SMALL–LARGE SUBGROUPS OF THE REALS

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Abstract. We are interested in subgroups of the reals that are small in one and large in another sense. We prove that, in ZFC, there exists a non–meager Lebesgue null subgroup of $\mathbb{R}$, while it is consistent that there there is no non–null meager subgroup of $\mathbb{R}$. This answers a question from Filipczak, Rosłanowski and Shelah [5].

1. Introduction

Subgroups of the reals which are small in one and large in another sense were crucial in Filipczak, Rosłanowski and Shelah [5]. If there is a non–meager Lebesgue null subgroup of $(\mathbb{R},+)$, then there is no translation invariant Borel hull operation on the σ–ideal $\mathcal{N}$ of Lebesgue null sets. That is, there is no mapping ψ from $\mathcal{N}$ to Borel sets such that for each null set $A\subseteq\mathbb{R}$:

- $A\subseteq\psi(A)$ and $\psi(A)$ is null, and
- $\psi(A+t)=\psi(A)+t$ for every $t\in\mathbb{R}$.

Parallel claims hold true if "Lebesgue null" is interchanged with "meager" and/or $(\mathbb{R},+)$ is replaced with $({}^{\omega}2,+_2)$.

If $\mathcal{M}$ is the σ–ideal of meager subsets of $\mathbb{R}$ (and $\mathcal{N}$ is the null ideal on $\mathbb{R}$) and $\{\mathcal{I},\mathcal{J}\}=\{\mathcal{N},\mathcal{M}\}$, then various set theoretic assumptions imply the existence of a subgroup of $\mathbb{R}$ which belongs to $\mathcal{I}$ but not to $\mathcal{J}$. But in [5, Problem 4.1] we asked if the existence of such subgroups can be shown in ZFC. This question is interesting *per se*, regardless of its connections to translation invariant Borel hulls.

The present paper presents two theorems. First, in Theorem 2.3 we give ZFC examples of null non-meager subgroups of $({}^{\omega}2,+_2)$ and $(\mathbb{R},+)$, respectively. Next in Theorem 4.1 we show that it is consistent with ZFC that every meager subgroup of $({}^{\omega}2,+_2)$ and/or $(\mathbb{R},+)$ has Lebesgue measure zero. This answers

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[5, Problem 4.1]. Also, our results give another example of a strange asymmetry between measure and category.

Notation Our notation is rather standard and compatible with that of classical textbooks (like Jech [6] or Bartoszyński and Judah [1]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

(1) The Cantor space ${}^{\omega}2$ of all infinite sequences with values 0 and 1 is equipped with the natural product topology, the product measure λ and the group operation of coordinate-wise addition $+_2$ modulo 2.
(2) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta$. Finite ordinals (non-negative integers) will be denoted by letters i, j, k, ℓ, m, n while integers will be called L, M.
(3) Most of our intervals will be intervals of non-negative integers, so $[m, n) = \{k \in \omega : m \leq k < n\}$ etc. They will be denoted by letter J (with possible indices). However, we will also use the notation $[0, 1)$ to denote the unit interval of reals.
(4) The Greek letter κ will stand for an uncountable cardinal such that $\kappa^{\aleph_0} = \kappa \geq \aleph_2$.
(5) For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$), and $\underset{\sim}{G}_{\mathbb{P}}$ will stand for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$.
(6) We fix a well ordering $\prec^*$ of all hereditarily finite sets.
(7) The set of all partial finite functions with domains included in ω and with values in 2 is denoted ${}^{\underline{\omega}}2$.

2. Null non–meager

Here we will give a ZFC construction of a non–meager Lebesgue null subgroup of the reals. The main construction is done in ${}^{\omega}2$ and then we transfer it to $\mathbb{R}$ using the standard binary expansion $\mathbf{E}$.

Definition 2.1. Let $D_0^\infty = \{x \in {}^{\omega}2 : (\exists^\infty i < \omega)(x(i) = 0)\}$ and for $x \in D_0^\infty$ let $\mathbf{E}(x) = \sum\limits_{i=0}^{\infty} x(i) 2^{-(i+1)}$.

Proposition 2.2. (1) *The function $\mathbf{E} : D_0^\infty \longrightarrow [0, 1)$ is a continuous bijection, it preserves both the measure and the category.*
(2) *Assume that*
  (a) *$x, y, z \in D_0^\infty$, $\mathbf{E}(z) = \mathbf{E}(x) + \mathbf{E}(y)$ modulo 1, and*
  (b) *$n < m < \omega$ and both $x{\restriction}[n, m]$ and $y{\restriction}[n, m]$ are constant.*
  Then $z{\restriction}[n, m-1]$ is constant.
(3) *Assume that*
  (a) *$x, y \in D_0^\infty$, $0 < \mathbf{E}(x)$ and $\mathbf{E}(y) = 1 - \mathbf{E}(x)$,*
  (b) *$n < m < \omega$ and $x{\restriction}[n, m]$ is constant.*

Then $y{\restriction}[n, m-1]$ is constant.

Proof. (1) Well known, cf. Bukovský [4, §2.4].
(2,3) Straightforward (just consider the possible constant values and analyze how the addition is performed). $\square$

Theorem 2.3. (1) *There exists a null non-meager subgroup of $({}^\omega 2, +_2)$.*
(2) *There exists a null non-meager subgroup of $(\mathbb{R}, +)$.*

Proof. (1) For $k \in \omega$ let $n_k = \frac{1}{2}k(k+1)$ and let D be a non-principal ultrafilter on ω. Define

$$H_D = \Big\{x \in {}^\omega 2 : \big(\exists m < \omega\big)\big(\exists j < 2\big)\big(\big\{k > m : x{\restriction}[n_k, n_{k+1} - m) \equiv j\big\} \in D\big)\Big\}.$$

(i) H_D is a subgroup of $({}^\omega 2, +_2)$.

Why? Suppose that $x_0, x_1 \in H_D$ and let $m_\ell < \omega$ and $j_\ell < 2$ be such that

$$A_\ell \stackrel{\rm def}{=} \big\{k > m_\ell : x_\ell{\restriction}[n_k, n_{k+1} - m_\ell) \equiv j_\ell\big\} \in D.$$

Let $m = \max(m_0, m_1)$ and $j = j_0 -_2 j_1$. Then $A_0 \cap A_1 \in D$ and for each $k \in A_0 \cap A_1$ we have $(x_0 -_2 x_2){\restriction}[n_k, n_{k+1} - m) \equiv j$. Hence $x_0 -_2 x_1 \in H_D$.

(ii) $H_D \in \mathcal{N}$.

Why? For each $m < k < \omega$ and $j < 2$ we have

$$\lambda(\{x \in {}^\omega 2 : x{\restriction}[n_k, n_{k+1} - m) \equiv j\}) = 2^{m-(k+1)}$$

and therefore for each $m < \omega$ and $j < 2$

$$\lambda(\{x \in {}^\omega 2 : (\exists^\infty k)(x{\restriction}[n_k, n_{k+1} - m) \equiv j)\}) = 0.$$

Now note that $H_D \subseteq \bigcup\limits_{m<\omega} \bigcup\limits_{j<2} \big\{x \in {}^\omega 2 : (\exists^\infty k)(x{\restriction}[n_k, n_{k+1} - m) \equiv j)\big\}$.

(iii) $H_D \notin \mathcal{M}$.

Why? Suppose that W is a dense Π^0_2 subset of ${}^\omega 2$. Then we may choose an increasing sequence $\langle k_i : i \in \omega\rangle$ and a function $f \in {}^\omega 2$ such that

$$\Big\{x \in {}^\omega 2 : \big(\exists^\infty i\big)\big(x{\restriction}[n_{k_i}, n_{k_{i+1}}) = f{\restriction}[n_{k_i}, n_{k_{i+1}})\big)\Big\} \subseteq W.$$

Let $A = \bigcup\{[k_{2i}, k_{2i+1}) : i \in \omega\}$ and $B = \bigcup\{[k_{2i+1}, k_{2i+2}) : i \in \omega\}$. Then either $A \in D$ or $B \in D$. Let $x_A, x_B \in {}^\omega 2$ be such that, for each $i \in \omega$,

$$\begin{array}{ll} x_A{\restriction}[n_{k_{2i}}, n_{k_{2i+1}}) \equiv 0, \quad x_A{\restriction}[n_{k_{2i+1}}, n_{k_{2i+2}}) = f{\restriction}[n_{k_{2i+1}}, n_{k_{2i+2}}) & \text{and} \\ x_B{\restriction}[n_{k_{2i+1}}, n_{k_{2i+2}}) \equiv 0, \quad x_B{\restriction}[n_{k_{2i}}, n_{k_{2i+1}}) = f{\restriction}[n_{k_{2i}}, n_{k_{2i+1}}). & \end{array}$$

Then $x_A, x_B \in W$ and either $x_A \in H_D$ or $x_B \in H_D$. Consequently, $W \cap H_D \neq \emptyset$.

(2) Consider $H^*_D = \mathbf{E}[H_D \cap D^\infty_0] + \mathbb{Z}$. It follows from 2.2(1) that H^*_D is a Lebesgue null meager subset of $\mathbb{R}$. We will show that it is a subgroup of $(\mathbb{R}, +)$.

Suppose that $x_0, x_1 \in H_D \cap D_0^\infty$ and $L_0, L_1 \in \mathbb{Z}$ and we will argue that $(\mathbf{E}(x_0) + L_0) + (\mathbf{E}(x_1) + L_1) \in H_D^*$. Let $m_\ell < \omega$ be such that

$$A_\ell \stackrel{\rm def}{=} \big\{k > m_\ell : x_\ell \restriction [n_k, n_{k+1} - m_\ell) \text{ is constant } \big\} \in D$$

and let $m = \max(m_0, m_1) + 1$. Choose $y \in D_0^\infty$ and $M \in \{0, 1\}$ such that $\mathbf{E}(x_0) + \mathbf{E}(x_1) = \mathbf{E}(y) + M$. It follows from 2.2(2) that for every $k \in A_0 \cap A_1$, $k > m$, we have that $y \restriction [n_k, n_{k+1} - m)$ is constant and since $A_0 \cap A_1 \in D$ we conclude $y \in H_D$. Consequently, $(\mathbf{E}(x_0)+L_0)+(\mathbf{E}(x_1)+L_1) = \mathbf{E}(y)+(M+L_0+L_1) \in H_D^*$.

Now assume that $x \in H_D \cap D_0^\infty$, $L \in \mathbb{Z}$ and we will argue that $-(\mathbf{E}(x) + L) \in H_D^*$. If $\mathbf{E}(x) = 0$ then the assertion is clear, so assume also $\mathbf{E}(x) > 0$. Let $m < \omega$ be such that

$$A \stackrel{\rm def}{=} \big\{k > m : x \restriction [n_k, n_{k+1} - m) \text{ is constant } \big\} \in D.$$

Choose $y \in D_0^\infty$ such that $1 - \mathbf{E}(x) = \mathbf{E}(y)$. It follows from 2.2(3) that for every $k \in A$, $k > m+1$, we have that $y \restriction [n_k, n_{k+1} - (m+1))$ is constant. Consequently, $y \in H_D$ and $-(\mathbf{E}(x) + L) = \mathbf{E}(y) - 1 - L \in H_D^*$. □

Remark 2.4. A somewhat simpler non–meager null subgroup of $({}^\omega 2, +_2)$ is

$$H_D^- = \Big\{x \in {}^\omega 2 : \big\{k \in \omega : x \restriction [n_k, n_{k+1}) \equiv 0\big\} \in D\Big\}.$$

The group H_D, however, was necessary for our construction of $H_D^* < \mathbb{R}$.

Corollary 2.5. *There exists no translation invariant Borel hull for the null ideal on* ${}^\omega 2$ *and/or on* $\mathbb{R}$.

3. Some technicalities

Here we prepare the ground for our consistency results.

3.1. **Moving from $\mathbb{R}$ to ${}^\omega 2$.** First, let us remind connections between the addition in $\mathbb{R}$ and that of ${}^\omega 2$ (via the binary expansion $\mathbf{E}$, see 2.1).

Definition 3.1. Let $J = [m, n)$ be a non-empty interval of integers and $c \in \{0, 1\}$. For sequences $\rho, \sigma \in {}^J 2$ we define $\rho \circledast_c \sigma$ as the unique $\eta \in {}^J 2$ such that

$$\Big(\sum_{i=m}^{n-1} \rho(i) 2^{-(i+1)} + \sum_{i=m}^{n-1} \sigma(i) 2^{-(i+1)} + c \cdot 2^{-n}\Big) - \sum_{i=m}^{n-1} \eta(i) 2^{-(i+1)} \in \{0, 2^{-m}\}.$$

For notational convenience we also set $\rho \circledast_2 \sigma = \rho +_2 \sigma$ (coordinate-wise addition modulo 2).

The operation $\circledast_c$ is defined on the set ${}^J 2$, so it does depend on J. We may, however, abuse notation and use that same symbol $\circledast_c$ for various J.

Observation 3.2. *Let m, ℓ, n be integers such that $m < \ell < n$ and let $J = [m, n)$.*

(1) *For each* $c \in \{0,2\}$, $({}^J 2, \circledast_c)$ *is an Abelian group.*
(2) *If* $\rho, \sigma \in {}^J 2$ *and* $\rho(\ell) = \sigma(\ell)$, *then* $(\rho \circledast_0 \sigma){\restriction}[m,\ell) = (\rho \circledast_1 \sigma){\restriction}[m,\ell)$.
(3) *If* $\rho, \sigma \in {}^J 2$ *and* $(\rho \circledast_0 \sigma)(\ell) = 0$, *then* $(\rho \circledast_0 \sigma){\restriction}[m,\ell) = (\rho \circledast_1 \sigma){\restriction}[m,\ell)$.
(4) *Suppose that* $r, s \in [0,1)$, $\rho, \sigma, \eta \in D_0^\infty$, $\mathbf{E}(\rho) = r$, $\mathbf{E}(\sigma) = s$ *and* $\mathbf{E}(\eta) = r + s$ *modulo 1. Then*
   - *if* $\sum\limits_{i \geq n} \big((\rho(i) + \sigma(i))/2^{i+1}\big) \geq 2^{-n}$, *then* $\eta{\restriction}J = (\rho{\restriction}J) \circledast_1 (\sigma{\restriction}J)$;
   - *if* $\sum\limits_{i \geq n} \big((\rho(i) + \sigma(i))/2^{i+1}\big) < 2^{-n}$, *then* $\eta{\restriction}J = (\rho{\restriction}J) \circledast_0 (\sigma{\restriction}J)$.

3.2. **The combinatorial heart of our forcing arguments.** For this subsection we fix a strictly increasing sequence $\bar{n} = \langle n_j : j < \omega \rangle \subseteq \omega$.

Definition 3.3. We define $\bar{m}[\bar{n}] = \langle m_i : i < \omega \rangle$, $\bar{N}[\bar{n}] = \langle N(i) : i < \omega \rangle$, $\bar{J}[\bar{n}] = \langle J_i : i < \omega \rangle$, $\bar{H}[\bar{n}] = \langle H_i : i < \omega \rangle$, $\pi[\bar{n}] = \langle \pi_i : i < \omega \rangle$ and $\mathbf{F}[\bar{n}]$ as follows.

We set $m_0 = 0$ and then inductively for $i < \omega$ we let

$(*)_1$ $m_{i+1} = 2^{n_{m_i} + 1081}$.

Next, for $i < \omega$,

$(*)_2$ $N(i) = n_{m_i}$, $J_i = \big[N(2^i), N(2^{i+1})\big)$, and

$(*)_3$ $H_i = \big\{a \subseteq {}^{J_i}2 : (1 - 2^{-N(2^i)}) \cdot 2^{|J_i|} \leq |a|\big\}$.

We also set $\pi_i : |H_i| \longrightarrow H_i$ to be the $\prec^*$–first bijection from $|H_i|$ onto H_i. Finally, for $\eta \in \prod\limits_{m<\omega} (m+1)$ we let

$(*)_4$ $\mathbf{F}_0[\bar{n}](\eta) = \big\{x \in {}^\omega 2 : \big(\forall i < \omega\big)\big(x{\restriction}J_i \in \pi_i(\eta(|H_i| - 1))\big)\big\}$ and
$\mathbf{F}[\bar{n}](\eta) = \big\{x \in {}^\omega 2 : \big(\forall^\infty i < \omega\big)\big(x{\restriction}J_i \in \pi_i(\eta(|H_i| - 1))\big)\big\}$.

Lemma 3.4. *For every* $\eta \in \prod\limits_{m<\omega} (m+1)$, $\mathbf{F}_0[\bar{n}](\eta) \subseteq {}^\omega 2$ *is a closed set of positive Lebesgue measure, and* $\mathbf{F}[\bar{n}](\eta)$ *is a* Σ^0_2 *set of Lebesgue measure 1.*

Proof. Note that $J_i \cap J_j = \emptyset$ and $|H_i| < |H_j|$ for $i < j$, and $\sum\limits_{i=0}^{\infty} 2^{-N(2^i)} < 1$. $\square$

Lemma 3.5. *Let* $i < \omega$, $c \in \{0,2\}$ *and let* $\eta \in {}^{J_i}2$. *Suppose that for each* $\ell < 2^i$ *and* $x < 2$ *we are given a function* $\mathcal{Z}^x_\ell : H_i \longrightarrow {}^{J_i}2$ *such that* $\mathcal{Z}^x_\ell(a) \in a$ *for each* $a \in H_i$. *Then there are* $a^0, a^1 \in H_i$ *such that for every* $\ell < 2^i$ *there is* $k \in [m_{2^i+\ell}, m_{2^i+\ell+1})$ *satisfying*

$$\big(\mathcal{Z}^0_\ell(a^0){\restriction}[n_k, n_{k+1})\big) \circledast^k_c \big(\mathcal{Z}^1_\ell(a^1){\restriction}[n_k, n_{k+1})\big) = \eta{\restriction}[n_k, n_{k+1}),$$

where $\circledast^k_c$ *denotes the operation* $\circledast_c$ *on* ${}^{[n_k, n_{k+1})}2$.

Proof. We start the proof with the following Claim.

Claim 3.5.1. *If* $\mathcal{A} \subseteq H_i$, $|\mathcal{A}| \leq 2^{|J_i| - N(2^i) - i}$ *and* $x < 2$, *then there is* $b \in H_i$ *such that* $\mathcal{Z}^x_\ell(b) \notin \{\mathcal{Z}^x_\ell(a) : a \in \mathcal{A}\}$ *for each* $\ell < 2^i$.

Proof of the Claim. Note that $|\{\mathcal{Z}^x_\ell(a) : \ell < 2^i \ \&\ a \in \mathcal{A}\}| \leq 2^i \cdot 2^{|J_i|-N(2^i)-i} = 2^{|J_i|-N(2^i)}$, so letting $b = {}^{J_i}2 \setminus \{\mathcal{Z}^x_\ell(a) : \ell < 2^i \ \&\ a \in \mathcal{A}\}$ we have $b \in H_i$. Since $\mathcal{Z}^x_\ell(b) \in b$ we see that b is as required in the claim. □

It follows from Claim 3.5.1 that we may pick sequences $\langle a^0_j : j < j^*\rangle \subseteq H_i$ and $\langle a^1_j : j < j^*\rangle \subseteq H_i$ with $\mathcal{Z}^x_\ell(a^x_{j_1}) \neq \mathcal{Z}^x_\ell(a^x_{j_2})$ for $j_1 < j_2 < j^*$, $\ell < 2^i$, $x < 2$ and such that $j^* > 2^{|J_i|-N(2^i)-i}$. Now, by induction on $\ell < 2^i$, we choose sets $X_\ell, Y_\ell \subseteq j^*$ and integers $k_\ell \in [m_{2^i+\ell}, m_{2^i+\ell+1})$ such that the following demands are satisfied.

(i) $X_{\ell+1} \subseteq X_\ell \subseteq j^*$, $Y_{\ell+1} \subseteq Y_\ell \subseteq j^*$,

(ii) if $j_0 \in X_\ell$ and $j_1 \in Y_\ell$ then
$$\big(\mathcal{Z}^0_\ell(a^0_{j_0})\restriction[n_{k_\ell}, n_{k_\ell+1})\big) \circledast^{k_\ell}_{\mathrm{c}} \big(\mathcal{Z}^1_\ell(a^1_{j_1})\restriction[n_{k_\ell}, n_{k_\ell+1})\big) = \eta\restriction[n_{k_\ell}, n_{k_\ell+1}),$$

(iii) $\min\big(|X_\ell|, |Y_\ell|\big) \geq j^* \cdot 2^{N(2^i)-N(2^i+\ell+1)-\ell-1}$.

We stipulate $X_{-1} = Y_{-1} = j^*$ and we assume that $X_{\ell-1}, Y_{\ell-1}$ have been already determined (and $\min\big(|X_{\ell-1}|, |Y_{\ell-1}|\big) \geq j^* \cdot 2^{N(2^i)-N(2^i+\ell)-\ell}$ if $\ell > 0$). Let

$$\begin{aligned}&X^* = \big\{j \in X_{\ell-1} : |X_{\ell-1}| \cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)-1} \leq \\ &\big|\{j' \in X_{\ell-1} : \mathcal{Z}^0_\ell(a^0_{j'})\restriction[N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}^0_\ell(a^0_j)\restriction[N(2^i+\ell), N(2^i+\ell+1))\}\big|\big\},\\ &Y^* = \big\{j \in Y_{\ell-1} : |Y_{\ell-1}| \cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)-1} \leq \\ &\big|\{j' \in Y_{\ell-1} : \mathcal{Z}^1_\ell(a^1_{j'})\restriction[N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}^1_\ell(a^1_j)\restriction[N(2^i+\ell), N(2^i+\ell+1))\}\big|\big\}.\end{aligned}$$

Claim 3.5.2. *$|X^*| \geq \frac{1}{2}|X_{\ell-1}|$ and $|Y^*| \geq \frac{1}{2}|Y_{\ell-1}|$.*

Proof of the Claim. Assume towards contradiction that $|X^*| < \frac{1}{2}|X_{\ell-1}|$. Then for some $\nu_0 \in {}^{[N(2^i+\ell), N(2^i+\ell+1))}2$ we have
$$\big|\{j \in X_{\ell-1} \setminus X^* : \nu_0 \subseteq \mathcal{Z}^0_\ell(a^0_j)\}\big| \geq |X_{\ell-1} \setminus X^*| \cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)} > \tfrac{1}{2}|X_{\ell-1}| \cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)}.$$

Let $j \in X_{\ell-1} \setminus X^*$ be such that $\nu_0 \subseteq \mathcal{Z}^0_\ell(a^0_j)$. Then $j \in X^*$, a contradiction.

Similarly for Y^*. □

Claim 3.5.3. *For some $k \in [m_{2^i+\ell}, m_{2^i+\ell+1})$ we have that both $\big|\{\mathcal{Z}^0_\ell(a^0_j)\restriction[n_k, n_{k+1}) : j \in X^*\}\big| > 2^{n_{k+1}-n_k-1}$ and $\big|\{\mathcal{Z}^1_\ell(a^1_j)\restriction[n_k, n_{k+1}) : j \in Y^*\}\big| > 2^{n_{k+1}-n_k-1}$.*

Proof of the Claim. Let
$$K^X = \big\{k \in [m_{2^i+\ell}, m_{2^i+\ell+1}) : |\{\mathcal{Z}^0_\ell(a^0_j)\restriction[n_k, n_{k+1}) : j \in X^*\}| \leq 2^{n_{k+1}-n_k-1}\big\}$$
and
$$K^Y = \big\{k \in [m_{2^i+\ell}, m_{2^i+\ell+1}) : |\{\mathcal{Z}^1_\ell(a^1_j)\restriction[n_k, n_{k+1}) : j \in Y^*\}| \leq 2^{n_{k+1}-n_k-1}\big\}.$$
Assume towards contradiction that $|K^X| \geq \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$. Then
$$|X^*| = |\{\mathcal{Z}^0_\ell(a^0_j) : j \in X^*\}| \leq 2^{-1/2(m_{2^i+\ell+1}-m_{2^i+\ell})} \cdot 2^{|J_i|} < 2^{|J_i|} \cdot 2^{-4N(2^i+\ell)}.$$

(Remember 3.3$(*)_1$.) Hence $|X_{\ell-1}| \leq 2^{|J_i|-4N(2^i+\ell)+1}$. If $\ell = 0$ then we get $2^{|J_i|-2N(2^i)} < j^* \leq 2^{|J_i|-4N(2^i)+1}$, which is impossible. If $\ell > 0$, then by the inductive hypothesis (iii) we know that $|X_{\ell-1}| \geq j^* \cdot 2^{N(2^i)-N(2^i+\ell)-\ell} > 2^{|J_i|-i-N(2^i+\ell)-\ell}$, so $3N(2^i+\ell)-1 < i+\ell$, a clear contradiction. Consequently $|K^X| < \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$, and similarly $|K^Y| < \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$. Pick $k \in [m_{2^i+\ell}, m_{2^i+\ell+1})$ such that $k \notin K^X \cup K^Y$. $\square$

Now, let $k_\ell \in [m_{2^i+\ell}, m_{2^i+\ell+1})$ be as given by Claim 3.5.3. Necessarily the sets $\big\{\rho \in {}^{[n_{k_\ell}, n_{k_\ell+1})}2 : (\exists j \in X^*)((\mathcal{Z}^0_\ell(a^0_j){\restriction}[n_{k_\ell}, n_{k_\ell+1})) \circledast^{k_\ell}_c \rho = \eta{\restriction}[n_{k_\ell}, n_{k_\ell+1}))\big\}$ and $\big\{\mathcal{Z}^1_\ell(a^1_j){\restriction}[n_{k_\ell}, n_{k_\ell+1}) : j \in Y^*\big\}$ have non-empty intersection. Therefore, we may find $j_X \in X^*$ and $j_Y \in Y^*$ such that

$$\big(\mathcal{Z}^0_\ell(a^0_{j_X}){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) \circledast^{k_\ell}_c \big(\mathcal{Z}^1_\ell(a^1_{j_Y}){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) = \eta{\restriction}[n_{k_\ell}, n_{k_\ell+1}).$$

Set

$$X_\ell = \big\{j \in X_{\ell-1} : \mathcal{Z}^0_\ell(a^0_j){\restriction}[N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}^0_\ell(a^0_{j_X}){\restriction}[N(2^i+\ell), N(2^i+\ell+1))\big\},$$

and

$$Y_\ell = \big\{j \in Y_{\ell-1} : \mathcal{Z}^1_\ell(a^1_j){\restriction}[N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}^1_\ell(a^1_{j_Y}){\restriction}[N(2^i+\ell), N(2^i+\ell+1))\big\}.$$

By the definition of X^*, Y^* and by the inductive hypothesis (iii) we have

$$|X_\ell| \geq |X_{\ell-1}| \cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)-1} \geq j^* \cdot 2^{N(2^i)-\ell-N(2^i+\ell+1)-1}$$

and similarly for Y_ℓ. Consequently, X_ℓ, Y_ℓ and k_ℓ satisfy the inductive demands (i)–(iii).

After the above construction is completed fix any $j_0 \in X_{2^i-1}$, $j_1 \in Y_{2^i-1}$ and consider $a^0 = a_{j_0}$ and $a^1 = a_{j_1}$. For each $\ell < 2^i$ we have $j_0 \in X_\ell$, $j_1 \in Y_\ell$ so

$$\big(\mathcal{Z}^0_\ell(a^0){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) \circledast^{k_\ell}_c \big(\mathcal{Z}^1_\ell(a^1){\restriction}[n_{k_\ell}, n_{k_\ell+1})\big) = \eta{\restriction}[n_{k_\ell}, n_{k_\ell+1}).$$

Hence $a^1, a^2 \in H_i$ are as required. $\square$

3.3. **The $*$–Silver forcing notion.** The consistency result of the next section will be obtained using CS product of the following forcing notion $\mathbb{S}_*$.

Definition 3.6. (1) We define the $*$–Silver forcing notion $\mathbb{S}_*$ as follows.
A condition in $\mathbb{S}_*$ is a partial function $p : \mathrm{dom}(p) \longrightarrow \omega$ such that $\mathrm{dom}(p) \subseteq \omega$ is coinfinite and $p(m) \leq m$ for each $m \in \mathrm{dom}(p)$.
The order $\leq = \leq_{\mathbb{S}_*}$ of $\mathbb{S}_*$ is the inclusion, i.e., $p \leq q$ if and only if $p \subseteq q$.

(2) For $p \in \mathbb{S}_*$ and $1 \leq n < \omega$ we let $u(n,p)$ be the set of the first n elements of $\omega \setminus \mathrm{dom}(p)$ (in the natural increasing order). Then for $p, q \in \mathbb{S}_*$ we let $p \leq_n q$ if and only if $p \leq q$ and $u(n,q) = u(n,p)$.
We also define $p \leq_0 q$ as equivalent to $p \leq q$.

(3) Let $p \in \mathbb{S}_*$. We let $S(n,p)$ be the set of all functions $s : u(n,p) \longrightarrow \omega$ with the property that $s(m) \leq m$ for all $m \in u(n,p)$.

(4) We let $\underset{\sim}{\eta}$ to be the canonical $\mathbb{S}_*$–name such that

$$\Vdash \underset{\sim}{\eta} = \bigcup\{p : p \in \underset{\sim}{G}_{\mathbb{S}_*}\}.$$

Remark 3.7. The forcing notion $\mathbb{S}_*$ may be represented as a forcing of the type $\mathbb{Q}^*_{\mathrm{w}\infty}(K,\Sigma)$ for some finitary creating pair (K,Σ) which captures singletons, see Rosłanowski and Shelah [8, Definition 2.1.10]. It is a close relative of the Silver forcing notion and, in a sense, it lies right above all $\mathbb{S}_n$'s studied for instance in Rosłanowski [7] and Rosłanowski and Steprāns [9].

Lemma 3.8. (1) *$(\mathbb{S}_*, \leq_{\mathbb{S}_*})$ is a partial order of size $\mathfrak{c}$. If $p \in \mathbb{S}_*$ and $s \in S(n,p)$ then $p \cup s \in \mathbb{S}_*$ is a condition stronger than p.*
(2) *$\Vdash_{\mathbb{S}_*} \underset{\sim}{\eta} \in \prod\limits_{m<\omega}(m+1)$ and $p \Vdash_{\mathbb{S}_*} p \subseteq \underset{\sim}{\eta}$ (for $p \in \mathbb{S}_*$).*
(3) *If $p \in \mathbb{S}_*$ and $1 \leq n < \omega$, then the family $\{p \cup s : s \in S(n,p)\}$ is an antichain pre-dense above p.*
(4) *The relations $\leq_n$ are partial orders on $\mathbb{S}_*$, $p \leq_{n+1} q$ implies $p \leq_n q$.*
(5) *Assume that $\underset{\sim}{\tau}$ is an $\mathbb{S}_*$–name for an ordinal, $p \in \mathbb{S}_*$, $1 \leq n, m < \omega$. Then there is a condition $q \in \mathbb{S}_*$ such that $p \leq_n q$, $\max\big(u(n+1,q)\big) > m$ and for all $s \in S(n,q)$ the condition $q \cup s$ decides the value of $\underset{\sim}{\tau}$.*
(6) *The forcing notion $\mathbb{S}_*$ satisfies Axiom A of Baumgartner [2, §7] as witnessed by the orders $\leq_n$, it is ${}^\omega\omega$–bounding and, moreover, every meager subset of ${}^\omega 2$ in an extension by $\mathbb{S}_*$ is included in a Σ^0_2 meager set coded in the ground model.*

Proof. Straightforward - the same as for the Silver forcing notion. □

Definition 3.9. Assume $\kappa^{\aleph_0} = \kappa \geq \aleph_2$.

(1) $\mathbb{S}_*(\kappa)$ is the CS product of κ many copies of $\mathbb{S}_*$. Thus
a condition p in $\mathbb{S}_*(\kappa)$ is a function with a countable domain $\mathrm{dom}(p) \subseteq \kappa$ and with values in $\mathbb{S}_*$, and
the order $\leq$ of $\mathbb{S}_*(\kappa)$ is such that
$p \leq q$ if and only if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $(\forall \alpha \in \mathrm{dom}(p))(p(\alpha) \leq_{\mathbb{S}_*} q(\alpha))$.
(2) Suppose that $p \in \mathbb{S}_*(\kappa)$ and $F \subseteq \mathrm{dom}(p)$ is a finite non-empty set and $\mu : F \longrightarrow \omega \setminus \{0\}$. Let $v(F,\mu,p) = \prod\limits_{\alpha \in F} u(\mu(\alpha), p(\alpha))$ and $T(F,\mu,p) = \prod\limits_{\alpha \in F} S(\mu(\alpha), p(\alpha))$.
If $\sigma \in T(F,\mu,p)$ then let $p|\sigma$ be the condition $q \in \mathbb{S}_*(\kappa)$ such that $\mathrm{dom}(q) = \mathrm{dom}(p)$ and $q(\alpha) = p(\alpha) \cup \sigma(\alpha)$ for $\alpha \in F$ and $q(\alpha) = p(\alpha)$ for $\alpha \in \mathrm{dom}(q) \setminus F$.
We let $p \leq_{F,\mu} q$ if and only if $p \leq q$ and $v(F,\mu,p) = v(F,\mu,q)$.
If μ is constantly n then we may write n instead of μ.

(3) Suppose that $p \in \mathbb{S}_*(\kappa)$ and $\bar{\underset{\sim}{\tau}} = \langle \underset{\sim}{\tau}_n : n < \omega \rangle$ is a sequence of names for ordinals. We say that p *determines* $\bar{\underset{\sim}{\tau}}$ *relative to* $\bar{F}$ if
- $\bar{F} = \langle F_n : n < \omega \rangle$ is a sequence of finite subsets of $\mathrm{dom}(p)$, and
- p forces a value to $\underset{\sim}{\tau}_0$ and for $1 \leq n < \omega$ and $\sigma \in T(F_n, n, p)$ the condition $p|\sigma$ decides the value of $\underset{\sim}{\tau}_n$.

Lemma 3.10. (1) *The forcing notion $\mathbb{S}_*(\kappa)$ satisfies $\mathfrak{c}^+$–chain condition.*

(2) *Suppose that $p \in \mathbb{S}_*(\kappa)$, $F \subseteq \mathrm{dom}(p)$ is finite non-empty, $\mu : F \longrightarrow \omega \setminus \{0\}$ and $\underset{\sim}{\tau}$ is a name for an ordinal. Then there is a condition $q \in \mathbb{S}_*(\kappa)$ such that $p \leq_{F,\mu} q$ and for every $\sigma \in T(F, \mu, q)$ the condition $q|\sigma$ decides the value of $\underset{\sim}{\tau}$.*

(3) *Suppose that $p \in \mathbb{S}_*(\kappa)$ and $\bar{\underset{\sim}{\tau}} = \langle \underset{\sim}{\tau}_n : n < \omega \rangle$ is a sequence of $\mathbb{S}_*(\kappa)$–names for objects from the ground model $\mathbf{V}$. Then there is a condition $q \geq p$ and a $\subseteq$–increasing sequence $\bar{F} = \langle F_n : n < \omega \rangle$ of finite subsets of $\mathrm{dom}(q)$ such that q determines $\bar{\underset{\sim}{\tau}}$ relative to $\bar{F}$.*

(4) *Assume $p, \bar{\underset{\sim}{\tau}}$ are as in (3) above and $p \Vdash$ "$\bar{\underset{\sim}{\tau}}$ is a sequence of elements of ${}^{\underline{\omega}}2$ with disjoint domains". Then there are a condition $q \geq p$ and an increasing sequence $\bar{F}$ of finite subsets of $\mathrm{dom}(q)$ and a function $f = (f_0, f_1) : \bigcup\limits_{1 \leq n < \omega} T(F_n, n, q) \longrightarrow \omega \times {}^{\underline{\omega}}2$ such that $q|\sigma \Vdash \underset{\sim}{\tau}_{f_0(\sigma)} = f_1(\sigma)$ (for all $\sigma \in \mathrm{dom}(f)$) and the elements of $\langle \mathrm{dom}(f_1(\sigma)) : \sigma \in \bigcup_{n<\omega} T(F_n, n, q) \rangle$ are pairwise disjoint.*

Proof. The same as for the CS product of Silver or Sacks forcing notions, see e.g. Baumgartner [3, §1]. □

Corollary 3.11. *Assume $\kappa = \kappa^{\aleph_0} \geq \aleph_2$. The forcing notion $\mathbb{S}_*(\kappa)$ is proper and every meager subset of ${}^{\omega}2$ in an extension by $\mathbb{S}_*(\kappa)$ is included in a Σ^0_2 meager set coded in the ground model.*
If CH holds, then $\mathbb{S}_*(\kappa)$ preserves all cardinals and cofinalities and $\Vdash_{\mathbb{S}_*(\kappa)} 2^{\aleph_0} = \kappa$.

4. Meager non–null

The goal of this section is to present a model of ZFC in which every meager subgroup of $\mathbb{R}$ or ${}^{\omega}2$ is also Lebesgue null.

Theorem 4.1. *Assume CH. Let $\kappa = \kappa^{\aleph_0} \geq \aleph_2$. Then*

(1) $\Vdash_{\mathbb{S}_*(\kappa)}$ "*$2^{\aleph_0} = \kappa$ and each meager subgroup of $({}^{\omega}2, +_2)$ is Lebesgue null.*"
(2) $\Vdash_{\mathbb{S}_*(\kappa)}$ "*every meager subgroup of $(\mathbb{R}, +)$ is Lebesgue null.*"

Proof. For $\alpha < \kappa$ let $\underset{\sim}{\eta}_\alpha$ be the canonical name for the $\mathbb{S}_*$–generic function in $\prod\limits_{m<\omega} (m+1)$ added on the α^{th} coordinate of $\mathbb{S}_*(\kappa)$.

(1) Suppose towards contradiction that for some $p_0 \in \mathbb{S}_*(\kappa)$ and a $\mathbb{S}_*(\kappa)$–name $\underset{\sim}{H}$ we have

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underset{\sim}{H} \text{ is a meager non–null subgroup of } ({}^\omega 2, +_2). \text{ ”}$$

By Corollary 3.11 (or, actually, Lemma 3.10(4)) we may pick a condition $p_1 \geq p_0$, a strictly increasing sequence $\bar{n} = \langle n_j : j < \omega \rangle \subseteq \omega$ and a function $f \in {}^\omega 2$ such that

$(*)_0$ $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$ “ $\underset{\sim}{H} \subseteq \big\{x \in {}^\omega 2 : \big(\forall^\infty j < \omega\big)\big(x{\restriction}[n_j, n_{j+1}) \neq f{\restriction}[n_j, n_{j+1})\big)\big\}$. ”

Let $\bar{m} = \bar{m}[\bar{n}]$, $\bar{N} = \bar{N}[\bar{n}]$, $\bar{J} = \bar{J}[\bar{n}]$, $\bar{H} = \bar{H}[\bar{n}]$, $\pi = \pi[\bar{n}]$ and $\mathbf{F} = \mathbf{F}[\bar{n}]$ be as defined in Definition 3.3 for the sequence $\bar{n}$. Also let $A = \{|H_i| - 1 : i < \omega\}$ and $r^+ \in \mathbb{S}_*$ be such that $\mathrm{dom}(r^+) = \omega \setminus A$ and $r^+(k) = 0$ for $k \in \mathrm{dom}(r^+)$.

Since, by Lemma 3.4, we have $\Vdash$ “ $\mathbf{F}(\underset{\sim}{\eta}_\alpha) \subseteq {}^\omega 2$ is a measure one set ”, we know that $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$ “ $(\forall \alpha < \kappa)(\mathbf{F}(\underset{\sim}{\eta}_\alpha) \cap \underset{\sim}{H} \neq \emptyset)$ ”. Consequently, for each $\alpha < \kappa$, we may choose a $\mathbb{S}_*(\kappa)$–name $\underset{\sim}{\rho}_\alpha$ for an element of ${}^\omega 2$ such that

$$p_1 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underset{\sim}{\rho}_\alpha \in \underset{\sim}{H} \ \& \ \underset{\sim}{\rho}_\alpha \in \mathbf{F}(\underset{\sim}{\eta}_\alpha) \text{ ”}.$$

Let us fix $\alpha \in \kappa \setminus \mathrm{dom}(p_1)$ for a moment. Let $p_1^\alpha \in \mathbb{S}_*(\kappa)$ be a condition such that $\mathrm{dom}(p_1^\alpha) = \mathrm{dom}(p_1) \cup \{\alpha\}$, $p_1^\alpha(\alpha) = r^+$ and $p_1 \subseteq p_1^\alpha$. Using the standard fusion based argument (like the one applied in the classical proof of Lemma 3.10(3) with 3.10(2) used repeatedly), we may find a condition $q^\alpha \in \mathbb{S}_*(\kappa)$, a sequence $\bar{F} = \langle F_n^\alpha : n < \omega \rangle$ of finite sets, a sequence $\langle \mu_n^\alpha : n < \omega \rangle$ and an integer $i^\alpha < \omega$ such that the following demands $(*)_1$–$(*)_6$ are satisfied.

- $(*)_1$ $q^\alpha \geq p_1^\alpha$, $\mathrm{dom}(q^\alpha) = \bigcup\limits_{n<\omega} F_n^\alpha$, $F_n^\alpha \subseteq F_{n+1}^\alpha$ and $F_0^\alpha = \{\alpha\}$.
- $(*)_2$ $\mu_n^\alpha : F_n^\alpha \longrightarrow \omega$, $\mu_n^\alpha(\alpha) = n+1$, $\mu_n^\alpha(\beta) = n$ for $\beta \in F_n^\alpha \setminus \{\alpha\}$.
- $(*)_3$ $\min\big(\omega \setminus \mathrm{dom}(q^\alpha(\alpha))\big) > |H_{i^\alpha}|$ and
  if $\max\big(u(n+1, q^\alpha(\alpha))\big) = |H_i| - 1$ and $n \geq 1$, then $|T(F_n, n, q^\alpha)|^2 < 2^i$,
- $(*)_4$ $q^\alpha \Vdash \big(\forall i \geq i^\alpha\big)\big(\underset{\sim}{\rho}_\alpha{\restriction}J_i \in \pi_i(\underset{\sim}{\eta}_\alpha(|H_i| - 1))\big)$, and
- $(*)_5$ q^α determines $\underset{\sim}{\rho}_\alpha$ relative to $\bar{F}$, moreover
- $(*)_6$ if $\sigma \in T(F_n^\alpha, \mu_n^\alpha, q^\alpha)$ and $\max\big(u(n+1, q^\alpha(\alpha))\big) = |H_i| - 1$, then $q^\alpha|\sigma$ decides the value of $\underset{\sim}{\rho}_\alpha{\restriction}J_i$.

Unfixing α and using a standard Δ–system argument with CH we may find distinct $\gamma, \delta \in \kappa \setminus \mathrm{dom}(p_1)$ such that $\mathrm{otp}(\mathrm{dom}(q^\gamma)) = \mathrm{otp}(\mathrm{dom}(q^\delta))$ and if $g : \mathrm{dom}(q^\gamma) \longrightarrow \mathrm{dom}(q^\delta)$ is the order preserving bijection, then the following demands $(*)_7$–$(*)_9$ hold true.

- $(*)_7$ $i^\gamma = i^\delta$, $g{\restriction}\big(\mathrm{dom}(q^\gamma) \cap \mathrm{dom}(q^\delta)\big)$ is the identity, $g(\gamma) = \delta$,
- $(*)_8$ $q^\gamma(\beta) = q^\delta(g(\beta))$ for each $\beta \in \mathrm{dom}(q^\gamma)$, and $g[F_n^\gamma] = F_n^\delta$,
- $(*)_9$ if $F \subseteq \mathrm{dom}(q^\delta)$ is finite, $\mu : F \longrightarrow \omega \setminus \{0\}$, $i < \omega$, $\sigma \in T(F, \mu, q^\delta)$, then

$$q^\delta|\sigma \Vdash \underset{\sim}{\rho}_\delta{\restriction}J_i = z \quad \text{ if and only if } \quad q^\gamma|(\sigma \circ g) \Vdash \underset{\sim}{\rho}_\gamma{\restriction}J_i = z.$$

Clearly $q^* \stackrel{\rm def}{=} q^\gamma\cup q^\delta$ is a condition stronger than both q^γ and q^δ. Let $F^*_n = F^\gamma_n\cup F^\delta_n$ for $n<\omega$.

Let $\langle k_\ell:\ell<\omega\rangle$ be the increasing enumeration of $\omega\setminus {\rm dom}(q^\gamma(\gamma))=\omega\setminus {\rm dom}(q^\delta(\delta))$. Note that by the choice of r^+ and p^γ_1, we have $\omega\setminus{\rm dom}(q^\gamma(\gamma))\subseteq A$, so each k_ℓ is of the form $|H_i|-1$ for some i. Now we will choose conditions $r_\delta,r_\gamma\in\mathbb{S}_*$ so that

$${\rm dom}(r_\delta)={\rm dom}(r_\gamma)={\rm dom}(q^\delta(\delta))\cup\{k_{2\ell}:\ell<\omega\},$$

$q^\delta(\delta)\le r_\delta$, $q^\gamma(\gamma)\le r_\gamma$ and the values of $r_\delta(k_{2\ell}),r_\gamma(k_{2\ell})$ are picked as follows.

Let i be such that $k_{2\ell}=|H_i|-1$. If $x\in\{\gamma,\delta\}$ and $\sigma\in T(F^x_{2\ell},\mu^x_{2\ell},q^x)$ then $q^x|\sigma$ decides the value of $\underaccent{\tilde}{\rho}_x{\restriction}J_i$ (by $(*)_6$) and this value belongs to $\pi_i\big(\sigma(x)(k_{2\ell})\big)$ (by $(*)_4+(*)_3$). Consequently, for $x\in\{\gamma,\delta\}$ and $\tau\in T(F^*_{2\ell},2\ell,q^*)$ we may define a function $\mathcal{Z}^x_\tau:H_i\longrightarrow {}^{J_i}2$ so that

$(*)_{10}$ **if** $a\in H(i)$, $\mu:F^*_{2\ell}\longrightarrow\omega$ is such that $\mu(x)=2\ell+1$ and $\mu(\alpha)=2\ell$ for $\alpha\neq x$, and $\tau_a\in T(F^*_{2\ell},\mu,q^*)$ is such that $\tau_a(\alpha)=\tau(\alpha)$ for $\alpha\in F^*_{2\ell}\setminus\{x\}$ and $\tau_a(x)=\tau(x)\cup\{(k_{2\ell},a)\}$,
then $q^*|\tau_a\Vdash_{\mathbb{S}_*(\kappa)}\underaccent{\tilde}{\rho}_x{\restriction}J_i=\mathcal{Z}^x_\tau(a)$ and $\mathcal{Z}^x_\tau(a)\in a$.

Since $|T(F^*_{2\ell},2\ell,q^*)|\le|T(F^\gamma_{2\ell},2\ell,q^\gamma)|^2<2^i$ (remember $(*)_3$), we may use Lemma 3.5 to find $r_\delta(k_{2\ell}),r_\gamma(k_{2\ell})\le k_{2\ell}$ such that

$(*)_{11}$ for every $\tau\in T(F^*_{2\ell},2\ell,q^*)$ there is $k\in[m_{2^i},m_{2^{i+1}})$ satisfying

$$\big(\mathcal{Z}^\gamma_\tau(\pi_i(r_\gamma(k_{2\ell}))){\restriction}[n_k,n_{k+1})\big)+_2\big(\mathcal{Z}^\delta_\tau(\pi_i(r_\delta(k_{2\ell}))){\restriction}[n_k,n_{k+1})\big)=f{\restriction}[n_k,n_{k+1}).$$

(Remember, f was chosen in $(*)_0$.)

This completes the definition of r_γ and r_δ. Let $q^+\in\mathbb{S}_*(\kappa)$ be such that ${\rm dom}(q^+)={\rm dom}(q^*)={\rm dom}(q^\gamma)\cup{\rm dom}(q^\delta)$ and $q^+(\alpha)=q^*(\alpha)$ for $\alpha\in{\rm dom}(q^+)\setminus\{\gamma,\delta\}$ and $q^+(\gamma)=r_\gamma$ and $q^+(\delta)=r_\delta$. Then q^+ is a (well defined) condition stronger than both q^γ and q^δ and such that

$(\clubsuit)$ $q^+\Vdash\Big(\exists^\infty k<\omega\Big)\Big(\big(\underaccent{\tilde}{\rho}_\gamma{\restriction}[n_k,n_{k+1})\big)+_2\big(\underaccent{\tilde}{\rho}_\delta{\restriction}[n_k,n_{k+1})\big)=f{\restriction}[n_k,n_{k+1})\Big)$

(by $(*)_{10}+(*)_{11}$). Consequently, by $(*)_0$,

$(\heartsuit)$ $q^+\Vdash$" $\underaccent{\tilde}{\rho}_\gamma,\underaccent{\tilde}{\rho}_\delta\in\underaccent{\tilde}{H}$ and $\underaccent{\tilde}{\rho}_\gamma+_2\underaccent{\tilde}{\rho}_\delta\notin\underaccent{\tilde}{H}$ and $(\underaccent{\tilde}{H},+_2)$ is a group",

a contradiction.

(2) The proof is a small modification of that for the first part, so we describe the new points only. Assume towards contradiction that for some $p_0\in\mathbb{S}_*(\kappa)$ and a $\mathbb{S}_*(\kappa)$–name $\underaccent{\tilde}{H}^*$ we have

$$p_0\Vdash_{\mathbb{S}_*(\kappa)}\text{“ }\underaccent{\tilde}{H}^*\text{ is a meager non–null subgroup of }(\mathbb{R},+)\text{ ”}.$$

Let $\underaccent{\tilde}{H}_0,\underaccent{\tilde}{H}_1$ be $\mathbb{S}_*$–names for subsets of D^∞_0 such that

$$p_0\Vdash_{\mathbb{S}_*(\kappa)}\text{“ }\underaccent{\tilde}{H}_0=\mathbf{E}^{-1}[\underaccent{\tilde}{H}^*\cap[0,1/2)]\text{ and }\underaccent{\tilde}{H}_1=\mathbf{E}^{-1}[\underaccent{\tilde}{H}^*\cap[0,1)]\text{ ”}.$$

Necessarily $p_0 \Vdash$" $\underset{\sim}{H}^* \cap [0,1/2)$ is not null ", so it follows from 2.2(1) that

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{" } \underset{\sim}{H}_0 \notin \mathcal{N} \text{ and } \underset{\sim}{H}_1 \in \mathcal{M} \text{ and } \underset{\sim}{H}_0 \subseteq \underset{\sim}{H}_1 \text{ ".}$$

Clearly we may pick a condition $p_1 \geq p_0$, a sequence $\bar{n} = \langle n_j : j < \omega \rangle \subseteq \omega$ and a function $f \in {}^\omega 2$ such that

- $(\oplus)_0$ $n_{j+1} > n_j + j + 1$ for each j,
- $(\oplus)_1$ $f(n_{j+1} - 1) = 0$ for each j, and
- $(\oplus)_2$ $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$"$\underset{\sim}{H}_1 \subseteq \big\{x \in {}^\omega 2 : \big(\forall^\infty j < \omega\big)\big(x{\restriction}[n_j, n_{j+1}-1) \neq f{\restriction}[n_j, n_{j+1}-1)\big)\big\}$."
  (Note: "$[n_j, n_{j+1} - 1)$" not "$[n_j, n_{j+1})$".)

Like in part (1), let $\bar{m} = \bar{m}[\bar{n}]$, $\bar{N} = \bar{N}[\bar{n}]$, $\bar{J} = \bar{J}[\bar{n}]$, $\bar{H} = \bar{H}[\bar{n}]$, $\pi = \pi[\bar{n}]$ and $\mathbf{F} = \mathbf{F}[\bar{n}]$. Let $A = \{|H_i| - 1 : i < \omega\}$ and $r^+ \in \mathbb{S}_*$ be such that $\mathrm{dom}(r^+) = \omega \setminus A$ and $r^+(k) = 0$ for $k \in \mathrm{dom}(r^+)$. Then each $\alpha < \kappa$ fix a $\mathbb{S}_*(\kappa)$–name $\underset{\sim}{\rho}_\alpha$ such that $p_1 \Vdash_{\mathbb{S}_*(\kappa)}$" $\underset{\sim}{\rho}_\alpha \in \underset{\sim}{H}_0 \cap \mathbf{F}(\underset{\sim}{\eta}_\alpha)$ ".

Now repeat the arguments of the first part (with $(*)_1$–$(*)_{11}$ there applied to our $\bar{n}, f, \underset{\sim}{\rho}_\alpha$ and the operation $\circledast_0$ here) to find $q^+ \geq p_1$ and $\gamma, \delta \in \mathrm{dom}(q^+)$ such that

$$(\Diamond)\ \ q^+ \Vdash \text{" } \big(\exists^\infty k < \omega\big)\big((\underset{\sim}{\rho}_\gamma{\restriction}[n_k, n_{k+1})) \circledast_0 (\underset{\sim}{\rho}_\delta{\restriction}[n_k, n_{k+1})) = f{\restriction}[n_k, n_{k+1})\big) \text{ ".}$$

Let $G \subseteq \mathbb{S}_*(\kappa)$ be a generic over $\mathbf{V}$ such that $q^+ \in G$ and let us work in $\mathbf{V}[G]$. Let $\eta \in D_0^\infty$ be such that $\mathbf{E}(\underset{\sim}{\rho}_\gamma^G) + \mathbf{E}(\underset{\sim}{\rho}_\delta^G) = \mathbf{E}(\eta)$ (remember $\mathbf{E}(\underset{\sim}{\rho}_\gamma^G), \mathbf{E}(\underset{\sim}{\rho}_\delta^G) < 1/2$). We know from $(\Diamond)$ that there are infinitely many $k < \omega$ satisfying

$$(\blacklozenge)\ \ (\underset{\sim}{\rho}_\gamma^G{\restriction}[n_k, n_{k+1})) \circledast_0 (\underset{\sim}{\rho}_\delta^G{\restriction}[n_k, n_{k+1})) = f{\restriction}[n_k, n_{k+1}).$$

Since $f(n_{k+1} - 1) = 0$ (see $(\oplus)_1$), we get from 3.2(3) that for each k as in $(\blacklozenge)$ we also have

$$\begin{aligned}&\big((\underset{\sim}{\rho}_\gamma^G{\restriction}[n_k, n_{k+1})) \circledast_0 (\underset{\sim}{\rho}_\delta^G{\restriction}[n_k, n_{k+1}))\big){\restriction}[n_k, n_{k+1} - 1) = \\ &\big((\underset{\sim}{\rho}_\gamma^G{\restriction}[n_k, n_{k+1})) \circledast_1 (\underset{\sim}{\rho}_\delta^G{\restriction}[n_k, n_{k+1}))\big){\restriction}[n_k, n_{k+1} - 1) = f{\restriction}[n_k, n_{k+1} - 1).\end{aligned}$$

Therefore (by 3.2(4)) for each k satisfying $(\blacklozenge)$ we have $\eta{\restriction}[n_k, n_{k+1} - 1) = f{\restriction}[n_k, n_{k+1} - 1)$, so

$$\big(\exists^\infty k < \omega\big)\big(\eta{\restriction}[n_k, n_{k+1} - 1) = f{\restriction}[n_k, n_{k+1} - 1)\big).$$

Consequently, by $(\oplus)_2$, we have that $\eta \notin \underset{\sim}{H}_1^G$, i.e., $\mathbf{E}(\eta) \notin (\underset{\sim}{H}^*)^G \cap [0,1)$. This contradicts the fact that $\mathbf{E}(\underset{\sim}{\rho}_\gamma^G), \mathbf{E}(\underset{\sim}{\rho}_\delta^G) \in (\underset{\sim}{H}^*)^G$, $\mathbf{E}(\eta) = \mathbf{E}(\underset{\sim}{\rho}_\gamma^G) + \mathbf{E}(\underset{\sim}{\rho}_\delta^G)$ and $(\underset{\sim}{H}^*)^G$ is a subgroup of $(\mathbb{R}, +)$. $\square$

Remark 4.2. Instead of the CS product of forcing notions $\mathbb{S}_*$ we could have used their CS iteration of length ω_2. Of course, that would restrict the value of the continuum in the resulting model.

5. Problems

Both theorems 2.3(1) and 4.1(1) can be repeated for other product groups. We may consider a sequence $\langle H_n : n < \omega\rangle$ of finite groups and their coordinate-wise product $H = \prod\limits_{n<\omega} H_n$. Naturally, H is equipped with product topology of discrete H_n's and the product probability measure. Then there exists a null non–meager subgroup of H but it is consistent that there is no meager non–null such subgroup. It is natural to ask now:

Problem 5.1. (1) Does every locally compact group (with complete Haar measure) admit a null non–meager subgroup?
(2) Is it consistent that no locally compact group has a meager non–null subgroup?

In relation to Theorem 4.1, we still should ask:

Problem 5.2. Is it consistent that there exists a translation invariant Borel hull for the meager ideal on ${}^{\omega}2$? On $\mathbb{R}$?

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