BIGNESS PROPERTIES FOR κ-TREES AND LINEAR ORDERS E81

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Abstract. This continues [?, Ch.VIII],[?], [?], [?] using class K of indescernible models. We deal with the case K is the class of linear order and the class of trees with $\delta + 1$ levels.

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The author thanks Alice Leonhardt for the beautiful typing. Here, e.g. [?, 1.1=L7.1] means Definition 1.1 in [?] which has label 7.1, so L stands for label. First typed December 2, 2015.

§ 0. Introduction

Our aim is to get general non-structure theorems as in [?, Ch.VIII], [?] and more [?], [?] but we try to make the paper self-contained. Those papers deal mainly with the class $K^\omega_{\rm tr}$ as the class of index models, where $K^\omega_{\rm tr}$ is the class of trees with $\omega+1$ levels and lexicographic orders. The thesis is that if we can build complicated sequences of index models, we can deduce various non-structure results for many classes.

In particular

- (a) using K = the class of linear order is suitable for e.g. dealing with the class of models of an unstable theory via EM models where the skeleton is linearly ordered by some formula $\varphi(x,y)$
- (b) using $K = K^\omega_{\rm tr}$ is suitable for e.g. dealizing with the class of models of an unsuperstable theorem T
- (c) using $K = K^\theta_{\rm tr}$, i.e. trees with $\kappa+1$ levels, is suitable for e.g. dealing with the model of T when $\kappa(T) > \theta$
- (d) in all those cases we can apply the methods to $\mathrm{PC}(T_1,T) := \{M_1{\restriction}\tau_T : M_1$ is a model of $T_2\}$ where $T_1 \supseteq T$ and, e.g. has Skolem functions; and to more general situations.

We have three aims:

(A) Linear Orders (as the class of index models)

In [?, 2.29=L2.25] we get the desired properties for the formula $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar x,\bar y)$, but we succeed to prove it, i.e. prove strong $(2^\lambda,\lambda,\mu,\aleph_0)$-bigness only for the case λ is a regular cardinal $> \mu$; for λ singular $(> \mu)$ we get here only 2^{λ_1} for any $\lambda_1 < \lambda$ rather than 2^λ. We like to get the full results. See §2.

(B) Concerning $K^\omega_{\rm tr}$, consider improving of [?, 2.20=L7.11]

The best is to get the full strong $(\lambda,\lambda,\mu,\kappa)^{6^+}$-bigness property or so; it seems that in the problematic case it suffices to have just:

$(*)$ $\mu^+ > \|N_n\|^{\aleph_0}$.

See below.

(C) $K^\theta_{\rm tr}$ with θ not necesarily $\aleph_0$

Do we have the full strong $(\lambda,\lambda,\mu,\kappa)$-bigness property? So using $\mathscr{M}_{\mu,\kappa}$ let us review cases and see what they cover:

- (a) $\lambda = \mathrm{cf}(\lambda) > \mu$
  - (α) $\kappa=\theta=\aleph_0$, see [?, 1.11=L7.6(1),p.12] get super$^{7^+}$, check, this implies "super$^{4^{++}}$"; moreover [?, 2.13=L7.8I] for $\ell = 7^+$
  - (β) for $\theta \ge \kappa+\aleph_1$ such that $\lambda \gg \kappa$ which means $(\forall\alpha<\lambda)(|\alpha|^{<\kappa} < \lambda)$
- (b) $\lambda > \mu + \mathrm{cf}(\lambda), \partial = \mathrm{cf}(\partial) \in [\kappa,\theta], \lambda \gg \kappa, \mathscr{T}$ a tree with ∂ levels $< \lambda$ nodes, $|\lim_\partial(\mathscr{T})| \ge \lambda$.

We choose $\chi \in (\mu + \mathrm{cf}(\lambda) + |\mathscr{T}| + \kappa + \partial)^+$ and use partial square $\bar C = \langle C_\alpha : \alpha \in S\rangle, S \subseteq S^{\chi^{++}}_{\le\chi}, \mathrm{otp}(C_\delta) \le \chi, \bar C{\restriction}S^\lambda_\chi$ guess clubs.

As in [?, 2.1=L7.8] we get

- $(*)$ $\ell = 5$ which has a weaker (vii), check - is $\ell = 5$ defined well? what is proved?

For $\theta = \kappa = \aleph_0$ we get (see 1.1) 4^{++}-bigness (so $M_n \cap \mu = N_n \cap \mu, \kappa \subseteq \mu_n$), $\nu \in I \cap \nu_\varepsilon \restriction k \in M_n \Rightarrow \nu \in M_n$.

Question 0.1. When do we ask $[M_n]^{<\kappa} \subseteq M_n$? (now $n < \theta$, so $\theta = \aleph_0$ the old case).

- (c) λ is strong limit of cofinality $> \theta$ or at least equal to $\sup\{\chi : \chi$ strong limit of cofinality $\kappa\}$.

If $\theta = \aleph_0$ see [**?**, 1.11(3)=L7.6] for $\ell = 6$ [**?**, 2.1=L7.8,pg.19] getting $\ell = 4^{++}$ (check). Choose $\langle \chi_i : i < \mathrm{cf}(\lambda)\rangle$, increasing to λ, χ_i strong limit, $\mathrm{cf}(\chi_i) = \kappa$ "or at least", not clera, but if $\mathrm{cf}(\lambda)$.

First, try to build $\langle I_\alpha : \alpha < 2^\lambda\rangle$ but choose $I_{\alpha,i} = I_{A_{\alpha,i}}, A_{\alpha,i} = A_\alpha \cap \chi_i$, etc.

Second, try to build $\langle I_\alpha : \alpha < \lambda\rangle$

- (d) $\lambda = \sup\{\chi : \mathrm{cf}(\chi) = \kappa < \chi < \lambda, \mathrm{pp}(\chi) > \chi^+\}$.

Like (a) using non-reflecting for $\theta = \aleph_0$, see [**?**, 1.16=L7.7].

Seems to generalize easily

- (e) $\mathrm{cf}(\lambda) < \chi = \chi^\theta < \lambda \leq 2^\chi$.

If $\theta = \aleph_0$; see [**?**, 1.11(2) = L7.6] we get super^6; what about getting 4^{++}? Check?; if $\kappa = \kappa^{\aleph_0}$ then we get super$^{6^+}$.

Assume $\mathrm{cf}([\chi]^\partial, \subseteq) = \chi, \theta < \partial = \mathrm{cf}(\partial)$, maybe $\mathrm{cf}([\partial]^\theta, \subseteq) = \partial$.

Can we get models of M_i, N_i of cardinality ∂?

Now if $\mu^{2^{\aleph_0}} < \lambda$ then let $\chi = \min\{\chi : \chi^{(2^{\aleph_0})} \geq \lambda\}$ so $\mathrm{cf}(\chi) \leq 2^{\aleph_0}$ we get $\ell = 4^{++}$ or more; find models of cardinality.

- (f) Assume $\mathrm{cf}(\lambda) \leq \theta < \chi < \lambda, \mathrm{cf}(\chi) = \theta, \mathrm{pp}(\chi) = \chi^+$, maybe $\chi = (2^\partial)^{+\kappa}, \partial < \lambda$ large enough, $\partial = \partial^\theta$.

See [**?**, 2.15=L7.9,pg.27] using $\langle \mathfrak{a}_i : i < \mathrm{cf}(\chi)\rangle$ pairwise disjoint unbounded subsets of $\mathrm{Reg} \cap \chi$, say $\ell = 4^+$. See also [**?**, 3.23=L7.14,pg.47]. Note that without loss of generality $\chi = \chi_*^{+\omega}, \mathrm{pp}(\chi) = \chi_*^+$ and $\chi_*^{\aleph_0} = \chi_*$.

- (g) The remaining case.

It $\theta = \aleph_0$ this means $\lambda = \aleph_{\alpha+\omega}$ is strong limit, see [**?**, 2.17=L7.109,pg.32] for $\ell = 6$, [**?**, 2.19=L7.10,pg.35] and $[M_n]^{\aleph_0} \subseteq M_n$ which gives 4^{++}; more?

Discussion 0.2. Can we improve [**?**, 2.15].

1) E.g., we assume only

- $(*)$ (a) $\lambda > \mathrm{cf}(\lambda)$
  - (b) $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$
  - (c) $\mathrm{cf}(\lambda) + \mu < \chi_n = \mathrm{cf}(\chi_n) = \chi_n^{\aleph_0} < \chi_{n+1} < \chi = \sum\limits_n \chi_n < \lambda$
  - (d) $\mathrm{pp}(\chi) = \chi^+$.

2) We choose

(∗) $\bar{\mu} = \langle \mu_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle$ increases $\mu_\varepsilon = \mathrm{cf}(\mu_\varepsilon) = \mu_\varepsilon^{\aleph_0} \in (\chi^+, \lambda), \mu_\varepsilon^{+7} < \mu_{\varepsilon+1}, \lambda_\varepsilon = \mu_\varepsilon^{+2}$ or $\lambda_\varepsilon = \mu_\varepsilon^{+(\varepsilon+1)}$.

3) Question: $2^{\aleph_0} = \aleph_1$
If not, then Υ_ε satisfies: every Υ and did not contain a perfect set.
4) Another direction:

(∗) (a) without loss of generality $\chi_n = \chi_0^{+n}$
(b) we choose $\bar{\eta} = \langle \eta_\alpha : \alpha < \chi^+\rangle, <_{J_\omega^{\mathrm{bd}}}$-increasing cofinal in $\prod\limits_n \chi_n$
(c) without loss of generality see [?], $\bar{\eta}$ is (χ, χ_0)-free r using $\lambda_\varepsilon = \mu_\varepsilon^{+(\omega+1)}$
(d) look for models of cardinality χ_0.

* * *

Definition 0.3. we define K_{tr}^δ as the class of trees with $\delta+1$ levels and lexicographic orders as in [?, §2] - FILL.

Definition 0.4. we define $\check{I}_\theta[\lambda]$ where $\theta < \lambda$ are regular as follows: (see [?, §0,y14]) $S \in \check{I}_\theta[\lambda]$ if there is $\langle a_\alpha : \alpha \in S^+\rangle$ which witness it meaning:

$(*)_1$ (a) $S \subseteq S^+ \subseteq S^\lambda_{\le\kappa}$
(b) $S^+ = \{\delta \in S^+ : \mathrm{cf}(\delta) = \kappa\} = \{\delta \in S^+ : \mathrm{otp}(a_\alpha) = \kappa\}$
(c) $a_\alpha \subseteq \alpha$ has order type $\le \kappa$
(d) $\delta \in S^+ \Rightarrow \delta = \sup(a_\delta)$
(e) for every $\alpha < \lambda$ the set $\{a_\beta \cap \alpha : \beta$ satisfies $\alpha \in a_B$ has cardinality $< \lambda$

By [?], [?], more [?].

Claim 0.5. *(Existence and Existence with guessing clubs) Let λ be regular uncountable.*
1) If $S \in \check{I}[\lambda]$ then we can find a witness $(E, \bar{a})$ for $S \in \check{I}[\lambda]$ such that:

(a) $\delta \in S \cap E \Rightarrow \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)$
(b) if $\alpha \notin S$ then $\mathrm{otp}(a_\alpha) < \mathrm{cf}(\delta)$ for some $\delta \in S \cap E$.

2) $S \in \check{I}[\lambda]$ iff there is a pair $(E, \bar{\mathscr{P}})$ such that:

(a) *E is a club of the regular uncountable λ*
(b) *$\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$, where $\mathscr{P}_\alpha \subseteq \{u : u \subseteq \alpha\}$ has cardinality $< \lambda$*
(c) *if $\alpha < \beta < \lambda$ and $\alpha \in u \in \mathscr{P}_\beta$ then $u \cap \alpha \in \mathscr{P}_\alpha$*
(d) *if $\delta \in E \cap S$ then some $u \in \mathscr{P}_\delta$ is an unbounded subset of δ (and δ is a limit ordinal).*

§ 1. On $K^\theta_{\rm tr}$

We consider a strengthening of [?, 1.1=L7.1]

Definition 1.1. 1) $I \in K^\theta_{\rm tr}$ is $(\mu,\kappa)-$super$^+$-unembeddable[1] (or super$^{4^{++}}$-unembeddable) into $J \in K^\theta_{\rm tr}$ <u>when</u> as in [?, 1.1=L7.1] with $n < \theta$, that is: for every regular large enough χ_* (in particular such that $\{I, J, \mu, \kappa, \theta\} \in \mathscr{H}(\chi^*)$) and well ordering $<^*_{\chi_*}$ of $\mathscr{H}(\chi_*)$ we have:

- $(*)$ there are η, M_i, N_i for $i < \theta$ such that:
  - (i) $M_i \prec N_i \prec M_j \prec N_j \prec (\mathscr{H}(\chi^*), \in, <^*_{\chi_*})$ for $i < j < \theta$
  - (ii) $M_i \cap \mu = M_0 \cap \mu$ for $i < t$
  - (iii) I, J, μ, κ belongs to M_0
  - (iv) $\eta \in P^I_\theta$
  - (v) for every $i < \theta$ for some $j < \theta$ we have $\eta{\restriction}j \in M_i, \eta(j) \in N_i \backslash M_i$ hence $\eta{\restriction}(j+1) \in N_i \backslash M_i$
  - (vi) if $\nu \in P^J_\theta$ and $\{\nu{\restriction}j : j < \theta\} \subseteq \bigcup\limits_i M_i$ then $\nu \in \bigcup\limits_i M_i$.

Definition 1.2. We repeat [?, 1.4=L7.2], defining [full] super$^+$-bigness, that is:

(1) $K^\omega_{\rm tr}$ has the $(\chi, \lambda, \mu, \kappa)$-super-bigness property <u>when</u>: there are $I_\alpha \in (K^\omega_{\rm tr})_\lambda$ for $\alpha < \chi$ such that for $\alpha \neq \beta, I_\alpha$ is (μ, κ)-super unembeddable into I_β

(2) $K^\omega_{\rm tr}$ has the full $(\chi, \lambda, \mu, \kappa)$-super-bigness property <u>when</u>: there are $I_\alpha \in (K^\omega_{\rm tr})_\lambda$ for $\alpha < \chi$ such that I_α is (μ, κ)-super unembeddable into $\sum\limits_{\beta < \chi, \beta \neq \alpha} I_\beta$, see [?]

(3) We may omit κ if $\kappa = \aleph_0$.

Exercise 1.3. Put 4^{++} in the diagram from [?] and see [?, 1.7=L7.4], [?, 1.8=L7.6].

Claim 1.4. *If λ is singular $> \mu$ <u>then</u> in* [?, 2.20=L7.11]*, not only $K^\omega_{\rm tr}$ have the $(\lambda, \lambda, \mu, \aleph_0)$-super-bigness property but we can add to Definition* [?, 1.1=L7.1]

- *for every $n < \omega$ for some $\mu' \in [\mu, \lambda)$ we have $\|M_n\| \leq \mu'$ and $M_n \cap \mu' = N_n \cap \mu'$.*

Proof. We should check the proof of [?, 2.20=L7.11], that is, the cases each treated by a claim there.

<u>Case 1</u>: λ regular $> \aleph_0$, see [?, 2.13=L7.8,pg.25].

We can find $\eta, \langle M_n : n < \omega \rangle$ as in "super$^{7^+}$" of Definition [?, 1.1=L7.1] so $M_n = N_{\alpha_{\delta,n}}$ (with guessing clubs) $\eta_\delta := \langle \alpha_{\delta,n} : n < \omega \rangle$ list C_δ.

Let N'_n be the Skolem hull of $N_{\alpha_{\delta,n}} \cup \{\eta_\delta(n)\}$ in $N_{\alpha_{\delta,n}+1}$.

<u>Case 2</u>: $\lambda = \aleph_1$, see [?, 1.11=L7.6(1),pg.12].

Similar.

<u>Case 3</u>: λ singular, $(\exists \chi)(\chi^{\aleph_0} < \lambda \leq 2^\chi)$.

See [?, 1.11(2)=L7.6(2),pg.12]. so prove the models are countable.

<u>Case 4</u>: λ is singular, $\lambda = \sup\{\chi : {\rm cf}(\chi) = \aleph_0 \text{ and } {\rm pp}(\chi) > \chi^+\}$.

[1] but below we omit the superscript + because we do not use any other version; similarly in Definition 1.2

See [?, 1.16=L7.7(1),pg.17].

Case 5: λ is singular and $(\exists \chi)(\chi < \lambda \leq \chi^{\aleph_0})$.
See [?, 2.1=L7.8,pg.19].

Case 6: $\lambda = \aleph_{\alpha+\omega}$ is strong limit.
By [?, 2.19=L7.10,pg.35]. $\square_{1.4}$

§ 2. Back to linear orders

We complete [?, 2.27=L2.23,2.31=L2.27], see there.

Definition 2.1. 1) For any $I \in K^\kappa_{\rm tr}$ we define $\mathbf{or}(I)$ as the following linear order (See Definition [?, 2.24=L2.20]).

set of elements is chosen as $\{(t,\ell) : \ell \in \{1,-1\}, t \in I\}$

the order is defined by $(t_1,\ell_1) < (t_2,\ell_2)$ if and only if $t_1 \triangleleft t_2 \wedge \ell_1 = 1$ or $t_2 \triangleleft t_1 \wedge \ell_2 = -1$ or $t_1 = t_2 \wedge \ell_1 = -1 \wedge \ell_2 = 1$ or $t_1 <_{\ell x} t_2 \wedge (t_1, t_2$ are $\triangleleft$-incomparable.

2) Let $\varphi_{\rm or} = \varphi_{\rm or}(x_0, x_1; y_0, y_1)$ be the formula $x_0 < x_1 \wedge y_1 < y_0$.

3) Let $\varphi^\kappa_{\rm tr} = \varphi^\kappa_{\rm tr}(x_0, x_1; y_0, y_1)$ be (this is for $K^\kappa_{\rm tr}$, for $\kappa = \aleph_0$ see example [?, 2.9=L2.4A])

$$\begin{aligned}\varphi_{\rm tr}(x_0, x_1 : y_0, y_1) := [x_0 = y_0] \text{ and } P_\kappa(x_0) \quad & \wedge \bigvee_{\epsilon<\kappa} [P_{\epsilon+1}(x_1) \\ & \wedge P_{\epsilon+1}(y_1) \wedge P_\epsilon(x_1 \cap y_1)] \\ & \wedge [x_1 \triangleleft x_0 \wedge \neg(y_1 \triangleleft y_0)] \text{ and } y_1 <_{\ell x} x_1].\end{aligned}$$

Recall [?, 2.28=L2.24]

Definition 2.2. We define the following (quantifier free infinitary) formulas for the vocabulary $\{<\}$. For any ordinal α, β and a one-to-one function π from α onto β, and we let $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar x, \bar y)$ where $\bar x = \bar x^\alpha = \langle x_i : i < \alpha\rangle$ and $\bar y = \bar y^\alpha = \langle y_i : i < \alpha\rangle$, be

$$\bigwedge\{x_i < x_j : i < j < \alpha\} \text{ and } \bigwedge\{y_i < y_j : i, j < \alpha \text{ and } \pi(i) < \pi(j)\}.$$

Claim 2.3. *Assume $\lambda > \mu$.*
1) For (α, β, π) as in 2.2, such that $\alpha, \beta \le \mu^+$, the class $K_{\rm or}$ has the full strong $(\lambda, \lambda, \mu, \kappa)$-bigness property for $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar x, \bar y)$.
2) For (α, β, π) as in 2.2 such that $\alpha, \beta \le \mu^+$, the class $K_{\rm or}$ has the strong $(2^\lambda, \lambda, \mu, \kappa)$ bigness property for $\varphi_{{\rm or},\alpha,\beta,\pi}$.
3) In fact in both part (1) and (2) we can find examples which satisfies the conclusion for all triples (α, β, π) as there simultaneously.

Proof. 1) By 2.4 below.
2) By part (1) and [?, 2.27=L2.23(1)], [?, 2.20=L2.8(1)] or here.
3) Check the proof. $\square_{2.3}$

Claim 2.4. *Assume $\mu < \lambda$.*

If $I, J \in K^\kappa_{\rm or}$ satisfies $\circledast$ below and $\alpha_*, \beta_* \le \mu^+$ and π is a one-to-one function from α_* onto β_* then recalling 2.1, $\mathbf{or}(I)$ is strongly $\varphi_{{\rm or},\alpha_*,\beta_*,\pi}(\bar x^{\alpha_*}, \bar y^{\alpha_*})$-unembeddable for (μ, κ) into $\mathbf{or}(J)$ where

$\circledast$ (a) $I, J \in K^\omega_{\rm tr}$
(b) *I is $(\mu, \aleph_0)$-super unembeddable into J, see Definition 1.1, check*
(c) *$I \in K^\kappa_{\rm tr}$ is $\{\eta_\delta{\restriction}i : i \le \partial, \delta \in S_1\} \cup \{\langle\alpha\rangle : \alpha < \lambda_1\}$*
(d) *$J \in K^\kappa_{\rm tr}$ is $\{\eta_\delta{\restriction}i : i \le \partial, \delta \in S_1\} \cup \{\langle\alpha\rangle : \alpha < \lambda_1\}$.*

Proof. So let f be a function from $\mathbf{or}(I)$ into $\mathscr{M}_{\mu,\kappa}$ $(\mathbf{or}(J))$ so actually a function from $I \times \{1,-1\}$ into $\mathscr{M}_{\mu,\kappa}(J \times \{1,-1\})$, and $<_*$ a well ordering of $\mathscr{M}_{\mu,\kappa}(J)$ but we "forget" to deal with it, as there are no problems, and let χ be large enough. Let χ^* be large enough and let η_*, M_η, N_n (for $n < \omega$) satisfies $(*)$ of Definition

?? with η_* here standing for η there. In particular $M_n \prec N_n \prec (\mathscr{H}(\chi), \in)$ such that $I, J, \lambda, \mu, \mathscr{M}_{\mu,\kappa}(J), f, <_*$ belong to N_0 and $M_n \cap \mu = N_n \cap \mu$; as it happens "$\alpha_*, \beta_*, \pi \in N_0$" is not needed. For any $\eta \in I$, clearly $f((\eta,1))$ is well defined and $\in \mathscr{M}_{\mu,\kappa}(J)$ so let $f((\eta,1)) = \sigma_\eta(\bar\nu_\eta), \bar\nu_\eta = \langle (\nu_{\eta,\epsilon}, \iota_{\eta,\epsilon}) : \epsilon < \epsilon_\eta \rangle, \nu_{\eta,i} \in J$ and $\iota_{\eta,\epsilon} \in \{1,-1\}, \epsilon < \omega$.

Let $\epsilon_* = \epsilon_{\eta_*}, \iota_\epsilon = \iota_{\eta_*,\epsilon}, i^*_\epsilon = \ell g(\nu_{\eta_*,\epsilon})$ for $\epsilon < \epsilon_*$ and let $j^*_\epsilon = \sup\{j \le i^*_\epsilon : \sup \mathrm{Rang}(\nu_{\eta_*,\epsilon}{\restriction} j) < \delta\}$. Let n_* be large enough such that:

- if $\varepsilon < \varepsilon_\eta$ and then $\{\nu_{\eta_*,\varepsilon}{\restriction} j : j \le \ell g(\nu_{\eta_*,\varepsilon})\} \cap N_{n_*} \subseteq M_{n_*}$.

[Why? This is by clause (v) of Definition 1.1, when for $j = \ell g(\nu_{\eta_*,\varepsilon}), \varepsilon < \varepsilon_*$ we do it "by hand" as ε_* is finite.]

Let $\nu^*_\epsilon = \nu_{\eta_*,\epsilon}{\restriction} j^*_\epsilon$, it belongs to M_{n_*}.

So $\{\nu^*_\epsilon : \epsilon < \epsilon_*\} \subseteq M_{n_*}$ is finite hence it follows that $\nu^* = \langle \nu^*_\epsilon : \epsilon < \epsilon_* \rangle \in M_{n_*}$. Let k_* be such that $\eta_*{\restriction} k_* \in M_{n_*}, \eta_*{\restriction}(k_*+1) \in N_{n_*} \backslash M_{n_*}$ hence $\eta_*(k_*) \in N_{n_*} \cap \lambda \backslash M_{n_*}$ and let $\nu_{\eta_*,\varepsilon_*} = \eta_*, j^*_{\varepsilon_*} = k_*$ and $\alpha^*_\lambda = \min(M_{n_*} \cap \lambda \backslash \eta_*(k_*))$ hence $\sup(M_{n_*} \cap \lambda) \le \eta_*(k_*) < \alpha_*$.

Let $u_* = u_1 = \{\epsilon < \epsilon_* : j^*_\epsilon < i^*_\epsilon\}$. For $\epsilon \in u_*$ let[2] $\alpha^*_\epsilon = \min(N_{n_*} \cap (\lambda+1) \setminus \nu_{\eta_*,\epsilon}(j^*_\epsilon))$, so also $\bar\alpha^* := \langle \alpha_\epsilon : \epsilon \in u_* \rangle$ belongs to M_{n_*}.

We define $\mathscr{U}_1$ as the set of $\eta \in I$:

$(*)_\eta$
(a) $\eta_*{\restriction} k_* {}^\frown \langle \beta \rangle \triangleleft \eta \in \ell g(\eta) = \omega$
(b) $\sigma_\eta = \sigma_*$ so $\epsilon_\eta = \epsilon_*$
(c) $\ell g(\nu_{\eta,\epsilon}) = i^*_\epsilon$ for $\epsilon < \epsilon_*$
(d) $\nu^*_\epsilon = \nu_{\eta_\beta,\epsilon}{\restriction} j^*_\epsilon$ for $\epsilon < \epsilon_*$
(e) $\iota_\epsilon = \iota_{\eta,\epsilon}$ for $\epsilon < \epsilon_*$

Note

$(*)_2$
(a) $\eta_* \in \mathscr{U}_1$ and $\mathscr{U}_1 \in M_{n_*}$
(b) $\mathrm{cf}(\alpha^*_\epsilon) \ge \mu^+$ for $\epsilon \in u_*$
(c) if $\bar\alpha \in \prod_{\epsilon \in u_*} \alpha^*_\epsilon$ <u>then</u> for some $\eta \in \mathscr{U}_1$ we have $\epsilon \in u_* \Rightarrow \nu_{\eta,\epsilon}(j^*_\epsilon) \in (\alpha_\epsilon, \alpha^*_\epsilon)$.

[Why? Clause (a) direct by our choice. If $\mathrm{cf}(\alpha^*_\varepsilon) \le \mu$ then α^*_ε is a limit ordinal and there is in $\mathscr{H}(\chi)$ an increasing function f_ε from $\mathrm{cf}(\alpha^*_\varepsilon)$ into α^*_ε with unbounded range. Without loss of generality $f_\varepsilon \in M_{n_*}$ so $\{f_\varepsilon(\beta) : \beta \in N_{n_*} \cap \mathrm{cf}(\alpha^*_\varepsilon)\}$ is an unbounded subsets of $N_{n_*} \cap \alpha^*_\varepsilon$; but this set is equal to $\{f_\varepsilon(\beta) : \beta \in M_{n_*} \cap \mathrm{cf}(\alpha^*_\varepsilon)\}$ so $\sup(\alpha^*_\varepsilon \cap M^*_{n_*}) = \sup(\alpha^*_\varepsilon \cap N_{n_*})$, but this contradicts the choice α^*_ε via $\nu_{\eta_*,\varepsilon}(j^*_\varepsilon)$. Clause (c) follows.]

$(*)_3$ let $\mathscr{U}_2$ be the set of $\beta < \alpha_*$ such that: for every $\bar\alpha \in \prod_{\varepsilon \in u_*} \alpha^*_\varepsilon$ there is η such that:
(a) $\eta \in \mathscr{U}_1$
(b) $\eta(k_*) = \beta$
(c) $\nu_{\eta,\varepsilon}(j^*_\varepsilon) \in (\alpha_\varepsilon, \alpha^*_\varepsilon)$ for $\varepsilon \in u_*$.

[2] Consider $u_2 = \{\varepsilon < \varepsilon_* : j^*_\varepsilon = i^*_\varepsilon = \omega$ but $\nu_{\eta_*,\varepsilon} \notin M_{n_*}\}$. Below we first assume $u_2 = \emptyset$. Second, if λ is regular or $\mu_1 < \lambda \le \mu_1^{\aleph_0}$ for some μ_1, by [**?**, xxx,yyy] this is to justify. If not, then by xxx without loss of generality $\|N_n\|^{\aleph_0}$. See §1.

$(*)_4$ $\eta_*(k_*) \in \mathscr{U}_2$.

[Why? Note that $\eta_*(k_*) \in N_{n_*}$ but $\varepsilon \in u_* \Rightarrow \nu_{\eta_*,\varepsilon}(j^*_\varepsilon) \notin N_{n_*}$.]

Let $\alpha, \beta \le \lambda$ and π be a one-to-one function from α onto β.

Now first we choose $\eta_\zeta \in I$ by induction on $\zeta < \alpha$ such that

$(*)_5$ (a) $\eta_{\zeta,1} \in \mathscr{U}_\eta$
(b) if $\epsilon \in u_*$ then $\nu_{\eta_{\delta(1,\zeta)},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \mathrm{sub}\{\nu_{\eta_{xi,1},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

This is easy.

Second we choose $\eta_{\zeta,2}$ by induction on $\zeta < \beta$ such that:

$(*)_6$ (a) $\eta_{\zeta,2} \in \mathscr{U}$
(b) if $\epsilon \in u_*$ then $\nu_{\eta_{\zeta,2},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \sup\{\nu_{\eta_{\xi,2},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

Let $\bar a = \langle a_\zeta : \zeta < \alpha\rangle$, $\bar b = \langle b_\zeta : \zeta < \alpha\rangle$ from ${}^\alpha I$ be chosen as follows: $a_\zeta = (\eta_{\delta(1,\zeta)}, 1), b_\zeta = (\eta_{\delta(1,\pi(\zeta))}, 1)$ for $\zeta < \alpha$.

Now check, e.g.:

$(*)_6$ $a_{\zeta(1)} <_{\mathrm{or}(I)} a_{\zeta(2)}$ <u>iff</u> $\gamma_{\zeta(1)} < \gamma_{\zeta(2)}$ <u>iff</u> $\zeta(1) < \zeta(2)$
$(*)_7$ $b_{\zeta(1)} <_{\mathrm{or}(I)} b_{\zeta(2)}$ <u>iff</u> $\gamma_{\pi(\zeta)(1)} < \gamma_{\pi(\zeta)(2)}$ <u>iff</u> $\pi(\zeta)(1) < \pi(\zeta)(2)$.

$\square_{2.4}$

Conclusion 2.5. *For $(\mu, \lambda, \alpha_*, \beta_*, \pi)$ as in 2.3(1), the class K_{or} has the full strong $(\lambda, \lambda_1, \mu, \kappa) - \varphi_{\mathrm{or},\alpha_*,\beta_*,\pi}$-bigness property and the strong $(2^\lambda, \lambda, \mu, \aleph_0) - \varphi_{\mathrm{or},\alpha_*,\beta_*,\pi}$-bigness property.*

Proof. By 2.4 + 1.4. $\square_{2.5}$

§ 3. Toward large "$\dot{I}(T_1, T, \kappa$-saturated) so large" when $\kappa_r(T) = \kappa$

We return to the non-super version (as in [?]).
We try to deal wih $K^\kappa_{\rm tr}, \kappa = {\rm cf}(\kappa) > \aleph_0$ using $\mathscr{M}_{\mu,\kappa}(-)$.
Compare with 2.1(3).

Definition 3.1. For the class of $I \in K^\kappa_{\rm tr}$

$$\begin{aligned}\varphi^\kappa_{\rm tr}(x_0, x_1 : y_0, y_1) := \quad & [x_0 = y_0 \text{ and } P_\kappa(x_0) \text{ and} \\ & \bigvee_{i<\kappa} [P_{i+1}(x_1) \text{ and } P_{i+1}(y_1) \text{ and } P_i(x_1 \cap y_1)] \text{ and} \\ & [x_1 \vartriangleleft x_0 \wedge y_1 \ntriangleleft y_0] \text{ and } y_1 <_{\rm lex} x_1].\end{aligned}$$

In other words, when for transparency we restrict ourselves to standard $I \subseteq {}^{\kappa \geq}\lambda :$ $x_0 = y_0 \in {}^\kappa\lambda$, and for some $n_i < \kappa$ and $\alpha < \beta < \lambda$ we have

$$x_1 = (x_0 \restriction i)\char`^\langle \alpha \rangle \vartriangleleft x_0$$

and

$$y_1 = (x_0 \restriction i)\char`^\langle \beta \rangle$$

Claim 3.2. *1) Let $\kappa = {\rm cf}(\kappa) > \aleph_0$. There is a sequence $\langle I_\alpha : \alpha < \lambda \rangle$ which witnesses full strong $\bar{\varphi}^\kappa_{\rm tr} - (\lambda, \lambda, \mu$,bigness) __when__ at least one of the of the following cases occurs (see inside the proof on the super version):*

- *(A) $\lambda = {\rm cf}(\lambda) > \kappa^{++}, \mu^{<\kappa}$ hence $\lambda \geq (2^{<\kappa})^+$*
- *(B)*
  - *(a) $\lambda > \mu + {\rm cf}(\lambda)$*
  - *(b) $\lambda > \chi > {\rm cf}(\chi) = \kappa$ and $\lambda \leq \chi^{<\kappa>_{\rm tr}}$.*

2) We have (A) $\Rightarrow \boxplus \Rightarrow$ the conclusion of part (1) where

$\boxplus$ there are $I_\varepsilon \in K^\kappa_{\rm tr}$ for $\varepsilon < \lambda, |I_\varepsilon| \leq \lambda$ and I_φ is super$^{7^+}$-unembeddable into $J_\varepsilon = \Sigma\{I_\zeta : \zeta \in \lambda \backslash \{\theta\}$ which means

$\boxplus_{\lambda,\mu,I,J}$ if $\chi_* >> \lambda, x \in \mathscr{H}(\chi_*)$ then we can find a pair $(\bar{M}, \eta)$ such that

- *(i) $\bar{M} = \langle M_i : i < \kappa \rangle, M_i \prec (\mathscr{H}(\chi_*), \in, <^*_{\chi_*}), M_i$ is $\prec$-increasing continuous, $\kappa + 1 \subseteq M_0, M_i \cap \mu = \mathscr{M}_0 \cap N$*
- *(ii) $M_i \cap \mu = \mu_0 \cap \mu$*
- *(iii) $\{I, J, \mu, \kappa, x\}$ belongs to M*
- *(iv) $\eta \in P^I_\kappa$ and $\{\eta \restriction i : i < \kappa\} \subseteq \bigcup_{j<\kappa} M_j$*
- *(v) $\eta \restriction j_i \in M_i, \eta(j_i) \in M_{i+1}$ for $i < \kappa$ successor*
- *(vi) if $\eta \in P^J_\kappa$ __then__ for some $j < \kappa, \{\eta \restriction i : i < \kappa\} \cap \bigcup_{i<\kappa} M_i \subseteq M_j$ (or $\in M_j$?)*
- *(vii) see [?, 1.5=L7.3(B)(vii)].*

Proof. 1) As $\lambda = {\rm cf}(\lambda) > \kappa^+$ there is a stationary $S \subseteq S^\lambda_\kappa$ which belongs to $\check{I}_\kappa[\theta]$, see 0.4, 0.5. We can find $\bar{a} = \langle a_\alpha : \alpha \in S^+ \rangle$ which witness it, see e.g. [?, 0.7=L0.5] so

$(*)_1$ (a) $S \subseteq S^+ \subseteq S^\lambda_{\leq\kappa}$

(b) $S = \{\delta \in S^+ : \mathrm{cf}(\delta) = \kappa\} = \{\delta \in S^+ : \mathrm{otp}(a_\alpha) = \kappa\}$
(c) $a_\alpha \subseteq \alpha$ has order type $\leq \kappa$
(d) $\delta \in S^+ \Rightarrow \delta = \sup(a_\delta)$
(e) for every $\alpha < \lambda$ the set $\{a_\beta \cap \alpha : \beta$ satisfies $\alpha \in a_\beta$ has cardinality $< \lambda$

(f) $\bar{a}$ guess clubs.

[Why? See 0.5 or [?, 0.8=L0.6].]

Let

$(*)_2$ (a) (α) let $\langle S_\varepsilon : \varepsilon < \lambda\rangle$ be a division of S to stationary subsets of λ
(β) without loss of generality $\langle \bigcup\limits_{\alpha \in S_\varepsilon} a_\alpha : \varepsilon < \lambda\rangle$ is a sequence of pairwise disjoint $\eta_\alpha \in {}^\kappa\delta$ list a_α in increasing order for $\delta \in S^+$

(c) let
(α) I_ε be the tree $\{\eta_\alpha : \alpha \in S^+ \backslash S\} \cup \{\eta_\delta : \delta \in S_\varepsilon\}$
(β) $I^+_\varepsilon = \{\eta : \eta \trianglelefteq \nu$ for some $\nu \in I_\varepsilon\}$
(γ) let $I_{\varepsilon,\alpha} = I_\varepsilon \cap {}^{\kappa\geq}\alpha, I^+_{\varepsilon,\alpha} = I^+_\varepsilon \cap {}^{\kappa\geq}\alpha$ for $\alpha < \lambda$

(d) for $\zeta < \lambda$ and $\alpha < \lambda$ let
(α) $J_\zeta = \Sigma\{I_\varepsilon : \varepsilon < \lambda$ and $\varepsilon \neq \zeta\}$
(β) $J_{\zeta,\alpha} = J_\zeta \restriction \{\nu$: for some $\xi \in \alpha \backslash \{\zeta\}$ we have $\nu \in I_\xi \cap {}^{\kappa>}\alpha\}$.

Note: if $\eta \in I^+_\varepsilon \backslash I_\varepsilon$ then $\ell g(\eta)$ is a limit ordinal and member of $I^+_\varepsilon \backslash I_\varepsilon$ cause problems as $I^+_{\varepsilon,\alpha}$ may have cardinality λ whereas

$(*)_{2.1}$ if $\alpha < \lambda, \varepsilon < \lambda$ then $I_\varepsilon \cap {}^{\kappa\geq}\alpha$ and $J_\varepsilon \cap {}^{\kappa\geq}\alpha$ have cardinality $< \lambda$ and to prove the claim assume:

$(*)_3$ $\zeta < \lambda$ and $f : I_\zeta \to {}^{\kappa>}\mathscr{M}_{\mu,\kappa}(J_\zeta)$

$(*)_4$ (a) $\eta \in I_\zeta$ let $f(\eta) = \sigma_\eta(\bar{\nu}_\eta), \bar{\nu}_\eta \in {}^{\kappa>}(J_\zeta)$
(b) let $\bar{\nu} = \langle \nu_{\eta,i} : i < i_\eta\rangle$ and $\nu_{\eta,i} \in I_{\varepsilon(\eta,i)}, \varepsilon(\eta,i) \in \lambda\backslash\{\zeta\}$
(c) for $\delta \in S_\zeta$ let $j_\delta = \sup\{\ell g(\eta_\delta \cap \nu_{\eta_\delta,i}) : i < i_{\eta_\delta}\}$
(d) let $E = \{\delta < \lambda : \delta$ a limit ordinal such that $\alpha \in \delta \wedge \eta \in I_\zeta \cap {}^{\kappa>}\alpha \Rightarrow f(\eta) \in {}^{\kappa>}(J_{\zeta,\delta})$

$(*)_5$ now for every $\nu \in I_\varepsilon$ of length $< \kappa$ let $u_\nu := \{\alpha < \lambda$: there is no $v \subseteq \lambda$ of cardinality $(2^{<\kappa})^+$ such that $\alpha \in v, \beta \in v \Rightarrow \sigma_{\nu\hat{}\langle\alpha\rangle} = \sigma_{\nu\hat{}\langle\beta\rangle}$ and $\langle \bar{\nu}_{\nu\hat{}\langle\beta\rangle} : \beta \in v\rangle$ is an indiscernible sequence in $K^\kappa_{\mathrm{tr}}\}$

$(*)_6$ above u_ν has cardinality $\leq 2^{<\kappa} + \mu^{<\kappa}$.

[Why? Let $E_{\delta,j} = \{(\alpha,\beta) : \alpha,\beta < \lambda$ and $\sigma_{(\eta_\delta\restriction 1)\hat{}\langle\alpha\rangle} = \sigma_{(\eta_\delta\restriction 1)\hat{}\langle\beta\rangle}\}$ is an equivalence relation with $\leq \mu^{<\kappa}$ equivalent classes, hence it suffices to prove $u_{\delta,j}$ has $\leq 2^{<\kappa}$ members in each equivalence class. So fix u equivalence class $\gamma/E_{\delta,j}$ if $v_i \subseteq \gamma/E_{\delta,j}$ has cardinality $(2^{<\kappa})^+$ then some $v \subseteq v_1$ of cardinality $(2^{<\kappa})^+$ satisfies the condition, in the " ... for no ... v ...", so we are done.]

$(*)_7$ (a) E is a club of λ where $E = \{\delta < \lambda : \delta$ a limit ordinal and if $\alpha < \delta, \nu \in I_{\varepsilon,\alpha}$ then $\sup(u_\nu) < \delta\}$
(b) E' is a club of λ where $E' = \{\delta \in E : \mathrm{otp}(E \cap \delta) = \delta > \kappa\}$.

Now choose $\delta \in S_\varepsilon \cap E'$ such that $a_\delta \subseteq E'$. Choose a successor ordinal $j \in [j_\delta, \kappa)$ so necessarily $u_{\eta_\delta \restriction j}$. $\square_{3.2}$

Comment 3.3. For the super version (i.e. as in [?, §1] rather than [?, §2]

- $(*)$ generalizing [?, 1.1], on $(\eta, \bar{M})$ we should add:
  - $\eta(i) \in M_{i+1} \backslash M_i$, moreover $\eta(i) \notin \cup\{u \in M_i : |u| \leq 2^{<\kappa}\}$
  - the proof above shows how the bigness implies the bigness properties.

Claim 3.4. *The conclusion of 3.2 or so holds* *when:*

- (B) (a) $\lambda > \theta = \mathrm{cf}(\theta) > \kappa^+ + \mu^{<\kappa}$ *and* λ *is singular*
  - (b) $\lambda = \Sigma\{\lambda_\zeta : \zeta < \mathrm{cf}(\lambda)\}, \lambda_\eta$ *increasing,* λ_ζ *regular*
  - (c) $S_\zeta \subseteq S^{\lambda_\zeta}_\kappa$ *is stationary,* $S_\zeta \in \check{I}_\kappa[\lambda_\zeta]$
  - (d) $\bar{\eta}_\zeta = \langle \eta_{\zeta,\delta} : \delta \in S_\zeta\rangle, \eta_{\zeta,\delta} \in {}^\kappa\delta$ *is increasing with limit* δ
  - (e) *if* $\zeta < \mathrm{cf}(\lambda)$ *and* $\alpha < \lambda$ *then* $\{\eta_{\zeta,\delta}\restriction i : \delta \in S_\zeta$ *satisfies* $\eta_{\zeta,\delta}(i) = \alpha\}$ *has cardinality* $< \lambda_\zeta$
  - (f) $\bar{\eta}_\zeta$ *guess clubs*
  - (g) $\bar{\eta}_\zeta$ *is* (θ,θ)*-free, see [?] and [?], moreover: if* $\mathscr{T} \subseteq {}^{\kappa>}(\lambda_\zeta)$ *a sub-tree* $|\mathscr{T}| = \theta$ *then* $\Lambda = \{\delta \in S_{\zeta(1)} : \eta_{\zeta(1),\delta} \in \lim(\mathscr{T})\}$ *has cardinality* $\leq \theta$ *and there is* $h : \Lambda \to \kappa$ *such that* $(\forall \eta \in \Lambda)(\exists^{<\theta}\nu \in \Lambda)[\nu\restriction h(\nu) = \eta\restriction h(\eta)]$.

Proof. Without loss of generality

- $(*)_1$ if $\zeta < \mathrm{cf}(\lambda), \delta \in S - \varepsilon$ and $i < \kappa$ then θ^+ divide $\eta_{\zeta,\delta}(i)$.

Let $S_* \in \check{I}_\kappa[\theta]$ be stationary and let $\bar{\rho} = \langle \rho_\delta : \delta \in S_*\rangle, \rho_\delta \in {}^\kappa\delta$ increases with limit δ guess clubs and $\theta > (\{\rho_\delta\restriction i : \rho_\delta(i) = \alpha\})$ for every $\alpha < \theta$. Choose $\langle S^*_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle$ as a sequence of pairwise disjoint stationary subets of S_i.

For $\delta \in S_\varepsilon$ let $\eta^*_{\varepsilon,\delta,\beta} = \langle \eta_{\varepsilon,\delta}(i) + \rho_{\varepsilon,\beta}(i) : i < \kappa\rangle$. Let $\langle S_{\varepsilon,\alpha} : \alpha < \lambda_\varepsilon\rangle$ be a partition of S_ε to stationary subsets of λ_ε.

Now if $\alpha \in [\lambda_{<\zeta}, \lambda_\zeta)$ we define

- $(*)_2$ (a) $I_\alpha = \{\eta^*_{\zeta,\delta,\alpha}\restriction i : i \leq \kappa$ and $\delta \in S_{\lambda,\alpha}\} \cup \{\langle\gamma\rangle : \gamma < \lambda\}$
  - (b) $J_\alpha = \Sigma\{I_\gamma : \gamma \in \lambda\backslash\{\alpha\}\}$.

So assume

- $(*)_3$ χ_* regular $>> \lambda, \alpha(1) \in [\lambda_{<\zeta(1)}, \lambda_{\zeta(1)})$ and $(I_{\alpha(1)}, J_{\alpha(1)}, \mu, \kappa, \ldots) \in \mathscr{H}(\chi_*)$.

We can choose N^1_β by induction on $\beta < \lambda_\zeta$ such that:

- $(*)_4$ (a) $N'_\beta \prec (\mathscr{H}(\chi_*), \in, <^*_{\chi_*})$ is increasing continuous with β
  - (b) if $\gamma' < \beta$ then $\langle N^1_\gamma : \gamma \leq \gamma'\rangle \in N_\beta$
  - (c) $\{I_\alpha, J_\alpha, \mu, \kappa\} \in N^1_\beta$
  - (d) $\|N^1_\beta\| < \lambda_{\zeta(1)}$ and $N^1_\beta \cap \lambda_\zeta \in \lambda_\zeta$
- $(*)_5$ choose $\delta(1) \in S_{\zeta(1),\alpha(1)}$ such that $N^1_{\delta(1)} \cap \lambda_{\zeta(1)} = \delta(1)$; moreovr, $\{\eta_{\zeta(1),\delta(1)}(i) : i < \kappa\} \subseteq \{\beta : N^1_\beta \cap \lambda = \beta\}$
- $(*)_6$ we choose $M_{\gamma,i}$ for $i < \kappa$ by induction on $\gamma < \theta$ such that:
  - (a) $M_{\gamma,i} \prec N_{\eta_{\zeta(1),\delta(1)}}(i+1)$ has cardinality $< \theta$

(b) $j < i \Rightarrow M_{\gamma,j} \cap N_{\eta_{\zeta(1),\delta(1)}}(i+1) \subseteq M_{\gamma,i}$

(c) $\eta_{\zeta(1),\delta(1)} \restriction i \in M_{\gamma,i}$

(d) $M_{\gamma,i} \supseteq \mu^{<\kappa} + 1$ and $j < i \Rightarrow M_{\gamma,j} \subseteq M_{\gamma,i}$.

Let $M_i = \cup\{M_{\gamma,i} : \gamma < \theta\}, M = V\}M_i : i < \kappa\}$.

Let $\Lambda = \{\eta \in P^J_\kappa : (\forall i < \kappa)[\eta \restriction i \in M)\}$ and be as in clause (B)(f) so $|\Lambda| \leq E = \{\gamma < \theta$: if $\eta \in \Lambda$ and $h(\eta) \in \bigcup\limits_i M_{\gamma,i}$ then $\{\eta \in \Lambda : (\eta \restriction h(\eta) \in \bigcup\limits_{i<\kappa} M_{\gamma,i}\} \subseteq \bigcup\limits_{i<\kappa} M_{\gamma,i}$.

So E is a club of θ and we can choose $\gamma(r) \in S^*_{\zeta(1)} \cap E$.

The rest should be filled. $\square_{3.4}$

Claim 3.5. *If $\lambda > \mu^{<\kappa}$ then (λ, λ, μ) has the super bigness property (see 3.2) except possible when*

(∗) $\lambda \leq (\mu^{<\kappa})^{+\kappa}$.

Proof. Case 1: λ is regular

Use 3.2.

Case 2: $\lambda > (\mu^{<\kappa})^{+\kappa}$

Let $\theta = (\mu^{<\kappa})^{+4}$. Now by [?] for arbitrarily large regular $\lambda' \in [\mu^{<\kappa}, \lambda)$ there are $S', \bar\eta'$ as in 3.4(B)(d) for $(S_\zeta, \bar\eta_\zeta)$ - FILL.

Case 3: $\lambda \in (\mu^{<\kappa}, (\mu^{<\kappa})^{+\kappa})$ is singular

If $\kappa = \aleph_0$ this is empty. Can we immitate [?, 2.19=L7.10,pg.35]? $\square_{3.5}$

Discussion 3.6. Can 3.4 be improved to get $\bar N \restriction (i+1) \in N_{i+1}$ for all/all successor i? Probably

(∗) $\alpha < \lambda_\zeta \Rightarrow \mathrm{cf}([\alpha]^\theta, \subseteq) < \lambda_\zeta$.

Claim 3.7. *1) If (A) then (B) where:*

(A) (a) $\lambda > \theta > \mu^{<\kappa} + \mathrm{cf}(\lambda)$

(b) $\mathrm{cf}(\theta) = \aleph_0$

(c) *optional: there is a sequence* $\langle \mathfrak{a}_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle, \mathfrak{a}_\alpha \subseteq \mathrm{Reg} \cap \theta \backslash (\mu < \kappa)^+, \sup(\mathfrak{a}_\varepsilon) = \theta^+, \mathrm{otp}(\mathfrak{a}_\varepsilon) = \omega, \varepsilon \neq \zeta \Rightarrow \aleph_0 > |\mathfrak{a}_\varepsilon \cap \geq_\zeta|, (\pi \mathfrak{a}_\varepsilon)$

(B) K^κ_{tr} *has the* $(\lambda, \lambda, \mu, \kappa)$*-bigness property.*

2) Debt: define a super-bigness version.

Proof. Step A:

$(*)_0$ (a) let $\bar{\mathfrak{a}} = \langle \mathfrak{a}_\zeta : \zeta < \mathrm{cf}(\lambda)\rangle$ be as in 3.7(A)(c)

(b) $S^*_\zeta \subseteq S^{\theta^+}_\kappa$ is stationary and belongs to $\check I_\theta[\theta^+]$ for $\zeta < \mathrm{cf}(\lambda)$

(c) $\rho_\zeta = \langle \rho_{\zeta,\gamma} : \gamma < \theta^+\rangle$ is a $<_{J^{\mathrm{bd}}_{\mathfrak{a}_\zeta}}$-increasing cofinal in $(\pi \mathfrak{a}_\zeta, <_{J^{\mathrm{bd}}_{\mathfrak{a}_\zeta}})$

(d) $\bar\rho_\zeta = \langle \rho_{\zeta,\gamma} : \gamma \in S^*_\zeta\rangle, \rho_{\zeta,\gamma} \in {}^\kappa\gamma$ is increasing with limit γ clearly

$(*)_1$ $\rho_\delta \restriction (i+1) \in N_{\rho_\delta(i+1)}$ for $\delta \in S, i < \kappa$.

[Why? Should be clear.]

Stage B:

Now we immitate the proof of 3.4

$(*)_2$ (a) we choose $\langle(\lambda_\zeta, S_\zeta) : \zeta < \mathrm{cf}(\lambda)\rangle$ and as in 3.4(B)(c)

(b) we choose $\bar\eta_\zeta$ such that:

(α) $\bar\eta_\zeta = \langle \eta^1_{\zeta,\delta} : \delta \in S_\zeta\rangle$

(β) $\eta^1_{\zeta,\delta} \in {}^{(\theta^+\cdot\kappa)}\delta$ is increasing with limit δ

(γ) $i < \theta^+ \cdot \kappa \wedge \alpha < \lambda_\zeta \Rightarrow \lambda_\zeta > |\{\eta^1_{\zeta,\delta}{\restriction} i : \delta$ satisfies $\eta_{\zeta,\delta}(i) = \alpha\})$

(δ) let $\langle S_{1,\alpha} = S_{\zeta,\alpha} : \alpha \in [\lambda_{<\zeta}, \lambda_\zeta)\rangle$ is a partition of S_ζ to stationary subsets

(ε) $\bar\eta_\zeta{\restriction} S_{\zeta,\alpha}$ guess clubs

$(*)_3$ (a) if $\zeta < \mathrm{cf}(\lambda), \alpha \in [\lambda_{<\zeta}, \lambda_\zeta)$ and $\delta \in S_{\zeta,\alpha}; \gamma \in S^*_\zeta$ then we let $\eta^*_{\zeta,\alpha,\delta,\gamma} \in {}^\kappa\delta$ be: $\eta^*_{\zeta,\alpha,\delta,\gamma}(\omega i + n) = \eta^1_{\zeta,\delta}(\theta^+ \cdot i + \theta \cdot \rho_{\zeta,\gamma}(i) + \varrho_{\zeta,\gamma}(n))$

(b) for $\zeta < \mathrm{cf}(\lambda)$ and $\alpha \in [\lambda_{<\zeta}, \lambda_\zeta)$ let $I_\alpha = \{\eta^*_{\zeta,\alpha,\delta,\gamma}{\restriction} i : i \le \kappa, \delta \in S_{\zeta,\alpha}$ and $\gamma \in S^*_\zeta\} \cup \{\langle\beta\rangle : \beta < \lambda\}$.

So it suffices to prove

$(*)_4$ if $\zeta(1) < \mathrm{cf}(\lambda), \alpha(1) \in [\lambda_{<\zeta(1)}, \lambda_{\zeta(1)}), J_{\alpha(1)} = \Sigma\{I_\beta : \beta \in \lambda\backslash\{\alpha(1)\}\}$ then I_α is (μ,κ)-unembeddable into $\mathscr{M}_{\mu,\kappa}(J_\alpha)$.

So assume

$\boxplus$ $p : I_{\alpha(1)} \to {}^{\kappa>}(\mathscr{M}_{\mu,\kappa}(J_\alpha))$.

Let $\chi_* > \lambda^+$ be regular.

$\boxplus$ choose N^1_β by induction on $\beta < \lambda_\alpha$ such that:

(a) $N^1_\beta \prec (\mathscr{H}^*(\chi_*), \in, <^*_{\chi_*})$

(b) if $\gamma^1 < \beta$ then $\langle N^1_\gamma : \gamma \le \gamma^1\rangle \in N_\beta$

(c) $\{I_\alpha, J_\alpha, f, \mu, \kappa\} \in N^1_\beta$

(d) $\|N^1_\beta\| < \lambda_{\zeta(1)}$ and $N^1_\beta \cap \lambda_{\zeta(1)} \in \lambda_{\zeta(1)}$

$\boxplus$ choose $\delta(1) \in S_{\zeta(1),\alpha(1)}$ such that $N^1_{\delta(1)} \cap \lambda_{\zeta(1)} = \delta(1)$ and moreover $\{\eta^1_{\zeta(1),\delta(1)}(i) : i < \theta^+ \cdot \kappa\} \subseteq \{\beta < \lambda_{\zeta(1)} : N^1_\beta \cap \lambda_{\zeta(1)} = \beta\}$

$(*)$ let

(a) $f(\eta^*_{\zeta(1),\alpha(1),\delta(1),\gamma}) = \sigma_\gamma(\bar\nu_\gamma)$

(b) $\bar\nu_\gamma = \langle \nu_i : i < i^*_\gamma\rangle$ so $\nu_{\gamma,i} \in J_{\alpha(1)}$

(c) $j_\gamma = \sup\{\ell g(\eta^*_{\zeta(1),\alpha(1),\delta(1),\gamma} \cap \nu_{\gamma,i}) + 1 : i < i_*\}$

(d) $E = \{(\gamma_1,\gamma_2) : \gamma_1, \gamma_2 \in S^*_\zeta$ and $\sigma_{\gamma_1} = \sigma_{\gamma_2}, i^*_{\gamma_1} = i^*_{\gamma_2}, \ell g(\nu_{\gamma_1,i}) = \ell g(\nu_{\gamma_2,i})$ for $i < i^*_{\gamma_1}, j_{\gamma_1} = j_{\gamma_2}$ maybe more

(e) $\gamma_* = \min\{\gamma : \gamma/E$ is stationary$\}$.

Now we can fix an interval of length θ in θ^+ and corresponding considering γ_*/E and immitate [**?**, 2.15=L7.9,pg.27]. $\square_{3.7}$

Discussion 3.8. 1) Is

$(*)_1$ $\mu^{<\kappa} + \mathrm{cf}(\lambda) < \theta < \lambda, \mathrm{cf}(\theta) = \aleph_0$ enough or do we need also

$(*)_2$ there are $\langle \mathfrak{a}_\varepsilon : \varepsilon < \mathrm{cf}(\lambda)\rangle$ as in 3.7?

If $(*)_2$ suffice, then $\lambda = (\mu^{<\kappa})^{+\omega}$ is the only open (for bigness, ignoring the super bigness version). Hence as $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ we have $(\mu^{<\kappa})^{\aleph_0} = \mu^{<\kappa}$. We may try to combine aspects of the last proof and [**?**, 2.19=L7.10,pg.35].
2) To prove $(*)_2$, we need then to use a fix $\mathfrak{a}$ for all ζ but $\bar{\rho}_\zeta = \bar{\rho} \restriction S^*$, $\langle S^*_\zeta : \zeta < \mathrm{cf}(\lambda)\rangle$ are pairwise disjoint stationary subsets of $S^{\theta^+}_{\aleph_0}$.
3) But, but if we let $\lambda = (\mu^{<\kappa})^{+\delta}$.

Case 1: $\delta > \kappa$
See xxx

Case 2: $\delta \leq \kappa$ and for some $\sigma < \mathrm{cf}(\delta)$ we have $\sigma^{\aleph_0} \geq \mathrm{cf}(\delta)$
By the version with $(*)_1 + (*)_2$.

Case 3: Neither Case 1 nor Case 2
So $\delta = \partial = \mathrm{cf}(\partial) \leq \kappa$ and $\alpha < \partial \Rightarrow |\alpha|^{\aleph_0} < \sigma$.

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