} = \bar{a}_t{\restriction}w_u$ where $w_u \subseteq \ell g(\bar{a}_t)$ is finite;
(c) $\ell g(\bar{a}_t) := I$ has cardinality $< \kappa$ and $\mathrm{Rang}(\bar{a}_t) \subseteq G$;
(d) if $G \subseteq H \subseteq L \in \mathbf{K}_{\rm lf}$ and $u \subseteq v \subseteq I$ are finite and $\bar{b} \in {}^v L$ realizes $q_{t_v}(H)$ then $\bar{b}{\restriction}u$ realizes $q_{t_u}(H)$.

2) We define $\Omega_{I,<\kappa}[\mathbf{K}, \mathfrak{S}]$ parallely and if $\mathfrak{S} = \Omega[\mathbf{K}]$ <u>then</u> we may omit it.
3) If $t \in \mathrm{Def}_{I,<\kappa}(G, \mathfrak{S})$ then $q_t(G) \in \mathbf{S}^I_{\rm bs}(G)$ is defined by $\cup\{q_{t_u}(\langle x_i : i \in u\rangle) : u \subseteq I$ finite$\}$.
4) Omitting κ means $\aleph_0$. We may replace "$< \kappa^+$" by κ and even a set I_1. We may replace I by "$< \mu$" meaning "some $\chi < \mu$". Similarly for "$\le \mu$".
5) For $n < \omega$ and $\mathfrak{s}_0, \dots, \mathfrak{s}_{n-1} \in \Omega_{<\mu,<\kappa}[\mathbf{K}]$ we define $\mathfrak{s}_0 \oplus \dots \oplus \mathfrak{s}_{n-1}$ and $\mathfrak{s}_0 \otimes \dots \otimes \mathfrak{s}_{n-1}$ naturally.

Claim 4.14. *1) If $G \in \mathbf{K}, \mathfrak{S} \subseteq \Omega[\mathbf{K}]$ and $t \in \mathrm{Def}_I(G, \mathfrak{S})$ <u>then</u> for some pair $(\bar{c}, H)$ we have $G \subseteq H \in \mathbf{K}_{\rm lf}, \bar{c} \in {}^I H, H = \langle G \cup \bar{c}\rangle_H$ and $\mathrm{tp}_{\rm bs}(\bar{c}, G, H) = q_t(G)$.*
2) If $\mathfrak{S}$ is closed <u>then</u> above $G \le_{\mathfrak{S}} H$.

Definition 4.15. Assume $\bar{H} = \langle H_i : i < \delta\rangle$ is $\subseteq$-increasing in $\mathbf{K}$ and $H_\delta = \cup\{H_i : i < \delta\}$. We say $\mathscr{A}$ is a one step $(< \mu, < \kappa, \delta, \mathfrak{S}) - \otimes$-construction (if $\delta = \omega$ we may omit it) <u>when</u>: as in 3.16 except that

(c)′ $t_{\alpha,i} \in \mathrm{Def}_{I_{\alpha,i},<\kappa}(H_i, \mathfrak{S})$ for some set $I_{\alpha,i}$ of cardinality $< \mu$.

The case we shall actually use in §5 is:

Claim 4.16. *Assume $K \subseteq L \in \mathbf{K}_{\rm lf}, K$ is finite and f embeds K into $G_1 \in \mathbf{K}_{\rm lf}$ and $\langle c_i : i < \mu\rangle$ list the members of L and $\{c_\ell : \ell < n\}$ is the set of elements of K. <u>Then</u> there is $t \in \mathrm{Def}_{\le\mu}(G_1, \mathfrak{S}[\mathbf{K}])$ such that: if $\bar{c}^* = \langle c^*_i : i < \mu\rangle \in {}^\mu(G_2)$ realizes $q_t(G_1)$ in G_2, so $G_1 \subseteq G_2$, <u>then</u> $c_i \mapsto c^*_i$ (for $i < \mu$) is an embedding of L into G_2 extending f.*

Proof. Straightforward by §2. $\square_{4.16}$

Discussion 4.17. Those definable types are still locally definable over finite sets.

§ 5. Constructing complete existentially closed G

Theorem 5.1. *Assume if $G \in \mathbf{K}_{\rm lf}$ and $|G| \le \mu = \mu^{\aleph_0}$.*
1) There is a complete $G' \in \mathbf{K}_{\rm lf}$ which extend G such that $|G'| = \mu^+$ and G' is existentially closed.
2) Moreover $G \le_{\Omega[\mathbf{K}_{\rm lf}]} G'$ and G' is full.
3) There is G' such that $G \le_{\mathfrak{S}} G'$ and $G' \in \mathbf{K}^{\rm exlf}_{\mu^+}$ is complete and $\mathfrak{S}$-full provided that $\mathfrak{S}$ satisfies:

- $(*)$ (α) $\mathfrak{S} \subseteq \Omega[\mathbf{K}_{\rm lf}]$
  - (β) *$\mathfrak{S}$ is dense and $\otimes$-closed (for $\mathbf{K}_{\rm lf}$)*
  - (γ) *some schemes introduced earlier belongs to $\mathfrak{S}$, specifically:*
    - *$\mathfrak{s}_{\rm ab(2)}$ from Definition 2.20, used in the paragraph before $\boxplus_3$*
    - *$\mathfrak{s}_{\rm cm}$ from Definition 4.9, used in $(*)_{4.3}$*
    - *$\mathfrak{s}_{\rm cg}$, from Definition 2.17(1), 2.18(2) used after $\boxplus_7$ Stage E*
    - *$\mathfrak{s}_{\rm gl}$ from Definition 2.17(2),2.18(3)*
    - *$\mathfrak{s}_{\rm gm}$ from Definition 4.10, see $(*)_{5.1}(f)$.*

Proof. Proof of 5.1
We let $\mathfrak{S} = \Omega[\mathbf{K}_{\rm lf}]$ for parts (1),(2) and fix $\mathfrak{S}$ for part (3) as there.

Stage A: Without loss of generality the universe of G is an ordinal $\le \mu$ and let $\lambda = \mu^+$.

Let $S \subseteq S^\lambda_{\aleph_0} := \{\delta < \lambda : {\rm cf}(\delta) = \aleph_0\}$ be a stationary subset of λ such that also $S^\lambda_{\aleph_0} \backslash S$ is stationary in λ and $\alpha \in S \Rightarrow (\mu$ divides $\alpha)$. Let $\langle S_\zeta : \zeta < \lambda\rangle$ be a partition of S to stationary sets. Let $S_* \subseteq \lambda\backslash S$ be stationary and a set of limit ordinals.

Let C_δ be an unbounded subset of δ of order type ω for $\delta \in S$ such that $\bar{C}_\zeta = \langle C_\delta : \delta \in S_\zeta\rangle$ guess clubs for each $\zeta < \lambda$, this means that for every club E of λ the set $\{\delta \in S_\zeta : C_\delta \subseteq E\}$ is a stationary subset of λ; such $\langle C_\delta : \delta \in S_\zeta\rangle$ exists by [**?**, Ch.III] = [**?**].

Let $\alpha_\delta(n)$ be the n-th member of C_δ.

Let $\bar{\tau}$ be such that:

- $\bar{\tau} = \langle \tau_\zeta : \zeta < \lambda\rangle$
- $\tau_\zeta \subseteq \mathscr{H}(\aleph_0)$ is a countable vocabulary
- if $\tau \subseteq \mathscr{H}(\aleph_0)$ is a countable vocabulary then $\{\zeta : \tau_\zeta = \tau\}$ has cardinality λ.

By [**?**, 3.26(3)=L6.11A,pg.31] there is $\mathbf{b}_\zeta$, a BB, black box for $(S_\zeta, \bar{C}_\zeta)$ say $\mathbf{b}_\zeta = \langle N^\delta_i : i \in \mathscr{T}_\delta, \delta \in S_\zeta\rangle$, that is:

$\boxplus_{0,\zeta}$ (a) N^δ_i is a model of cardinality $\aleph_0$ with universe $\subseteq \delta = \sup(N^\delta_i)$ and vocabulary $\tau_\zeta \subseteq \mathscr{H}(\aleph_0)$

(b) if N is a τ_ζ-model with universe λ then for stationarily many $\delta \in E_N \cap S_\zeta$ for some $i \in \mathscr{T}_\delta$ we have $C_\delta \subseteq E_N\backslash S$ where $E_N := \{\alpha : N{\restriction}\alpha \prec N\}$ and $N^\delta_i \prec N$; moreover

(b)$^+$ if $\tau = \tau_\zeta, \bar{N} = \langle N_\eta : \eta \in \mathscr{T}\rangle, \mathscr{T}$ a non-empty subtree of ${}^{\omega>}\lambda$ such that $\tau(N_\eta) = \tau_\zeta, \eta \triangleleft \nu \Rightarrow N_\eta \prec N_\nu$ and $|N_\eta| \in [\lambda]^{\aleph_0}$ and E a club of $\lambda, \eta \in \mathscr{T} \Rightarrow (\exists^\lambda \alpha)(\eta\hat{}\langle\alpha\rangle \in \mathscr{T})$ and $\eta \triangleleft \nu \in \mathscr{T} \Rightarrow \sup(N_\eta) < \sup(N_\nu)$

then for some $\delta \in S_\zeta \cap E$ we have $C_\delta \subseteq E, i \in \mathscr{T}_\delta$ and $\eta \in \lim_\omega(\mathscr{T})$ we have $N^\delta_i = \cup\{N_{\eta\restriction n} : n < \omega\}$;

(c) if $i \neq j \in \mathscr{T}_\delta$ then $N^\delta_i \cap N^\delta_j$ is bounded in δ (used just after $(*)_{5.5}$), moreover:

(c)$^+$ if $i \neq j \in \mathscr{T}_\delta$ then the set $\{\beta < \delta : \beta$ a limit ordinal such that $\sup(N^\delta_i \cap \beta) = \beta = \sup(N^\delta_j \cap \beta)\}$ is bounded in δ;

(d) $N^\delta_i \cap (\alpha_\delta(n), \alpha_\delta(n+1)) \neq \emptyset$ and $N^\delta_i \restriction \alpha_\delta(n) \prec N^\delta_i$ for $n < \omega, \delta \in S, i \in \mathscr{T}_\delta$;

(e) for notational simplicity we assume $\mathscr{T}_\delta \subseteq \mu$.

Stage B: By induction on $\gamma < \lambda$ we shall choose the following:

$\boxplus_1$ (a) $G_\gamma \in \mathbf{K}_{\rm lf}$ of cardinality μ and the universe of G_γ is an ordinal $< \lambda$;

(b) $G_0 = G$;

(c) $\langle G_\beta : \beta \le \gamma\rangle$ is increasing continuous;

(d) if $\beta \in \gamma\backslash S$ then $G_\beta \le_{\mathfrak{S}} G_\gamma$

(e) if $\gamma = \beta+1, \beta \notin S$, then:

(α) G_γ is generated by $\{\bar c_{\beta,i} : i \in \mathscr{T}_\beta\} \cup G_\beta$, where $\mathscr{T}_\beta$ is a set of cardinality $\le \mu$ (to be chosen),

(β) $t_{\beta,i} \in {\rm Def}_{\le\mu}(G_\beta, \mathfrak{S})$, non-trivial (see Definition 4.13(5)) for $i \in \mathscr{T}_\beta$,

(γ) ${\rm tp}_{\rm bs}(\bar c_{\beta,i}, G_\beta, G_\gamma) = q_{t_{\beta,i}}(G_\beta)$ for $i \in \mathscr{T}_\beta$,

(δ) if $n < \omega$ and $i(0), \ldots, i(n-1) \in \mathscr{T}_\beta$ are pairwise distinct, then ${\rm tp}_{\rm bs}(\bar c_{\beta,i(0)}{}^\frown \ldots {}^\frown \bar c_{\beta,i(n-1)}, G_\beta, G_\gamma) = q_t(G_\beta)$, where $t = t_{\beta,i(0)} \otimes \ldots \otimes t_{\beta,i(n-1)}$,

(ε) if $t = (\mathfrak{s}, \bar a) \in {\rm def}_{\mathfrak{S}}(G_\beta)$ is non-trivial then for some $i \in \mathscr{T}_\beta$ we have $t_{\beta,i} = t$;

(f) if $\gamma = \delta+1, \delta \in S$ then:

(α) G_γ is generated by $\{\bar c_{\delta,i} : i \in \mathscr{T}_\delta\} \cup G_\delta$,

(β) $\mathscr{A}_\gamma = (G_{\delta+1}, G_\delta, \langle \bar c_\delta, t_{\delta,i,n} : i \in \mathscr{T}_\delta\rangle)$ is a one step $(< \aleph_0, < \aleph_0, \mathfrak{S}) - \otimes$-construction over $\langle G_{\alpha_\delta(n)} : n < \omega\rangle$, see 3.16; used in $(*)_{5.2}$'s proof[22],

(g) $t_{\beta,i} = (\mathfrak{s}_{\beta,i}, \bar a_{\beta,i})$ for $\beta \in \gamma\backslash S$.

First we shall show:

$\boxplus_2$ we can carry the induction.

Why? For $\gamma = 0$ we have nothing to do by clause (b).

For γ limit we let $G_\gamma = \cup\{G_\beta : \beta < \gamma\}$.

For $\gamma = \beta+1, \beta \notin S$ we have some freedom, as we have $t_{\beta,i} \in {\rm Def}_{\le\mu}(G_\beta, \mathfrak{S})$ not just ${\rm def}(G_\beta, \mathfrak{S})$. So let $\mathscr{T}_\beta = \mu, \{t_{\beta,i} : i \in \mathscr{T}_\delta\} \subseteq {\rm Def}_{\le\mu}(G_\beta, \mathfrak{S})$ be of cardinality μ and including ${\rm def}(G_\beta, \mathfrak{S})$ and so $\langle t_{\beta,i} = (\mathfrak{s}_{\beta,i}, \bar a_{\beta,i}) : i < \mu\rangle$, possibly with repetitions. Clearly $\boxplus_1(e)(\varepsilon)$ holds.

Now as in Claim 3.13 we can find $G_\gamma, \langle \bar c_{\beta,i} : i < \mu\rangle$ such that:

[22] Actually can use a one step $(\le \mu, < \aleph_0, \mathfrak{S}) - \otimes$-construction.

- $G_\beta \leq_{\mathfrak{S}} G_\gamma$;
- $G_\gamma = \langle \{\bar c_{\beta,i} : i < \mu\} \cup G_\beta \rangle_{G_\gamma}$;
- if $n < \omega$ and $i_k < \mu$ for $k < n$ and $\langle i_k : \ell < n\rangle$ is with no repetitions <u>then</u> $\bar c_{\beta,i_0} \hat{} \ldots \hat{} \bar c_{\beta,i_{n+1}}$ realizes $q_t(G_\beta)$ where $t = t_{\beta,i_0} \otimes \ldots \otimes t_{\beta,i_{n-1}}$.

If $\gamma = \delta+1, \delta \in S$ we can let $\mathfrak{s}_{\delta,i} = \mathfrak{s}_{\mathrm{ab}(2)}$, clearly we satisfy clause (f); but we may act differently. Clearly, as in the previous case, there is some freedom left: what we do for $\gamma = \delta+1, \delta \in S$ and this will depend on the $\langle N^\delta_i : i \in \mathscr{T}_\delta\rangle$ from $\boxplus_0$. During the rest of the proof we shall use (some of the freedom left) to guarantee that G_* (see below) is as required.

Of course, we let:

$\boxplus_3$ $G_* = G_\lambda = \cup\{G_\alpha : \alpha < \lambda\}$.

We now point out some useful properties of the construction:

$(*)_{3.1}$ there is a model N_* expanding G_*, so with universe λ, and a countable vocabulary such that for any $N \subseteq N_*$ we have:
- (a) $G_* {\restriction} N$ is a subgroup of G_*;
- (b) $\beta \in N$ <u>iff</u> $N \cap G_{\beta+1} \backslash G_\beta \neq \emptyset$ <u>iff</u> $\beta+1 \in N$;
- (c) if $\gamma = \beta+1, \gamma \in N$ then $N \cap G_\gamma = \langle \cup\{\bar c_{\beta,i} : i \in N \cap \mathscr{T}_\beta\} \cup (N \cap G_\beta)\rangle_{G_\gamma}$;
- (d) if $i \in N \cap \mathscr{T}_\beta$ and $\beta \in N$, <u>then</u> $|\ell g(\bar c_{\beta,i})| \leq \omega \Rightarrow \bar c_{\beta,i} \subseteq N \cap G_{\beta+1}$ and $|\ell g(\bar a_{t_{\beta,i}})| \leq \omega \Rightarrow \bar a_{t_{\beta,i}} \subseteq N \cap G_\beta$;
- (e) $\tau(N_*) \subseteq \mathscr{H}(\aleph_0)$, but $\mathscr{H}(\aleph_0) \backslash \tau(G_*)$ is infinite;
- (f) if $\delta \in N \cap S$ then $C_\delta \subseteq N$.

Now note

$(*)_{3.2}$ if $\alpha < \lambda$ is a limit ordinal, <u>then</u> $G_\alpha \in \mathbf{K}_{\mathrm{exlf}}$.

[Why? Recall clause $(e)(\varepsilon)$ of $\boxplus_1$ noting that S is a set of limit ordinals, hence $\alpha = \sup(\alpha \backslash S)$.]

We now assume:

$\boxplus_4$ $\mathbf{h}$ is an automorphism of G_*.

We shall eventually prove that (if we suitably use the freedom left in $\boxplus_1$, then) $\mathbf{h}$ is an inner automorphism, i.e. $b \in G_* \Rightarrow \mathbf{h}(b) = a^{-1}ba$ for some $a \in G_*$, this clearly suffices noting that G_* has trivial center as $\mathfrak{s}_{\mathrm{cg}} \in \mathfrak{S}$.

We shall often use

$(*)_{4.1}$ for limit $\beta \in \lambda \backslash S$ let $L^*_\beta = \mathrm{cp}(G_\beta, G_{\beta+\omega})$ (see Definition 4.12(3)), i.e. $c \in L^*_\beta$ <u>iff</u> for some finite $K \subseteq \mathbf{C}_{G_{\beta+\omega}}(G_\beta)$ with trivial center we have $c \in K$ and $K \cap G_\beta = \{e_{G_\beta}\}$.

Note that

$(*)_{4.2}$ the last demand in $(*)_{4.1}$, "$K \cap G_\beta = \{e_{G_*}\}$", is redundant.

[Why? Recall β is a limit ordinal hence by $(*)_{3.2}$, G_β has trivial center.]

Note:

$(*)_{4.3}$ if $a \in L^*_\beta$ and K witnesses it, then $K \subseteq L^*_\beta, K \cap G_\beta = \{e\}$, and moreover there is $L \in \mathbf{K}_{\rm exlf}$ included in L^*_β and including K.

[Why? We can choose $\bar K = \langle K_n : n < \omega\rangle$ such that $K_0 = K, K_n$ is a finite group with trivial center, $K_n \subseteq K_{n+1}$ and $\bigcup\limits_n K_n \in \mathbf{K}_{\rm exlf}$. We now choose by induction on n an embedding f_n of K_n into $G_{\beta+\omega}$ such that $f_0 = {\rm id}_K, f_n \subseteq f_{n+1}$ and ${\rm Rang}(f_n) \subseteq L^*_\beta$; the induction step is possible by 4.9. Now $\bigcup\limits_n f_n(K_n)$ is as required.]

We shall use:

$(*)_{4.4}$ let $E_{\mathbf h} = \{\delta : \delta$ is a limit ordinal and $\mathbf h$ maps G_δ onto G_δ and $(N^*{\restriction}\delta, \mathbf h{\restriction}\delta) \prec (N^*, \mathbf h)\}$.

Now

$(*)_{4.5}$ $E_{\mathbf h}$ is a club of λ.

[Why? Just look at $(*)_{4.4}$.]

Stage C: We shall prove

$\boxplus_5$ for some $\alpha(*) < \lambda$, for every $\beta \in S_* \cap E_{\mathbf h} \backslash \alpha(*)$ and $c \in L^*_\beta$ we have $\mathbf h(c) \in c\ell(G_{\alpha(*)} \cup \{c\}, G_*)$.

Why? If not, for every $\alpha < \lambda$ there are $\beta_\alpha \in S_* \cap E_{\mathbf h}\backslash\alpha, m(\alpha) = m_\alpha \in \{2,3,\ldots\}$ and $c_\alpha \in L^*_{\beta_\alpha}$ of order m_α such that $\mathbf h(c_\alpha) \notin c\ell(G_\alpha \cup \{c_\alpha\}, G_*)$. Now let $\bar c_\alpha$ witness that $c_\alpha \in L^*_{\beta_\alpha}$ with $c_{\alpha,0} = c_\alpha$, i.e. $\bar c_\alpha$ list the members of a finite subgroup of $G_{\beta_\alpha+\omega}$ commuting with G_{β_α} with trivial center and so included in $L^*_{\beta_\alpha}$.

$(*)_{5.0}$ without loss of generality $\mathbf h(c_\alpha) \in c\ell(G_\alpha \cup \bar c_\alpha)$.

[Why? Let K_0 be the subgroup of $G_{\beta_\alpha+\omega}$ with universe $\bar c_\alpha$; we can find K_1, K_2, K_3 such that K_3 is a finite group and for $\ell = 0,1,2$ and $K_1 \cap K_2 = \langle c_0\rangle_{K_0}$ without loss of generality $K_3 \subseteq G_{\beta_\alpha+\omega}$, so we can replace K_0 by K_1 or by K_2.]

Let $\mathbf x_\alpha \in X_2$ be such that $\bar c_\alpha$ realizes $q_{\mathfrak s_{\rm ab}[\mathbf x_\alpha]}(\langle\rangle, G_{\beta_\alpha})$, see 4.7(2) + 4.11. But if $\alpha_1 < \alpha_2$ then $(\beta_{\alpha_2}, c_{\alpha_2}, m_{\alpha_2})$ can serve as $(\beta_{\alpha_1}, c_{\alpha_1}, m_{\alpha_1})$, hence, without loss of generality, $\mathbf x_\alpha = \mathbf x, m_\alpha = m_*$ for every α.

$(*)_{5.1}$
- (a) Let $\bar b_{\alpha,1} = \bar c_\alpha$; let $k_{\alpha,1} < \omega$ be such that $\bar b_{\alpha,1} \subseteq G_{\beta_\alpha+k_{\alpha,1}+1}, \bar b_{\alpha,1} \nsubseteq G_{\beta_\alpha+k_{\alpha,1}}$;
- (b) let $k_{\alpha,*} \in (k_{\alpha,1}+1, \omega)$ be such that: ${\rm tp}_{\rm bs}(\mathbf h(\bar b_{\alpha,1}), G_{\beta_\alpha+\omega}, G_*) = q_{\mathfrak s_\alpha}(\bar a^\bullet_\alpha, G_{\beta_\alpha+\omega})$ for some $\mathfrak s_\alpha \in \mathfrak S$ with $\bar a^\bullet_\alpha \subseteq G_{\beta_\alpha+k_{\alpha,*}}$;
- (c) let $\bar b_{\alpha,2} \subseteq G_{\beta_\alpha+\omega}$ realize $q_{\mathfrak s_{\rm ab}[\mathbf x]}(\langle\rangle, G_{\beta_\alpha+k_{\alpha,*}})$;
- (d) let $k_{\alpha,2} < \omega$ be such that $\bar b_{\alpha,2} \subseteq G_{\beta_\alpha+k_{\alpha,2}+1}, \bar b_{\alpha,2} \nsubseteq G_{\beta_\alpha+k_{\alpha,2}}$, so actually without loss of generality $k_{\alpha,2} = k_{\alpha,*}+1$;
- (e) note that $\bar b_{\alpha,1}{}^\frown \bar b_{\alpha,2}$ realizes $p_{\mathfrak s_{\rm gm}}(\bar x)$, see 4.11;
- (f) let $k_{\alpha,3} < \omega$ be $> k_{\alpha,1}, k_{\alpha,2}$ and let $b_{\alpha,3} \in G_{\beta_\alpha+k_{\alpha,3}+1}$ realizes $q_{\mathfrak s_{\rm gm}[\mathbf x]}(\bar b_{\alpha,1}{}^\frown \bar b_{\alpha,2}, G_{\beta_\alpha+k_{\alpha,3}}, G_{\beta_\alpha})$, (see Definition 4.11); so it commutes with $\mathbf C_{G_{\beta_\alpha+k_{\alpha,2}+1}}(\bar b_{\alpha,1}{}^\frown \bar b_{\alpha,2})$, hence with G_{β_α} and conjugating by it interchange $\bar b_{\alpha,1}, \bar b_{\alpha,2}$;

(g) without loss of generality $\mathfrak{s}_\alpha = \mathfrak{s}_*$ and $(\ell g(\bar{b}_{\alpha,1}), k_{\alpha,2}, \ell g(\bar{b}_{\alpha,2}))$ does not depend on α.

Our intention (in this stage) is to find $\alpha_n < \lambda$ increasing with n satisfying $\beta_{\alpha_n} < \alpha_{n+1}$ and element d such that, on the one hand, conjugating with d maps $c_{\alpha_n} = b_{\alpha_n,1,0}$ to $b_{\alpha_n,2,0}$ for each n, and on the other hand, $\mathrm{tp}_{\mathrm{bs}}(d, G_{\beta_n+\omega}, G_\lambda)$ does not split over $G_{\beta_n} + b_{\alpha_n,3}$, a contradiction.

Let N be such that:

$(*)_{5.2}$ (a) N is a model with universe λ;

(b) N is with countable vocabulary;

(c) N expands N_* from $(*)_{3.1}$;

(d)
- $F_0^N = \mathbf{h}$, so F_0 is a unary function symbol,
- $F_{1,\iota,\ell}^N(\alpha) = b_{\alpha,\iota,\ell}$ for $\iota = 1,2$ and $\ell < \ell g(\bar{b}_{\alpha,\iota})$, (if $\ell = 0$ we may omit it),
- $F_{1,3}^N(\alpha) = b_{\alpha,3}$,
- $F_2^N(\alpha) = \beta_\alpha$,
- $F_{2,\iota}(\alpha) = \beta_\alpha + k_{\alpha,\iota}$ for $\iota = 1,2,3$,
- $F_3^N(\alpha) = \beta_\alpha + \omega$,

(e) $F_{4,n}^N$ is an $(n+1)$-place function such that: if $\alpha_0 < \ldots < \alpha_n, c_{\alpha_\ell} \in G_{\alpha_{\ell+1}}$, each α_ℓ is a limit ordinal then $F_{4,n}^N(\alpha_0, \ldots, \alpha_n)$ is the product of $a_0 a_1 \ldots a_n$ where $a_k = F_{1,1,0}(\alpha_k)$;

(f) $P^N = \{(\alpha, c) : \alpha < \lambda \text{ and } c \in G_\alpha\}$.

Without loss of generality $\tau_N \subseteq \mathscr{H}(\aleph_0)$, choose $\zeta(1) < \lambda$ such that $\tau_{\zeta(1)} = \tau_N$ and for each $\delta \in S_{\zeta(1)}$ we use the amount of freedom we are left with (see before $\boxplus_3$), choosing $G_{\delta+1}$ such that:

$(*)_{5.3}$ if $\delta \in S_{\zeta(1)}, i \in \mathscr{T}_\delta$, letting $\alpha_{\delta,i,n} := \min(N_i^\delta \backslash \alpha_\delta(n))$ <u>then</u> $(a) \Rightarrow (b)$ where:

(a)
- $\beta_{\delta,i,n} := F_2^{N_i^\delta}(\alpha_{\delta,i,n})$ is $\geq \alpha_{\delta,i,n}$ but $< \alpha_\delta(n+1)$,
- $F_3^{N_i^\delta}(\alpha_{\delta,i,n}) = \beta_{\delta,i,n} + \omega$,
- $b_{\delta,i,n,\iota} = F_{1,\iota,\ell}^N(\alpha_{\delta,i,n})$ for $\iota = 1,2$ and $\ell = 0$,
- $k_{\delta,i,n,\iota} = F_{2,\iota}^{N_i^\delta}(\alpha_{\delta,i,n}) - \alpha_{\delta,i,n}$ for $\iota = 1,2,3$,
- $b_{\delta,i,n,3} = F_3^{N_i^\delta}(\beta_{\delta,i,n})$,
- $b_{\delta,i,n,\iota} \in G_{\beta_{\delta,i,n}+k_{\delta,i,n,\iota}+1}$ commute with $G_{\beta_{\delta,i,n}}$ and conjugating by $b_{\delta,i,n,3}$ interchange $b_{\delta,i,n,1,\ell}, b_{\delta,i,n,2,\ell}$,
- δ is the set of elements of G_δ, similarly $\alpha_{\delta,i,n}$ (as they $\in E_{\mathbf{h}}$),
- for every $\beta < \delta$ we have $(G_{\beta+1} \backslash G_\beta) \cap N_i^\delta \neq \emptyset \Leftrightarrow \beta \in N_{\delta,i}$,
- if $\beta \in N_i^\delta \backslash S$ and $\bar{c} \in {}^{\omega>}(N_i^\delta)$, so $\bar{c} \in {}^{\omega>}(G_\delta)$, then $\mathrm{tp}_{\mathrm{bs}}(\bar{c}, G_\beta, G_\delta) \in q_t(G_\beta)$ for some $t \in \mathrm{def}(G_\beta)$ satisfying $\bar{a}_t \in {}^{\omega>}(N_i^\delta \cap G_\beta)$,

(b) $\bar{c}_{\delta,i} = \langle c_{\delta,i} \rangle$ and $\mathrm{tp}_{\mathrm{bs}}(c_{\delta,i}, G_\delta, G_{\delta+1})$ is as in claim 2.19 with $G_{\alpha_\delta(n)}(n < \omega), G_\delta, b_{\beta_{\delta,i,n,3}} \in N_i^\delta(n < \omega)$ here standing for $G_n(n < \omega), G_\omega, a_n^t(n < \omega)$ with $I = \{t\}$ there;

$(*)_{5.4}$ let $\mathscr{T}'_\delta = \{i \in \mathscr{T}_\delta$: clause (a) of $(*)_{5.3}$ holds$\}$;

$(*)_{5.5}$ let $\beta_{\delta,i,n} = \alpha_{\delta,i,n} + \omega$ and $b_{\delta,i,n} = b_{\delta,i,n,\iota} \in G_{\beta_{\delta,i,n}+1}$ for $\iota = 1,2,3$ realizes $q_{\mathfrak{s}_{\rm ab(2)}}(\langle\rangle, G_{\beta_{\delta,i,n}})$ <u>when</u> the assumption of clause (a) fails.

Why can we fulfill $(*)_{5.3}$? Let $\langle i_\ell : \ell < \ell(*)\rangle$ be a finite sequence of members of $\mathscr{T}_\delta$. For $\ell < \ell(*)$ and $n < \omega$ let $d_{\ell,n} = b_{\delta,i_\ell,n,3}$.

Now

$(*)_{5.6}$ $\langle d_{\ell,n} : n < \omega\rangle$ pairwise commute if $i(\ell) \in \mathscr{T}'_\delta$ for each $\ell < \ell(*)$.

[Why? As $b_{\delta,i(\ell),n,3} \in \mathbf{C}(G_{\beta_{\delta,i(\ell),n}}, G_{\beta_{\delta,i(\ell),n}+\omega})$ for $n < \omega$ and $\beta_{\delta,i(\ell),n} + \omega < \alpha_\delta(n+1) \le \alpha_{\delta,i(\ell),n+1} \le \beta_{\delta,i(\ell),n+1}$, recalling $N^i_\delta{\restriction}\alpha_\delta(n+1) \prec N^i_\delta$ and $N^i_\delta \cap (\alpha_\delta(n), \alpha_\delta(n+1)) \ne \emptyset$.]

$(*)_{5.7}$ $\langle d_{\ell,n} : n < \omega\rangle$ pairwise commute when $i(\ell) \notin \mathscr{T}'_\delta$.

[Why? Even easier.]

$(*)_{5.8}$ if $\ell(1) \ne \ell(2)$ <u>then</u> for every $n(1) < n(2)$ the elements $b_{\delta,i(\ell(1)),n(1),3}, b_{\delta,i(\ell(2)),n(2),3}$ commute.

[Why? Recall that $b_{\delta,i(\ell(1)),n(1),3} \in G_{\alpha_\delta(n(2))} \subseteq G_{\beta_{\delta,i(\ell(2)),n(2)}}$; note that $b_{\delta,i,n,\iota} \in G_{\beta_{\delta,i,n,\iota}+\omega}$ commute with $G_{\beta_{\delta,i,n}}$ rather than with $G_{\alpha_{\delta,i,n}}$ but not used.]

$(*)_{5.9}$ if $\ell(1), \ell(2) < \ell(*)$, <u>then</u> for n large enough, for every $n(1), n(2) \in (n,\omega)$ the elements $d_{\ell(1),n(1)}, d_{\ell(2),n(2)}$ of G_δ commute.

[Why? Similarly, as $N^\delta_{i_{\ell(1)}} \cap N^\delta_{i_{\ell(2)}}$ is bounded in δ, but not used.]

$(*)_{5.10}$ The conditions in 2.19 hold hence we can fulfill $(*)_{5.3}, (*)_{5.4}$, i.e. we can carry the induction in $\boxplus_1$.

[Why? Think.]

Next let

$(*)_{5.11}$ $E = \{\delta < \lambda : \delta$ a limit ordinal is the universe of G_δ and $N{\restriction}\delta \prec N$, hence $\mathbf{h}$ maps G_δ onto itself$\}$.

Clearly E is a club of λ, hence by $\boxplus_{0,\zeta(1)}$ from stage A, there is a pair $(\delta, i_*) = (\delta, i(*))$ such that

$(*)_{5.12}$ $\delta \in E \cap S_{\zeta(1)}$ and $i_* \in \mathscr{T}_\delta$ and $N^\delta_{i_*} \prec N$.

Let $d = \mathbf{h}(c_{\delta,i_*}) \in G_*$, so:

$(*)_{5.13}$
(a) the pair (δ, i_*) satisfies the demands in $(*)_{5.3}(a)$;
(b) for some finite set $u_* \subseteq \mathscr{T}_\delta$ and $\bar{b}_* \in {}^{\omega>}(G_\delta)$, the type $\mathrm{tp_{bs}}(d, G_{\delta+1}, G_*)$ does not split over $\{c_{\delta,i} : i \in u_*\} \cup \bar{b}_*$;
(c) without loss of generality $i_* \in u_*$.

[Why? For clause (a), as $\delta \in E$ and $N^\delta_{i(*)} \prec N$, recalling the choice of N (including $\mathbf{h} = F^N_0$). For clause (b), apply properties of the construction in $\boxplus_1$, i.e. $G_{\delta+1} \le_{\mathfrak{S}} G_*$.]

$(*)_{5.14}$ conjugating by d in G_* interchange $b_{\delta,i(*),n,1}$ with $b_{\delta,i(*),n,2}$ for $n < \omega$.

[Why? Should be clear as for $m \in \omega \backslash \{n\}$ and $\iota(1), \iota(2) \in \{1,2,3\}$, the element $b_{\delta,i(*),m,\iota(1)}$ commutes with $b_{\delta,i(*),m,\iota(2)}$.]

Recalling $\boxplus_0(c)$ there is $n(*) < \omega$ large enough such that:

$(*)_{5.15}$ $\bar{b}_* \subseteq G_{\beta_{\delta,i(*),n(*)}}$ and $j_1 \neq j_2 \in u_* \Rightarrow N_{j_1} \cap N_{j_2} \subseteq G_{\alpha_\delta(n(*))}$ and j_1, j_2 are like i, j as in $\boxplus_{0,\zeta(1)}(c)^+$.

Clearly for some $\beta(*) < \lambda$ we have $\mathbf{h}(b_{\delta,i(*),n(*),1}) \in G_{\beta(*)+1} \backslash G_{\beta(*)}$. As $\alpha_{\delta,i(*),n(*)} = \min(N^\delta_{i(*)} \backslash \alpha_\delta(n(*)) \in N^\delta_{i(*)}$, clause (d) of $\boxplus_0$ and $N^\delta_{i_*} \prec N$, clearly:

$(*)_{5.16}$ (a) $\mathbf{h}$ maps $G_{\alpha_{\delta,i(*),n(*)}} \cap N^\delta_{i_*}$ onto itself and so $\beta(*) \in N^\delta_{i_*} \backslash \alpha_{\delta,i(*),n(*)}$

(b) $\mathbf{h}$ maps $G_{\alpha_{\delta,i(*),n(*)+1}}$ onto itself hence $\beta(*) \in N^\delta_{i(*)} \cap \alpha_{\delta,i(*),n(*)+1} \backslash \alpha_{\delta,i(*),n(*)}$.

Also,

$(*)_{5.17}$ if $\beta(*) < \beta_{\delta,i(*),n(*)} + \omega$ then $\beta(*) \leq \beta_{\delta,i(*),n(*)} + k_{\delta,i(*),n(*),2}$.

[Why? By $(*)_{5.1}$.]

Now,

$(*)_{5.18}$ there is $\beta \in N^\delta_{i_*} \cap (\beta(*)+1) \backslash \alpha_\delta(n(*)) \backslash S$ such that $[\beta, \beta(*)+\omega) \cap N^\delta_{i_*}$ is disjoint from N^δ_j if $j \in u_*$ but $j \neq i_*$.

[Why? First assume $\beta(*) \notin S$, let $\beta = \beta(*)$, so clearly $\beta \in N^\delta_{i_*}$ by $(*)_{5.14}, \beta \in (\beta(*)+1)$, also $\beta \notin \alpha_\delta(n(*))$ as by $(*)_{5.6}$ and the fact that $\beta \notin S$ by its choice. Also $[\beta, \beta(*)+\omega) = [\beta(*), \beta(*)+\omega) \subseteq N^\delta_{i_*}$ as N is closed under $\alpha \mapsto \alpha+1$ by $(*)_{3.1}(b)$. If $j \in u_*$ but $j \neq i_*$ then $N^\delta_j \cap N^\delta_{i_*} \subseteq \alpha_\delta(n(*)) \leq \beta$, hence $[\beta, \beta(*)+\omega) \cap N^\delta_j = \emptyset$, so we are done.

Second, assume $\beta(*) \in S$, hence $\mathrm{cf}(\delta) = \aleph_0$, and by $(*)_{3.1}(f), \{\alpha_{\beta(*)}(n) : n < \omega\} \subseteq N^\delta_{i_*}$. But by $\boxplus_0(c)^+$ we have $j \in u_* \wedge j \neq i_* \Rightarrow \sup(N^\delta_j \cap \beta(*)) < \beta(*)$. As u_* is finite there is $\beta \in \{\alpha_{\beta(*)}(n) : n < \omega\}$ such that $(\beta, \beta(*)) \cap N^\delta_j = \emptyset$; hence as before also $(\beta, \beta(*)+\omega) \cap N^\delta_j = \emptyset$, whenever $j \in u_* \wedge j \neq i_*$. So $(*)_{5.16}$ holds indeed.]

We finish the proof of $\boxplus_5$ by getting a contradiction as follows.

<u>Case 1</u>: $\beta(*) \geq \beta_{\delta,i(*),n(*)} + \omega$.

So by the choice of β and the proof of $(*)_{5.3}$ the type $\mathrm{tp}_{\mathrm{bs}}(d, G_{\beta(*)+\omega}, G_*)$ does not split over G_β, and even over some finite subset of it.

Now by $\boxplus_1(e)$ in $G_{\beta(*)+\omega}$ there is $d' \neq \mathbf{h}(b_{\beta_{\delta,i(*),n(*),1}})$ realizing

$$\mathrm{tp}_{\mathrm{bs}}(\mathbf{h}(b_{\beta_{\delta,i(*),n(*),1}}), G_\beta, G_{\beta(*)+\omega}) \text{ so } \mathbf{h}(b_{\beta_{\delta,i(*),n(*),3}}) \notin c\ell(G_\beta \cup \{d\}).$$

However, $G_* \models d^{-1}\, \mathbf{h}(c_{\beta_{\delta,i(*),n(*),1}})d = \mathbf{h}(c_{\beta_{\delta,i(*),n(*),2}})$, contradiction.

<u>Case 2</u>: $\beta(*) < \beta_{\delta,i(*),n(*)} + \omega$.

Hence $\beta(*) \leq \beta_{\delta,i(*),n(*)} + k_{\delta,i(*),n(*),2}$ and so $\{\mathrm{tp}_{\mathrm{bs}}(d), G_{\beta_{\delta,i(*),n(*)}+\omega}, G_*)\}$ does not split over $G_{\beta_{\delta,i(*),n(*)}} \cup \{b_{\delta,i(*),n(*),3}\})$ but $\mathrm{tp}(b_{\delta,i(*),n(*),3}, G_{\beta_{\delta,i(*),n(*)}+k_{\delta,i(*),n(*),3}}, G_*)$ does not split over $G_{\beta_{\delta,i(*),n(*)}} \cup \mathrm{Rang}(\bar{b}_{\delta,i(*),n(*),1}) \cup \mathrm{Rang}(\bar{b}_{\delta,i(*),n(*),2})$.

It follows that $\mathrm{tp}_{\mathrm{bs}}(d, G_{\beta_{\delta,i(*),n(*)}+k_{\delta,i(*),n(*),2}}, G_*)$ does not split over $G_{\beta_{\delta,i(*),n(*)}} \cup \bar{b}_{\delta,i(*),n(*),1}$ and recall $\mathbf{h}(\bar{b}_{\beta_{\delta,i(*),n(*),1}}) \subseteq G_{\beta_{\delta,i(*),n(*)}+k_{\delta,i(*),n(*),2}}$, contradiction by $(*)_{5.0}$.

So we have finished proving $\boxplus_5$.

Stage D:

$\boxplus_6$ (a) for some stationary $S^*_1 \subseteq S_*(\subseteq \lambda\backslash S)$ for every $\beta \in S^*_1\backslash\alpha(*)$ if $b \in L^*_\beta$ then $\mathbf{h}(b) = \sigma^{G_*}(b,\bar a)$ for some $\bar a \in {}^{\omega>}(G_\beta)$ and group-term $\sigma(x,\bar y)$

(b) moreover $\mathbf{h}(b) = \sigma^{G_*}(b)$ if $b \in L^*_\beta$.

Why?

$(*)_{6.1}$ clause (a) of $\boxplus_6$ holds even for every $\beta \in S^*_2 := S_* \cap E_{\mathbf{h}}\backslash\alpha(*)$.

[Why? By $\boxplus_5$.]

$(*)_{6.2}$ Without loss of generality if $\beta \in S^*_2$ and $b \in L^*_\beta$, then $\mathbf{h}(b) = \sigma_b(b)a_b$ for some $a_b \in G_\beta$.

[Why? This by $(*)_{6.1}$ because $\mathbf{h}$ maps G_β onto itself, b commutes with G_β whereas $\bar a_b \in {}^{\omega>}(G_\beta)$.]

$(*)_{6.3}$ (a) $b \mapsto \sigma_b(b)$ is a homomorphism from the set L^*_β into L^*_β (but we did not claim L^*_β is a subgroup);

(b) $b \mapsto a_b$ induces a homomorphism from the set L^*_β into the group G_β, that is if $\sigma(x_0,\dots,x_{n-1})$ is a group term and $b_0,\dots,b_{n-1} \in L^*_\beta$ and $G_{\beta+\omega} \models \sigma(b_0,\dots,b_{n-1}) = e$ then $G_\beta \models \sigma(a_{b_0},\dots,a_{b_{n-1}}) = e$.

[Why? As $\mathbf{h}$ is an automorphism of G_* and as $a_{b_1},\sigma_{b_2}(b_2)$ commute for $b_1,b_2 \in L^*_\beta$.]

We try to get rid of the homomorphism from $(*)_{6.3}(b)$ in order to prove $\boxplus_6(b)$. Toward contradiction assume (for the rest of this stage):

$(*)_{6.4}$ $\gamma \in S^*_2 \subseteq \lambda\backslash S^*_1$ is a limit ordinal and $b_* \in L^*_\gamma$ and $a_{b_*} \neq e$.

Now as $\gamma \in S_* \subseteq \lambda\backslash S$ we can find a sequence $\bar f^\gamma = \langle f^\gamma_\eta : \eta \in {}^\omega\mu\rangle$ satisfying f^γ_η is a function from $\{\eta{\restriction}n : n < \omega\}$ into G_γ such that for every $f : {}^{\omega>}\mu \to G_\gamma$ for some $\eta \in {}^\omega\mu$ we have $f^\gamma_\eta \subseteq f$; i.e. a simple black box, see [**?**, Fact 1.5=L4.5A], it exists as $\mu = \mu^{\aleph_0}$. Now generally for $\gamma \in \lambda\backslash S$ let $\mathscr{W}_\gamma = \{\eta \in {}^\omega\mu$: for some $c \in G_\gamma$ of order 2 we have $n < \omega \Rightarrow c^{-1} f^*_\eta(\eta{\restriction}(2n))c = f^*_\eta(\eta{\restriction}(2n+1))\}$.

Let K_* be the group of permutations of $I = {}^{\omega>}\mu \times \{0,1\}$ with finite support, i.e. $\{f \in \mathrm{Sym}(I) : (\exists^{<\aleph_0} t \in I)(f(t) \neq t)\}$. For $\eta \in {}^{\omega>}\mu$ let $h_\eta \in K_*$ be such that $h_\eta((\eta,\iota)) \equiv (\eta, 1-\iota)$, for $\iota = 0,1$, and is the identity otherwise. Let K_γ be the group of permutations of $I = {}^{\omega>}\mu \times \{0,1\}$ generated by $K_* \cup \{y_\eta : \eta \in {}^{\omega>}\mu\}$, where:

$(*)_{6.5}$ (a) if $\eta \in \mathscr{W}_\gamma$ then y_η interchanges $(\eta{\restriction}(2n+1),\iota),(\eta{\restriction}(2n+2),\iota)$ for $n < \omega, \iota = 0,1$ and otherwise is the identity;

(b) if $\eta \in {}^\omega\mu\backslash\mathscr{W}_\gamma$ then y_η interchanges $(\eta{\restriction}(2n),\iota)$ and $(\eta{\restriction}(2n+1),\iota)$ for $n < \omega, \iota = 0,1$, and is the identity otherwise.

Let

- d be the permutation of I interchanging $(<>,0),(<>,1)$ and being the identity otherwise.

Now we shall use some of the amount of freedom left, clearly:

$(*)_{6.6}$ (a) there is $K \subseteq \mathbf{C}_{G_{\gamma+\omega}}(G_\beta)$ finite with trivial center such that $b_* \in K$;

(b) there is $\bar{b}$ which lists the member of K such that $d_0 = b_*$;

(c) there is $\bar{d}$, a finite sequence from K_γ realizing $\mathrm{tp}(\bar{b}, \emptyset, G_*)$;

(d) there is $n(*)$ such that $K \subseteq G_{\gamma+n(*)}$.

$(*)_{6.7}$ There is an embedding g_γ of K_γ into $\mathbf{C}_{G_{\gamma+n(*)+1}}(G_\gamma)$ mapping $\bar{d}$ to $\bar{b}$ hence d_0 to b_*;

$(*)_{6.8}$ $b \mapsto a_b$ (for $b \in g_\gamma(K_\gamma)$) is a homomorphism from $g_\gamma(K_\gamma)$ into G_γ;

$(*)_{6.9}$ let $f : {}^{\omega>}\lambda \to G_\beta$ be defined by $f(\eta) = a_{g_\gamma(h_\eta)}$.

By the choice of $\langle f_\eta : \eta \in {}^{\omega>}\mu\rangle$ for some $\eta \in {}^\omega\mu$ we have $n < \omega \Rightarrow f^\gamma_\eta(\eta{\restriction}n) = f(\eta{\restriction}n)$.

Now does $\eta \in \mathscr{W}_\gamma$? First, assume $\eta \notin \mathscr{W}_\gamma$, then (by the choice of K_γ) ($g_\gamma(y_\eta) \in G_\gamma$ and) conjugating by $g_\gamma(y_\eta)$ for each n, interchanges $g_\gamma(h_{\eta\restriction(2n)}), g_\gamma(h_{\eta\restriction(2n+1)})$ which means that in K_γ, conjugating by h_η interchanges $f^\gamma_\eta(\eta{\restriction}(2n)), f^\gamma_\eta(\eta{\restriction}(2n+1))$, but by the choice of $\mathscr{W}_\gamma$ this means $\eta \in \mathscr{W}_\gamma$.

Second, assume $\eta \in \mathscr{W}_\gamma$, by the definition of $\mathscr{W}_\gamma$ there is $c \in G_\gamma$ of order 2 such that conjugating by c for each n interchanges $g_\gamma(h_{\eta\restriction(2n)}), g_\gamma(h_{\eta\restriction(2n+1)})$. But conjugating by $g_\gamma(y_\eta)$ for n interchange $g_\gamma(h_{\eta\restriction(2n+1)}), g_\gamma(h_{\eta\restriction(2n+2)})$. So in G_*, the subgroup generated by $\{c, g_\gamma(y_\eta), g_\gamma(h_{\eta\restriction 1})\}$ includes $g_\gamma(h_{\eta\restriction n})$ for $\eta = 1, 2, \ldots$; why? just prove it by induction on n. But $\{g_\gamma(h_{\eta\restriction n}) : n = 1, 2, \ldots\} \subseteq G_*$ is infinite, contradiction.

<u>Stage E</u>:

$\boxplus_7$ there is a finite sequence $\bar{a}_*$ such that for every $b \in G_*$ we have $\mathbf{h}(b) \in c\ell(\bar{a}_* \cup \{(b, G_*)\}$.

[Why? For $\beta \in S^*_1$ let $d_\beta \in G_{\beta+1}$ realize $\mathfrak{s}_{\mathrm{cg}}(<>, G_\beta)$ in $G_{\beta+1}$. So for every $a \in G_\beta$ of order m as G_β is existentially closed there is a finite $K_a \subseteq G_\beta$ with trivial center to which a belongs. Hence the element $d_\beta a d_\beta^{-1}$ commute with G_β and belongs to $G_{\beta+1}$ and moreover to L^*_β. Hence, by $\boxplus_6(b)$, for some $k(a) < m$ we have:

$(*)_{7.1}$ $\mathbf{h}(d_\beta^{-1} a d_\beta) = (d_\beta^{-1} a d_\beta)^{k(a)}$.

Hence

$(*)_{7.2}$ $\mathbf{h}(a) = \mathbf{h}(d_\beta^{-1})\mathbf{h}(d_\beta^{-1} a d_\beta)\mathbf{h}(d_\beta^{-1}) = \mathbf{h}(d_\beta^{-1})(d_\beta^{-1} a d_\beta)^{k(a)}\mathbf{h}(d_\beta)$.

Also, as $\beta \notin S$, there is a finite $K_\beta \subseteq G_\beta$ such that $\mathrm{tp}_{\mathrm{bs}}(\langle \mathbf{h}(d_\beta), d_\beta\rangle, G_\beta, G_*; \mathbf{K}_{\mathrm{lf}})$ does not split over K_β. By $(*)_{7.2}, \mathrm{tp}_{\mathrm{bs}}(\mathbf{h}(a), G_\beta, G_*; \mathbf{K}_{\mathrm{lf}})$ does not split over $K_\beta \cup \{d\}$, but $\mathbf{h}(a) \in G_\beta$ hence $\mathbf{h}(a) \in \langle K \cup \{d\}\rangle_{G_*}$. By Fodor's lemma this is enough for $\boxplus_7$.

Clearly we are done by 2.23. $\square_{5.1}$

* * *

Question 5.2. 1) In 5.1 we can easily get 2^λ pairwise non-isomorphic groups G'. But can they be pairwise far? (i.e. no $G \in \mathbf{K}_\lambda$, can be embedded in two of them)?
2) Even more basically can we demand G_* has no uncountable Abelian subgroup (when G does not)? Or at least no Abelian group of cardinality λ?
3) Can we prove 5.1 for every $\lambda > \aleph_0$? or at least $\lambda \geq \beth_\omega$?

Discussion 5.3. 1) Concerning 5.2(1), the problem with our approach is using $p \in \mathbf{S}_{\mathfrak{S}}(G)$, so as λ is regular we will get subgroups generated by indiscernible sequences, but let us elaborate. Assume $G_* \in \mathbf{K}_\lambda, G_* = \cup\{G_\alpha : \alpha < \lambda\}, G_\alpha$ increases with α and $|G_\alpha| < \lambda$. Further, assume $\mathfrak{s} \in \Omega[\mathbf{K}]$ and $\bar{a} \in {}^{n(\mathfrak{s})}G_*$ and $S = \{\alpha < \lambda : \bar{a} \subseteq G_\alpha$ and the type $q_{\mathfrak{s}}(\bar{a}, G_\alpha)$ is realized in $G_*\}$ is unbounded in λ and thus it is an end segment. Let $\bar{c}_\alpha \in {}^{k(\mathfrak{s})}G_*$ realize $q_{\mathfrak{s}}(\bar{a}, G_\alpha)$ and so for some club E of $\lambda, \alpha \in S \cap E \Rightarrow \bar{c}_\alpha \in G_{\min(E\setminus(\alpha+1))}$. Now $\bar{\mathbf{c}} = \langle \bar{c}_\alpha : \alpha \in S \cap E\rangle$ satisfies: if h is a partial increasing finite function from $S \cap E$ to $S \cap E$, then it induces a partial automorphism of $G_* : \bar{c}_\alpha \mapsto \bar{c}_{h(\alpha)}$. This is a case of indiscernible sequences. Hence the isomorphism type of $c\ell(\cup\{\bar{c}_\alpha : \alpha \in S \cap E\}, G_*)$ depends only on $\mathfrak{s}$ (and $\mathrm{tp}_{\mathrm{bs}}(\bar{a}, \emptyset, G_*)$. Hence the number of pairwise far such G_*'s is $\leq |\mathfrak{S}| + \aleph_0$.
2) Concerning 5.2(2), the problem with our approach is that we use $\mathfrak{s} = \mathfrak{s}_{\mathrm{ab}(k)}$ and more generally $\mathfrak{s} \in \Omega[\mathbf{K}]$ such that if $q_{\mathfrak{s}}(\bar{a}, G) = \mathrm{tp}_{\mathrm{bs}}(\bar{c}, G, H)$ <u>then</u> some $c \in H\setminus G$ commute with every (or simply many) members of G. Hence in the construction above, G_* has Abelian subgroups of cardinality λ.
3) What about considering the class of $(G, F_h)_{h\in H}, F_h \in \mathrm{aut}(G), G \in \mathbf{K}_{\mathrm{lf}}, h \mapsto F_h$ a homomorphism? We intend to deal with it in [**?**].

Discussion 5.4. 1) Naturally the construction in the proof of 5.1 is not unique, the class has many complicated models. In the construction in the proof of 5.1 we choose one where we realize many definable types.
2) We may like in $\boxplus_5$ of Stage C in the proof of 5.1 to consider $c \in G_\lambda$, not necessarily from $G_{\beta+\omega}$; (so later the role of $\mathfrak{s}_{\mathrm{cg}}$ in translating knowledge on $\mathbf{h}{\restriction}G_{\beta+\omega}$ to knowledge on G_β + use of Fodor is not necessary). Presently the way we combine $\langle b_{\delta,i(\ell),n,3} : n < \omega, \ell < \ell(*)\rangle$ to one n-type in $\mathbf{S}_{\mathrm{bs}}(G_\delta)$ works using 2.19.

Concerning the existence of complete groups in $\mathbf{K}^{\mathrm{lf}}_\lambda$ extending any $G \in \mathbf{K}^{\mathrm{lf}}_\lambda$ there are some restrictions.

Claim 5.5. *Assume $\lambda > \mathrm{cf}(\lambda) = \aleph_0, \chi = \lambda^{\aleph_0}$.*
1) If $G \in \mathbf{K}^{\mathrm{lf}}_\lambda$ is full, <u>then</u> its outer automorphism group has cardinality $\geq \chi$.
2) G has $\geq \chi$ outer automorphisms <u>when</u> $G \in \mathbf{K}^{\mathrm{lf}}_\lambda$ and for some sequence $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ listing the elements of G, letting $G_\alpha = c\ell(\{a_\beta : \beta < \alpha\}, G)$ we have:

- (a) *for every $\alpha < \lambda$ for λ ordinals $\beta < \lambda, a_\beta$ commutes with G_α*
- (b) *for every $a \in G\setminus\{e_G\}$ some element $b \in G, a$ does not commute with b.*

3) Like (2) but G_α has center of cardinality $< \lambda$.
4) Instead of (a),(b) we can use:

- (a)′ *for every $\alpha < \lambda$ we have $\lambda = |\{a/\mathrm{Cent}(G) : a \in G$ commute with $G_\alpha\}|$.*

Proof. 1) We reduce it to part (2). Let $\bar{\mathbf{a}} = \langle a_\alpha : \alpha < \lambda\rangle$ witness fullness (so $\lambda \geq 2^{\aleph_0}$). Now using the schemes $\mathfrak{s} = s_{\mathrm{ab}(2)}$, the pair $(G, \bar{a})$ satisfies clause (a) of part (2). Using, e.g. the scheme $\mathfrak{s} = \mathfrak{s}_{\mathrm{cg}}$ and the claim on non-commuting, 2.21, also clause (b) there holds.
2) Let $\lambda = \sum\limits_n \lambda_n, \lambda_n < \lambda_{n+1}$. For each n, by clause (a) we have $|S^1_n| = \lambda$ where $S^1_n := \{\alpha : a_\alpha$ commute with $c\ell(\{a_\beta : \beta < \lambda_n\}, G)\}$. Hence for some $k_n > n$ we have $S^3_n = \{\alpha < \lambda_{k_n} : \alpha \in S^1_n\}$ has cardinality $> \lambda_n$.

Replacing $\langle \lambda_n : n < \omega\rangle$ by a subsequence without loss of generality $\bigwedge\limits_n k_{2n} = 2n + 1$. Let $\langle \alpha_{n,i} : i < \lambda_n\rangle$ be a sequence of pairwise distinct members of $S^3_{2n}\setminus\lambda_{2n}$.

Now for each $\eta \in \prod\limits_{\ell<n} \lambda_{2\ell}$ let $b_\eta = a_{\eta(0)} a_{\eta(1)} \dots a_{\eta(n-1)} \in G$ and so $h_\eta := \square_{b_\eta}$, conjugation by b_η, is an inner automorphism of H. Also $\nu \triangleleft \eta \in \prod\limits_{\ell<n} \lambda_{2\ell} \Rightarrow \square_{b_\eta}, \square_{b_\nu}$ agree on $\{a_\beta : \beta < \lambda_{2\ell g(\nu)}\}$.

Hence if $\eta \in \prod\limits_n \lambda_{2n}$ then $\langle h_{\eta \restriction n} : n < \omega\rangle$ converge, i.e. for every $a \in G$, the sequence $\langle h_{\eta\restriction n}(a) : n < \omega\rangle$ is eventually constant and called the eventual value $h_\eta(a)$.

So h_η is an automorphism of G (for each $\eta \in \prod\limits_n \lambda_{2n}$). Now if $\eta_1, \eta_2 \in \prod\limits_n \lambda_{2n}, \eta_1(k) \neq \eta_2(k), \eta_1 \restriction k = \eta_2 \restriction k$ and for some $\alpha < \lambda_{2k}, a_\alpha$ does not commute with $a_{\eta_1(k)} a^{-1}_{\eta_2(k)}$ then $h_{\eta_1} \neq h_{\eta_2}$. Hence we can easily find $2^{\aleph_0}$ pairwise distinct h_η's. So if $\lambda < 2^{\aleph_0}$ we are done; otherwise, let $\mu = \min\{\mu : \mu^{\aleph_0} \geq \lambda$ equivalently $\mu^{\aleph_0} = \lambda^{\aleph_0}\}$, so $2^{\aleph_0} < \mu < \lambda$ and $\alpha < \mu \Rightarrow |\alpha|^{\aleph_0} < \mu$.

Choose $\bar{\mu} = \langle \mu_n : n < \omega\rangle$ such that $\sum\limits_n \mu_n = \mu, \mu_n < \mu_{n+1}$; moreover each μ_n regular and $\alpha < \mu_n \Rightarrow |\alpha|^{\aleph_0} < \mu_n$. Now for $n < k$ let $E_{n,k} = \{(i,j) : i,j < \mu_n$ and the conjugation $\square_{a_{\alpha_{n,i}}}, \square_{a_{\alpha_{n,j}}}$ agree on $\{a_\beta : \beta < \lambda_{2k}\}\}$, an equivalence relation. By clause (b) in the assumption, $\bigcap\limits_{k>n} E_{n,k}$ is the equality on μ_n, hence for some $k(n) > n, \mu_n/E_{n,k}$ has μ_n equivalence class. The rest should be clear.

3),4) Similarly. $\square_{5.5}$

§ 6. Other Classes

Note that

Theorem 6.1. *The results of §1 holds for any universal class* $\mathbf{K}$ *- see* [?].

However, we cannot in general prove the existence of dense $\mathfrak{S} \subseteq \Omega[\mathbf{K}]$, in fact, possibly $\Omega[\mathbf{K}] = \emptyset$. We refer the reader to §0 before 0.13, and to 0.17, 2.1. We may expand an lf group by choosing representations for left cosets bK, for K a finite subgroup of $G, b \in G$. Then the density of $\Omega[\mathbf{K}]$ is easy.

Definition 6.2. 1) Let $\mathbf{K}_{\text{clf}}$ be the class of structures M such that M is an expansion of an lf group $G = G_M$ by $F_n = F_n^M$ for $n \geq 1$ such that:

- (a) F_n^M is a partial $(n+1)$-place function from G to G;
- (b) if $(a_0, \ldots, a_n) \in \text{Dom}(F_n^M)$ then $(a_0, \ldots, a_{n-1})$ list without repetitions the elements of a subgroup of G_M and $a_n \in G_M$, of course;
- (c) if $F_n^M(a_0, \ldots, a_n) = b$ then $b \in \{a_n a_\ell : \ell < n\}$;
- (d) if K is a finite subgroup of G_M with n elements and for some $(a_0, \ldots, a_{n-1})$ listing its elements with no repetitions and b we have $(a_0, \ldots, a_{n-1}, b) \in \text{Dom}(F_n^M)$, then for every $(a'_0, \ldots, a'_{n-1})$ listing the members of K and $b' \in bK \subseteq G_M$ we have $(a'_0, \ldots, a'_{n-1}, b') \in \text{Dom}(F_n^M)$ and $b'K = bK \Rightarrow F_n^M(a_0, \ldots, a_{n-1}, b') = F_n^M(a_0, \ldots, a_{n-1}, b)$;
- (e) if K_1, K_2 are as in clause (d) then also $K_1 \cap K_2$ is;
- (f) if $A \subseteq G_M$ is finite then there is a minimal K as in clause (d) which contains A and if A is empty then $K = \{e_{G_M}\}$.

Definition 6.3. Let $\mathbf{K}_{\text{plf}}$ be the class of structures M such that: M expands a lf group G by P_n^M for $n < \omega$ and F_n^M for $n < \omega$ (actually definable from the rest) such that:

- (a) P_n^M is an $(n+3)$-place relation;
- (b) if $\bar{a} = (a_0, \ldots, a_{n+2}) \in P_n^M$ then $\{a_0, \ldots, a_{n-1}\}$ list with no repetitions the elements of a finite subgroup of G_M;
- (c) if $\{a_0, \ldots, a_{n-1}\} = \{a'_0, \ldots, a'_{n-1}\}$ are as above and moreover $b, b' \in M$ and $\{ba_0, \ldots, ba_{n-1}\} = \{b'a'_0, \ldots, b'a'_{n-1}\}$ then $M \models$ "$P_n(a_0, \ldots, a_{n-1}, b, c, d) = P_n(a'_0, \ldots, a'_{n-1}, b', c, d)$" for every $c, d \in M$;
- (d) if $(a_0, \ldots, a_{n-1})$ list the members of a finite subgroup K of G with no repetitions and $b \in G$ then $\{(c,d) : (a_0, \ldots, a_{n-1}, b, c, d) \in P_n^M\}$ is a linear order on the right coset bK, which we denote by $<^M_{K,b}$;
- (e) if the sequence $(a_0, \ldots, a_{n-1})$ is as above and $b \in G$ then $F_n^M(a_0, \ldots, a_{n-1}, b)$ is the first element by the order there in $\{ba_0, \ldots, ba_{n-1}\}$.

Definition 6.4. 1) For $M \in \mathbf{K}_{\text{clf}}$ let $\text{fsb}(M)$ be the set of finite subgroups K of G_M such that for some $a_0, \ldots, a_{n-1}$ listing with no repetitions the elements of K and for some $b \in G_M$ we have $(a_0, \ldots, a_{n-1}, b) \in \text{Dom}(F_n^M)$, i.e. they are as in clause (d) of Definition 6.2.
2) In this case we may write $F_K^M(b) = F_n^M(a_0, \ldots, a_{n-1}, b)$.
3) For $M, N \in \mathbf{K}_{\text{clf}}$ let $M \leq_{\text{elf}} N$ or $M \subseteq N$ mean that $G_M \subseteq G_N$ and $F_n^M = F_n^N {\restriction} M$ hence $K \in \text{sfb}(N) \wedge K \subseteq M \Rightarrow K \in \text{fsb}(M)$. We define similarly $\leq_{\text{plf}}, \leq_{\text{olf}}$, see Definition 6.3, 0.15. We may write $M \leq_{\mathbf{K}} N$ for the appropriate $\mathbf{K}$, etc.

4) "$M \in \mathbf{K}_{\rm clf}$ is (existentially closed)" is defined as in 0.13(2).
5) Let clf-group mean a member of $K_{\rm clf}$ and similarly an olf-group.
6) Similarly for "olf-groups" and "plf-groups".

Convention 6.5. 1) Let $\mathbf{K}$ denote one of the classes defined above, but let it be $\mathbf{K}_{\rm clf}$ if not said otherwise.

Definition/Claim 6.6. 1) For $M \in \mathbf{K}_{\rm olf}$ let $M^{[\rm clf]}$ be the unique $N \in \mathbf{K}_{\rm clf}$ such that: $G_N = G_M$ and $\mathrm{fsb}(N) = \{K : K \subseteq G_M$ is finite$\}$ and $F^M_K(b)$ is the $<_M$-first member of bK $\subseteq G$ (well defined as bK is finite non-empty).
1A) For $M \in \mathbf{K}_{\rm olf}$ we define $M^{[\rm plf]}$ and for $M \in \mathbf{K}_{\rm plf}$ we define $M^{[\rm clf]}$ parallely.
2) For $M \in \mathbf{K}_{\rm clf}$ and $A \subseteq M$, there is $N \subseteq M$ from $\mathbf{K}_{\rm clf}$ with universe A iff for every finite $A \subseteq B$ there is $K \in \mathrm{fsb}(M)$ such that $A \subseteq K \subseteq B$.
2A) So if $M \in \mathbf{K}_{\rm clf}$ and $K \in \mathrm{fsb}(M)$ then $M{\restriction}K \in \mathbf{K}_{\rm clf}$ and is finite.
3) For $A \subseteq M \in \mathbf{K}$ let $c\ell(A, M)$ be the minimal $N \subseteq M$ such that $A \subseteq N$, equivalently $\cup\{K : K \in \mathrm{fsb}(M)$ and there is no $L \in \mathrm{fsb}(M)$ such that $A \cap K \subseteq L \subset K\}$.
4) For $A \subseteq M \in \mathbf{K}$ let $c\ell_{\rm gr}(A, M)$ be the closure of A under the group operations.
5) We call $M \in \mathbf{K}_{\rm clf}$ full when $\mathrm{fsb}(M)$ is the set of finite $K \subseteq G_M$.

Claim 6.7. *1) The objects in 6.6 are well defined (in the right class).*
2) If $M \in \mathbf{K}_{\rm olf}$ or $M \in \mathbf{K}_{\rm plf}$ <u>then</u> $M^{[\rm clf]} \in \mathbf{K}_{\rm clf}$ is full.
3) $\mathfrak{S}(\mathbf{K}_{\rm olf})$ is dense.
4) $\mathfrak{S}(\mathbf{K}_{\rm clf})$ is dense.

Proof. 1) Straightforward, e.g. in part (3) for $K_{\rm clf}$ the closure is well defined because $\mathrm{fsb}(M)$ is closed under intersections.
2) Easy, too.
3),4) As in §2. $\square_{6.7}$

Remark 6.8. Call $M \in \mathbf{K}_{\rm clf}$ invariant <u>when</u> for every finite $K \subseteq G_M$ there is a function $F^M_K : G \to G$ such that $F^M_K(g) \in gK$ and is equal to $F^M_n(a_0, \ldots, a_{n-1})$ when $a_0, \ldots, a_{n-1}$ list the members of K with no repetitions. Restricting ourselves to such M seems to cause problems in amalgamations, whereas for $\mathbf{K}_{\rm plf}$ this is not so.

Definition 6.9. For $M \in \mathbf{K}$ and $n < \omega$ let $\mathbf{S}^n_{\rm gd}(M)$ be the set of good n-types $p(\bar{x}) \in \mathbf{S}^n_{\rm bs}(M)$ which means: $p = \mathrm{tp}(\bar{a}, M, N)$ where $M \subseteq N \in \mathbf{K}$ and $\bar{a} \in {}^n N$ and $c\ell_{\rm gr}(\bar{a} + M, N) = c\ell(\bar{a} + M, N)$.

Claim 6.10. *The classes $\mathbf{K} = \mathbf{K}_{\rm cfl}, \mathbf{K}_{\rm plf}, \mathbf{K}_{\rm olf}$ have dense closed $\mathfrak{S} \subseteq \Omega[K]$.*

Proof. Straightforward. $\square_{6.10}$

* * *

Definition 6.11. 1) Let $\mathbf{K}_{\rm sl}$ be the class of locally finite semi-groups, i.e. G, it has only one operation, binary which is associative.
2) Let $\mathbf{K}_{\rm usl}$ be defined similarly with an individual constant e such that $G \models ge_G = g = e_G g$ for every $g \in G \in \mathbf{K}_{\rm usl}$.

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