THE NONSTATIONARY IDEAL ON $P_\kappa(\lambda)$ FOR λ SINGULAR

Pierre MATET * and Saharon SHELAH †

Abstract

We give a new characterization of the nonstationary ideal on $P_\kappa(\lambda)$ in the case when κ is a regular uncountable cardinal and λ a singular strong limit cardinal of cofinality at least κ.

1 Introduction

Let κ be a regular uncountable cardinal and $\lambda \geq \kappa$ be a cardinal. As [10] and [11] of which it is a continuation, this paper investigates ideals on $P_\kappa(\lambda)$ with some degree of normality. For $\delta \leq \lambda$, let $\mathrm{NS}^\delta_{\kappa,\lambda}$ denotes the least δ-normal ideal on $P_\kappa(\lambda)$. Thus $\mathrm{NS}^\delta_{\kappa,\lambda}$ = the noncofinal ideal $I_{\kappa,\lambda}$ for any $\delta < \kappa$, and $\mathrm{NS}^\lambda_{\kappa,\lambda}$ = the nonstationary ideal $\mathrm{NS}_{\kappa,\lambda}$. $\mathrm{NSS}_{\kappa,\lambda}$ denotes the least seminormal ideal on $P_\kappa(\lambda)$. It is simple to see that $\mathrm{NSS}_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda}$ in case $\mathrm{cf}(\lambda) < \kappa$. If λ is regular, then by a result of Abe [1], $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\delta<\lambda} \mathrm{NS}^\delta_{\kappa,\lambda}$.

One problem we address in the paper is whether for $\lambda > \kappa$ $\mathrm{NS}_{\kappa,\lambda}$ is the restriction of a smaller ideal, i.e. whether $\mathrm{NS}_{\kappa,\lambda} = J|A$ for some ideal $J \subset \mathrm{NS}_{\kappa,\lambda}$ and some $A \in \mathrm{NS}^*_{\kappa,\lambda}$. The question as stated has a positive answer (see [2]) with $J = \nabla^\lambda I_{\kappa,\lambda}$. By a result of Abe [1] we can also take $J = \mathrm{NSS}_{\kappa,\lambda}$ in case $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. We investigate the possibility of taking $J = \bigcup_{\delta<\xi} \mathrm{NS}^\delta_{\kappa,\lambda}$ for some $\xi \leq \lambda$. If λ is regular, no such J will work since then, by an argument of [11], there is no A such that $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda} \mid A$.

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*Publication 33.

†Research supported by the United States - Israel Binational Science Foundation (Grant no. 2002323). Publication 869.

2010 Mathematics Subject Classification : 03E05, 03E55

Key words and phrases : $P_\kappa(\lambda)$, nonstationary ideal, precipitous ideal

Let $\mathcal{H}_{\kappa,\lambda}$ assert that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) \leq \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$, where $\overline{\mathrm{cof}}(\mathrm{NS}^{\tau}_{\kappa,\lambda})$ denotes the reduced cofinality of $\mathrm{NS}^{\tau}_{\kappa,\lambda}$. Clearly, $\mathcal{H}_{\kappa,\lambda}$ follows from $2^{<\lambda} = \lambda$. But there are other situations in which $\mathcal{H}_{\kappa,\lambda}$ holds. For instance, if in V, GCH holds, λ is a limit cardinal, χ is a regular uncountable cardinal less than κ, and $\mathbb{P}$ is the forcing notion to add λ^+ Cohen subsets of χ, then in $V^{\mathbb{P}}, 2^{\chi} > \lambda$ but, by results of [11], for every cardinal τ with $\kappa \leq \tau < \lambda, \mathrm{cof}(\mathrm{NS}_{\kappa,\tau}) = \tau^+$ and hence $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) \leq \lambda$.

It is known ([16], [10]) that if $\mathrm{cf}(\lambda) < \kappa$, then $\mathcal{H}_{\kappa,\lambda}$ holds just in case $\mathrm{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$ for some A. We will prove the following.

Theorem 1.1. *Suppose that $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ and $\mathcal{H}_{\kappa,\lambda}$ holds. Then (a) $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A, but (b) there is no B such that $\mathrm{NS}_{\kappa,\lambda} = (\bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}^{\delta}_{\kappa,\lambda})|B$.*

It is not known whether the converse holds :

Question. Suppose that $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ and $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A. Does it follow that $\mathcal{H}_{\kappa,\lambda}$ holds ?

If λ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds, then by the results above $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A. The following problem is open.

Question. Is it consistent that "λ is singular but $\mathrm{NS}_{\kappa,\lambda} \neq \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for every $\delta < \lambda$ and every $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$" ?

For any infinite cardinal $\tau < \lambda$, let $u(\tau, \lambda) =$ the least size of any cofinal subset of $(P_\tau(\lambda), \subset)$.
Now suppose $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. Then by results of [10], there is no A such that $\mathrm{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$. And it is shown in [11] that for any δ such that $\kappa \leq \delta < \mathrm{cf}(\lambda)$ and $u(|\delta|^+, \lambda) = \lambda$, there is no A such that $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$. Thus assuming Shelah's Strong Hypothesis (SSH), $\mathrm{NS}_{\kappa,\lambda} \neq \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for every $\delta < \mathrm{cf}(\lambda)$ and every $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$.

Question. Is it consistent relative to some large cardinal that "$\kappa < \mathrm{cf}(\lambda) < \lambda$ and $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for some $\delta < \mathrm{cf}(\lambda)$ and some $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$"?

Another problem we consider is whether $\mathrm{NS}^{\delta}_{\kappa,\lambda}$ is nowhere precipitous, where $\delta \leq \lambda$. As shown by Matsubara and Shioya [14], $I_{\kappa,\lambda}$ is nowhere precipitous, and in fact so is any ideal J on $P_\kappa(\lambda)$ of cofinality $u(\kappa, \lambda)$. Thus for every ideal J on $P_\kappa(\lambda)$,

$$\overline{\mathrm{cof}}(J) \leq \lambda \Rightarrow \mathrm{cof}(J) = u(\kappa, \lambda) \Rightarrow J \text{ is nowhere precipitous.}$$

We establish the following.

Proposition 1.2. *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and let $\xi > \kappa$ be such that*

- *ξ is either a successor ordinal, or a limit ordinal of cofinality at least κ ;*
- *$\xi \leq \eta$, where η equals $\lambda + 1$ if $\mathrm{cf}(\lambda) < \kappa$, and $\mathrm{cf}(\lambda)$ otherwise.*

Then $\overline{\mathrm{cof}}\Big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\Big) \leq \lambda$.

It follows from Theorem 1.1 and Proposition 1.2 that if $\mathcal{H}_{\kappa,\lambda}$ holds, then $\mathrm{NSS}_{\kappa,\lambda}|A = \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for some $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$, where δ equals $\mathrm{cf}(\lambda)$ if $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and 0 otherwise.

Let us next consider cases when $\kappa \leq \delta \leq \lambda$ and $\mathrm{cof}\big(\mathrm{NS}^{\delta}_{\kappa,\lambda}\big) > u(\kappa, \lambda)$. Goldring [7] and the second author proved that if λ is regular and $\mu > \lambda$ is Woodin, then in $V^{\mathrm{Col}(\lambda,<\mu)}$ $\mathrm{NS}_{\kappa,\lambda}$ is precipitous. On the other hand Matsubara and the second author [13] showed [(1)] that if λ is a strong limit cardinal with $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $\mathrm{NS}_{\kappa,\lambda}$ is nowhere precipitous. We establish the following.

Theorem 1.3. *Let σ be a cardinal such that $\kappa \leq \mathrm{cf}(\lambda) \leq \sigma < \lambda$ Then the following hold :*

(i) *If $\sigma = \mathrm{cf}(\lambda)$ and $\tau^{\mathrm{cf}(\lambda)} < \lambda$ for every cardinal $\tau < \lambda$, then $\mathrm{NS}^{\sigma}_{\kappa,\lambda}$ is nowhere precipitous.*

(ii) *If $\mathrm{cf}(\lambda) < \sigma$ and $\tau^{c(\kappa,\sigma)} < \lambda$ for every cardinal $\tau < \lambda$, where $c(\kappa, \sigma)$ denotes the least size of any closed unbounded subset of $P_\kappa(\sigma)$, then $\mathrm{NS}^{\sigma}_{\kappa,\lambda}$ is nowhere precipitous.*

Note that if $\kappa \leq \mathrm{cf}(\lambda) \leq \sigma < \lambda$ and the hypothesis of (i) (respectively, (ii)) of Theorem 1.3. holds, then $\lambda^{<\mathrm{cf}(\lambda)} = \lambda$, so by results of [10],

$$\mathrm{cof}(\mathrm{NS}^{\sigma}_{\kappa,\lambda}) \geq \overline{\mathrm{cof}}(\mathrm{NS}^{\sigma}_{\kappa,\lambda}) > \lambda = u(\kappa, \lambda).$$

By combining Theorems 1.1 and 1.3, we obtain the following.

Theorem 1.4. *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and $\tau^{\mathrm{cf}(\lambda)} < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}$ is nowhere precipitous.*

It is not clear whether Theorem 1.4 constitutes a real improvement in comparison to the result of Matsubara and the second author quoted above.

Question. Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and $\tau^{\mathrm{cf}(\lambda)} < \lambda$ for every cardinal $\tau < \lambda$. Does it then follow that λ is a strong limit cardinal ?

1 At some point the first author claimed to have found an error in the proof but it turned out that the mistake was his.

The paper is organized as follows. Section 2 collects basic definitions and facts concerning ideals on $P_\kappa(\lambda)$. It is shown in Section 3 that $\overline{\mathrm{cof}}(\mathrm{NS}^\pi_{\kappa,\lambda})$ is a non-decreasing function of π. In Section 4 we establish that if λ is regular, then $\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda}) = \lambda$ just in case $\mathcal{H}_{\kappa,\lambda}$ holds. In Section 5, Proposition 1.2 is proved. In Section 6 we show that it is consistent relative to a large cardinal that "λ is regular and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}|A) < \lambda$ for some A". It is shown in Section 7 that if λ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A. Finally in Section 8 we prove Theorem 1.3 and note that it is consistent relative to a large cardinal that "there is an ideal J on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) > \lambda$ but $\mathrm{cof}(J) = u(\kappa, \lambda)$."

2 Ideals on $P_\kappa(\lambda)$

In this section we collect basic material concerning ideals on $P_\kappa(\lambda)$.
NS_κ denotes the nonstationary ideal on κ.
For a set A and a cardinal ρ, let $P_\rho(A) = \{a \subseteq A :\mid a \mid< \rho\}$.
Given four cardinals τ, ρ, χ and σ, we define $\mathrm{cov}(\tau, \rho, \chi, \sigma)$ as follows. If there is $X \subseteq P_\rho(\tau)$ with the property that for any $a \in P_\chi(\tau)$, we may find $Q \in P_\sigma(X)$ with $a \subseteq \bigcup Q$, we let $\mathrm{cov}(\tau, \rho, \chi, \sigma)$ = the least cardinality of any such X. Otherwise we let $\mathrm{cov}(\tau, \rho, \chi, \sigma) = 0$.
We let $\mathrm{cov}(\tau, \rho, \chi, \sigma) = u(\tau, \chi)$ in case $\rho = \chi$ and $\sigma = 2$.

FACT 2.1. ([15, pp. 85-86]) *Let τ, ρ, χ and σ be four cardinals such that $\tau \geq \rho \geq \chi \geq \omega$ and $\chi \geq \sigma \geq 2$. Then the following hold :*

(i) *If $\tau > \rho$, then $\mathrm{cov}(\tau, \rho, \chi, \sigma) \geq \tau$.*

(ii) $\mathrm{cov}(\tau, \rho, \chi, \sigma) = \mathrm{cov}(\tau, \rho, \chi, \max\{\omega, \sigma\})$.

(iii) $\mathrm{cov}(\tau^+, \rho, \chi, \sigma) = \max\{\tau^+, \mathrm{cov}(\tau, \rho, \chi, \sigma)\}$.

(iv) *If $\tau > \rho$ and $\mathrm{cf}(\tau) < \sigma = \mathrm{cf}(\sigma)$, then $\mathrm{cov}(\tau, \rho, \chi, \sigma) = \sup\{\mathrm{cov}(\tau', \rho, \chi, \sigma) : \rho \leq \tau' < \tau\}$.*

(v) *If τ is a limit cardinal such that $\tau > \rho$ and $\mathrm{cf}(\tau) \geq \chi$, then $\mathrm{cov}(\tau, \rho, \chi, \sigma) = \sup\{\mathrm{cov}(\tau', \rho, \chi, \sigma) : \rho \leq \tau' < \tau\}$.*

$I_{\kappa,\lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ such that $\{a \in A : b \subseteq a\} = \emptyset$ for some $a \in P_\kappa(\lambda)$.
By an *ideal* on $P_\kappa(\lambda)$, we mean a collection J of subsets of $P_\kappa(\lambda)$ that is closed under subsets (i.e. $P(A) \subseteq J$ for all $A \in J$), κ-complete (i.e. $\bigcup X \in J$ for every $X \in P_\kappa(J)$), fine (i.e. $I_{\kappa,\lambda} \subseteq J$) and proper (i.e. $P_\kappa(\lambda) \notin J$).
Given an ideal J on $P_\kappa(\lambda)$, let $J^+ = \{A \subseteq P_\kappa(\lambda) : A \notin J\}$ and $J^* = \{A \subseteq P_\kappa(\lambda) : P_\kappa(\lambda) \setminus A \in J\}$. For $A \in J^+$, let $J|A = \{B \subseteq P_\kappa(\lambda) : B \cap A \in J\}$. Given a cardinal $\chi > \lambda$ and $f : P_\kappa(\lambda) \to P_\kappa(\chi)$, we let

$$f(J) = \{X \subseteq P_\kappa(\chi) : f^{-1}(X) \in J\}.$$

$\mathcal{M}_J$ denotes the collection of all maximal antichains in the partially ordered set $(J^+, \subseteq)$, i.e. of all $Q \subseteq J^+$ such that

- $A \cap B \in J$ for any distinct $A, B \in Q$;
- for every $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.

For a cardinal ρ, J is *ρ-saturated* if $|Q| < \rho$ for every $Q \in \mathcal{M}_J$.
$\mathrm{cof}(J)$ denotes the least cardinality of any $X \subseteq J$ such that $J = \bigcup_{A \in X} P(A)$.
$\overline{\mathrm{cof}}(J)$ denotes the least size of any $Y \subseteq J$ with the property that for every $A \in J$, there is $y \in P_\kappa(Y)$ with $A \subseteq \bigcup y$.
$\mathrm{non}(J)$ denotes the least cardinality of any $A \in J^+$.
Note that $\mathrm{cof}(J) \geq \mathrm{non}(J) \geq \mathrm{non}(I_{\kappa,\lambda}) = u(\kappa, \lambda)$.
The following is well-known (see e.g. [10] and [11]).

FACT 2.2.

(i) $\lambda^{<\kappa} = \max\{2^{<\kappa}, u(\kappa, \lambda)\}$.

(ii) $\overline{\mathrm{cof}}(I_{\kappa,\lambda}) = \lambda$.

(iii) *Let J be an ideal on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) \leq \lambda$. Then $\mathrm{cof}(J) = u(\kappa, \lambda)$.*

Shelah's Strong Hypothesis (SSH) asserts that for any two uncountable cardinals χ and ρ with $\chi \geq \rho = \mathrm{cf}(\rho)$, $u(\rho, \chi)$ equals χ if $\mathrm{cf}(\chi) \geq \rho$, and χ^+ otherwise.

FACT 2.3. ([8])

(i) *Suppose that there is a π-saturated ideal on $P_\nu(\lambda)$, where π and ν are two cardinals such that $\omega < \nu = \mathrm{cf}(\nu) \leq \lambda$ and $\pi < \nu \cap \kappa^+$. Then $u(\kappa, \lambda)$ equals λ if $\mathrm{cf}(\lambda) \geq \kappa$, and λ^+ otherwise.*

(ii) *Suppose that there is a regular uncountable cardinal $\nu < \lambda$ that is mildly π^+-ineffable for every cardinal π with $\nu \leq \pi < \lambda$. Then the following hold :*

- *$u(\kappa, \lambda)$ equals λ if $\mathrm{cf}(\lambda) \geq \kappa$, and λ^+ if $\omega < \mathrm{cf}(\lambda) < \kappa$.*
- *$\mathrm{cov}(\lambda, \kappa, \kappa, \omega_1) = \lambda$ if $\mathrm{cf}(\lambda) = \omega$.*

Numerous variations on the original notion of ideal normality have been considered over the years. One such variant is the concept of δ-normality which has been studied by Abe [1].

Let $\delta \leq \lambda$. An ideal J on $P_\kappa(\lambda)$ is *δ-normal* if given $A \in J^+$ and $f : A \to \delta$ with the property that $f(a) \in a$ for all $a \in A$, there exists $B \in J^+ \cap P(A)$ such that f is constant on B.

$\mathrm{NS}^\delta_{\kappa,\lambda}$ denotes the smallest δ-normal ideal on $P_\kappa(\lambda)$.
Note that λ-normality is the same as normality, so $\mathrm{NS}^\lambda_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda}$.

$c(\kappa, \lambda)$ denotes the least size of any closed unbounded subset of $P_\kappa(\lambda)$.

FACT 2.4.

(i) ([1]) *Let δ be an ordinal such that $\delta + \kappa \leq \lambda$. Then* $\mathrm{NS}^{\delta+\kappa}_{\kappa,\lambda} \setminus \mathrm{NS}^{\delta}_{\kappa,\lambda} \neq \emptyset$.

(ii) ([11]) *Suppose $\kappa \leq \delta < \lambda$. Then* $\mathrm{NS}^{\delta}_{\kappa,\lambda} = \mathrm{NS}^{|\delta|}_{\kappa,\lambda}|A$ *for some A.*

(iii) ([11]) *Let δ and η be two ordinals such that $|\delta| < |\eta| < \lambda$ and $\kappa \leq \eta$. Then there is no A such that* $NS^{\eta}_{\kappa,\lambda} = \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$.

FACT 2.5.

(i) ([10]) $\overline{\mathrm{cof}}(\mathrm{NS}^{\delta}_{\kappa,\lambda}) \geq \lambda$ *for every $\delta \leq \lambda$.*

(ii) ([8], [10]) *Let $\delta \leq \lambda$. Then* $\overline{\mathrm{cof}}(\mathrm{NS}^{\delta}_{\kappa,\lambda}|A) = \overline{\mathrm{cof}}(\mathrm{NS}^{\delta}_{\kappa,\lambda})$ *for every* $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$.

(iii) ([10]) $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}) \geq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\rho})$ *for every cardinal ρ with $\kappa \leq \rho < \lambda$.*

(iv) ([10]) *Suppose* $\mathrm{cf}(\lambda) \geq \kappa$. *Then* $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}) > \lambda$.

The concept of $[\delta]^{<\theta}$-normality generalizes that of δ-normality.

Let $\delta \leq \lambda$, and let θ be a cardinal with $\theta \leq \kappa$. An ideal J on $P_\kappa(\lambda)$ is *$[\delta]^{<\theta}$-normal* if given $A \in J^+$ and $f : A \to P_\theta(\delta)$ with the property that $f(a) \in P_{|a\cap\theta|}(a \cap \delta)$ for all $a \in A$, there exists $B \in J^+ \cap P(A)$ such that f is constant on B.
Note that for $\theta = \kappa$, $[\lambda]^{<\theta}$-normality is the same as strong normality.
We set $\overline{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and κ is a limit cardinal, and $\overline{\theta} = \nu$ if $\theta = \kappa = \nu^+$.

FACT 2.6. ([11])

(i) *Suppose that $\delta < \kappa$, or $\theta < \kappa$, or κ is not a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if $|P_{\overline{\theta}}(\rho)| < \kappa$ for every cardinal $\rho < \kappa \cap (\delta + 1)$.*

(ii) *Suppose that $\delta \geq \kappa, \theta = \kappa$ and κ is a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if κ is a Mahlo cardinal.*

(iii) *Suppose that there exists a $[\kappa]^{<\theta}$- normal ideal on $P_\kappa(\lambda)$. Then $\kappa^{<\overline{\theta}} = \kappa$, and $(\pi^{<\overline{\theta}})^{<\overline{\theta}} = \pi^{<\overline{\theta}}$* for every cardinal $\pi > \kappa$.

Assuming that there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, $\mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\lambda}$ denotes the smallest such ideal.

FACT 2.7. ([11])

(i) *Suppose that $\theta < 2$ or $\delta < \kappa$. Then* $\mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\lambda} = I_{\kappa,\lambda}$.

(ii) *Suppose that* $2 \le \theta \le \omega$. *Then* $\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = \mathrm{NS}_{\kappa,\lambda}^{\delta}$.

(iii) *Suppose that* $|\delta|^{<\overline{\theta}} = |\eta|^{<\overline{\pi}}$, *where* $\kappa \le \eta \le \lambda$ *and* π *is a cardinal with* $2 \le \pi \le \kappa$. *Then* $\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} | A = \mathrm{NS}_{\kappa,\lambda}^{[\eta]^{<\pi}} | A$ *for some* $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\gamma]^{<\rho}}\right)^*$, *where* $\gamma = \max\{\delta, \eta\}$ *and* $\rho = \max\{\theta, \pi\}$.

Given an ordinal η, a cardinal π and $f : P_\pi(\eta) \to P_\kappa(\lambda)$, let $C(f, \kappa, \lambda)$ be the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \pi \ne \emptyset$ and $f(e) \subseteq a$ for every $e \in P_{|a\cap\pi|}(a \cap \eta)$.

FACT 2.8. ([11]) *Suppose that* $A \subseteq P_\kappa(\lambda)$, $\kappa \le \delta \le \lambda$, *and* θ *is a cardinal with* $2 \le \theta \le \kappa$. *Then the following are equivalent :*

(i) $A \in \mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

(ii) $A \cap C(f, \kappa, \lambda) = \emptyset$ *for some* $f : P_{\max\{\overline{\theta},3\}}(\delta) \to P_\kappa(\lambda)$.

(iii) $A \cap \{a \in C(g, \kappa, \lambda) : a \cap \kappa \in \kappa\} = \emptyset$ *for some* $g : P_{\max\{\overline{\theta},3\}}(\delta) \to P_3(\lambda)$.

FACT 2.9. ([10]) *Let* χ *and* θ *be two cardinals such that* $2 \le \theta \le \kappa \le \chi \le \lambda$. *Then the following hold :*

(i) *Let* J *be a* $[\chi]^{<\theta}$*- normal ideal on* $P_\kappa(\lambda)$. *Then either* $\mathrm{cf}\left(\overline{\mathrm{cof}}(J)\right) < \kappa$, *or* $\mathrm{cf}\left(\overline{\mathrm{cof}}(J)\right) > \chi^{<\overline{\theta}}$.

(ii) *If* $\chi^{<\overline{\theta}} < \lambda$, *then* $\overline{\mathrm{cof}}\left(NS_{\kappa,\lambda}^{[\chi]^{<\theta}}\right) \ge \lambda$.

FACT 2.10. ([10], [11]) *Suppose that* $\kappa \le \delta < \lambda$, *and* θ *is a cardinal with* $2 \le \theta \le \kappa$. *Then the following hold :*

(i) $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \max\{\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,|\delta|}^{|\delta|^{<\theta}}), \mathrm{cov}\left(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, \kappa\right)\}$ *and* $\mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \max\{\mathrm{cof}(\mathrm{NS}_{\kappa,|\delta|}^{|\delta|^{<\theta}}), \mathrm{cov}\left(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, 2\right)\}$.

(ii) *If* λ *is a limit cardinal and either* $\mathrm{cf}(\lambda) < \kappa$ *or* $\mathrm{cf}(\lambda) > |\delta|^{<\overline{\theta}}$, *then* $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}) = \sup\{\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\delta]^{<\theta}}) : \delta < \tau < \lambda\}$.

For a cardinal $\tau, \mathfrak{d}_{\kappa,\lambda}^{\tau}$ denotes the smallest cardinality of any family F of functions from τ to $P_\kappa(\lambda)$ with the property that for any $g : \tau \to P_\kappa(\lambda)$, there is $f \in F$ such that $g(\alpha) \subseteq f(\alpha)$ for every $\alpha < \tau$.

FACT 2.11. ([11])

(i) *For any cardinal* $\tau > 0$, $\mathrm{cf}(\mathfrak{d}_{\kappa,\lambda}^{\tau}) > \tau$.

(ii) *Suppose that* $0 < \delta \le \lambda$, *and* θ *is a cardinal with* $0 < \theta \le \kappa$. *Then* $\mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A) = \mathfrak{d}_{\kappa,\lambda}^{|P_{\overline{\theta}}(\delta)|}$ *for every* $A \in (\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}})^+$.

Next let us recall a few facts concerning the notion of precipitousness.

An ideal J on $P_\kappa(\lambda)$ is *precipitous* if whenever $A \in J^+$ and $\langle Q_n : n < \omega\rangle$ is a sequence of members of $\mathcal{M}_{J|A}$ such that $Q_{n+1} \subseteq \bigcup_{B \in Q_n} P(B)$ for all $n < \omega$, there exists $f \in \prod_{n\in\omega} Q_n$ such that $f(0) \supseteq f(1) \supseteq \dots$ and $\bigcap_{n<\omega} f(n) \neq \emptyset$. J is *nowhere precipitous* if for each $A \in J^+$, $J|A$ is not precipitous.
Let $G(J)$ denote the following two-player game lasting ω moves, with player I making the first move : I and II alternately pick members of J^+, thus building a sequence $\langle X_n : n < \omega\rangle$, subject to the condition that $X_0 \supseteq X_1 \supseteq \dots$. II wins $G(J)$ just in case $\bigcap_{n<\omega} X_n = \emptyset$.

FACT 2.12. ([5]) *An ideal J on $P_\kappa(\lambda)$ is nowhere precipitous if and only if* II *has a winning strategy in the game $G(J)$.*

The following is a straightforward generalization of a result of Foreman [4] :

PROPOSITION 2.13. *Let χ and θ be two cardinals such that $\chi \le \lambda$ and $\theta \le \kappa$. Then every $[\chi]^{<\theta}$-normal, $(\chi^{<\overline{\theta}})^+$-saturated ideal on $P_\kappa(\lambda)$ is precipitous.*

FACT 2.14. ([14]) *Suppose that J is an ideal on $P_\kappa(\lambda)$ such that* $\mathrm{cof}(J) =$ *non*(J). *Then J is nowhere precipitous.*

Thus for an ideal J on $P_\kappa(\lambda)$,

$$\overline{\mathrm{cof}}(J) \le \lambda \Rightarrow \mathrm{cof}(J) = u(\kappa,\lambda) \Rightarrow J \text{ is nowhere precipitous.}$$

Let τ be a cardinal with $\kappa \le \tau \le \lambda$. It is simple to see that if GCH holds and either $\mathrm{cf}(\lambda) < \kappa$ or $\tau < \mathrm{cf}(\lambda)$, then $\mathrm{cof}(\mathrm{NS}^\tau_{\kappa,\lambda}) = u(\kappa,\lambda)$. Note that if SSH holds and $\kappa \le \mathrm{cf}(\lambda) \le \tau$, then by Facts 2.5 (i) and 2.9, $\mathrm{cof}(\mathrm{NS}^\tau_{\kappa,\lambda}) > u(\kappa,\lambda)$.

PROPOSITION 2.15. *Suppose that σ is a strong limit cardinal with* $\mathrm{cf}(\sigma) <$ $\kappa < \sigma \le \lambda \le 2^\sigma$. *Then the following hold :*

(i) $\mathrm{cof}(\mathrm{NS}^\tau_{\kappa,\lambda}) = u(\kappa,\lambda)$ *for every cardinal τ with $\kappa \le \tau \le \sigma$.*

(ii) *Suppose $2^\lambda = 2^\sigma$. Then* $\mathrm{cof}(\mathrm{NS}^\tau_{\kappa,\lambda}) = u(\kappa,\lambda)$ *for every cardinal τ with $\sigma < \tau \le \lambda$.*

Proof.

(i) : Let τ be a cardinal with $\kappa \le \tau \le \sigma$. If $\tau = \lambda$, then

$$\mathrm{cof}(\mathrm{NS}^\tau_{\kappa,\lambda}) \le 2^\lambda = \lambda^{<\kappa} = u(\kappa,\lambda).$$

Otherwise by Fact 2.10, $\mathrm{cof}(\mathrm{NS}^\tau_{\kappa,\lambda}) = \max\{\mathrm{cof}(\mathrm{NS}_{\kappa,\tau}), u(\tau^+,\lambda)\} \le \lambda^\tau = \sigma^\tau = \sigma^{\mathrm{cf}(\sigma)} \le \lambda^{<\kappa} = u(\kappa,\lambda)$.

(ii) : Given a cardinal τ with $\sigma < \tau \leq \lambda$,

$$\mathrm{cof}(\mathrm{NS}^{\tau}_{\kappa,\lambda}) \leq 2^{\lambda} = 2^{\sigma} = \sigma^{\mathrm{cf}(\sigma)} = u(\kappa, \lambda). \qquad \square$$

3 $\overline{\mathbf{cof}}(\mathrm{NS}^{\chi}_{\kappa,\lambda})$

By Fact 2.11 (ii), $\mathrm{cof}(\mathrm{NS}^{\chi}_{\kappa,\lambda}) = \mathfrak{d}^{\chi}_{\kappa,\lambda}$ for any cardinal χ with $\kappa \leq \chi \leq \lambda$. We now derive a similar formula for $\overline{\mathrm{cof}}(\mathrm{NS}^{\chi}_{\kappa,\lambda})$.

For a cardinal τ, $\overline{\mathfrak{d}}^{\tau}_{\kappa,\lambda}$ denotes the smallest cardinality of any family F of functions from τ to $P_\kappa(\lambda)$ with the property that for any $g : \tau \to P_\kappa(\lambda)$, there is $Z \in P_\kappa(F)$ such that $g(\alpha) \subseteq \bigcup_{f \in Z} f(\alpha)$ for every $\alpha < \tau$.

LEMMA 3.1. *Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}) \leq \overline{\mathfrak{d}}^{\chi^{<\overline{\theta}}}_{\kappa,\lambda}$.*

Proof. Select a collection G of functions from $P_{\max\{\overline{\theta},3\}}(\chi)$ to $P_\kappa(\lambda)$ so that $|G| = \overline{\mathfrak{d}}^{\chi^{<\overline{\theta}}}_{\kappa,\lambda}$ and for any $k : P_{\max\{\overline{\theta},3\}}(\chi) \to P_\kappa(\lambda)$, there is $Z_k \in P_\kappa(G)$ such that $k(e) \subseteq \bigcup_{g \in Z_k} g(e)$ for all $e \in P_{\max\{\overline{\theta},3\}}(\chi)$. Then clearly for each $k : P_{\max\{\overline{\theta},3\}}(\chi) \to P_\kappa(\lambda), \bigcap_{g \in Z_k} C(g, \kappa, \lambda) \subseteq C(k, \kappa, \lambda)$. Hence $\overline{\mathrm{cof}}(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}) \leq |G|$. $\square$

LEMMA 3.2. *Let θ and χ be two cardinals such that $\omega \leq \theta = \mathrm{cf}(\theta) < \kappa \leq \chi \leq \lambda$. Then $\overline{\mathfrak{d}}^{\chi^{<\theta}}_{\kappa,\lambda} \leq u\big(\theta, \overline{\mathrm{cof}}(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda})\big)$.*

Proof. Pick a collection H of functions from $P_\theta(\chi) \to P_3(\lambda)$ so that $|H| = \overline{\mathrm{cof}}(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda})$ and for any $A \in (NS^{[\chi]^{<\theta}}_{\kappa,\lambda})^*$, there is $Q \in P_\kappa(H) \setminus \{\emptyset\}$ with $\big\{b \in \bigcap_{h \in Q} C(h, \kappa, \lambda) : b \cap \kappa \in \kappa\big\} \subseteq A$. Select $\mathfrak{X} \subseteq P_\theta(H) \setminus \{\emptyset\}$ so that $|\mathfrak{X}| = u(\theta, |H|)$ and for any $Z \in P_\theta(H)$, there is $X \in \mathfrak{X}$ with $Z \subseteq X$. For $X \in \mathfrak{X}$, define $t_X : P_\theta(\chi) \to P_\kappa(\lambda)$ by $t_x(e) = \bigcap T_{X,e}$, where

$$T_{X,e} = \Big\{b \in \textstyle\bigcap_{h \in X} C(h, \kappa, \lambda) : e \cup \theta \subseteq b \text{ and } b \cap \kappa \in \kappa\Big\}.$$

Note that $t_X(e) \in T_{X,e}$, and $t_Y(e) \subseteq t_X(e)$ for all $Y \in \mathfrak{X} \cap P(X)$.
Now fix $f : P_\theta(\chi) \to P_\kappa(\lambda)$. We may find $W \in P_\kappa(\mathfrak{X})$ such that

$$\Big\{b \in \textstyle\bigcap_{h \in \bigcup W} C(h, \kappa, \lambda) : b \cap \kappa \in \kappa\Big\} \subseteq C(f, \kappa, \lambda),$$

$\theta \leq |W|$ and for any $K \in P_\theta(W)$, there is $Z \in W$ with $\bigcup K \subseteq Z$. For $e \in P_\theta(\chi)$, put $b_e = \bigcup_{X \in W} t_X(e)$. Note that $b_e \cap \kappa \in \kappa$.

Claim. *Let $k \in \bigcup W$. Then $b_e \in C(k, \kappa, \lambda)$.*

Proof of Claim. Fix $d \in P_\theta(b_e \cap \chi)$. Pick $\varphi : d \to W$ so that $\beta \in t_{\varphi(\beta)}(e)$ for every $\beta \in d$. Select $Y \in W$ with $k \in Y$. There must be $Z \in W$ such that $Y \cup \left(\bigcup_{\beta\in d}\varphi(\beta)\right) \subseteq Z$. Then $d \in P_\theta(t_Z(e))$ and $t_Z(e) \in C(k,\kappa,\lambda)$, so $k(d) \subseteq t_Z(e) \subseteq b_e$. This completes the proof of the claim.

Thus $b_e \in \bigcap_{h\in\cup W} C(h,\kappa,\lambda)$. Hence $b_e \in C(f,\kappa,\lambda)$, and consequently $f(e) \subseteq b_e$. $\square$

PROPOSITION 3.3. *Let χ be a cardinal with $\kappa \le \chi \le \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NS}^\chi_{\kappa,\lambda}) = \bar{\mathfrak{d}}^\chi_{\kappa,\lambda}$.*

Proof. By Lemmas 3.1 and 3.2. $\square$

COROLLARY 3.4. *Let π and χ be two cardinals such that $\kappa \le \pi < \chi \le \lambda$.*

Then $\overline{\mathrm{cof}}(\mathrm{NS}^\pi_{\kappa,\lambda}) \le \overline{\mathrm{cof}}(\mathrm{NS}^\chi_{\kappa,\lambda})$.

4 $\mathrm{NSS}_{\kappa,\lambda}$

An ideal J on $P_\kappa(\lambda)$ is *seminormal* if it is δ-normal for every $\delta < \lambda$.
$\mathrm{NSS}_{\kappa,\lambda}$ denotes the smallest seminormal ideal on $P_\kappa(\lambda)$.

FACT 4.1.

(i) (Folklore) *Suppose $\mathrm{cf}(\lambda) < \kappa$. Then $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda}$.*

(ii) ([1]) *Suppose $\kappa \le \mathrm{cf}(\lambda) < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda}|A$ for some A.*

PROPOSITION 4.2. *Suppose $\kappa \le \mathrm{cf}(\lambda) < \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda}) > \lambda$.*

Proof. By Facts 2.5 (iv) and 4.1. $\square$

We will see that "$\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda}) > \lambda$" needs not hold in case λ is regular. Note that if λ is regular, then by Fact 2.5 (iv), $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}) > \lambda$.

FACT 4.3. ([1]) *Suppose that λ is regular. Then $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\delta<\lambda}\mathrm{NS}^\delta_{\kappa,\lambda}$.*

Proof. It is immediate that $\bigcup_{\delta<\lambda}\mathrm{NS}^\delta_{\kappa,\lambda} \subseteq \mathrm{NSS}_{\kappa,\lambda}$. To show the reverse inclusion, fix $A \in \left(\bigcup_{\delta<\lambda}\mathrm{NS}^\delta_{\kappa,\lambda}\right)^+$, η with $\kappa \le \eta < \lambda$, and $f : A \longrightarrow \eta$ with the property that $f(a) \in a$ for all $a \in A$. For ξ with $\eta \le \xi < \lambda$, we may find $B_\xi \in (\mathrm{NS}^\xi_{\kappa,\lambda})^+ \cap P(A)$ and $\gamma_\xi < \eta$ such that f takes the constant value γ_ξ on B_ξ. There must be $\beta < \eta$ and $Z \subseteq \{\xi : \eta \le \xi < \lambda\}$ such that $|Z| = \lambda$ and

$\gamma_\xi = \beta$ for all $\xi \in Z$. Now set $C = \bigcup_{\xi \in Z} B_\xi$. Then clearly $C \in \left(\bigcup_{\delta<\lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda}\right)^+$. Moreover f is identically β on C. □

FACT 4.4. ([10]) *Suppose that θ is a cardinal with $2 \leq \theta \leq \kappa$, and J is an ideal on $P_\kappa(\lambda)$ such that $J \subseteq \mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}$ and $\overline{\mathrm{cof}}(J) \leq \lambda^{<\overline{\theta}}$. Then $J|A = I_{\kappa,\lambda}|A$ for some $A \in \left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right)^*$.*

In particular, if $J \subseteq \mathrm{NS}_{\kappa,\lambda}$ and $\overline{\mathrm{cof}}(J) \leq \lambda$, then $J|D = I_{\kappa,\lambda}|D$ for some $D \in \mathrm{NS}^*_{\kappa,\lambda}$.

FACT 4.5. ([10]) *Suppose that θ is a cardinal with $2 \leq \theta \leq \kappa$, and let σ be the least cardinal τ such that $\tau^{<\overline{\theta}} \geq \lambda$. Then $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \geq \sigma$ for every $A \in \left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right)^*$.*

PROPOSITION 4.6. *Suppose that θ is a cardinal with $2 \leq \theta \leq \kappa$, and J is an ideal on $P_\kappa(\lambda)$ with $J \subseteq \mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}$. Let σ be the least cardinal τ such that $\tau^{<\overline{\theta}} \geq \lambda$. Then $\overline{\mathrm{cof}}(J) \geq \sigma$.*

Proof. If $\overline{\mathrm{cof}}(J) > \lambda^{<\overline{\theta}}$, there is nothing to prove. Otherwise, there is by Fact 4.4 $A \in \left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right)^*$ such that $J|A = I_{\kappa,\lambda}|A$. Then by Fact 4.5, $\sigma \leq \overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \leq \overline{\mathrm{cof}}(J)$. □

In particular, $\overline{\mathrm{cof}}(J) \geq \lambda$ for any ideal $J \subseteq \mathrm{NS}_{\kappa,\lambda}$.

FACT 4.7. ([8])

(i) *Suppose that λ is a successor cardinal, say $\lambda = \nu^+$. Then $\mathrm{NSS}_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for some $C \in \mathrm{NS}^*_{\kappa,\lambda}$ if and only if $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\nu}) \leq \lambda$.*

(ii) *Suppose that λ is a regular limit cardinal. Then $\mathrm{NSS}_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for some $C \in \mathrm{NS}^*_{\kappa,\lambda}$ if and only if $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) \leq \lambda = \mathrm{cov}(\lambda, \tau^+, \tau^+, \kappa)$ for every cardinal τ with $\kappa \leq \tau < \lambda$.*

Recall from the introduction that $\mathcal{H}_{\kappa,\lambda}$ is said to hold if $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) \leq \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$.

PROPOSITION 4.8. *Suppose that λ is a regular cardinal. Then the following are equivalent :*

(i) *$\mathcal{H}_{\kappa,\lambda}$ holds.*

(ii) *$\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda}) = \lambda$.*

(iii) *$\mathrm{NSS}_{\kappa,\lambda}|C = I_{\kappa,\lambda} \mid C$ for some $C \in \mathrm{NS}^*_{\kappa,\lambda}$.*

Proof.
(i)$\longrightarrow$ (ii) : By Proposition 4.6, $\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda}) \geq \lambda$. For the reverse inequality, we consider two cases. First suppose that λ is a successor cardinal, say $\lambda = \nu^+$. Then by Fact 4.3 $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\nu \leq \delta < \lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda}$. Now for $\nu \leq \delta < \lambda$, $\overline{\mathrm{cof}}(\mathrm{NS}^{\delta}_{\kappa,\lambda}) \leq \overline{\mathrm{cof}}(\mathrm{NS}^{\nu}_{\kappa,\lambda}) = \max\{\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\nu}), \mathrm{cov}(\lambda,\lambda,\lambda,\kappa)\} \leq \max\{\lambda,\lambda\} = \lambda$ by Facts 2.4 (ii) and 2.10. Hence $\overline{\mathrm{cof}}\big(\bigcup_{\nu \leq \delta < \lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda}\big) \leq \lambda$.
Next suppose that λ is a limit cardinal. Given a cardinal χ with $\kappa \leq \chi < \lambda$, by Corollary 3.4 $\overline{\mathrm{cof}}(\mathrm{NS}^{\chi}_{\kappa,\tau}) \leq \lambda$ for every cardinal τ with $\chi \leq \tau < \lambda$, so by Fact 2.10 $\overline{\mathrm{cof}}(\mathrm{NS}^{\chi}_{\kappa,\lambda}) \leq \lambda$. It follows that $\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda}) \leq \lambda$ since by Fact 4.3 $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\kappa \leq \chi < \lambda} \mathrm{NS}^{\chi}_{\kappa,\lambda}$.

(ii)$\longrightarrow$ (iii) : By Fact 4.4.
(iii) $\longrightarrow$ (i) : By Facts 2.5 (iii) and 4.7. $\square$

5 Ideals J on $P_\kappa(\lambda)$ with $\overline{\mathbf{cof}}(J) = \lambda$

In this section we look for cases when $\overline{\mathrm{cof}}\big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\big) = \lambda$, where $\kappa < \xi \leq \lambda+1$. We start with the following observation.

LEMMA 5.1. *Suppose that $K \subseteq \mathrm{NS}_{\kappa,\lambda}$ is an ideal on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(K) \leq \lambda$, and ξ is an ordinal such that*

- $\kappa < \xi \leq \lambda + 1$ *;*
- *ξ is either a successor ordinal, or a limit ordinal of cofinality at least κ ;*
- $\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda} \subseteq K$.

Then $\overline{\mathrm{cof}}\big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\big) = \lambda$.

Proof. By Fact 4.5 we may find $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$ such that $K|A = I_{\kappa,\lambda}|A$. For any cardinal χ with $\kappa \leq \chi < \xi$, $\mathrm{NS}^{\chi}_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$, so by Lemma 2.5 (ii) $\overline{\mathrm{cof}}(\mathrm{NS}^{\chi}_{\kappa,\lambda}) \leq \lambda$. Hence by Fact 2.4 (ii) $\overline{\mathrm{cof}}(\mathrm{NS}^{\delta}_{\kappa,\lambda}) \leq \lambda$ for every δ with $\kappa \leq \delta < \xi$. It easily follows that $\overline{\mathrm{cof}}\big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\big) \leq \lambda$. The reverse inequality holds by Proposition 4.6. $\square$

So we are looking for a large $K \subseteq \mathrm{NS}_{\kappa,\lambda}$ with $\overline{\mathrm{cof}}(K) \leq \lambda$. Assuming that $\mathcal{H}_{\kappa,\lambda}$ holds, we can take $K = \bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}^{\delta}_{\kappa,\lambda}$ if λ is a singular cardinal of cofinality at least κ, and $K = \mathrm{NSS}_{\kappa,\lambda}$ otherwise.

FACT 5.2. ([10]) *Let θ be a cardinal with $2 \leq \theta \leq \kappa$. Suppose $\overline{\theta} \leq \mathrm{cf}(\lambda) < \kappa$. Then for any cardinal ν with $\kappa \leq \nu < \lambda$, $\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\big) \leq \bigcup_{\nu \leq \tau < \lambda} \overline{\mathrm{cof}}\big(\mathrm{NS}^{[\tau]^{<\theta}}_{\kappa,\tau}\big)$.*

PROPOSITION 5.3. *Let θ be a cardinal with $2 \le \theta \le \kappa$. Suppose that $\overline{\theta} \le \mathrm{cf}(\lambda) < \kappa$ and there is a cardinal ν with $\kappa \le \nu < \lambda$ such that for any cardinal τ with $\nu \le \tau < \lambda$, $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\tau]^{<\theta}}) \le \lambda$ and $\tau^{<\overline{\theta}} < \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}) = \lambda$.*

Proof. By Proposition 4.6 and Fact 5.2. $\square$

In particular, if $\mathrm{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}) = \lambda$.

Note that if $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}) = \lambda$, then by Fact 4.4 $\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}} = I_{\kappa,\lambda}|C$ for some C.

FACT 5.4. ([11]) *Let $A \in I_{\kappa,\lambda}^+$ be such that $|\{a \in A : b \subseteq a\}| = |A|$ for every $b \in P_\kappa(\lambda)$. Then A can be decomposed into $|A|$ pairwise disjoint members of $I_{\kappa,\lambda}^+$.*

PROPOSITION 5.5. *Let θ be a cardinal with $2 \le \theta \le \kappa$. Suppose that there is C such that $\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}} = I_{\kappa,\lambda}|C$. Then $P_\kappa(\lambda)$ can be split into π members of $\left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^+$, where π is the least size of any member of $\left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$.*

Proof. Pick $D \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$. Then by Fact 5.4, $C \cap D$ can be decomposed into π pairwise disjoints members of $\left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^+$. $\square$

In particular, if $\mathrm{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|C$ for some C, then $P_\kappa(\lambda)$ can be split into $c(\kappa, \lambda)$ disjoint stationary sets.

PROPOSITION 5.6. *Suppose that θ and ρ are two cardinals such that $\omega \le \theta = \mathrm{cf}(\theta) < \kappa \le \rho \le \lambda$, $u(\theta, \lambda) = \lambda$, and either $\mathrm{cf}(\lambda) < \kappa$ or $\mathrm{cf}(\lambda) > \rho^{<\theta}$. Suppose further that for every cardinal τ with $\rho \le \tau < \lambda$, $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\tau]^{<\theta}}) \le \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\rho]^{<\theta}}) \le \lambda$.*

Proof. It suffices to show that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\rho]^{<\theta}}) \le \lambda$ for any cardinal τ with $\rho \le \tau < \lambda$ since by Facts 2.1 and 2.10 $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\rho]^{<\theta}}) = \bigcup_{\rho < \tau < \lambda} \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\rho]^{<\theta}})$ if λ is a limit cardinal, and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\rho]^{<\theta}}) = \max\{\lambda, \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\nu}^{[\rho]^{<\theta}})\}$ if $\lambda = \nu^+$. Now for any cardinal τ with $\rho \le \tau < \lambda$,

$$\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\rho]^{<\theta}}) \le \overline{\mathfrak{d}_{\kappa,\tau}^{\rho^{<\overline{\theta}}}} \le \overline{\mathfrak{d}_{\kappa,\tau}^{\tau^{<\theta}}} \le u\big(\theta, \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\tau]^{<\theta}})\big) \le u(\theta, \lambda) = \lambda$$

by Lemmas 3.1 and 3.2. $\square$

PROPOSITION 5.7. *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and ξ is an ordinal such that*

- *$\kappa < \xi \le \eta$, where η equals $\lambda + 1$ if $\mathrm{cf}(\lambda) < \kappa$, and $\mathrm{cf}(\lambda)$ otherwise ;*
- *ξ is either a successor ordinal, or a limit ordinal of cofinality at least κ.*

Then $\overline{\mathrm{cof}}\Big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\Big) = \lambda$.

Proof. By Facts 2.4 (ii) and 5.1 and Propositions 4.8, 5.3 and 5.6. □

In particular if $\mathcal{H}_{\kappa,\lambda}$ holds and $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $\overline{\mathrm{cof}}\Big(\bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}^{\delta}_{\kappa,\lambda}\Big) = \lambda$ (and hence by Fact 1.5 (iv) there is no A such that $\mathrm{NS}_{\kappa,\lambda} = \Big(\bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}^{\delta}_{\kappa,\lambda}\Big)|A$).

6 Ideals J on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(J) < \lambda$

There may exist ideals J on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) < \lambda$. Some examples were presented in [10]. We now give some more.

Given two cardinals $\pi \leq \kappa$ and $\chi \geq \lambda$, $\mathcal{A}_{\kappa,\lambda}(\pi,\chi)$ asserts the existence of $Z \subseteq P_\pi(\lambda)$ with $|Z| = \chi$ such that $|Z \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

FACT 6.1. ([10]) *Let θ and χ be two cardinals such that*

- $2 \leq \theta \leq \kappa$, $\lambda < \chi$ *and there is a* $[\chi]^{<\theta}$*-normal ideal on* $P_\kappa(\chi)$ *;*
- $\mathcal{A}_{\kappa,\lambda}(\pi,\chi)$ *holds for some regular uncountable cardinal* $\pi < \kappa$.

Then $\overline{\mathrm{cof}}(I_{\kappa,\chi}|A) \leq \lambda$ *for some* $A \in \big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\big)^+$.

FACT 6.2. ([9]) *Let τ be the largest limit cardinal less than or equal to κ. Assume $\mathrm{cf}(\lambda) < \kappa$ and one of the following conditions is satisfied :*

(a) $\tau = \kappa$.

(b) $\tau > \mathrm{cf}(\lambda)$ *and* $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\tau)$.

(c) $\tau > \mathrm{cf}(\lambda) = \mathrm{cf}(\tau)$ *and* $\min\{\mathrm{pp}(\tau), \tau^{+3}\} < \kappa$.

(d) $\tau \leq \mathrm{cf}(\lambda)$ *and* $\min\{2^{\mathrm{cf}(\lambda)}, (\mathrm{cf}(\lambda))^{+3}\} < \kappa$.

Then $\mathcal{A}_{\kappa,\lambda}\big((\mathrm{cf}(\lambda))^+, \lambda^+\big)$ *holds.*

Suppose for instance that κ is a limit cardinal and $\mathrm{cf}(\lambda) < \kappa$. Then by Facts 6.1 and 6.2, $\overline{\mathrm{cof}}(I_{\kappa,\lambda^+}|B) \leq \lambda$ for some $B \in \mathrm{NS}^+_{\kappa,\lambda^+}$.

Note that in case κ is the successor of a cardinal of cofinality $\mathrm{cf}(\lambda)$, Fact 6.2 does not apply, as none of the conditions (a) - (d) is satisfied. To handle this case, we introduce the following principle.

Given a cardinal $\chi \geq \lambda$, $\mathcal{B}_{\kappa,\lambda}(\chi)$ asserts the existence of $Z \subseteq P_\kappa(\lambda)$ with $|Z| = \chi$ such that for every $e \subseteq Z$ with $|e| = \kappa$, there is a $< \kappa$ -to-one function in $\prod_{z\in e} z$.

FACT 6.3. ([10]) *Let θ and χ be two cardinals such that*

- $2 \leq \theta \leq \kappa$, $\lambda < \chi$ *and there is a* $[\chi]^{<\theta}$*-normal ideal on* $P_\kappa(\chi)$;
- $\mathcal{B}_{\kappa,\lambda}(\chi)$ *holds.*

Then $\overline{\text{cof}}(I_{\kappa,\chi}|A) \leq \lambda$ *for some* $A \in \left(\text{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\right)^+$.

Note that in case $\text{cf}(\lambda < \kappa$, $\mathcal{B}_{\kappa,\lambda}(\lambda^+)$ follows from ADS_λ, where ADS_λ asserts the existence of $y_\alpha \subseteq \lambda$ for $\alpha < \lambda^+$ such that

- for any $\alpha < \lambda^+$, $\sup y_\alpha = \lambda$ and $\text{o.t.}(y_\alpha) = \text{cf}(\lambda)$;
- given $\beta < \lambda^+$, there is $g : \beta \longrightarrow \lambda$ such that
$$\big(y_\alpha \setminus g(\alpha)\big) \cap \big(y_{\alpha'} \setminus g(\alpha')\big) = \emptyset$$
for any $\alpha, \alpha' \in \beta$ with $\alpha \neq \alpha'$.

For more on the existence of $A \in \left(\text{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\right)^+$ such that $\overline{\text{cof}}(I_{\kappa,\chi}|A) < \chi$, see [9] and [10].

PROPOSITION 6.4. *Suppose that θ and χ are two cardinals such that $2 \leq \theta \leq \kappa$ and $\lambda < \chi$, and $A \in \left(\text{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\right)^+$ is such that $\overline{\text{cof}}(I_{\kappa,\chi}|A) \leq \lambda$. Then there is $B \in \left(\text{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\right)^+$ and a function f such that*

- *f is an isomorphism from $(P_\kappa(\lambda), \subset)$ onto $(B, \subset)$;*
- *for any $\delta \leq \lambda$, $f\big(\text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\big) = \text{NS}_{\kappa,\chi}^{[\delta]^{<\theta}}|B$ (and hence $\overline{\text{cof}}\big(\text{NS}_{\kappa,\chi}^{[\delta]^{<\theta}}|B\big) \leq \overline{\text{cof}}\big(\text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\big)$ and $\text{cof}\big(\text{NS}_{\kappa,\chi}^{[\delta]^{<\theta}}\big) \leq \text{cof}\big(\text{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\big)$).*

Proof. Select $x_\beta \in P_\kappa(\chi)$ for $\beta < \lambda$ so that for each $X \in I_{\kappa,\chi}$, there is $z \in P_\kappa(\lambda)$ with $X \cap \{y \in A : \bigcup_{\beta \in z} x_\beta \subseteq y\} = \emptyset$. For $\lambda \leq \alpha < \chi$, pick $z_\alpha \in P_\kappa(\lambda)$ with
$$\{y \in A : \textstyle\bigcup_{\beta \in z_\alpha} x_\beta \subseteq y\} \subseteq \{t \in P_\kappa(\chi) : \alpha \in t\}.$$
Let C be the set of all $x \in P_\kappa(\chi)$ such that $\big(\bigcup_{\beta \in x \cap \lambda} x_\beta\big) \cup \big(\bigcup_{\alpha \in x \setminus \lambda} z_\alpha\big) \subseteq x$. Note that $C \in \text{NS}^*_{\kappa,\chi}$.

Claim 1. *Let $x \in A \cap C$. Then $x \setminus \lambda = \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda\}$.*

Proof of Claim 1. Since $x \in C$, $x \setminus \lambda \subseteq \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda\}$. To show the reverse inclusion, fix $\alpha \in \chi \setminus \lambda$ with $z_\alpha \subseteq x \cap \lambda$. Then $\bigcup_{\beta \in z_\alpha} x_\beta \subseteq x$, and hence $\alpha \in x$, which completes the proof of Claim 1.

Claim 2. Let $a \in P_\kappa(\lambda)$. Then $|\{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq a\}| < \kappa$.

Proof of Claim 2. Pick $x \in A \cap C$ with $a \subseteq x$. Then by Claim 1,

$$\{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq a\} \subseteq \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq x \cap \lambda\} \subseteq x,$$

which completes the proof of Claim 2.

Using Claim 2, define $f : P_\kappa(\lambda) \longrightarrow P_\kappa(\chi)$ by $f(a) = a \cup \{\alpha \in \chi \setminus \lambda : z_\alpha \subseteq a\}$. Put $B = \operatorname{ran}(f)$. By Claim 1, $x = f(x \cap \lambda)$ for any $x \in A \cap C$, so $A \cap C \subseteq B$. It follows that $B \in \big(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\big)^+$.

As is easily seen, f is an isomorphism from $\big(P_\kappa(\lambda), \subset\big)$ onto $(B, \subset)$, and moreover $f^{-1}(X) \in I_{\kappa,\lambda}$ for any $X \in I_{\kappa,\chi}$. Now fix $\delta \leq \lambda$. Set $J = \mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}$. It is simple to see that $f(J)$ is an ideal on $P_\kappa(\chi)$ with the property that $B \in \big(f(J)\big)^*$.

Claim 3. $f(J)$ is $[\delta]^{<\theta}$-normal.

Proof of Claim 3. Fix $X \in \big(f(J)\big)^+ \cap P(B)$ and $h : X \longrightarrow P_\theta(\delta)$ such that $h(x) \in P_{|x \cap \theta|}(x)$ for every $x \in X$. Define $k : f^{-1}(X) \longrightarrow P_\theta(\delta)$ by $k(a) = h\big(f(a)\big)$. There must be $A \in J^+ \cap P\big(f^{-1}(X)\big)$ such that k is constant on A. Then clearly $f``A \in \big(f(J)\big)^+ \cap P(X)$, and moreover $h$ is constant on $f``A$, which completes the proof of the claim.

It immediately follows from Claim 3 that $\mathrm{NS}_{\kappa,\chi}^{[\delta]^{<\theta}} | B \subseteq f(J)$.

To establish the reverse inclusion fix $Y \in f(J)$. Since $f^{-1}(Y \cap B) \in J$, we may find $g : P_\theta(\delta) \longrightarrow P_\kappa(\lambda)$ such that $f^{-1}(Y \cap B) \cap C(g, \kappa, \lambda) = \emptyset$. Then clearly $(Y \cap B) \cap C(g, \kappa, \chi) = \emptyset$ and hence $Y \cap B \in \mathrm{NS}_{\kappa,\chi}^{[\delta]^{<\theta}}$. $\square$

Let $\kappa = (2^\rho)^+$, where ρ is an infinite cardinal, and suppose that λ is a strong limit cardinal with $\mathrm{cf}(\lambda) \leq \rho$. Then $\mathcal{A}_{\kappa,\lambda}(\rho^+, 2^\lambda)$ holds, since $|P_{\rho^+}(\lambda) \cap P(a)| \leq 2^\rho$ for any $a \in P_\kappa(\lambda)$. Hence by Facts 5.2 and 6.1 and Proposition 6.4, $\overline{\mathrm{cof}}\big(\mathrm{NS}_{\kappa,2^\lambda}^\lambda | B\big) \leq \lambda$ for some $B \in \mathrm{NS}_{\kappa,2^\lambda}^+$.

PROPOSITION 6.5. *Suppose that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}) \leq \lambda^+$, and there is $A \in \mathrm{NS}_{\kappa,\lambda^+}^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda^+} | A) \leq \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda^+} | B) < \lambda^+$ for some $B \in \mathrm{NS}_{\kappa,\lambda^+}^+$.*

Proof. By Fact 4.7 (i), there is $C \in \mathrm{NS}_{\kappa,\lambda^+}^*$ such that $\mathrm{NSS}_{\kappa,\lambda^+} | C = I_{\kappa,\lambda^+} | C$. Then $B = A \cap C$ is as desired. $\square$

For example, suppose that $\kappa = \omega_4$ and $\lambda = \beth_\alpha$ for some infinite limit ordinal α of cofinality ω. Then by Facts 5.2, 6.1 and 6.2 and Proposition 6.5, $\overline{\mathrm{cof}}\big(\mathrm{NSS}_{\kappa,\lambda^+} | B\big) \leq \lambda$ for some $B \in \mathrm{NS}_{\kappa,\lambda^+}^+$.

If λ is singular, then by Fact 4.1 $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda} | B$ for some B, so $\overline{\mathrm{cof}}(\mathrm{NSS}_{\kappa,\lambda} | A) < \lambda$ for some $A \in \mathrm{NS}_{\kappa,\lambda}^+$ just in case $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda} | D) < \lambda$ for some $D \in \mathrm{NS}_{\kappa,\lambda}^+$.

Suppose that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}|D) < \lambda$ for some $D \in \mathrm{NS}^+_{\kappa,\lambda}$. Then setting $\sigma = \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}|D)$,

$$\mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}) \leq u(\kappa,\sigma) \leq u(\kappa,\lambda) \leq \mathrm{cof}(\mathrm{NS}_{\kappa,\lambda})$$

by Fact 2.11 (ii), so $\mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}) = u(\kappa,\sigma) = u(\kappa,\lambda)$. Hence by Fact 2.5 (iv), SSH does not hold.

PROPOSITION 6.6. *Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa$ and $\lambda < \chi$. Suppose that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}) \leq \chi^{<\overline{\theta}}$, and there is $A \in \left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\chi}|A| \leq \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}|B) \leq \lambda$ for some $B \in \left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$.*

Proof. By Fact 4.4 there is $C \in \left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^*$ such that $\mathrm{NS}_{\kappa,\chi}|C = I_{\kappa,\chi}|C$. Then $B = A \cap C$ is as desired. $\square$

Here is an example of a situation where Proposition 6.6 applies. Starting from a $\mathcal{P}^3(\nu)$-hypermeasurable, Cummings [3] constructs a generic extension W of V in which for any infinite cardinal $\rho, 2^\rho$ equals ρ^+ if ρ is a successor cardinal, and ρ^{++} otherwise. In W, let σ be a regular uncountable cardinal, and $\mu > \sigma$ be a cardinal of cofinality less than σ. Suppose that

- σ is not the successor of a cardinal τ with $\mathrm{cf}(\tau) \leq \mathrm{cf}(\mu)$;
- σ is not the successor of the successor of a limit cardinal π with $\mathrm{cf}(\pi) \leq \mathrm{cf}(\mu)$.

Then by Facts 6.1 and 6.2 and Proposition 6.6, $\overline{\mathrm{cof}}\big(\mathrm{NS}_{\sigma,\mu^+}|B\big) \leq \mu$ for some $B \in \left(\mathrm{NS}^{[\mu^+]^{<(\mathrm{cf}(\mu))^+}}_{\sigma,\mu^+}\right)^+$.

7 Cases when $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A

In this section we establish that if $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A. Note that if $\mathrm{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then by Facts 4.5 and 5.1, $\mathrm{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$ for some A. Note further that if λ is regular, then trivially $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\lambda}_{\kappa,\lambda}|P_\kappa(\lambda)$. By combining the three cases, we obtain that if $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) \leq \lambda$ for every cardinal τ with $\max\{\kappa, \mathrm{cf}(\lambda)\} \leq \tau < \lambda$, then $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A.

THEOREM 7.1. *Let π, θ and χ be three cardinals with $\kappa \leq \pi < \lambda$ and $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Suppose that*

- *λ is singular ;*
- *$\overline{\theta} \leq \mathrm{cf}(\lambda)$ in case $\chi = \lambda$;*
- *$\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\min\{\chi,\tau\}]^{<\theta}}_{\kappa,\tau}\big) \leq \lambda^{<\overline{\theta}}$ for every cardinal τ with $\pi \leq \tau < \lambda$.*

Then there is $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$ *such that* $\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}} \subseteq \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$.

Proof. Set $\mu = \mathrm{cf}(\lambda)$ and select an increasing sequence of cardinals $\langle \lambda_\eta : \eta < \mu \rangle$ so that

- $\sup\{\lambda_\eta : \eta < \mu\} = \lambda$;
- $\lambda_0 > \max\{\pi, \mu\}$;
- $\lambda_0 \geq \chi$ in case $\chi < \lambda$.

For $\eta < \mu$, pick a family G_η of functions from $P_{\max\{\overline{\theta},3\}}(\min\{\chi, \lambda_\eta\})$ to $P_3(\lambda_\eta)$ so that $|G_\eta| \leq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda_\eta}^{[\min\{\chi,\lambda_\eta\}]^{<\theta}})$ and for every $H \in (\mathrm{NS}_{\kappa,\lambda_\eta}^{[\min\{\chi,\lambda_\eta\}]^{<\theta}})^*$, there is $y \in P_\kappa(G_\eta) \setminus \{\emptyset\}$ such that $\{b \in \bigcap_{g \in y} C(g, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq H$. Let $\bigcup_{\eta<\mu} G_\eta = \{g_e : e \in P_{\overline{\theta}}(\lambda)\}$.
Let A be the set of all $a \in P_\kappa(\lambda)$ such that

- $\overline{\theta} \subseteq a$ in case $\overline{\theta} < \kappa$;
- $\omega \subseteq a$;
- $a \cap \kappa \in \kappa$;
- $k(\alpha) \in a$ for every $\alpha \in a$, where $k : \lambda \to \mu$ is defined by $k(\alpha) =$ the least $\eta < \mu$ such that $\alpha \in \lambda_\eta$;
- if $\chi = \lambda$, then $i(v) \in a$ for every $v \in P_{|a \cap \max\{\overline{\theta},3\}|}(a)$, where $i : P_{\max\{\overline{\theta},3\}}(\lambda) \to \mu$ is defined by $i(v) =$ the least $\eta < \mu$ such that $v \subseteq \lambda_\eta$;
- $g_e(u) \subseteq a$ whenever $e \in P_{|a\cap\overline{\theta}|}(a)$ and $u \in P_{|a\cap\max\{\overline{\theta},3\}|}(a) \cap \mathrm{dom}(g_e)$.

It is immediate that $A \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$. Let us check that A is as desired. Thus fix $f : P_{\max\{\overline{\theta},3\}}(\chi) \to P_3(\lambda)$. Given $\eta < \mu$, define $p_\eta : P_{\max\{\overline{\theta},3\}}(\min\{\chi, \lambda_\eta\}) \to P_2(\lambda_\eta)$ by $p_\eta(v) = \{\zeta\}$, where $\zeta =$ the least σ such that $\eta \leq \sigma < \mu$ and $f(v) \subseteq \lambda_\sigma$. Also define $q_\eta : P_{\max\{\overline{\theta},3\}}(\min\{\chi, \lambda_\eta\}) \to P_3(\lambda_\eta)$ by $q_\eta(v) = \lambda_\eta \cap f(v)$. Select $x_\eta, y_\eta \in P_\kappa(P_{\overline{\theta}}(\lambda)) \setminus \{\emptyset\}$ so that

- $\{g_e : e \in x_\eta \cup y_\eta\} \subseteq G_\eta$;
- $\{b \in \bigcap_{e \in x_\eta} C(g_e, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(p_\eta, \kappa, \lambda_\eta)$;
- $\{b \in \bigcap_{e \in y_\eta} C(g_e, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(q_\eta, \kappa, \lambda_n)$.

Finally define $u : \mu \to P_\kappa(\lambda)$ by $u(\eta) = \bigcup(x_\eta \cup y_\eta)$, and $t : P_2(\mu) \to P_\kappa(\lambda)$ so that for any $\eta \in \mu$, $t(\{\eta\})$ equals $u(\eta)$ if $\overline{\theta} < \kappa$, and $u(\eta) \cup |u(\eta)|^+$ otherwise.
We claim that $A \cap C_t^{\kappa,\lambda} \subseteq C_f^{\kappa,\lambda}$. Thus let $a \in A \cap C_t^{\kappa,\lambda}$ and $v \in P_{|a\cap\max\{\overline{\theta},3\}|}(a \cap \chi)$. There must be $\eta \in a \cap \mu$ such that $v \subseteq \lambda_\eta$. Then $a \cap \lambda_\eta \in C(p_\eta, \kappa, \lambda_\eta)$ since $x_\eta \subseteq P_{|a\cap\overline{\theta}|}(a)$. It follows that $v \cup f(v) \subseteq \lambda_\sigma$ for some $\sigma \in a \cap \mu$. Now

$a \cap \lambda_\sigma \in C(q_\sigma, \kappa, \lambda_\sigma)$, since $y_\sigma \subseteq P_{|a \cap \overline{\theta}|}(a)$, so $f(v) \subseteq a$. $\square$

In Theorem 7.1 we assumed that $\overline{\theta} \leq \mathrm{cf}(\lambda)$ in case $\chi = \lambda$. Some condition of this kind is necessary. In fact if $\mathrm{cf}(\lambda) < \kappa$ and $u(\kappa, \lambda^{<\overline{\theta}}) = \lambda^{<\overline{\theta}}$, then for each $A \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*, \mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}} \neq \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$ since by Fact 2.11,

$$\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\overline{\theta}}}) > \lambda^{<\overline{\theta}} \geq \lambda \geq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)} \mid A).$$

COROLLARY 7.2. *Suppose that one of the following holds :*

(i) SSH *holds.*

(ii) There exists a σ-saturated ideal on $P_\nu(\lambda)$, where σ and ν are two cardinals such that $\omega < \nu = \mathrm{cf}(\nu) < \lambda$ and $\sigma < \nu$.

(iii) There is a regular uncountable cardinal $\tau < \lambda$ that is mildly π-ineffable for every cardinal π with $\tau \leq \pi < \lambda$.

Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa$, $\max\{\kappa, \mathrm{cf}(\lambda)\} \leq \chi < \lambda$ and $\overline{\mathrm{cof}}\big(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\big) \leq \lambda^{<\overline{\theta}}$. Then $\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}|A = \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$ for some $A \in \big(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$.

Proof. Use Facts 2.3 and 2.10. $\square$

COROLLARY 7.3. *Suppose that $\mathrm{cf}(\lambda) < \kappa$, and θ and χ are two cardinals such that $2 \leq \theta \leq \kappa \leq \chi < \lambda$ and $\overline{\mathrm{cof}}\big(\mathrm{NS}_{\kappa,\tau}^{[\chi]^{<\theta}}\big) \leq \lambda^{<\overline{\theta}}$ for every cardinal τ with $\chi \leq \tau < \lambda$. Then $\mathrm{cof}\big(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}\big) = u(\kappa, \lambda)$.*

Proof. By Theorem 7.1 there is $A \in \big(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\big)^*$ such that $\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}|A = I_{\kappa,\lambda}|A$. Then by Fact 2.11 (ii), $\mathrm{cof}\big(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}\big) = \mathrm{cof}\big(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}|A\big) = \mathrm{cof}(I_{\kappa,\lambda}|A) = \mathrm{cof}(I_{\kappa,\lambda}) = u(\kappa, \lambda)$. $\square$

COROLLARY 7.4.

(i) *Suppose that λ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds. Then $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$ for some A.*

(ii) *Let χ be a cardinal such that $\max\{\kappa, \mathrm{cf}(\lambda)\} \leq \chi < \lambda$ and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{\chi}) \leq \lambda$ for every cardinal τ with $\chi \leq \tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}^{\chi}|A = \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)} \mid A$ for some $A \in \mathrm{NS}_{\kappa,\lambda}^*$.*

COROLLARY 7.5.

(i) *Let $\chi \geq \kappa$ be a cardinal, and $\alpha < \kappa$ be a limit ordinal such that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}) \leq \chi^{+\alpha}$. Then $\mathrm{NS}^{\chi}_{\kappa,\chi^{+\alpha}}|A = I_{\kappa,\chi^{+\alpha}}|A$ for some $A \in \mathrm{NS}^*_{\kappa,\chi^{+\alpha}}$.*

(ii) *Let $\chi > \kappa$ be a cardinal such that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}) < \chi^{+\kappa}$. Then $\mathrm{NS}^{\chi}_{\kappa,\chi^{+\kappa}}|A = \mathrm{NS}^{\kappa}_{\kappa,\chi^{+\kappa}}|A$ for some $A \in \mathrm{NS}^*_{\kappa,\chi^{+\kappa}}$.*

Proof. Use Facts 2.1 and 2.10. $\square$

Note that we do get a better result by considering the reduced cofinality ($\overline{\mathrm{cof}}$) instead of the usual one (cof). For example, suppose that GCH holds in V. By a result of [12], there is a $< \kappa$-closed, κ^+-cc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$, $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\kappa}) = \kappa^{+\omega}$ and $\mathrm{cof}(\mathrm{NS}_{\kappa,\kappa}) = \kappa^{+(\omega+1)}$. Then in $V^{\mathbb{P}}$, there is by Corollary 7.5 (i) $A \in \mathrm{NS}^*_{\kappa,\kappa^{+\omega}}$ such that $\mathrm{NS}^{\kappa}_{\kappa,\kappa^{+\omega}}|A = I_{\kappa,\kappa^{+\omega}}|A$.

Let us next discuss the condition in Theorem 7.1 that $\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\tau]^{<\theta}}_{\kappa,\tau}\big) \leq \lambda^{<\overline{\theta}}$ for almost all cardinals $\tau < \lambda$.

PROPOSITION 7.6. *Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi < \lambda$. Suppose that $\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\big) \leq \lambda^{<\overline{\theta}}$ and $\chi^{<\overline{\theta}} \geq \lambda$. Then $\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}\big) = \overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\big) \leq \chi$.*

Proof. By Fact 2.6 (iii) $\chi^{<\overline{\theta}} = \lambda^{<\overline{\theta}}$, so by Fact 4.5 $\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi} = I_{\kappa,\chi}|A$ for some A. It follows that $\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\big) \leq \chi$. Moreover by Fact 2.10

$$\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}\big) = \max\{\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\big), \mathrm{cov}\big(\lambda, (\lambda^{<\overline{\theta}})^+, (\lambda^{<\overline{\theta}})^+, \kappa\big)\} = \overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\big).$$

$\square$

COROLLARY 7.7. *Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi < \chi^{<\overline{\theta}} = \lambda$. Suppose $\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\big) \leq \chi^{<\overline{\theta}}$. Then there is $A \in \big(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\big)^*$ such that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda} \mid A) \leq \chi$.*

Proof. By Fact 2.7 we may find $A \in \big(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\big)^*$ such that $\mathrm{NS}_{\kappa,\lambda}|A = \mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}|A$. Then by Proposition 7.6, $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}|A) \leq \overline{\mathrm{cof}}\big(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}\big) \leq \chi$. $\square$

Question. Suppose that θ and χ are two cardinals such that $2 \leq \theta \leq \kappa \leq \chi$ and $\overline{\mathrm{cof}}\big(NS^{[\chi]^{<\theta}}_{\kappa,\chi}\big) \leq \chi^{<\overline{\theta}}$. Does then $\chi^{<\overline{\theta}} = \chi$ hold ?

PROPOSITION 7.8.

(i) *Suppose that θ and σ are two cardinals such that $2 \leq \theta \leq \kappa \leq \sigma < \lambda$, $\overline{\theta} \leq \mathrm{cf}(\lambda)$ and $\overline{\mathrm{cof}}\big(\mathrm{NS}^{[\tau]^{<\theta}}_{\kappa,\tau}\big) \leq \lambda^{<\overline{\theta}}$ for every cardinal τ with $\sigma \leq \tau < \lambda$.*

Then there is a cardinal π *with* $\sigma \leq \pi < \lambda$ *such that* $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \lambda$ *for every cardinal* χ *with* $\pi \leq \chi < \lambda$.

(ii) *Let* θ *and* π *be two cardinals with* $2 \leq \theta \leq \kappa \leq \pi < \lambda$. *Suppose that* $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, *and* $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \lambda$ *for every cardinal* χ *with* $\pi \leq \chi < \lambda$. *Then* $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\rho}^{[\rho]^{<\theta}}) < \lambda$ *for every cardinal* ρ *with* $\max\{\pi, \mathrm{cf}(\lambda)\} \leq \rho < \lambda$.

Proof.

(i) : If $\nu^\rho < \lambda$ for every cardinal $\nu < \lambda$ and every cardinal $\rho < \overline{\theta}$, then $\lambda^{<\overline{\theta}} = \lambda$, and $\pi = \sigma$ is as desired. Now suppose there are two cardinals $\nu < \lambda$ and $\rho < \overline{\theta}$ such that $\nu^\rho \geq \lambda$. Set $\pi = \max\{\nu, \sigma\}$. Let χ be a cardinal with $\pi \leq \chi < \lambda$. Then $\chi^{<\overline{\theta}} = \lambda^{<\overline{\theta}}$, so by Proposition 7.6 $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \chi$.

(ii) : By Fact 2.9. □

In particular, if $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $\mathcal{H}_{\kappa,\lambda}$ holds just in case $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) < \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$.

Suppose that λ is a limit cardinal and χ is a cardinal with $\kappa \leq \chi \leq \lambda$. If either $\mathrm{cf}(\lambda) < \kappa$ or $\mathrm{cf}(\lambda) > \chi$, then by Fact 2.10 and Lemma 5.1,

$$\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{\chi}) \leq \sup\{\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{\min\{\chi,\tau\}}) : \pi \leq \tau < \lambda\},$$

where π equals κ if $\chi = \lambda$, and χ otherwise. We will now deal with the case when $\kappa \leq \mathrm{cf}(\lambda) \leq \chi$. The proof of the following is a modification of that of Theorem 7.1.

PROPOSITION 7.9 *Let* χ *be a cardinal such that* $\max\{\kappa, \mathrm{cf}(\lambda)\} \leq \chi \leq \lambda$. *Set* $\pi = \kappa$ *if* $\chi = \lambda$, *and* $\pi = \chi$ *otherwise. Then* $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{\chi}) \leq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\rho}^{\mathrm{cf}(\lambda)})$ *and* $\mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}^{\chi}) \leq \mathrm{cof}(\mathrm{NS}_{\kappa,\rho}^{\mathrm{cf}(\lambda)})$ *where* $\rho = \sup\{\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{\min\{\chi,\tau\}}) : \pi \leq \tau < \lambda\}$.

Proof. We can assume that $\mathrm{cf}(\lambda) < \chi$ since otherwise the result is trivial. We show that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{\chi}) \leq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\rho}^{\mathrm{cf}(\lambda)})$ and leave the proof of the other assertion to the reader. Put $\mu = \mathrm{cf}(\lambda)$ and pick an increasing sequence $\langle \lambda_\eta : \eta < \mu \rangle$ of cardinals cofinal in λ so that $\lambda_0 > \max\{\kappa, \mu\}$, and $\lambda_0 \geq \chi$ in case $\chi < \lambda$. For $\eta < \mu$, select a family G_η of functions from $P_3(\min\{\chi, \lambda_\eta\})$ to $P_3(\lambda_\eta)$ so that $|G_\eta| \leq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda_\eta}^{\min\{\chi,\lambda_\eta\}})$ and for any $H \in (\mathrm{NS}_{\kappa,\lambda_\eta}^{\min\{\chi,\lambda_\eta\}})^*$, there is $y \in P_\kappa(G_\eta) \setminus \{\emptyset\}$ with $\{b \in \bigcap_{g \in y} C(g, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq H$. Let $\bigcup_{\eta<\mu} G_\eta = \{g_\xi : \xi < \rho\}$. For $\xi < \rho$, let $g_\xi \in G_{\eta_\xi}$. Let A be the set of all $a \in P_\kappa(\lambda)$ such that $\omega \subseteq a$, $a \cap \kappa \in \kappa$ and $k(\alpha) \in a$ for all $\alpha \in a$, where $k : \lambda \to \mu$ is defined by $k(\alpha) =$ the least $\eta < \mu$ such that $\alpha \in \lambda_\eta$. Clearly $A \in \mathrm{NS}_{\kappa,\lambda}^*$, so by Fact 2.5 (ii) $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{\chi} | A) = \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{\chi})$.

By Proposition 2.3 we may find a collection T of functions from μ to $P_\kappa(\rho)$ such that $|T| = \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\rho}^{\mu})$ and for any $u : \mu \to P_\kappa(\rho)$, there is $z \in P_\kappa(T)$ with the property that $u(\eta) \subseteq \bigcup_{t\in z} t(\eta)$ for every $\eta \in \mu$. For $t \in T$, let D_t be the set of all $a \in P_\kappa(\lambda)$ such that for any $\eta \in a \cap \mu$ and any $\xi \in t(\eta)$, $a \cap \lambda_{n_\xi} \in C(g_\xi, \kappa, \lambda_{\eta_\xi})$. Note that $D_t \in (\mathrm{NS}^\chi_{\kappa,\lambda})^*$.

Now fix $f : P_3(\chi) \to P_3(\lambda)$. Given $\eta < \mu$, define $p_\eta : P_3(\min\{\chi, \lambda_\eta\}) \to P_2(\lambda_\eta)$ by $p_n(v) = \{\zeta\}$, where $\zeta =$ the least σ such that $\eta \leq \sigma < \mu$ and $f(v) \leq \lambda_\sigma$, and $q_n : P_3(\min\{\chi, \lambda_\eta\}) \to P_3(\lambda_\eta)$ by $q_\eta(v) = \lambda_\eta \cap f(v)$. Select $x_\eta, y_\eta \in P_\kappa(\rho) \setminus \{\emptyset\}$ so that

- $\{g_\xi : \xi \in x_\eta \cup y_\eta\} \subseteq G_\eta$;
- $\{b \in \bigcap_{\xi\in x_\eta} C(g_\xi, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(p_\eta, \kappa, \lambda_\eta)$;
- $\{b \in \bigcap_{\xi\in y_\eta} C(g_\xi, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(q_\eta, \kappa, \lambda_\eta)$.

We may find $z \in P_\kappa(T)$ such that $x_\eta \cup y_\eta \subseteq \bigcup_{t\in z} t(\eta)$ for every $\eta \in \mu$.

Let us show that $A \cap \left(\bigcap_{t\in z} D_t\right) \subseteq C(f, \kappa, \lambda)$. Thus let $a \in A \cap \left(\bigcap_{t\in z} D_t\right)$ and $v \in P_3(a \cap \chi)$. There must be $\eta \in a \cap \mu$ such that $v \subseteq \lambda_\eta$. Then $a \cap \lambda_\eta \in \bigcap_{\xi\in x_\eta} C(g_\xi, \kappa, \lambda_\eta)$, so $v \cup f(v) \subseteq \lambda_\sigma$ for some $\sigma \in a \cap \mu$. Now $a \cap \lambda_\sigma \in \bigcap_{\xi\in y_\sigma} C(g_\xi, \kappa, \lambda_\sigma)$, and therefore $f(v) \subseteq a$. $\square$

8 Nowhere precipitousness of $\mathrm{NS}^\nu_{\kappa,\lambda}$

Throughout this section it is assumed that $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. **Let** ν **be a cardinal with** $\mathrm{cf}(\lambda) \leq \nu < \lambda$. We will show that under certain conditions, $\mathrm{NS}^\nu_{\kappa,\lambda}$ is nowhere precipitous. Our proof will follow that of Theorem 2.1 in [13], except that we do not appeal to pcf theory.
Set $\mu = \mathrm{cf}(\lambda)$. **We assume that** $c(\kappa,\nu) < \lambda$ **in case** $\nu > \mu$. **Let** $\rho < \lambda$ **be a regular cardinal such that** $\rho > \mu$ **if** $\nu = \mu$, **and** $\rho > c(\kappa,\nu)$ **otherwise**. Select a continuous, increasing sequence $\langle\lambda_\beta : \beta < \mu\rangle$ of cardinals so that $\sup\{\lambda_\beta : \beta < \mu\} = \lambda$ and $\lambda_0 > \rho$. Let E be the set of all infinite limit ordinals $\alpha < \mu$ with $\mathrm{cf}(\alpha) < \kappa$. We define D as follows. If $\nu = \mu$, we set $D = E$. Otherwise we pick D in $\mathrm{NS}^*_{\kappa,\nu}$ so that

- for any $d \in D$, $\sup(d \cap \mu)$ is an infinite limit ordinal ;
- $|D| = c(\kappa,\nu)$.

We will show that if $\tau^{|D|} < \lambda$ for every cardinal $\tau < \lambda$, then $\mathrm{NS}^\nu_{\kappa,\lambda}$ is nowhere precipitous.

For $d \in D$, put $\alpha(d) = \sup(d \cap \mu)$. Note that $\alpha(d) \in E$. Moreover $\alpha(d) = d$ in case $\nu = \mu$.

Let W be the set of all $a \in P_\kappa(\lambda)$ such that

- $0 \in a$;
- $\gamma + 1 \in a$ for every $\gamma \in a \cap \nu$;
- $a \cap \kappa \in \kappa$;
- $\lambda_\beta \in a$ for every $\beta \in a \cap \mu$;
- $a \cap \nu \in D$ in case $\nu > \mu$.

Then clearly, $W \in \left(\mathrm{NS}^{\nu}_{\kappa,\lambda}\right)^*$. For $d \in D$, define W_d by letting $W_d = \{a \in W : \sup(a \cap \mu) = d\}$ if $\nu = \mu$, and $W_d = \{a \in W : a \cap \nu = d\}$ otherwise. Note that W is the disjoint union of the W_d's. Moreover, $\sup(a \cap \lambda_{\alpha(d)}) = \lambda_{\alpha(d)}$ for every $a \in W_d$.

LEMMA 8.1. *Suppose that there is $T \subseteq P_\kappa(\lambda)$ such that*

(a) $|T \cap P(a)| < \rho$ *for any* $a \in P_\kappa(\lambda)$;

(b) $u(\rho, \tau) \leq |T|$ *for every cardinal* τ *with* $\rho \leq \tau < \lambda$.

Then for every $R \in (\mathrm{NS}^{\nu}_{\kappa,\lambda})^+$,

$$\{d \in D : |\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})\}$$

lies in NS^+_μ *if* $\nu = \mu$, *and in* $\mathrm{NS}^+_{\kappa,\nu}$ *otherwise.*

Proof. For $\beta \in \mu$, select $Z_\beta \in I^+_{\rho,\lambda_\beta}$ with $|Z_\beta| \leq |T|$. Then clearly there is $Q \subseteq T$ with $|\bigcup_{\beta<\mu} Z_\beta| = |Q|$. Pick a bijection $i : \bigcup_{\beta<\mu} Z_\beta \to Q$ and let j denote the inverse of i. For $\alpha \in E$, define $k_\alpha : P_\kappa(\lambda_\alpha) \to P_\rho(\lambda_\alpha)$ by $k_\alpha(b) = \bigcup_{e \in Q \cap P(b)} (j(e) \cap \lambda_\alpha)$.

Claim. *Let* $S \in \left(\mathrm{NS}^{\nu}_{\kappa,\lambda}\right)^+$. *Then there is* $d \in D$ *such that*

$$\{k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) : a \in S \cap W_d\} \in I^+_{\rho,\lambda_{\alpha(d)}}.$$

Proof of the claim. Assume otherwise. For $d \in D$, select $y_d \in P_\rho(\lambda_{\alpha(d)})$ so that $y_d \setminus k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) \neq \emptyset$ for every $a \in S \cap W_d$. Set $y = \bigcup_{d \in D} y_d$. Note that $y \in P_\rho(\lambda)$. For $\beta \in \mu$, pick $z_\beta \in Z_\beta$ so that $y \cap \lambda_\beta \subseteq z_\beta$. Now let H be the set of all $a \in P_\kappa(\lambda)$ such that $i(z_\beta) \in \bigcup_{\zeta \in a \cap \mu} P(a \cap \lambda_\zeta)$ for every $\beta \in a \cap \mu$. Since $H \in (\mathrm{NS}^{\mu}_{\kappa,\lambda})^*$, we can find a in $S \cap B \cap H$. Set $d = \sup(a \cap \mu)$ if $\nu = \mu$, and $d = a \cap \nu$ otherwise. Then $a \in W_d$ and $y_d \subseteq y \cap \lambda_{\alpha(d)} = \bigcup_{\beta \in a \cap \mu} (y \cap \lambda_\beta) \subseteq \bigcup_{\beta \in a \cap \mu} z_\beta = \bigcup_{\beta \in a \cap \mu} j(i(z_\beta)) \subseteq k_{\alpha(d)}(a \cap \lambda_{\alpha(d)})$. This contradiction completes the proof of the claim.

It is now easy to show that the conclusion of the lemma holds: Fix $R \in (\mathrm{NS}^{\nu}_{\kappa,\lambda})^+$, and A such that $A \in \mathrm{NS}^*_\mu$ if $\nu = \mu$, and $A \in \mathrm{NS}^*_{\kappa,\nu}$ otherwise. Set $Y = \bigcup_{d \in D \cap A} W_d$. Since $Y \in (\mathrm{NS}^{\nu}_{\kappa,\lambda})^*$, there must be some $d \in D$ such that

$$\{k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) : a \in (R \cap Y) \cap W_d\} \in I^+_{\rho,\lambda_{\alpha(d)}}.$$

Then clearly, $d \in A$ and $|\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})$. □

Consider for instance the following situation : In V, GCH holds, σ is a strong cardinal with $\rho < \sigma < \lambda$, and η a cardinal greater than λ. Then by a result of Gitik and Magidor [6], there is a cardinal preserving, σ^+-cc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$,

- no new bounded subsets of σ are added ;
- σ changes its cofinality to ω ;
- $2^\sigma \geq \eta$.

Now working in $V^{\mathbb{P}}$, let $T = P_{\omega_1}(\sigma)$. Then clearly $|T \cap P(a)| \leq 2^{|a|} \leq \kappa < \rho$ for any $a \in P_\kappa(\lambda)$. Moreover for any two uncountable cardinals χ and θ with $\mathrm{cf}(\chi) = \chi < \sigma \leq \theta \leq \eta$,

$$u(\chi, \theta) = \max\{2^{<\chi}, u(\chi, \theta)\} = \theta^{<\chi} = \sigma^{<\chi} = \sigma^{\aleph_0} = |T|.$$

Hence $u(\rho, \tau) \leq |T|$ for every cardinal τ with $\rho \leq \tau < \lambda$, so by Lemma 8.1 for any $R \in (\mathrm{NS}^\nu_{\kappa,\lambda})^+$,

$$\big\{d \in D : |\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})\big\}$$

lies in NS^+_μ if $\nu = \mu$, and in $\mathrm{NS}^+_{\kappa,\nu}$ otherwise.
Note that for any cardinal χ with $\kappa \leq \chi \leq \sigma$, $\mathrm{cof}\big(\mathrm{NS}^\chi_{\kappa,\lambda}\big) = u(\kappa, \lambda)$ since $\mathrm{cof}(\mathrm{NS}^\chi_{\kappa,\lambda}) \leq (\lambda^{<\kappa})^\chi = (2^\sigma)^\chi = 2^\sigma$, and moreover, by Fact 2.9 and Proposition 4.6, $\overline{\mathrm{cof}}(\mathrm{NS}^\chi_{\kappa,\lambda}) > \lambda$ in case $\mu \leq \chi$.

Let us observe that if $T \subseteq P_\kappa(\lambda)$ is, as in condition (a) of Lemma 8.1, such that $|T \cap P(a)| \leq u(\kappa, \lambda)$ for any $a \in P_\kappa(\lambda)$, then it is easy to see that $|T| \leq u(\kappa, \lambda)$.

PROPOSITION 8.2. *Suppose that there is $T \subseteq P_\kappa(\lambda)$ and a cardinal π with $\rho \leq \pi < \lambda$ such that*

- *$|T \cap P(a)| < \rho$ for any $a \in P_\kappa(\lambda)$;*
- *$\tau^\nu \leq u(\rho, \tau) \leq |T|$ for every cardinal τ with $\pi < \tau < \lambda$.*

Then $\mathrm{NS}^\nu_{\kappa,\lambda}$ is nowhere precipitous.

Proof. By Fact 2.12 it suffices to show that II has a winning strategy in the game $G(\mathrm{NS}^\nu_{\kappa,\lambda})$. We can assume without loss of generality that $\lambda_0 > \pi$. For $g : P_3(\nu) \to P_3(\lambda)$ and $\alpha < \mu$, define $g_\alpha : P_3(\nu) \to P_3(\lambda_\alpha)$ by $g_\alpha(e) = g(e) \cap \lambda_\alpha$.

Claim 1. *Let $g : P_3(\nu) \to P_3(\lambda)$. Then*

$$\{d \in D : \big\{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})\big\} \subseteq C(g, \kappa, \lambda)\}$$

lies in $(\mathrm{NS}_\mu|E)^*$ if $\nu = \mu$, and in $\mathrm{NS}^*_{\kappa,\nu}$ otherwise.

Proof of Claim 1. We prove the claim in the case when $\nu > \mu$, and leave the proof in the case when $\nu = \mu$ to the reader. Define $h : P_3(\nu) \to \mu$ by $h(e) =$ the least $\beta < \mu$ such that $g(e) \subseteq \lambda_\beta$. Let Q be the set of all $d \in D$ such that $h(e) \in d$ for every $e \in P_3(d)$. Then clearly $Q \in \mathrm{NS}^*_{\kappa,\nu}$. Now fix $d \in Q$ and $a \in W_d$ such that $a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})$. Let $e \in P_3(a \cap \nu)$. Then $h(e) \in d$, so $g(e) \subseteq \lambda_{\alpha(d)}$. It follows that $g(e) \subseteq a$, since $g(e) \cap \lambda_{\alpha(d)} \subseteq a$. Thus $a \in C(g, \kappa, \lambda)$. This completes the proof of Claim 1.

Claim 2. *Let $X \in (\mathrm{NS}^\nu_{\kappa,\lambda})^+$ and $Y \subseteq W$. Suppose that*

$$Y \cap \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(k, \kappa, \lambda_{\alpha(d)})\} \neq \emptyset$$

whenever $d \in D$ and $k : P_3(\nu) \to P_3(\lambda_{\alpha(d)})$ are such that

$$|\{a \cap \lambda_{\alpha(d)} : a \in X \cap W_d \text{ and } a \cap \lambda_{\alpha(d)} \in C(k, \kappa, \lambda_{\alpha(d)})\}| \geq u(\rho, \lambda_{\alpha(d)}).$$

Then $Y \in (\mathrm{NS}^\nu_{\kappa,\lambda})^+$.

Proof of Claim 2. Fix $g : P_3(\nu) \to P_3(\lambda)$. By Lemma 8.1 and Claim 1, there must be $d \in D$ such that

$$|\{a \cap \lambda_{\alpha(d)} : a \in \big(X \cap C(g, \kappa, \lambda)\big) \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})$$

and

$$\big\{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})\big\} \subseteq C(g, \kappa, \lambda).$$

Then

$$Y \cap \big\{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})\big\} \neq \emptyset$$

since $a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})$ for every $a \in X \cap C(g, \kappa, \lambda) \cap W_d$. Hence $Y \cap C(g, \kappa, \lambda) \neq \emptyset$. This completes the proof of the claim.

Now to describe a strategy τ for player II in the game $G(\mathrm{NS}^\nu_{\kappa,\lambda})$, let

$$X_0, Y_0, X_1, \ldots, Y_{n-1}, X_n$$

be a partial play of the game. We may assume $X_0 \subseteq W$. We define a subset of X_n, $Y_n \in (\mathrm{NS}^\nu_{\kappa,\lambda})^+$ and its 1-1 enumeration $\langle y^n_{d,\xi} : d \in D \text{ and } \xi < |K^n_d|\rangle$. Here K^n_d is the set of all $k : P_3(\nu) \to P_3(\lambda_{\alpha(d)})$ such that

$$|X_n \cap \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(k, \kappa, \lambda_{\alpha(d)})\}| \geq u(\rho, \lambda_{\alpha(d)}).$$

Fix $d \in D$ with K^n_d nonempty. Enumerate K^n_d as $\langle k^n_{d,\xi} : \xi < |K^n_d|\rangle$. Note that $|K^n_d| \leq \lambda^\nu_{\alpha(d)} \leq u(\rho, \lambda_{\alpha(d)})$ (and $K^n_d \subseteq K^{n-1}_d$ by $X_n \subseteq X_{n-1}$). So by induction on $\xi < |K^n_d|$ we can choose $y^n_{d,\xi}$ from

$$X_n \cap \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(k^n_{d,\xi}, \kappa, \lambda_{\alpha(d)})\} \setminus \big(\{y^n_{d,\zeta} : \zeta < \xi\} \cup \{y^{n-1}_d : \zeta \leq \xi\}\big).$$

Define $Y_n = \{y^n_{d,\xi} : d \in D \text{ and } \xi < |K^n_d|\}$. Then Y_n is a subset of X_n by construction, and is an element of $(\mathrm{NS}^\nu_{\kappa,\lambda})^+$ by Claim 2. Moreover the enumeration is 1-1 by construction and the definition of W_d.

To see that τ is a winning strategy, suppose that $X_0, Y_0, X_1, \ldots$ is a play during which player II obeyed the strategy τ. We claim that $\bigcap_{n<\omega} Y_n = \emptyset$. Suppose to the contrary that $x \in \bigcap_{n<\omega} Y_n$. Let d be $\sup(x \cap \mu)$ if $\nu = \mu$, and $x \cap \nu$ otherwise. Then $d \in D$ and for each $n < \omega$, there is $\xi(n)$ such that $x = y^n_{d,\xi(n)}$. By the choice of $y^n_{d,\xi}$ we have $\xi(n) < \xi(n-1)$ for each $0 < n < \omega$, a contradiction. $\square$

Let us observe the following. Suppose that there exist T and π as in the statement of Proposition 8.2. Then either $\mathrm{cof}(\mathrm{NS}^\nu_{\kappa,\lambda}) = u(\kappa,\lambda)$, or $\lambda^{<\mu} = \lambda$. To establish this, note that $u(\kappa,\lambda) \leq \lambda^{<\kappa} \leq \lambda^{<\mu} \leq |T| \leq u(\kappa,\lambda)$, so $|T| = u(\kappa,\lambda) = \lambda^{<\mu}$. It is now simple to see that $|T| = \lambda$ if $\tau^\nu < \lambda$ for every cardinal $\tau < \lambda$, and $|T| = \lambda^\nu$ otherwise.

THEOREM 8.3.

(i) *Suppose that $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}^\mu_{\kappa,\lambda}$ is nowhere precipitous.*

(ii) *Suppose that $\nu > \mu$, and $\tau^{c(\kappa,\nu)} < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}^\nu_{\kappa,\lambda}$ is nowhere precipitous.*

(iii) *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}$ is nowhere precipitous.*

Proof.

(i) : Put $\nu = \mu$, $\rho = \mu^+$, $\pi = 2^\mu = 2^{<\rho}$ and $T = P_2(\lambda)$. Then clearly, $|T \cap P(a)| \leq |a| < \kappa < \rho$ for any $a \in P_\kappa(\lambda)$. Moreover, $\tau^\nu = \tau^{<\rho} = u(\rho,\tau) < \lambda = |T|$ for any cardinal τ with $\pi < \tau < \lambda$. Hence by Proposition 8.2, $\mathrm{NS}^\nu_{\kappa,\lambda}$ is nowhere precipitous.

(ii) : Put $\rho = c(\kappa,\nu)^+$, $\pi = 2^{<\rho}$ and $T = P_2(\lambda)$. Then clearly, $|T \cap P(a)| \leq |a| < \kappa < \rho$ for all $a \in P_\kappa(\lambda)$. Moreover, $\tau^\nu \leq \tau^{<\rho} = u(\rho,\tau) < \lambda = |T|$ for every cardinal τ with $\pi < \tau < \lambda$. Now apply Proposition 8.2.

(iii) : Use (i) and Corollary 7.4 (i).

$\square$

Acknowledgments. The authors would like to express their gratitude to the referee for a number of helpful suggestions.

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Université de Caen - CNRS
Laboratoire de Mathématiques
BP 5186
14032 Caen Cedex
France

pierre.matet@unicaen.fr

Institute of Mathematics
The Hebrew University of Jerusalem
91904 Jerusalem
Israel
shelah@math.huji.ac.il

and

Department of Mathematics
Rutgers University
New Brunswick, NJ 08854
USA
shelah@math.huji.ac.il