MODELS OF PA: WHEN TWO ELEMENTS ARE NECESSARILY ORDER AUTOMORPHIC
SH924

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Abstract. We are interested in the question of how much the order of a non-standard model of PA can determine the model. In particular, for a model M, we want to characterize the complete types $p(x,y)$ of non-standard elements (a,b) such that the linear orders $\{x : x < a\}$ and $\{x : x < b\}$ are necessarily isomorphic. It is proved that this set includes the complete types $p(x,y)$ such that if the pair (a,b) realizes it (in M) then there is an element c such that for all standard $n, c^n < a, c^n < b, a < bc$ and $b < ac$. We prove that this is optimal, because if $\diamondsuit_{\aleph_1}$ holds, then there is M of cardinality $\aleph_1$ for which we get equality. We also deal with how much the order in a model of PA may determine the addition.

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§ 0. Introduction

Let M be a model of Peano Arithmetic (PA). For an $a \in M$, by $M_{<a}$ we denote the set $\{c \in M : M \models c < a\}$ with the inherited linear order. For any pair (a, b) of non-standard elements of M, let $(*)_{M,a,b}$ be the condition dfeined by

$(*)_{M,a,b}$ $(M_{<a}, <_M) \cong (M_{<b}, <_M)$.

We will also consider

$(*)^{\rm pot}_{M,a,b}$ for every N, if $M \prec N$, then $(*)_{N,a,b}$

$(*)^{\rm tp}_{M,a,b}$ for every model N, if $M \equiv N$ and for some $M_0 \prec N$ we have $\{a, b\} \subseteq M_0 \prec M$, then $(*)_{N,a,b}$.

Looking recently at models of PA, we wonder:

Question 0.1. What is the set of complete types $p(x, y)$ such that: if the pair (a, b) realizes the type $p(x, y)$ then $(*)_{M,a,b}$ holds? Another variant is, given a model M and $a, b \in M$ when do we have $(*)^{\rm pot}_{M,a,b}$ or just $(*)^{\rm tp}_{M,a,b}$?

Our main aim is sorting this out. For the problem as stated, on the one hand we give a sufficient condition, and on the other hand, for $(*)^{\rm tp}_{M,a,b}$, we prove its necessity, assuming $\diamondsuit_{\aleph_1}$. However, we may consider a relative:

Question 0.2. Like question 0.1, but we restrict ourselves to $\aleph_1$-saturated models.

It seems natural to ask:

Question 0.3. Generally, how much does the linear order of a non-standard model M of PA determine M? Is there a non-standard model M of PA such that for every model N of PA, if $(M{\restriction}\{<\}) \cong (N, {\restriction}\{<\})$, <u>then</u> are $M \cong N$?

We discussed those problems with Gregory Cherlin and he asked:

Question 0.4. [Cherlin] Show that $\{M{\restriction}\{<\} : M \models \text{PA}\}$ is complicated.

This question is too vague for my taste.

Recall the much earlier problem 14 from Kossak-Schmerl [**?**] asked by Friedman:

Question 0.5. Is there a model of PA such that for every model N of PA, if $(M{\restriction}\{<\}) \cong (N, {\restriction}\{<\})$, then $M \equiv N$?

This seems of different character, speaking just of the theory of another model, but of course, a positive answer to Question 0.3 would also give an answer to Question 0.5.

We may go half way: maybe the linear order of M does not determine M, say up to isomorphism, but just the additive structure (from which the order is definable). This means

Question 0.6. How much the order of M, a non-standard model of $T, T \in \text{PA}^{\rm com}$, determines the isomorphism type of $(|M|, <_M, +_M)$?

A more general version of our question is

Question 0.7. Can we construct a non-standard model M of PA with few order automorphisms in some sense?

Recall that for any countable non-standard model M of PA, it is recursively saturated hence has "lots" of order automorphisms (see [**?**]). Much has been done on other classes. Concerning Abelian groups and modules, see the book Göbel-Trlifaj [**?**]. For general first order see [**?**] and history in both. In particular, there are non-standard models of PA with no automorphism, this motivating the "order-automorphism" in 0.7. Now answer to 0.1 sheds some light on 0.6.

Let us review the present work. First, in §1 we introduce and deal with some relevant equivalence relations, and in 1.8 it is proved that the so called $aE^2_M b$ implies $(*)_{M,a,b}$ so $aE^2_M b$ is a sufficient condition for a positive answer to Question 0.1(1), while for the so called 2-order rigid models M, we prove that the isomorphism type of $M{\restriction}\{<\}$ determines that of $M{\restriction}\{<,+\}$ but only up to almost isomorphism, shedding some light on 0.6.

Second, in §2 we get that even $aE^3_M b$ implies $(*)_{M,a,b}$. This shows that 1.8 is not so interesting but its proof is a warm-up for §2. Moreover this is only part of the picture, see §4. In §2 we also show that if M is 3-o.r. then $M{\restriction}\{<,+\}$ is unique up to almost isomorphism.

In §3 we show that E^3_M is the right notion as if $\diamondsuit_{\aleph_1}$ holds then every countable model of PA has elementary extension M of cardinality $\aleph_1$ such that for $a,b \in M\backslash\mathbb{N}$ we have $aE^3_M b \Leftrightarrow M_{<a} \cong M_{<b}$. We comment there on the case $\neg aE^4_M b$.

Naturally, for most results some weaker version of PA suffices. We comment on this in §4; so usually when a result supercedes an earlier one, normally it has a harder proof and really use more axioms of PA.

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Convention 0.8. Models are models of PA, even ordinary ones if not said otherwise where M is ordinary when $\mathbb{N} \subseteq M$. It is used e.g. in Definition 1.2(b),(c). We may circumvent it, defining $a +_M n$ by repeated addition of 1_M and $a \times_M n$ by repeated additions of a, but it seems to me less convenient.

§ 1. Somewhat rigid order

We define (in 1.2) some equivalence relations E^ℓ_M for models M (of PA). We shall deal with their basic properties in 1.5, 1.13, the relations between them (in 2.1), cofinalities of equivalence classes (in 1.12, 2.5), on order isomorphism/almost $\{<,+\}$-isomorphism in 1.1, 1.7, 2.4; including noting for ℓ-o.r. models that $E^\ell_M \subseteq E^5_M$.

Lastly, we prove versions of "if $M_1{\restriction}\{<\}, M_2{\restriction}\{<\}$ are isomorphic then $M_1{\restriction}\{<,+\}, M_2{\restriction}\{<,+\}$ are almost isomorphic", see Theorems 1.8, 2.6.

Definition 1.1. 1) We say M, N are almost $\{<,+\}$-isomorphic <u>when</u>: some f is an almost $\{<,+\}$-isomorphism from M onto N which means f is an isomorphism from $M{\restriction}\{<\}$ onto $N{\restriction}\{<\}$ such that for all $a, b \in M$ there is an $n \in \mathbb{N}$ such that the distance between $f(a +_M b)$ and $f(a) +_N f(b)$ is n.
2) An equivalence relation E on a model $M = (|M|, <^M, \ldots)$ is convex <u>when</u> $a <_M b <_M c, aEc$ implies $aEb \wedge bEc$.
3) Given M for $x, y \in M$ we say λ is an additively/multiplicativity/exponentially small relative if $x\lambda y$ implies that for every $n \in \mathbb{N}$ we have $x + n <_M y, x \cdot n <_M y, x^n <_M y$, respectively.

Definition 1.2. We define the following equivalence relations on $M \backslash \mathbb{N}$

(a) $E^0_M : aE^0_M b$ <u>iff</u> $\bigvee\limits_{n\in\mathbb{N}} (a <_M b + n \wedge b <_M a + n)$

(b) $E^1_M : aE^1_M b$ <u>iff</u> for some $c \in M$ satisfying $n \in \mathbb{N} \Rightarrow nc <_M a, b$ we have $a < b + c$ and $b < a + c$

(c) $E^2_M : aE^2_M b$ <u>iff</u> $\bigvee\limits_{n\in\mathbb{N}} (a <_M b \times_M n \wedge b <_M a \times_M n)$

(d) $E^3_M : aE^3_M b$ <u>iff</u> for some $c \in M$ we have $\bigwedge\limits_{n\in\mathbb{N}} [c^n <_M a \wedge c^n <_M b$ and $a < b \times_M c \wedge b < a \times_M c]$

(e) $E^4_M : aE^4_M b$ <u>iff</u> $\bigvee\limits_{n\in\mathbb{N}} (a < b^n \wedge b < a^n)$

(f) $E^5_M : aE^5_M b$ <u>iff</u> some order-automorphism of M maps a to b

(g) $E^6_M : aE^6_M b$ <u>iff</u> a, b are equivalent for the minimal convex equivalent relations on M which E^5_M refines.

Definition 1.3. For $\ell \in \{0, 1, \ldots, 6\}, M$ is called ℓ-o.r. (for order-rigid) if for all $a, b \in M, (M_{<a}, <_M) \cong (M_{<b}, <_M)$ implies that $aE^\ell_M b$.

Discussion 1.4. While we know that there are rigid linear orders and we know that there are rigid models of PA, it is harder to build ℓ-.o.r. models of PA, our relevant result will be Theorem 3.10.

Claim 1.5. *1) For $\ell \in \{0, \ldots, 6\}$ the two place relation E^ℓ_M is an equivalence relation on $M \backslash \mathbb{N}$ and is convex except possibly for $\ell = 5$.*
1A) Moreover, if $aE^3_M b$ <u>then</u> for every $c \in M$ we have $M \models (\bigwedge\limits_n c^n < a) \Leftrightarrow M \models (\bigwedge\limits_n c^n < b)$ and the set of such c's is closed under products and sums and is convex.
1B) Also if $aE^1_M b$ <u>then</u> for every $c \in M$ we have $M \models$ "$(\bigwedge\limits_n c \times n < a) \Leftrightarrow (\bigwedge\limits_n (c \times n < b)$" and the set of such c's is closed under sums and is convex.

2) If $a, b \in M \backslash \mathbb{N}$ and $aE^2_M b$ <u>then</u> $(M_{<a}, <_M) \cong (M_{<b}, <_M)$; moreover there is an automorphism of $M{\restriction}\{<\}$ mapping a to b, that is, $aE^5_M b$.
3) Assume $a, b \in M \backslash \mathbb{N}$ <u>then</u> $(M_{<a}, <_M) \cong (M_{<b}, <_M)$ iff $aE^5_M b$.
4) $aE^6_M b$ iff there is $c \le_M \min\{a, b\}$ and an order-automorphism f of M such that $\max\{a, b\} \le_M f(c)$ <u>iff</u> this holds for $c = \min\{a, b\}$.

Remark 1.6. Note that in §2 we establish a stronger version of 1.5(2), see 2.4 but the proof of 2.4 uses 1.5(2), also the proof of 1.5(2) applies to weaker versions of PA than the proof of 2.4, see §4.

Proof. 1) Let $\ell = 3$. If $a_1 E^3_M a_2$ and let c witness it, then c witnesses also $a_2 E^3_M a_2$ and $a_2 E^3_M a_1$, so reflexivity and symmetry holds.

Lastly, assume $M \models$ "$a_1 < a_2 < a_3$"; if $a_k E^3_M a_{k+1}$ and let c_k witness this for $k = 1, 2$, then the product $c_1 c_2$ witness $a_1 E^3_M a_3$ by part (1A) proved below and if $a_1 E^3_M a_3$ then the same witness gives $a_1 E^3_M a_2 \wedge a_2 E^3_M a_3$ so transitivity and convexity holds.

For $\ell = 1$ the proof is similar (using part (1B) instead of part (1A)), also for $\ell = 0, 2, 4$ the proof is even easier and for $\ell = 5, 6$ it holds by the definition.
1A), 1B) Check.
2) Without loss of generality, assume that $a < (n-1)a < b < na$, where $2 \le n \in \mathbb{N}$, and as we can replace b by any $b' \in b/E^0_M$ also there is a c such that $(n-1)c = na - b$. Then $c < a$ because otherwise $(n-1)a \le (n-1)c = na - b$ hence $b \le a$, contradiction. Let X be a set of representatives for M/E_0 and, without loss of generality, assume $a, c \in X$. Now define $f : M \to M$ by first defining it on X and then extending it to all of M in the obvious way. (The obvious way is: if $y = x + k$, where $x \in X$ and $k \in \mathbb{Z}$, then $f(y) = f(x) + k$ and f is the identity on $\mathbb{N}$.) If $x \in X$, then let

$$f(x) = \begin{cases} x & \text{if } x \le c, \\ n(x-a) + b & \text{otherwise.} \end{cases}$$

Clearly, $f(c) = c$ and $f(a) = b$. Now check that f is as required.
3) First, if f exemplifies $aE^5_M b$, i.e. is an automorphism of M mapping a to b then $f{\restriction}M_{<a}$ is an isomorphism from $(M_{<a}, <_M)$ onto $(M_{<b}, <_M)$. Second, if f is an isomorphism from $(M_{<a}, <_M)$, onto $(M_{<b}, <_M)$ we define a function $g : M \to M$ by: $g(c)$ is $f(c)$ if $c <_M a$ and is $b + (c - a)$ if $a \le_M c$. Now check.
4) Let $E'_M = \{(a, b)$: for some order-automorphism f of M we have $f(\min\{a, b\}) \ge \max\{a, b\}\}$; clearly this is symmetric (by definition), reflexive (using $f =$ the identity) and as f is monotonic also convex (i.e. $a \le a_1 \le b_1 \le b \wedge aE'_M b \Rightarrow a_1 E'_M b_1$). To prove transitivity it is now enough to show $a_1 < a_2 < a_3 \wedge a_1 E'_M a_2 \wedge a_2 E'_M a_3 \Rightarrow a_1 E'_M a_3$ which hold by composing the automorphisms f_1, f_2 witnessing $a_1 E'_M a_2, a_2 E'_M a_3$ respectively. So E'_M is a convex equivalence relation and obviously $aE^5_M b \Rightarrow aE'_M b$.

Lastly, E'_M is refined by any convex equivalence relation refining E^5_M, so it follows that $E^6_M = E'_M$ so we are done. $\square_{1.5}$

Claim 1.7. *1) If f is an order-isomorphism from M_1 onto M_2 <u>then</u> f maps $E^0_{M_1}$ onto $E^0_{M_2}$.*
2) If f is an almost $\{<, +\}$-isomorphism from M_1 onto M_2 <u>then</u> f maps $E^1_{M_1}$ onto $E^1_{M_2}$ and $E^2_{M_1}$ onto $E^2_{M_2}$.
3) Similarly for embeddings (not used).

Proof. Straight. $\square_{1.7}$

Recalling Definition 1.3(2)

Theorem 1.8. *If M_1 is 2-o.r. and $M_1{\restriction}\{<\}, M_2{\restriction}\{<\}$ are isomorphic* then *M_1, M_2 are almost $\{<,+\}$-isomorphic.*

Remark 1.9. 1) We have not said "by the same isomorphism".
2) The assumption is too strong to be true for models of full PA. But it makes sense for weaker versions of PA, see 4.5 and part of the proof serves as proof to 2.7 so indirectly serves 2.6.

Question 1.10. Are M_1, M_2 isomorphic when (the main case is $\ell = 3$):

- (a) M_1, M_2 are isomorphic as linear orders
- (b) M_1 is ℓ-o.r.

Remark 1.11. In 1.10, it is less desirable but we may consider adding that also M_2 is l.o.r.

Proof. Easily by 1.5(1)

$(*)_0$ each $E^2_{M_\ell}$-equivalence class is convex.

Without loss of generality

$(*)_1$ $<_{M_1} = <_{M_2}$ hence M_1, M_2 have the same universe and let $M := M_1{\restriction}\{<\} = M_2{\restriction}\{<\}$.

Also as usual

$(*)_2$ $\mathbb{N} \subseteq M_\ell$ for $\ell = 1, 2$.

Now

$(*)_3$ if $a, b \in M \backslash \mathbb{N}$ and $a <_M b$ and $a +_{M_2} b = c$ and $a' = c -_{M_1} b$, that is $M_1 \models$ "$c = a' + b$" then $a E^2_{M_1} a'$.

[Why? Now $[b, c)_{M_2} = [b, c)_M$ is isomorphic to $[0, a)_{M_2}$ as linear orders (as $M_2 \models$ "$a + b = c$" as M_2 satisfies PA) hence $[b, c)_{M_1}$ is isomorphic to $[0, a)_{M_1}$ as linear orders. Of course $[b, c)_{M_1}$ is isomorphic to $[0, c -_{M_1} b) = [0, a')_{M_1}$ as linear orders. So $[0, a)_{M_1}, [0, a')_{M_1}$ are isomorphic as linear orders. But M_1 is 2-o.r. hence $a E^2_{M_1} a'$ as required].

$(*)_4$ if $a <_M b$ and $b \in M \backslash \mathbb{N}$ then $b, a +_{M_1} b, a +_{M_2} b$ are $E^2_{M_1}$-equivalent.

[Why? Similar proof: as by the proof of 1.5(3) as trivially $b E^2_{M_\ell} (a +_{M_\ell} b)$ for $\ell = 1, 2$ and $(a +_{M_1} b) E^5_{M_1} (a +_{M_2} b)$ so use "M_1 is 2-o.r." to deduce $(a +_{M_1} b) E^2_{M_1} (a +_{M_2} b)$, together we are done.]

$(*)_5$ if $a, b \in M \backslash \mathbb{N}$ and $a \times_{M_\ell} b = c_\ell$ for $\ell = 1, 2$ then $c_1 E^2_{M_1} c_2$.

[Why? For $\ell = 1, 2$, as M_ℓ is a model of PA it follows that $(M_{<c_\ell}, <_M)$ is isomorphic to $(M_{<a}, <_M) \times (M_{<b}, <_M)$ ordered lexicographically. Hence $((M_1)_{<c_1}, <_M) = (M_{<c_1}, <_M)$ and $((M_2)_{<c_2}, <_M) = (M_{<c_2}, <_M)$ are isomorphic (and trivially $c_1, c_2 \notin \mathbb{N}$) hence by "$M_1$ is 2-o.r." we have $c_1 E^2_{M_1} c_2$ as promised.]

$(*)_6$ for $a, b \in M \backslash \mathbb{N}$ we have:
$aE^2_{M_2} b$ iff $aE^2_{M_1} b$.

[Why? First, assume $aE^2_{M_2} b$; now without loss of generality $a <_M b$, and say $k \in \mathbb{N}, k \neq 0, b < k \times_{M_2} a$ so $b < a +_{M_2} \ldots +_{M_2} a$ (k summands). By $(*)_4$ we can prove by induction on k that $bE^2_{M_1} a$ as required in the "only if" direction. (Alternatively by 1.5(2) we have $aE^5_M b$ hence by "M_1 is 2-o.r.", we get $bE^2_{M_1} a$.)

Second, assume $\neg(aE^2_{M_2} b)$ and without loss of generality $a <_M b$; note that we cannot use the same argument as above interchanging M_1, M_2, because only on M_1 we assume its being 2 - o.r. As $M_2 \models$ PA there is $c \in M_2$ such that $M_2 \models$ "$a \times c \leq b < a \times c + a$", now $c \notin \mathbb{N}$ because we are assuming $\neg(aE^2_{M_2} b)$. By $(*)_5$ we have $(a \times_{M_1} c) E^2_{M_1} (a \times_{M_2} c)$. But by $(*)_4$, we have $a \times_{M_2} c, a \times_{M_2} c +_{M_2} a$ are $E^2_{M_1}$-equivalent hence by the choice of c and $(*)_0$ also $a \times_{M_2} c, b$ are $E^2_{M_1}$-equivalent; so together with the previous sentence $(a \times_{M_1} c) E^2_{M_1} b$. But by the definitions, as $c \in M \backslash \mathbb{N}$ clearly $\neg(aE^2_{M_1}(a \times_{M_1} c))$ hence $\neg(aE^2_{M_1} b)$ as required in the "if" direction.]

$(*)_7$ if $a \in M \backslash \mathbb{N}$ then $(a/E^2_{M_1}, <_M)$ has cofinality $\aleph_0$ and also its inverse has cofinality $\aleph_0$.

[Why? As $M_1 \models$ PA the sequence $\langle a \times_{M_1} 2^n : n \in \mathbb{N} \rangle$ is increasing, the members form an unbounded subset of $a/E^2_{M_1}$; similarly $\langle \min\{b \in M : a \leq b \times_{M_1} 2^n\} : n \in \mathbb{N} \rangle$ is decreasing, the members form a subset of $a/E^2_{M_1}$ unbounded from below, recalling the definition of $E^2_{M_1}$.]

$(*)_8$ let $\ell \in \{1, 2\}$ if $a \in M \backslash \mathbb{N}$ and $X_\ell = \{(2^b)^{M_\ell} : b \in M_\ell\}$ for $\ell = 1, 2$ <u>then</u> $X_\ell \cap (a/E^2_{M_1})$ has order-type $\mathbb{Z}$ and is unbounded in $(a/E^2_{M_1}, <_M)$ from above and from below.

[Why? If $a' \in M \backslash \mathbb{N}$ and $\ell \in \{1, 2\}$, <u>then</u> as $M_\ell \models$ PA for some b_ℓ we have $M_\ell \models$ "$2^{b_\ell} \leq a' < 2^{b_\ell + 1} = 2^{b_\ell} + 2^{b_\ell}$" so recalling the definition of $E^2_{M_\ell}$ we are done.]

We may hope that: if $a \in M \backslash \mathbb{N}$ and $M_\ell \models$ "$2^a = b_\ell$" for $\ell = 1, 2$ <u>then</u> $b_1 E^2_{M_1} b_2$. Anyhow

$\circledast_1$ for $\ell = 1, 2$
  (a) define $f_\ell : M_\ell \to M_\ell$ by $f_\ell(a) = (2^a)^{M_\ell}$,
  (b) define: M^*_ℓ is the model with universe $X_\ell :=$ Rang(f_ℓ) such that f_ℓ is an isomorphism from M_ℓ onto M^*_ℓ

$\circledast_2$ for $\ell = 1, 2$ if $a, b \in X_\ell$ then $a +_{M^*_\ell} b = a \times_{M_\ell} b$.

[Why? As PA $\vdash 2^x 2^y = 2^{x+y}$.]

$\circledast_3$ if $\ell = 1, 2$ and $a, b \in M_1 \backslash \mathbb{N}$ <u>then</u>
  (a) $aE^0_{M_\ell} b$ iff $f_\ell(a) E^2_{M^*_\ell} f_\ell(b)$
  (b) $aE^1_{M_\ell} b$ iff $f_\ell(a) E^3_{M^*_\ell} f_\ell(b)$
  (c) $aE^2_{M_\ell} b$ iff $f_\ell(a) E^4_{M^*_\ell} f_\ell(b)$.

[Why? Look at the definitions and do basic arithmetic.]

$\circledast_4$ there is an order isomorphism h from X_1 onto X_2 such that

(a) $h \restriction \{(2^n)^{\mathbb{N}} : n \in \mathbb{N}\}$ is the identity

(b) if $a \in M \backslash \mathbb{N}$ then h maps $X_1 \cap (a/E^2_{M_1})$ onto $X_2 \cap (a/E^2_{M_2})$.

[Why? By $(*)_8 + (*)_6$.]

$\circledast_5$ if $a, b \in X_1$ then $aE^0_{M^*_1} b \Leftrightarrow h(a) E^0_{M^*_2} h(b)$.

[Why? By $\circledast_3$ and $\circledast_4$ and the definition of $E^0_{M^*_\ell}$.]

$\circledast_6$ if $M^*_\ell \models$ "$a_\ell + b_\ell = c_\ell$" for $\ell = 1,2$ and $h(a_1) = a_2$ and $h(b_1) = b_2$ then

(a) $c_1 E^2_{M_\ell} c_2$ for $\ell = 1,2$

(b) $c_1 E^0_{M^*_\ell} c_2$ for $\ell = 1,2$.

[Why? If $a_1 \in \mathbb{N}$ or $b_1 \in \mathbb{N}$ the conclusion follows easily so we assume $a_1, b_1 \notin \mathbb{N}$. For $\ell = 1,2$ by $\circledast_2$ we have $M_\ell \models$ "$a_\ell \times b_\ell = c_\ell$". Also by $\circledast_4(b)$ we have $x \in X_1 \backslash \mathbb{N} \Rightarrow x E^2_{M_1} h(x)$ recalling $E^2_{M_1} = E^2_{M_2}$ by $(*)_6$ we have $a_1 E^2_{M_2} a_2, b_1 E^2_{M_2} b_2$ hence for some $n \in \mathbb{N}$ we have $M_2 \models$ "$a_1 < n \times a_2 \wedge a_2 < n \times a_1 \wedge b_1 < n \times b_2 \wedge b_2 < n \times b_1$" hence $M_2 \models$ "$a_1 \times b_1 < n^2 \times a_2 \times b_2 \wedge a_2 \times b_2 < n^2 \times a_1 \times b_1$" hence $(a_1 \times_{M_2} b_1), (a_2 \times_{M_2} b_2)$ are $E^2_{M_2}$-equivalent and also are $E^2_{M_1}$-equivalent.

So by $(*)_5$ we have $(a_1 \times_{M_2} b_1), (a_1 \times_{M_1} b_1)$ are $E^2_{M_1}$-equivalent and also $(a_2 \times_{M_1} b_2), (a_2 \times_{M_2} b_2)$ are $E^2_{M_1}$-equivalent hence together with the previous paragraph by $(*)_6$ they are $E^2_{M_\ell}$-equivalent, in particular c_1, c_2 are $E^2_{M_\ell}$-equivalent as required in clause (a) of $\circledast_6$. By $\circledast_3 + \circledast_1(b)$ also clause (b) of $\circledast_6$ there follows.]

So by $\circledast_4 + \circledast_6(b)$ we are done. $\square_{1.8}$

We have used

Observation 1.12. Assume $a \in M \backslash \mathbb{N}$

1) $\langle a + n : n \in \mathbb{N} \rangle$ is increasing and cofinal in a/E^0_M.

2) $\langle a - n : n \in \mathbb{N} \rangle$ is decreasing and unbounded from below in a/E^0_M.

3) $\langle n \times a : n \in \mathbb{N} \rangle$ is increasing and cofinal in a/E^2_M.

4) Moreover $\langle \min\{b : n \times_M b \geq a\} : n \in \mathbb{N}\} \rangle$ is decreasing and unbounded from below in a/E^2_M.

5) Moreover for some $b, 2^b \leq a < 2^{b+1}$ hence we can use in (3),(4) the sequence $\langle 2^{b+n} : n \in \mathbb{N} \rangle, \langle 2^{b-n} : n \in \mathbb{N} \rangle$.

Observation 1.13. 1) Assume $M_k \models$ "$a \times b = c_k$" for $k = 1,2$ and $M_1 \restriction \{<\} = M_2 \restriction \{<\}$. <u>Then</u> $c_1 E^5_{M_2} c_2$ for $k = 1,2$.

2) Assume $M_k \models$ "$a_1 \times a_2 \times \ldots \times a_m = c_k$" for $k = 1,2$ and $M_1 \restriction \{<\} = M_2 \restriction \{<\}$. <u>Then</u> $c_1 E^5_{M_k} c_2$ for $k = 1,2$.

Proof. 1) See $(*)_5$ in the proof of 1.8.

2) Similar proof. $\square_{1.13}$

§ 2. More for E^3_M

Here we say more on the equivalence relations E^ℓ_M. In 2.1 we deal with basic properties: when $E^\ell_\mu \subseteq E^{\ell+1}_\mu$, when ℓ-o.r. implies $(\ell+1)$-o.r., preservation under $+$ and $\times$. We also prove one half of our answer to 0.1 that is in 2.4 we prove $a_1 E^3_M b$ implies $a_1 E^5_\mu a_2$. Concerning the weak form of uniqueness of the additive structure in Theorem 2.6 we prove e.g. if M_1, M_2 are order isomorphic and M_1 is 3-o.r. then $M_1 {\restriction} \{<,+\}, M_2 {\restriction} \{<,+\}$ are almost isomorphic (i.e. the "error" in $+$ is finite) but not necessarily by the same isomorphism. We end (in 2.9) that a/E^4_μ is divided by E^3_μ if $I = \{b/E^3_\mu : b \in a/E^4_\mu\}$ is naturally ordered, is isomorphic to a subset of $\mathbb{R}$, even one which is an additive subgroup (a "translation" of the product in M).

Claim 2.1. *1) E^ℓ_M refines $E^{\ell+1}_M$ for $\ell = 0,1,2,3,5$.*
2) If $a_k E^\ell_M b_k$ for $k = 1,2$ and $\ell = 0,1,2,3,4,5,6$ then $(a_1 + a_2) E^\ell_M (b_1 + b_2)$.
3) If $a_k E^\ell_M b_k$ for $k = 1,2$ and $\ell = 2,3,4$ then $(a_1 \times_M a_2) E^\ell_M (b_1 \times_M b_2)$.
4) Part (3) holds also for $\ell = 5,6$.
5) If M is ℓ-o.r. then M is $(\ell+1)$-o.r. for $\ell = 0,1,2,3$.

Remark 2.2. Concerning 2.1 recall that E^3_M refines E^5_M by 1.5(3),(4).

Proof. 1) Read the definitions.
2) First, assume $\ell = 0$, so by the assumption for $k = 1,2$ there are $m_k, n_k \in \mathbb{N}$ such that $M \models$ "$a_k + m_k = b_k + n_k$". Now let $m := m_1 + m_2 \in \mathbb{N}$ and $n := n_1 + n_2 \in \mathbb{N}$ hence $M \models$ "$(a_1+a_2)+(m_1+m_2) = (b_1+b_2)+(n_1+n_2)$" hence $(a_1+a_2) E^0_M (b_1+b_2)$ as required.

Second, assume $\ell = 2$, so by the assumption, for $k = 1,2$ there is $n_k \in \mathbb{N}$ such that $M \models$ "$a_k < n_k \times b_k \wedge b_k < n_k \times a_k$". Let $n = \max\{n_1, n_2\} \in \mathbb{N}$ hence $M \models$ "$(a_1 + a_2) < n_1 b_1 + n_2 b_2 \le n b_1 + n b_2 = n(b_1 + b_2)$" and similarly $M \models$ "$(b_1 + b_2) < n(a_1 + a_2)$" hence $(a_1 + a_2) E^2_M (b_1 + b_2)$.

Third, assume $\ell = 1,3$; without loss of generality $a_1 <_M b_1$ and as E^ℓ_M is convex (see 1.5(1)) without loss of generality $a_2 \le_M b_2$. Letting c_k witness $a_k E^\ell_M b_k$ for $k = 1,2$ easily $c = \max\{c_1, c_2\}$ witness $(a_1 + a_2) E^\ell_M (b_1 + b_2)$.

Fourth, the case $\ell = 4$ is easy, too.

Fifth, assume $\ell = 5$ and f_k is an order-automorphism of M mapping a_k to b_k for $k = 1,2$. Define a function f from M to M by

$(*)$ (a) $f(x) = f_1(x)$ if $x <_M a_1$
(b) $f(x) = b_1 + f_2(x - a_1)$ if $a_1 \le_M x$.

Now check.

Sixth, assume $\ell = 6$, let $c_k = \min\{a_k, b_k\}$; by 1.5(4) there is $d_k \ge \max\{a_k, b_k\}$ such that $c_k E^5_M d_k$ so $a_k, b_k \in [c_k, d_k]$ for $k = 1,2$. But $(c_1 + c_2) E^5_M (d_1 + d_2)$ by the result for $\ell = 5$ and $a_1 + b_1, a_2 + b_2 \in [c_1 + c_2, d_1 + d_2]$ so we are done.
3) First, assume $\ell = 2$ and for $k = 1,2$ let n_k witness $a_k E^2_M b_k$ and choose $n = n_1 n_2$ noting that $n_1, n_2 > 0$ by Definition 1.2(c). Now $M \models$ "$a_1 \times a_2 < (n_1 \times b_1) \times (n_2 \times b_2) = n \times (b_1 \times b_2)$" and similarly $M \models (b_1 \times b_2) < n(a_1 \times a_2)$.

The proof for $\ell = 3$ is easy, too. For $\ell = 4$ by the convexity of E^4_M without loss of generality $a_1 \le a_2, b_1 \le b_2$ and so there are $n, m \in \mathbb{N}$ such that $a_2 \le a^n_1, b_2 \le b^m_1$, so $a_1 \times b_1 \le a_2 \times b_2 \le a^n_1 \times b^m_1 \le (a_1 \times b_1)^{n+m}$ hence $(a_1 \times b_1) E^4_M (a_2 \times b_2)$.
4) For $\ell = 5$, as in the proof of 1.8, i.e. if $c_k = a_k \times_M b_k$ for $k = 1,2$ there is an order isomorphism h_k from $M_{<a_k} \times M_{<b_k}$ onto $M_{<c_k}$ and let f_k be an order automorphism

of M mapping a_k to b_k. Combining there is an order-isomorphism g_1 from $M_{<c_1}$ onto $M_{<c_2}$ and let g be the order automorphism of M such that g extends g_1 and $c_1 \leq d \in M \Rightarrow g(d) = c_2 + f_1(d - c_1)$; so g witness $(a_1 \times b_1) E^5_M (a_2 \times b_2)$ as promised.

For $\ell = 6$ it follows in the proof of part (3).

5) By the definition of m-o.r. and part (1). $\square_{2.1}$

Question 2.3. Is E^5_M convex for every M?

Claim 2.4. *If $a_1 E^3_M a_2$ then there is an order-automorphism of M mapping a_1 to a_2, i.e. $a_1 E^5_M a_2$.*

Proof. Without loss of generality $a_1 <_M a_2$. If $a_1 E^2_M a_2$ then $a_1 E^5_M a_2$ by 1.5(2), so without loss of generality $\neg(a_1 E^2_M a_2)$ hence $n \times a_1 < a_2$ for $n \in \mathbb{N}$. So by the definition of E^3_M and the assumption $a_1 E^3_M a_2$, clearly for some $c \in M \backslash \mathbb{N}$ we have

$(*)_1$ $c <_M a_1, M \models$ "$c^n < a_1$" for $n \in \mathbb{N}$ and $(c-1) \times_M a_1 <_M a_2 \leq_M c \times_M a_1$.

Clearly $a_2 E^2_M (c \times_M a_1)$ hence again by 1.5(2) without loss of generality

$(*)_2$ $M \models$ "$a_2 = c \times_M a_1$".

We define an equivalence relation E on $M \backslash \mathbb{N}$:

$(*)_3$ xEy iff $\bigvee\limits_n |x - y| < c^n$.

Clearly E is a convex equivalence relation. We choose a set X of representatives for E such that $0, a_1, a_2 \in X$, can be done as $0 + c^n = c^n < a_1$ and $c^n + a_1 < 2 \times a_1 < a_2$ for any $n \in \mathbb{N}$.

Note

$(*)_4$ if $b_1, b_2 \in M_{\leq a_1}$ then $(b_1 E b_2) \Leftrightarrow (c \times b_1) E (c \times b_2)$.

[Why? As we have

$$|(c \times b_2) - (c \times b_1)| = c \times (|b_2 - b_1|)$$

so for $n \in \mathbb{N}$:

$$|(c \times b_2) - (c \times b_1)| < c^{n+1} \Leftrightarrow |b_2 - b_1| < c^n$$

so $(*)_4$ is true indeed.]

Now we define a function f from M into M as follows:

$(*)_5$ (a) if $x \in 0/E$, i.e. $\bigvee\limits_n x <_M c^n$ then $f(x) = x$

(b) if $y \in X$ and $y \neq 0$ then $f(y) = c \times y$

(c) if $x \in X \backslash (0/E) \wedge x \leq_M y \in x/E$ then $f(y) = f(x) + (y - x)$

(d) if $x \in X \backslash (0/E) \wedge y <_M x \wedge y \in x/E$ then $f(y) = f(x) + (x - y)$

Note that f is well defined and is one-to-one order preserving and onto M by $(*)_4$. As $f(a_2) = c \times a_1 = a_2$ we have $a_1 E a_2$ so we are done. $\square_{2.4}$

Comparing with $(*)_7$ of the proof of 1.8

Observation 2.5. 1) For any $a \in M \backslash \mathbb{N}$ we have:

(a) the sequence $\langle \lfloor a^{1+2^{-n}} \rfloor : n \in \mathbb{N}\rangle$, that is $\langle \max\{b : b$ in M, a divides b and $(\lfloor b/a \rfloor)^{2^n} \le a\} : n \in \mathbb{N}\rangle$ is a decreasing sequence from $\{b : a' < b$ for every $a' \in a/E^3_M\}$ unbounded from below in it

(b) the sequence $\langle \lfloor a^{1-2^{-n}} \rfloor : n \in \mathbb{N}\rangle$, that is $\langle \max\{b : (\lfloor a/b \rfloor)^{2^n} \le a\} : n \in \mathbb{N}\rangle$ is an increasing sequence included in $\{b : b < a'$ for every $a' \in a/E^3_M\}$ and unbounded from above in it.

2) For $a \in M\backslash\mathbb{N}$ we have:

(a) the sequence $\langle \lfloor (1+2^{-n})a \rfloor : n \in \mathbb{N}\rangle$, is a decreasing sequence in $\{b \in M : b$ above $a/E^1_M\}$ cofinal in it

(b) the sequence $\langle \lfloor (1+2^{-n})a \rfloor : n \in \mathbb{N}\rangle$, is an increasing sequence in $\{b \in M : b$ below $a/E^1_M\}$ cofinal in it.

Proof. Straight. $\square_{2.5}$

Theorem 2.6. *1) If M_1 is 3-o.r. and f is an order-isomorphism from M_1 onto M_2 then f maps $E^k_{M_1}$ onto $E^k_{M_2}$ for $k = 3, 4$.*
2) In part (1), moreover $M_1{\restriction}\{<,+\}, M_2{\restriction}\{<,+\}$ are almost isomorphic.
3) For any M let $E^7_M = \{(a,b) : (\lfloor \log_2(a) \rfloor) E^4_{M_1} (\lfloor \log_2(b) \rfloor)\}$. Assume there is an order-isomorphism f from M_1 onto M_2 mapping $E^4_{M_1}$ to $E^4_{M_2}$, e.g. as in the conclusion of part (1) and f maps $E^7_{M_1}$ onto $E^7_{M_2}$ then $M_1{\restriction}\{<,+\}, M_2{\restriction}\{<,+\}$ are almost $\{<,+\}$ isomorphic.

Proof. 1) By the assumption and by 2.4 respectively

$(*)_0$ (a) $E^3_{M_1} \supseteq E^5_{M_1}$
(b) $E^3_{M_\ell} \subseteq E^5_{M_\ell}$ for $\ell = 1, 2$.

Easily by 1.5(1)

$(*)_1$ each $E^3_{M_\ell}$-equivalence class is convex.

Without loss of generality

$(*)_2$ (a) $<_{M_1} = <_{M_2}$ hence
(b) M_1, M_2 have the same universe and
(c) $E^5_{M_1} = E^5_{M_2}$ so $E^3_{M_2} \subseteq E^5_{M_2} \subseteq E^5_{M_1} = E^3_{M_1}$
(d) let $M := M_1{\restriction}\{<\} = M_2{\restriction}\{<\}$ hence $E^5_{M_1} = E^5_M = E^5_{M_2} = E^3_{M_1}$.

Also as usual

$(*)_3$ $\mathbb{N} \subseteq M_k$ for $k = 1, 2$.
$(*)_4$ if $a <_M b$ and $b \in M\backslash\mathbb{N}$ then $b, a +_{M_1} b, a +_{M_2} b$ are $E^3_{M_1}$-equivalent.

[Why? By the definition of E^3_M.]

$(*)_5$ if $a \in M, b \in M\backslash\mathbb{N}$ and $a \times_{M_k} b = c_k$ for $k = 1, 2$ then $c_1 E^3_{M_1} c_2$ and $c_1 E^5_M c_2$.

[By 1.13(1) we have $c_1 E^5_M c_2$ and use $(*)_0(a)$ to deduce $c_1 E^3_{M_1} c_2$.]

$(*)_6$ if $M_\ell \models$ "$a_1 \times a_2 \times \ldots \times a_n = b_\ell$" for $\ell = 1, 2$ then $b_1 E^3_{M_1} b_2$ and $b_1 E^5_M b_2$.

[Why? Similarly to $(*)_5$, i.e. by 1.13(2).]

$(*)_7$ $E^4_{M_1} = E^4_{M_2}$.

[Why? Let $a, b \in M \backslash \mathbb{N}$ be given. For $\ell = 1, 2$ and $n \in \mathbb{N}$ let $a_{\ell,n}$ be such that $M_\ell \models$ "$a^n = a_{\ell,n}$".

First, assume $aE^4_{M_2}b$ and without loss of generality $a < b$. So for some $n \in \mathbb{N}$ we have $M_2 \models$ "$a < b < a^n$" so $M_2 \models$ "$a < b < a_{2,n}$". Also $a_{1,n}E^3_{M_1}a_{2,n}$ by $(*)_6$ so by 2.1(1) we have $a_{1,n}E^4_{M_1}a_{2,n}$ hence for some $m, M_1 \models$ "$(a_{1,n})^m \geq a_{2,n}$", in fact, even $m = 2$ is O.K. So $M_1 \models$ "$b < a_{1,mn}$" but $a <_M b$, so together $aE^4_{M_1}b$.

Second, assume $\neg(aE^4_{M_2}b)$ and without loss of generality $a <_M b$. So for every $n \in \mathbb{N}, a_{2,n} <_M b$ and by $(*)_6$ we have $a_{2,n}E^3_{M_1}a_{1,n}$ hence $a_{1,n}/E^3_{M_1}$ has a member $< b$, and so in particular $a_{1,n+1}/E^3_{M_1}$ has a member $< b$, but $a_{1,n}/E^3_{M_1}$ is below $a_{1,n+1}/E^3_{M_1}$ (just think on the definitions) so $a_{1,n} <_M b$. As this holds for every $n \in \mathbb{N}$ we conclude $\neg(aE^4_{M_1}b)$.]

Let

$(*)_8$ $I^k_d = \{c \in M : M_k \models$ "$c^n < d$" for every $n \in \mathbb{N}\}$ for $d \in M$ and $k = 1, 2$.

Now

$(*)_9$ $I^1_d = I^2_d$ for $d \in M$.

[Why? By $(*)_7$ as $I^\ell_d = \{c : c/E^4_{M_\ell}$ is below $d\}$.]

$(*)_{10}$ $E^3_{M_2} = E^3_{M_1}$.

[First, if $a_1E^3_{M_2}a_2$ then by 1.5(3),(4) we have $a_1E^5_{M_2}a_2$ which, recalling $M_1 {\restriction} \{<\} = M = M_2 {\restriction} \{<\}$, is equivalent to $a_1E^5_{M_1}a_2$ which implies (by M_1 being 3-o.r. which we are assuming) $a_1E^3_{M_1}a_2$.

Second, assume $\neg(a_1E^3_{M_2}a_2)$ and without loss of generality $a_1 <_M a_2$. As $M_2 \models$ PA, for some $c \in M$ we have $M_2 \models$ "$ca_1 \leq a_2 < (c+1)a_2$" so by the previous sentence $c \notin I^2_{a_1}$ hence by $(*)_9$ also $c \notin I^1_{a_1}$.

By $(*)_5$ we have $(c \times_{M_1} a_1)E^3_{M_1}(c \times_{M_2} a_1)$ and as $c \times_{M_2} a_1 \leq_M a_2$ clearly $(c \times_{M_2} a_1)/E^3_{M_1} \leq a_2/E^3_{M_1}$ so together $(c \times_{M_1} a_1)/E^3_{M_1}$ is smaller or equal to $a_2/E^3_{M_1}$ so for some $a_3 \in a_2/E^3_{M_1}$ we have $c \times_{M_1} a_1 < a_3$. As $c \notin I^1_{a_1}$ this implies a_1, a_3 are not $E^3_{M_1}$-equivalent, so by the choice of a_3 also $\neg(a_1E^3_{M_1}a_2)$, so we are done proving $(*)_{10}$.]

Hence by $(*)_7 + (*)_{10}$ part (1) holds.

2),3) By part (1) and 2.7 below for the function $x \mapsto 2^{2^x}$, and the equivalence relation E^4_M. $\square_{2.6}$

Claim 2.7. *The models M_1, M_2 are almost $\{<, +\}$-isomorphic* *<u>when</u>*

- (a) *E_k is a convex equivalence relation on $M_k \backslash \mathbb{N}$ for $k = 1, 2$*
- (b) *h is an order-isomorphism from M_1 onto M_2 mapping E_1 onto E_2*
- (c) *f_k is a function definable in M_k, is increasing, maps $\mathbb{N}$ into $\mathbb{N}$, for each E_k-equivalence class Y the set $\{a \in M : f_k(a) \in Y\}$ has the order type of $\mathbb{Z}$ and is unbounded from below and from above in Y, for $k = 1, 2$, of course*
- (d) *for $k = 1, 2$, if $a_1E_ka_2$ and $b_1E_kb_2$ then $(a_1 + b_1)E_k(a_2 + b_2)$.*

Proof. As in the proof of 1.8. $\square_{2.7}$

Claim 2.8. *Assume h is an order-isomorphism from M_1 onto M_2 and M_1 is 4-o.r. Then for $a, b \in M$ we have $h(a) E^4_{M_2} h(b) \Rightarrow a E^4_{M_1} b$.*

Proof. Without loss of generality h is the identity and let $M_1 \restriction \{<\} = M = M_2 \restriction \{<\}$. Define $E_M = \{(a,b)$: there are $n \in \mathbb{N}, c_1 \in M, c_2 \in M$ such that $c_1 \leq_M \min_M\{a,b\}$ and $\max_M\{a,b\} \leq c_2$ and $(M_{<c_1})^n \cong M_{<c_2}\}$. Easily E_M is a convex equivalence relation, $E^4_{M_\ell} \subseteq E_M$ for $\ell = 1,2$ and $E^4_{M_1} = E_M$. $\square$

Claim 2.9. *1) Assume $a \in M$ is non-standard and $I = \{b/E^3_M : b \in a/E^4_M\}$, naturally ordered. Then the linear order I can be embedded into $\mathbb{R}_{>0}$ with dense image.*
2) If M is $\aleph_1$-saturated then the embedding is onto $\mathbb{R}_{>0}$.
3) Moreover defining $+_I$ by $(b_1/E^3_M) +_I (b_2/E^3_M) = (b_3/E^3_M)$ when $b_1 \times_M b_2 = b_3$, the embedding commute with addition so the image is an additive sub-semi-group of $\mathbb{R}$. Also $1_\mathbb{R}$ belongs to the image.

Proof. As in Definition 3.2 but we elaborate.

Fix $a \in M$ and for $b \in a/E^4_M$ let $\mathscr{S}_{a,b} = \{\frac{m_1}{m_2} : m_1, m_2 \in \mathbb{N}\backslash\{0\}$ and $M \models$ "$b^{m_2} \geq a^{m_1}$"$\}$. Clearly S_b is a subset of $\mathbb{Q}_{>0}$ and as M is a model of PA clearly S_b is an initial segment of $\mathbb{Q}_{>0}$. By the definition of "$b \in a/E^4_M$ necessarily $S_b \neq \emptyset$ and $S_b \neq \mathbb{Q}_{>0}$, so together $r_b = \sup(\mathscr{S}_b)$ belongs to $\mathbb{R}_{>0}$.

Again by PA

(a) $r_{b_1} \leq r_{b_2}$ when $b_1 \leq_M b_2$ are from a/E^4_M.

[Why? See the definition of $\mathscr{S}_{b_1}, S_{b_2}$.]

(b) $r_{b_1} = r_{b_1} \Leftrightarrow b_1 E^3_M b_2$ for $b_1, b_2 \in a/E^4_M$.

[Why? By the definition of E^3_M.]

(c) if $\mathbb{Q} \models$ "$\frac{m_1}{m_2} < \frac{m_3}{m_4}$" where $m_\ell \in \mathbb{N}\backslash\{0\}$ for $\ell = 1,2,3,4$ then for some $b \in a/E^4_M$ we have $\mathbb{R} \models$ "$\frac{m_1}{m_2} \leq r_b < \frac{m_3}{m_4}$".

[Why? Let $n \in \mathbb{N}\backslash\{0\}$ be such that $M \models$ "$b < a^n$", exists as $b \in a/E^4_M$. Without loss of generality $m_2 = m_4$ call it m, so necessarily $m_1 < m_3$ and without loss of generality $m_1 + n < m_3$. Now by the definition of $b \mapsto r_b$ the demand on b means that $M \models$ "$b^m \geq a^{m_1}$ and $b^m < a^{m_3}$". Let b be the minimal member of M such that $M \models$ "$b^m \geq a^{m_1}$" hence $M \models$ "$b^{m-1} < a^{m_1}$ hence $b^m < a^{m_1} b \leq a^{m_1} a^n = a^{m_1+n} \leq a^{m_3}$" so b is as required.]

(d) $\{r_b : b \in a/E^4_M\}$ is a dense subset of $\mathbb{R}_{>0}$.

[Why? By (c).]

(e) If M is $\aleph_1$-saturated then $\{r_b : b \in a/E^4_M\} = \mathbb{R}_{>0}$, i.e. part (2).

[Why? For any real r and n we can find $b_1, b_2 \in a/E^4_M$ such that $r - \frac{1}{n} < r_{b_2} < r < r_{b_2} < \frac{1}{n}$ by (d) and "$\mathbb{Q}$ is dense in $\mathbb{R}$".]

(f) part (3) of the claim holds.

[Why? By the rules of exponentiation which can be phrased in PA.]

Together we are done. $\square_{2.10}$

Remark 2.10. So the "distance" between E^3_M and E^4_M is small.

§ 3. Constructing somewhat rigid models

Hypothesis 3.1. λ is regular.

Definition 3.2. For any M (model of PA)

- (a) let $\mathbb{Z}_M = \mathbb{Z}[M]$ be the ring M generates (so $a \in \mathbb{Z}_M$ iff $a = b \vee a = -b$ for some $b \in M$, of course $\mathbb{Z}_M$ is determined only up to isomorphism over M; similarly below); when, as usual, M is ordinary without loss of generality $\mathbb{Z}_M \supseteq \mathbb{Z}$
- (b) let $\mathbb{Q}_M = \mathbb{Q}[M]$ be the field of quotients of $\mathbb{Z}_M$; in fact, it is an ordered field, if M is ordinary then without loss of generality $\mathbb{Q}_M \supseteq \mathbb{Q}$
- (c) let $\mathbb{R}_M = \mathbb{R}[M]$ be the closure of $\mathbb{Q}_M$ adding all definable cuts, so in particular it is a real closed field, see 3.3 below
- (d) let $S_M = \mathbb{R}^{\rm bd}_M/\mathbb{R}^{\rm infi}_M$ where (bd stands for bounded, infi stands for infitesimal)
  - $\mathbb{R}^{\rm bd}_M = \{a \in \mathbb{R}_M : \mathbb{R}_M \models -n < a < n \text{ for some } n \in \mathbb{N}\}$
  - $\mathbb{R}^{\rm infi}_M = \{a \in \mathbb{R}^{\rm bd}_M : \mathbb{R}_M \models \text{“} - 1/n < a < 1/n\text{” for every } n \in \mathbb{N}\}$
  - $\mathbf{j}_M$ is the function from $\mathbb{R}^{\rm bd}_M$ into $\mathbb{R}$ such that $M \models$ “$n_1/m_1 < a < n_2/m_2$” $\Rightarrow \mathbb{R} \models$ “$n_1/m_1 < \mathbf{j}_M(a) < n_2/m_2$ for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$” such that $m_1, m_2 > 0$.

Remark 3.3. 1) On 3.2(d) see Claim 2.9.
2) Concerning 3.2(c) note that $\mathbb{R}[M]$ is a sub-field of the Scott-Cauchy completion $\bar{\mathbb{R}}[M]$ of $\mathbb{Q}[M]$ and that for so called “rather classless” models $M, \mathbb{R}[M]$ coincide with $\bar{\mathbb{R}}[M]$. See more on completions in [**?**].

Definition 3.4. Let AP $=$ AP$_\lambda$ be the set of $\mathbf{a}$ such that

- (a) $\mathbf{a} = (M, \Gamma) = (M_{\mathbf{a}}, \Gamma_{\mathbf{a}})$ but we may[1] write $M^{\mathbf{a}}_{<b}$ instead $(M_{\mathbf{a}})_{<b}$
- (b) M is a model of PA
- (c) $|M|$, the universe of M, is an ordinal $< \lambda^+$
- (d) Γ is a set of $\le \lambda$ of types over M
- (e) each $p \in \Gamma$ has the form $\{a_{p,\alpha} < x < b_{p,\alpha} : \alpha < \lambda\}$ where $\alpha < \beta \Rightarrow M \models a_{p,\alpha} < a_{p,\beta} < b_{p,\beta} < b_{p,\alpha}$
- (f) M omits every $p \in \Gamma$.

Definition 3.5. 1) $\le_{\rm AP}$ is the following two-place relation on AP:
$\mathbf{a} \le_{\rm AP} \mathbf{b}$ iff $M_{\mathbf{a}} \prec M_{\mathbf{b}}$ and $\Gamma_{\mathbf{a}} \subseteq \Gamma_{\mathbf{b}}$.
2) Let AP$_T = \{\mathbf{a} \in$ AP $: M_{\mathbf{a}}$ is a model of $T\}$.
3) Let AP$^{\rm sat} = \{\mathbf{a} \in$ AP $: M_{\mathbf{a}}$ is saturated$\}$ and AP$^{\rm sat}_T =$ AP$_T \cap$ AP$^{\rm sat}$.

Claim 3.6. *1) $\le_{\rm AP}$ is a partial order of* AP.
2) If $\langle \mathbf{a}_\alpha : \alpha < \delta\rangle$ is $\le_{\rm AP}$-increasing, δ a limit ordinal $< \lambda^+$ then $\mathbf{a}_\delta = \cup\{\mathbf{a}_\alpha : \alpha < \delta\}$ defined by $M_{\mathbf{a}_\delta} = \cup\{M_{\mathbf{a}_\alpha} : \alpha < \delta\}, \Gamma_{\mathbf{a}_\delta} = \cup\{\Gamma_{\mathbf{a}_\alpha} : \alpha < \delta\}$, is a $\le_{\rm AP}$-lub of $\langle \mathbf{a}_\alpha : \alpha < \delta\rangle$.
3) Assume $\lambda = \lambda^{<\lambda} > \aleph_0$. If $\mathbf{a} \in$ AP then there is $\mathbf{b}$ such that $\mathbf{a} \le_{\rm AP} \mathbf{b}$ and $M_{\mathbf{b}}$ is saturated (of cardinality λ).

Proof. Easy. $\square_{3.6}$

[1] alternatively, replace M^0_b by $(M_{\mathbf{a}})_{<b}$ in the proof of 3.7 below.

Main Claim 3.7. *1) If (A) then (B) where:*

(A) (a) $\lambda = \aleph_0, \mathbf{a} \in \mathrm{AP}$

(b) $M_{\mathbf{a}} \models$ "$a_* > b_* > n$" *for* $n \in \mathbb{N}$ *and* a_*, b_* *are not* $E^3_{M_{\mathbf{a}}}$*-equivalent*

(c) F *is an order isomorphism from* $M^{\mathbf{a}}_{<a_*}$ *onto* $M^{\mathbf{a}}_{<b_*}$

(B) *there are* $\mathbf{b}, c_*$ *satisfying*

(a) $\mathbf{a} \le_{\mathrm{AP}} \mathbf{b}$

(b) $c_* <_{M_{\mathbf{b}}} a_*$ *so* $c_* \in M_{\mathbf{b}}$ *but* $c_* \notin M_{\mathbf{a}}$

(c) *some* $p \in \Gamma_{\mathbf{b}}$ *is equivalent to* $\{F(a_1) < x < F(a_2) : a_1, a_2 \in M_{\mathbf{a}}$ *and* $M_{\mathbf{b}} \models a_1 < c_* < a_2 \le a_*\}$ *recalling* F *is the isomorphism from (A)(c).*

Remark 3.8. 1) We use a_*, b_*, c_* as they are constant during the proof of 3.7, and we like to let a, a_i, etc. be free to denote other things.
2) Below in the Discussion 3.9 we try to explain the proof of 3.7; of course, it cannot be fully digested per se, but the reader can try to look at it from time to time, particularly when you lose track in the proof, hopefully this will help.
3) Note: usually the hypothesis on Φ is that it consists of "big formulas/types" but here we use "φ_1 is a big relative to φ_2", a finer condition.

Proof. Stage A:
Let

$\boxplus_1$ (a) $\Phi = \Phi_{\mathbf{a}}$ is the set of formulas $\varphi(x) = \varphi(x, \bar a)$ with $\varphi(x, \bar y) \in \mathbb{L}(\tau_{\mathrm{PA}})$ and $\bar a \in {}^{\ell g(\bar y)}(M_{\mathbf{a}})$

(b) $\varphi'(x) \vdash \varphi''(x)$ means both are from Φ and $M_{\mathbf{a}} \models$ "$(\forall x)(\varphi'(x) \to \varphi''(x)$"

(c) $\varphi(M_{\mathbf{a}}) = \varphi(M_{\mathbf{a}}, \bar a) = \{b \in M_{\mathbf{a}} : M_{\mathbf{a}} \models \varphi[b, \bar a]\}$ if $\varphi = \varphi(x) = \varphi(x, \bar a) \in \Phi$

(d) we define $\Sigma^k_{\mathbf{a}} = \Sigma^k_{M_{\mathbf{a}}}$ as the set of $\sigma(x_0, \dots, x_{k-1}) = \sigma(x_0, \dots, x_{k-1}, \bar a)$ satisfying $\sigma(\bar x, \bar y)$ is a definable function in $M_{\mathbf{a}}$ and $\bar a \in {}^{\ell g(\bar y)}(M_{\mathbf{a}})$ and $k \in \mathbb{N}$, we may omit k when it is 1 and so may write $\sigma(x, \bar y)$ and $\sigma(x)$

(e) let[2] $\xi(\varphi) = \xi(\varphi(x)) = \mathbf{j}_M(\log_2(|\varphi(M_{\mathbf{a}})|)/\log(a_*))$ for $\varphi = \varphi(x) \in \Phi_{\mathbf{a}}$ such that $\varphi(M) \subseteq [0, a_*)_M$, see 3.2(d)

(f) if $\varphi_1, \varphi_2 \in \Phi, \varphi_2(M) \ne \emptyset$ and $\sigma \in \Sigma_{\mathbf{a}}$ then[3] let $\sigma' := \sigma[\varphi_1, \varphi_2]$ be the following function from $\varphi_1(M)$ to $\varphi_2(M) \cup \{0\}$, definable in $M : M \models \sigma'(a) = b$ iff $a \in \varphi_1(M)$ and $b = \max\{b$: either $b = 0$ and $\sigma(a) < \min(\varphi_2(M))$ or $b \in \varphi_2(M)$ and $b \le \sigma(a)\}$.

We define $\mathbb{P}$, it will serve as a set of approximations to $\mathrm{tp}(c_*, M_{\mathbf{a}}, M_{\mathbf{b}})$, as follows:

$\boxplus_2$ the quasi-order $\mathbb{P}$ is defined by:

(a) a member of $\mathbb{P}$ is a pair $\bar\varphi = (\varphi_1, \varphi_2)$ such that:

(α) $\varphi_\ell = \varphi_\ell(x)$ are from Φ

(β) $\varphi_1(x) \vdash x < a_*$ and $\varphi_2(x) \vdash x < b_*$

[2]the $|\varphi(M_{\mathbf{a}})|$ and $\log_2$ has natural meanings; of course we can translate it to a formula in $\mathbb{L}(\tau_{\mathrm{PA}})$.

[3]So σ' is σ restricting the domain to $\varphi_1(M)$ and rounding the image to be in $\varphi_2(M) \cup \{0\}$.

(γ) if $a_1 < a_2$ are from $\varphi_1(M_{\mathbf{a}})$ then $[F(a_1), F(a_2)]_M \cap \varphi_2(M) \neq \emptyset$

(δ) $\xi(\varphi_1(M)) > \xi(\varphi_2(M))$

(b) $\mathbb{P} \models \bar{\varphi}' \leq \bar{\varphi}''$ iff $\varphi''_\ell(x) \vdash \varphi'_\ell(x)$, for $\ell = 1, 2$.

Note that $(\varphi_1(x, \bar{a}), \varphi_2(x, \bar{a}_2)) \in \mathbb{P}$ is not definable in $M_{\mathbf{a}}$ mainly because of $(a)(\gamma)$ of $\boxplus_2$.

Obviously observe

$\boxplus_3$ if $\bar{\varphi} \in \mathbb{P}$ and $\varphi(x) \in \Phi$ then for some $\mathbf{t} \in \{0, 1\}$ we have $(\varphi_1(x) \wedge \varphi(x)^{\mathbf{t}}, \varphi_2(x)) \in \mathbb{P}$ (and is $\leq_{\mathbb{P}}$-above $\bar{\varphi}$; recall that $\varphi^{\mathbf{t}}$ is φ is $\mathbf{t} = 1$ and is $\neg\varphi$ if $\mathbf{t} = 0$).

[Why? $M_{\mathbf{a}} \models$ "$|\varphi_1(M) \cap \varphi(M)| \geq \frac{1}{2}|\varphi_1(M)|$ or $M_{\mathbf{a}} \models$ "$|\varphi_1(M) \backslash \varphi(M)| \geq \frac{1}{2}|\varphi_2(M)|$".

As $a_* \notin \mathbb{N}$ clearly $\xi(\varphi_1(x) \wedge \varphi(x)) = \xi(\varphi_2(x))$ or $\xi(\varphi_2(x) \wedge \neg\varphi(x)) = \xi(\varphi_2(x))$. So clearly we are done proving $\boxplus_3$.]

Stage B:

We arrive at a major point: how to continue to omit members of $\Gamma_{\mathbf{a}}$

$\boxplus_4$ if $\bar{\varphi} \in \mathbb{P}, \sigma(x) \in \Sigma_{\mathbf{a}}$ and $p(x) \in \Gamma_{\mathbf{a}}$ then for some $\bar{\varphi}'$ and n we have

(a) $\bar{\varphi} \leq \bar{\varphi}' \in \mathbb{P}$

(b) $\varphi'_0(x) \vdash$ "$\sigma(x) \notin (a_{p,n}, b_{p,n})$".

The rest of this stage is dedicated to proving this. We use a "wedge question".

Case 1: There is $d \in M_{\mathbf{a}}$ such that $\bar{\varphi}' := (\varphi_1(x) \wedge \sigma(x) = d, \varphi_2(x)) \in \mathbb{P}$.

In this case obviously $\bar{\varphi} \leq \bar{\varphi}' \in \mathbb{P}$. Also as $d \in M_{\mathbf{a}}$ and $M_{\mathbf{a}}$ omit $p(x)$ recalling $p(x) \in \Gamma_{\mathbf{a}}$, clearly d does not realize $p(x)$ hence for some $n, d \notin (a_{p,n}, b_{p,n})_{M_{\mathbf{a}}}$; so $\bar{\varphi}', n$ are as promised.

Case 2: Not case 1.

So

$(*)_{4.1}$ $\xi(\varphi(x) \wedge \sigma(x) = d) \leq \xi(\varphi_2(x))$ for every d from $M_{\mathbf{a}}$.

Clearly there is a minimal $d_* \in M_{\mathbf{a}}$ satisfying

$(*)_{4.1}$ $M_{\mathbf{a}} \models$ "$|\{c \in \varphi_1(M) : \sigma(c) \leq d_*\}| \geq \frac{1}{2}|\varphi_1(M_{\mathbf{a}})|$".

So

$(*)_{4.2}$ $M_{\mathbf{a}} \models$ "$|\{c \in \varphi_1(M) : \sigma(c) \geq d_*\}| \geq \frac{1}{2}|\varphi_2(M_{\mathbf{a}})|$".

But $M_{\mathbf{a}}$ omits the type $p(x)$ as $p(x) \in \Gamma_{\mathbf{a}}$ and $d_* \in M_{\mathbf{a}}$, so for some $n, d_* \notin (a_{p,n}, b_{p,n})$.

So one of the following sub-cases occurs:

Sub-case 2a: $d_* \leq a_{p,n}$.

Let $\varphi'_1(x) = \varphi_1(x) \wedge (\sigma(x) \leq a_{p,n+1})$ and $\varphi'_2(x) = \varphi_2(x)$. Now the pair $\bar{\varphi}' = (\varphi'_1, \varphi'_2)$ is as required, noting (by $(*)_{4.1}$) that

$$M_{\mathbf{a}} \models \text{"}|\varphi'_1(M)| \geq |\{a \in \varphi_1(M) : \sigma(c) \leq d_*\}| \geq \frac{1}{2}|\varphi_1(M)|\text{"}$$

hence

$$\xi(\varphi'_1(x)) = \xi(\varphi_1(x)) > \xi(\varphi_2(x)) = \xi(\varphi'_2(x)).$$

Sub-case 2b: $d_* \geq a_{p,n}$.

Let $\varphi'_1(x) = \varphi_2(x) \wedge (\sigma(x) \geq b_{p,n+1})$ and $\varphi'_2(x) = \varphi_2(x)$. Now $\bar{\varphi}' = (\varphi'_1, \varphi'_2)$ is as required noting (by $(*)_{4.2}$) that

$$M_a \models \text{“}|\varphi'_1(M)| \geq |\{c \in \varphi_1(M) : \sigma(c) \geq d_*\}| \geq \frac{1}{2}|\varphi_1(M)|\text{”}$$

hence

$$\xi(\varphi'_1(x)) = \xi(\varphi_1(x)) > \xi(\varphi_2(x)) = \xi(\varphi'_2(x)).$$

So we are done proving $\boxplus_4$.

Stage C:

How do we omit the new type? Recall that c_* will realize a type to which $\bar{\varphi} \in \mathbb{P}$ is an approximation and we have to omit the relevant type from clause $(B)(c)$ of the Claim.

This stage is dedicated to proving the relevant statement:

$\boxplus_5$ if $\bar{\varphi} \in \mathbb{P}$ and $\sigma(x) \in \Sigma_{\mathbf{a}}$ then for some $\bar{\varphi}'$:
- (a) $\bar{\varphi} \leq \bar{\varphi}' \in \mathbb{P}$
- (b) for some $a_1 < a_2 \leq a_*$ we have
  - $\varphi'_1(x) \vdash a_1 \leq x < a_2$
  - $\varphi'_1(x) \vdash \neg(F(a_1) \leq \sigma(x) < F(a_2))$.

First note

$(*)_{5.1}$ if there is $\bar{\varphi}' \in \Phi$ such that $\bar{\varphi} \leq \bar{\varphi}'$ and $\xi(\varphi'_1)/\xi(\varphi'_2) > 2$ then the conclusion of $\boxplus_5$ holds.

[Why? As $|\bigcup\limits_i A_i| = \sum\limits_i |A_i|$ for pairwise disjoint sets, i.e. the version provable in PA we can find $\varphi''_1(x) \in \Phi$ such that $\varphi''_1(M_{\mathbf{a}}) \subseteq \varphi'_1(M_{\mathbf{a}}), M_{\mathbf{a}} \models$ “$|\varphi'_1(M_{\mathbf{a}})|/|\varphi''_1(M_{\mathbf{a}})| \leq |\varphi_2(M_{\mathbf{a}})|$” and $\sigma[\varphi''_1, \varphi_2]$, which was defined in Clause $\boxplus_1(f)$, is constant say constantly e, hence $e \in \varphi_2(M_{\mathbf{a}})$. So $\xi(\varphi''_1(x)) \geq \xi(\varphi'_1(x)) - \xi(\varphi'_2(x)) > 2\xi(\varphi'_2(x)) - \xi(\varphi'_2(x)) = \xi(\varphi'_2(x))$ hence $(\varphi''_1, \varphi'_2)$ belongs to $\mathbb{P}$ and $\bar{\varphi} \leq \bar{\varphi}' \leq (\varphi''_1, \varphi'_2)$. As $e \in \varphi_2(M_{\mathbf{a}}) \subseteq [0, b_*)_{M_{\mathbf{a}}}$ and F is onto $M^{\mathbf{a}}_{<b_*}$ for some $d <_{M_{\mathbf{a}}} a_*$ we have $F(d) = e$. By $\boxplus_3$ without loss of generality $\varphi''(x) \vdash$ “$x < d$” or $\varphi''(x) \vdash$ “$d \leq x$” so we can choose (a_1, a_2) as $(0, d)$ or as (d, a_*). So we are done.]

So we can assume $(*)_{5.1}$ does not apply. Hence it is natural to deduce (can replace $\frac{1}{8}$ by any fixed $\varepsilon > 0$).

$(*)_{5.2}$ Without loss of generality for no $\bar{\varphi}' \in \Phi$ do we have $\bar{\varphi} \leq \bar{\varphi}'$ and

$$\xi(\varphi'_1)/\xi(\varphi'_2) > (1 + \frac{1}{8})\xi(\varphi_1)/\xi(\varphi_2).$$

[Why? We try to choose $\bar\varphi^n$ by induction on $n \in \mathbb{N}$ such that $\bar\varphi^n \in \mathbb{P}, \bar\varphi^0 = \bar\varphi, \bar\varphi^n \le \bar\varphi^{n+1}$ and $\xi(\varphi^n_1)/\xi(\varphi'_2) \ge (1+\frac{1}{8})^n \xi(\varphi_1)/\xi(\varphi_2)$. So for some n we have $\xi(\varphi^n_1)/\xi(\varphi'_2) > 2$ and we can apply $(*)_{5.1}$, contradiction. But $\bar\varphi^0$ is well defined, hence for some $n, \bar\varphi^n$ is well defined but we cannot choose $\bar\varphi^{n+1}$. Now $\bar\varphi^n$ is as required in $(*)_{5.2}$.]

Clearly

$(*)_{5.3}$ if $a_1 < a_2$ are from $\varphi_1(M)$ and $b_1 \le F(a_1) < F(a_2) \le b_2$ are from $\varphi_2(M)$ and $\xi(\varphi_1(x) \wedge a_1 \le x < a_2 \wedge \sigma(x) \notin [b_1, b_2)) > \xi(\varphi_2(x) \wedge b_1 \le x < b_2)$ <u>then</u> we are done.

So from now on we assume that there are no a_1, a_2, b_1, b_2 as in $(*)_{5.3}$.

Let

$(*)_{5.4}$ (a) $k_* \in \mathbb{N}\backslash\{0\}$ be large enough such that $(\xi(\varphi_1) - \xi(\varphi_2))/\xi(\varphi_2) > 2/k_*$
  (b) let $n(1) \in \mathbb{N}\backslash\{0\}$ be large enough such that:
   - $\xi(\varphi_1) - \xi(\varphi_2) > 1/n(1)$
   - $\xi(\varphi_2) > (k_* + 1)/n(1)$
  (c) let $n(2) \in \mathbb{N}$ be $> n(1)$

$(*)_{5.5}$ let
  (a) $\psi_1(x_1, x_2; y_1, y_2) = x_1 < x_2 \wedge \varphi_1(x_1) \wedge \varphi_1(x_2) \wedge y_1 < y_2 < b_*$
  (b) $\psi_2(x_1, x_2; y_1, y_2)$ is the conjunction of:
   - $\psi_1(x_1, x_2, y_1, y_2)$
   - $|\{x : \varphi_1(x) \wedge x_1 \le x < x_2\}|^{n(1)} \ge |\{y : \varphi_2(y) \wedge y_1 \le y < y_2\}|^{n(1)} \times a_*$
  (c) $\vartheta_2(x_1, x_2) = (\exists y_1, y_2)(\psi_2(x_1, x_2; y_1, y_2)$
  (d) $\psi_3(x_1, x_2; y_1, y_2)$ is the conjunction of
   - $\psi_2(x_1, x_2; y_1, y_2)$
   - $|\{x : \varphi_1(x) \wedge x_1 \le x < x_2 \wedge \sigma(x) \notin [y_1, y_2)\}|^{n(2)} < |\{y : \varphi_2(a) \wedge y_1 \le y < y_2\}|^{n(2)} \times a_*$
  (e) $\vartheta_3(x_1, x_2) = (\exists y_1, y_2)\psi_3(x_1, x_2, y_1, y_2)$.

So by our assumptions (for clause (b) use "$(*)_{5.3}$ does not apply") we have

$(*)_{5.6}$ (a) if $a_1 < a_2$ are from $\varphi_1(M_{\mathbf{a}})$ then $M_{\mathbf{a}} \models \psi_1[a_1, a_2; F^{[\varphi_2]}(a_1), F^{[\varphi_2]}(a_2)]$

  (b) if $M_{\mathbf{a}} \models \psi_2[a_1, a_2; F^{[\varphi_2]}(a_1), F^{[\varphi_2]}(a_2)]$ then
   - $M_{\mathbf{a}} \models \psi_3[a_1, a_2; F^{[\varphi_2]}(a_1), F^{[\varphi_2]}(a_2)]$
   - $M_{\mathbf{a}} \models \vartheta_3[a_1, a_2]$.

Clearly

$(*)_{5.7}$ if $M_{\mathbf{a}} \models \psi_3[a_1, a_2; b^\iota_1, b^\iota_2]$ for $\iota = 1, 2$ then $[b^1_1, b^1_2)_{M_{\mathbf{a}}} \cap [b^2_1, b^2_2)_{M_{\mathbf{a}}} \ne \emptyset$.

It is well known that for a linear order, for any finite family of intervals, their intersection is non-empty iff the intersection of any two is non-empty.

Now a version of this can be proved in PA hence

$(*)_{5.8}$ for some $\sigma(x_1, x_2) \in \Sigma^2_{\mathbf{a}}$, we have: if $M_{\mathbf{a}} \models \vartheta_3[a_1, a_2]$ <u>then</u> $\sigma(a_1, a_2) \in \varphi_2(M)$ and for every $b_1, b_2 \in \varphi_2(M)$ we have $M_{\mathbf{a}} \models \psi_3[a_1, a_2; b_1, b_2]$ implies $\sigma(a_1, a_2) \in [b_1, b_2)_{M_{\mathbf{a}}}$.

Now

$(*)_{5.9}$ (a) let $\varepsilon \in \mathbb{Q}_M \subseteq \mathbb{R}_M$ be a true rational such that $\xi(\varphi_2) > \varepsilon > \xi(\varphi_2)k_*/(k_*+1) + 1/n(1)$

(b) let $d_* = \lfloor (a_*)^\varepsilon \rfloor \in M_{\mathbf{a}}$ computed in $\mathbb{R}_{\mathbf{a}}$ and $c_* = \lfloor (a_*)^{\varepsilon - 1/n(1)} \rfloor$

$(*)_{5.10}$ in $M_{\mathbf{a}}$ we can define an increasing sequence $\langle a_{1,i} : i < i(*)\rangle$, so $i(*) \in M_{\mathbf{a}}$ such that

- $a_{1,0} = 0, a_{1,i(*)} = a_*$
- $a_{1,i+1} = \min\{a : \varphi_1(a) \text{ and } a_{1,i} < a \text{ and } |\varphi_1(M_{\mathbf{a}}) \cap [a_{1,i}, a)| \text{ is } d_*\}$
- $|\{a : \varphi_1(a) \text{ and } a_{1,i(*)-1} \leq a < a_*\}|$ is $\geq d_*$ but $\leq 2d_*$

$(*)_{5.11}$ (a) in $M_{\mathbf{a}}$ we can define
$u = \{i < i(*) : M_{\mathbf{a}} \models \vartheta_3[a_{1,i}, a_{1,i+1}]\}$
$v = \{i < i(*) : i \notin u\}$

(b) let $\varphi_{1,i}(x) := \varphi_1(x) \wedge a_{1,i} \leq x < a_{1,i+1}$.

So (will be used in Case 1 below)

$(*)_{5.12}$ (a) $\xi(\varphi_{1,i}(x)) = \varepsilon$ for $i < i(*)$

(b) if $i < i(*)$ and $i \in v$ then
$M_{\mathbf{a}} \models$ "$|\varphi_2(M_{\mathbf{a}}) \cap (F(a_{1,i}), F(a_{1,i+1}))_{M_{\mathbf{a}}}| \geq |\varphi_{1,i}(M_{\mathbf{a}})| \times a_*^{-1/n(1)}$"
$= b_* \times a_*^{-1/n(1)}$".

[Why? Clause (a) is obvious by the definition of $\xi(-)$ and $a_{1,i+1}$. For clause (b) note that by the definition of ϑ_3 in $(*)_{5.5}(e)$ we have $M_{\mathbf{a}} \models \neg\psi_3[a_{1,i}, a_{2,i}; F^{[\varphi_2]}(a_{1,i}), F^{[\varphi_2]}(a_{2,i})]$, but by $(*)_{5.6}(a)$ we have $M_{\mathbf{a}} \models \psi_3[a_{1,i}, a_{2,i}; F(a_{1,i}), F(a_{1,i})]$. By the definition of ψ_3 in $(*)_{5.5}(d)$ we are done.]

Now towards Case 2 note

$(*)_{5.13}$ if $i_1 < i_2$ are from u then $F(a_{1,i_1}) < \sigma^M(a_{1,i_2}, a_{1,i_2+1})$.

[Why? Obvious by $(*)_{5.8}$.]

$(*)_{5.14}$ we define terms $\sigma_1(x_1,x_2), \sigma_2(x_1,x_2) \in \Sigma^2_{\mathbf{a}}$ such that if $i < i(*)$ then:

(a) $M_{\mathbf{a}} \models \sigma_1(a_{1,i}, a_{1,i+1}) < \sigma(a_{1,i}, a_{1,i+1}) < \sigma_2(a_{1,i}, a_{1,i+1})$

(b) $\varphi_2(M_{\mathbf{a}}) \cap [\sigma^{M_{\mathbf{a}}}(a_{1,i}, a_{1,i+1}), \sigma_2^{M_{\mathbf{a}}}(a_{1,i}, a_{1,i+1}))_{M_{\mathbf{a}}}$ has $\leq c_*$ elements, in $M_{\mathbf{a}}$'s-sense

(c) $\varphi_2(M_{\mathbf{a}}) \cap [\sigma_1^{M_{\mathbf{a}}}(a_{1,i}, a_{1,i+1}), \sigma^{M_{\mathbf{a}}}((a_{1,i}, a_{1,i+1}))_{M_{\mathbf{a}}}$ has $\leq c_*$ elements in $M_{\mathbf{a}}$'s-sense

(d) if $i < j$ are from u then $M_{\mathbf{a}} \models$ "$\sigma_2(a_{1,i}, a_{1,i+1}) < \sigma(a_{1,j}, a_{1,j+1})$"

(e) $M_{\mathbf{a}} \models$ "$\sigma_1(a_{1,i}, a_{1,i+1}) < \sigma(a_{1,i}, a_{1,i+1}) < \sigma_2(a_{1,i}, a_{1,i+1})$"

(f) if $i \in u$ then $M_{\mathbf{a}} \models$ "$\sigma_1(a_{1,i}, a_{1,i+1}) < F(a_{1,i}) < \sigma(a_{1,i}, a_{1,i+1}) < F(a_{1,i+1}) < \sigma_2(a_{1,i}, a_{1,i+1})$".

[Why? Let $\sigma_2(a_{1,i}, a_{1,i+1})$ be maximal such that the relevant part of (a) and (b),(d) holds and $\sigma_1(a_{1,i}, a_{1,i+1})$ be minimal such that the other part of (a) and $(c),(e)$ holds. By $(*)_{5.8}$ we can finish.]

Now the proof splits.

<u>Case 1</u>: $M_{\mathbf{a}} \models$ "$|v| \geq i(*)/3$". Here we shall use clause (γ) of $\boxplus_2(a)$.

Let $v_1 = \{i \in v : M_{\mathbf{a}} \models$ "$|\{j \in v : j < i\}|$ is even$\}$, so $M_{\mathbf{a}} \models$ "$|v_1| \geq i(*)/6$". Let $\varphi'_1(x) := \varphi_1(x) \wedge (\exists z)[z \in v_1 \wedge x \in [a_{1,z}, a_{1,z+1}) \wedge \neg(\exists y)(\varphi_2(y) \wedge y \in [a_{1,z}, a_{1,z+1}) \wedge y < x)]$.

Let $\varphi'_2(x) := \varphi_2(x) \wedge$ (the number $|\{y : \varphi_2(a) \wedge y < x\}|$ is divisible by c_*).

Now

$(*)_{5.15}$ (a) $\bar{\varphi}' := (\varphi'_1(x), \varphi'_2(x)) \in \mathbb{P}$

(b) $\xi(\varphi'_1(x)) = \xi(\varphi_1(x)) - \varepsilon$

(c) $\xi(\varphi'_2(x)) = \xi(\varphi_2(x)) - \varepsilon + 1/n(1)$.

So

$$\begin{aligned} \xi(\varphi'_1(x))/\xi(\varphi'_2(x)) \quad & = (\xi(\varphi_1(x)) - \varepsilon)/(\xi(\varphi_2(x) - \varepsilon + 1/n(1)) \\ & \geq (\xi(\varphi_1(x) - \xi(\varphi_2(x))/(\xi(\varphi_2) - \xi(\varphi_2)k_*/(k_* + 1)) \\ & = (k_* + 1)(\xi(\varphi_1(x)) - \xi(\varphi_2(x)))/\xi(\varphi_2) > 2 \end{aligned}$$

and we fall under $(*)_{5.1}$ finishing the proof of $\boxplus_5$.

Case 2: $M_{\mathbf{a}} \models$ "$|u| \geq i(*)/3$.

Define in $M_{\mathbf{a}}$

$u_2 = \{i \in u :$ "$|\{j \in \varphi_1(M) : j < i\}|$ is even"$\}$.

So $M_{\mathbf{a}} \models$ "$|u_1| \geq i(*)/6$". Now $M_{\mathbf{a}}$ satisfies

$$\begin{aligned} |\varphi_1(M)| \leq (i(*)+1)d_* \quad & \leq 7|\cup\{\varphi_1(M_{\mathbf{a}}) \cap [a_{1,i}, a_{1,i+1}) : i \in u_1\}| \\ & = 7\Sigma\{|\varphi_1(M_{\mathbf{a}}) \cap [a_{1,i}, a_{1,i+1})| : i \in u_1\}| \\ & \leq 7\Sigma\{|\varphi_2(M_{\mathbf{a}}) \cap [\sigma_1(a_{1,i}, a_{1,i+1}), \sigma_2(a_{1,i}, a_{1,i+1}))| \times a_*^{1/n(1)} : i \in u_1\}| \\ & 7|\cup\{\varphi_2, M_{\mathbf{a}}) \cap [\sigma_1(a_{1,i}, a_{2,i+1}), \sigma_2(a_{1,i}, a_{1,i+1})) : i \in u_2\}| \times a_*^{1/n(1)} \\ & < 7 \times |\varphi_2(M_{\mathbf{a}})| \times a_*^{1/n(1)}. \end{aligned}$$

Together $|\varphi_2(M)| \leq 7 \times |\varphi_2(M)| \times a_{a_*}^{1/n(1)}$. But as $n(1)$ was chosen large enough, i.e. $1/n(1) < \xi(\varphi_2(M_{\mathbf{a}}) - \xi(\varphi_2(M_{\mathbf{a}}))$, contradiction. So we are done proving $\boxplus_5$.

Stage D:

We define the sets $\mathbf{T} = \mathbf{T}_1 \cup \mathbf{T}_2 \cup \mathbf{T}_3$ of tasks where

$\boxplus_6$
- $\mathbf{T}_1 = \{(1, \varphi(x)) : \varphi(x) \in \Phi\}$, toward completeness
- $\mathbf{T}_2 = \{(2, \sigma(x), p(x)) : \sigma(x) \in \Sigma_{\mathbf{a}}$ and $p(x) \in \Gamma_{\mathbf{a}}\}$, toward preserving $p(x)$ is omitted
- $\mathbf{T}_3 = \{(3, \sigma(x)) : \sigma(x) \in \Sigma_{\mathbf{a}}\}$, toward "stopping F", "omitting the new type".

Clearly $\mathbf{T}$ is countable, let $\langle \mathbf{s}_n : n < \omega \rangle$ list it.

We now choose $\bar{\varphi}^n$ by induction on n such that:

$\boxplus_7$ (a) $\bar{\varphi}^n \in \mathbb{P}$

(b) $\bar{\varphi}^m \leq \bar{\varphi}^n$ for $m < n$

(c) if $n = m+1$ and $\mathbf{s}_m = (1, \varphi(x))$ then $\varphi_n(x) \vdash \varphi(x)$ or $\varphi_n(x) \vdash \neg\varphi(x)$

(d) if $n = m+1$ and $\mathbf{s}_m = (2, \sigma(x), p(x))$ then for some $k, \varphi_n(x) \vdash$ "$\sigma(x) \notin (a_{p,k}, b_{p,k})$"

(e) if $n = m+1$ and $\mathbf{s}_m = (3, \sigma(x))$ then for some $a_{1,m} < a_{2,m} \le a_*$ we have $\varphi'_n(x) \vdash a_{1,m} \le x < a_{2,m} \wedge \neg(F(a_{1,n}) \le \sigma(x) < a_{2,m})$.

Why can we carry the induction? $\bar\varphi^0$ trivial, for $\bar\varphi^{m+1}$ if $\mathbf{s}_m \in \mathbf{T}_1$ by $\boxplus_3$, if $\mathbf{s}_m \in \mathbf{T}_2$ by $\boxplus_4$ and if $\mathbf{s}_m \in \mathbf{T}_3$ by $\boxplus_5$, more fully, let $\mathbf{s}_m = (3, \sigma(x))$, let $\sigma'(x)$ be defined by $\sigma'(x) = \min\{y : y = b_* \text{ or } \sigma(x) \le y \wedge \varphi(y)\}$, now apply $\boxplus_5$ to $(\bar\varphi^m, \sigma'(-))$.

Note

$(*)_{7.1}$ $\{\varphi_n(x) : n < \omega\}$ is a complete type over $M_{\mathbf{a}}$.

[Why? By clause (c) of $\boxplus_7$ and the choice of $\mathbf{T}_1$.]

By compactness there are N and c_* such that

$(*)_{7.2}$ (a) $M_{\mathbf{a}} \prec N$

(b) c_* realizes $\{\varphi_n(x) : n < \omega\}$.

As T being a completion of PA has definable Skolem functions, without loss of generality

$(*)_{7.3}$ N is the Skolem hull of $M_{\mathbf{a}} \cup \{c_*\}$.

Now by $\boxplus_7(d)$

$(*)_{7.4}$ N omit every $p \in \Gamma_{\mathbf{a}}$.

Also by $\boxplus_7(c)$

$(*)_{7.5}$ N omits $\{F(a) < x < F(a_1) : a_0 < c_* < a_1 \le a_* \text{ and } \{a_0, a_1\} \subseteq M_{\mathbf{a}}\}$.

By renaming, without loss of generality the set of elements of N is a countable ordinal, so we can finish the proof of the claim. $\square_{3.7}$

Discussion 3.9. 1) Note that a natural approach is to approximate the type of c_* by formulas $\varphi(x)$ with parameters from M such that $\varphi(x) \vdash$ "$x < a_*$" and $M_{\mathbf{a}}$ "think" $|\varphi(M_{\mathbf{a}})|$ is large enough than b_*. Then for $\sigma(x) \in \Sigma_{\mathbf{a}}$ (i.e. a term with parameters from M), which maps $\{c : c <_M a_*\}$ into $\{d : d <_M b_*\}$ we have to ensure $\sigma(c_*)$ will not realize the undesirable type, so it is natural to "shrink" $\varphi(x)$ to $\varphi'(x)$ such that "$|\varphi'(M)|$ is large enough then $|\varphi(M)|/b_*$" in the sense of M and $\sigma(-)$ is constant on $\varphi'(M)$. This requires that also $|\varphi(M)|/b_*$ is large so a natural choice for "$\varphi(x)$ is large" means, e.g. $M \models$ "$|\varphi(M)| \ge b_*^n$" for every $n \in \mathbb{N}$. This is fine if $\neg(a_* E^4_M b_*)$, but not if we know $\neg(a_* E^3_M b_*)$ but possibly $a_* E^4_M b_*$. This motivates the main definition of $\mathbb{P}$ above.

2) We shall say that $(\varphi_1, \varphi_2) \in \mathbb{P}$ is a "poor man's substitute" to the original problem when:

(a) $[0, a_*)_{M_{\mathbf{a}}}$ is replaced by $\varphi_1(M)$

(b) $[0, b_*)_{M_{\mathbf{a}}}$ is replaced by $\varphi_2(M)$

(c) $F{\restriction}[0, a_*)_{M_{\mathbf{a}}}$ is replaced by $F{\restriction}\varphi_1(M_{\mathbf{a}})$, really rounded to $\varphi_2(M)$

(d) $a_* > b_* \wedge \neg(a_* E^3_{M_{\mathbf{a}}}, b_*)$ is replaced by $\xi(\varphi_1) > \xi(\varphi_2)$, see $\boxplus_1(e)$ in the proof.

3) Why we demand condition (δ) of $\boxplus_2(a)$ in the proof of 3.7?

Assume $M_{\mathbf{a}} \prec M$ and $a \in M \backslash M_{\mathbf{a}}, a <_M a_*$ and F^+ is an automorphism of $M \restriction \{<\}$ extending F then $\{f(a_1) : a_1 <_M a_*, a_1 <_M a\}, \{F(a_2) : a_2 \in M_{\mathbf{a}}, a_* < a_2 < a_*\})$ is a cut of $M_{\mathbf{a}}$, which $F^+(a)$ realizes in M. If M "thinks" $|\varphi_2(M)|$ is $\ll b_*, F$ may be one-to-one from $\varphi_2(M)$ onto some definable subset $\varphi'_2(M) \subseteq [0, b_*)_M$. A reasonable suggestion is to demand $|\varphi_2(M)| \gg b_*$. Consider for transparency the case $M_{\mathbf{a}} \models$ "$a_* < b_* b_*$".

But then let E be the definable convex equivalent relation on $[0, a_*)$ such that each equivalence class is of size b_{**}, then the cut the new element realizes is really a cut of $[0, a_*)/E$. Now F^+ maps every E-equivalence class to some E'-equivalence class, E' a definable convex equivalence relation on $[0, b_{**})$ and F as a map from $[0, a_*)/E$ into $[0, b_*)/E'$ is defined, possible if $|[0, a_*)/E| = |[0, b_*)/E'|$.

The solution is via clause (δ), which tells us that in part (1),(2) of the discussion, $\xi(\varphi_1) > \xi(\varphi_2)$ is a real substitute, see clause (d) in part (2).

4) Why clause (γ) in $\boxplus_2(a)$, defining $\mathbb{P}$? Otherwise $\varphi_2(-)$ may be irrelevant to the type we like to omit, so impossible.

5) By such approximations, i.e. member of $\mathbb{P}$,

(A) why can we arrive to a complete type?

Answer: As if we divide φ_1 to two sets at least one has the same $\xi(-)$:

(B) why can we continue to omit $p(x) \in \Gamma_{\mathbf{a}}$?

Answer: As if $\sigma(-)$ is a definable (in $M_{\mathbf{a}}$) function with domain φ_1 let d_* be maximal such that $|\{a \in \varphi_1(M) : \sigma(a) < d_*\}| \leq \frac{1}{2}|\varphi_2(M)|$, i.e. is in the middle in the right sense.

If $\sigma^{-1}\{d_*\}$ is large enough we easily finish; otherwise for some n we have $d_* \notin (a_{p,n}, b_{p,n})$, so $\varphi_1(M) \wedge \sigma(x) \notin (a_{p,n}, b_{p,n})$ has $\geq \frac{1}{2}(\varphi_2(M))$ elements

(C) why can guarantee that such $\sigma(x)$ does not realize the forbidden new type?

Answer: This is a major point. If $\xi(\varphi_1) > 2\xi(\varphi_2)$ this is easy (as in the case we use $\neg a_* E^4_M b_*$) and if for some $a_1 < a_2$ we have $\xi(\varphi'_2) > \xi(\varphi'_2)$ we let

$$\varphi'_2(x) = (\varphi_2(x) \wedge a_1 \leq x < a_2 \wedge \sigma(x) \notin (F(a_1), F(a_2))$$

and we let

$$\varphi''_2(x) := (\varphi_2 \wedge F(a_1) \leq x < F(a_2))$$

we are done, so assume there are no such a_1, a_2.

We consider two possible reasons for the "failure" of a suggested pair (a_1, a_2). One reason is that maybe the length of the interval $[F(a_1), F(a_2))$ of $\varphi_2(M_1)$ is too large. The second is that it is small enough but $\sigma(-)$ maps the large majority of $\varphi_1(M) \cap [a_1, a_2)$ into $[F(a_1), F(a_2))$. In the second version we can define a version of it's property satisfied by $(a_1, a_2, F(a_1), F(a_2))$. So we have enough intervals of pseudo second kind (pseudo means using the definable version of the property). So dividing $\varphi_1(M)$ to convex subsets of equal (suitable) size (essentially $a_*^{\xi(\varphi_2)}, \zeta \in \mathbb{R}_{>0}$ small enough) by $\langle a_i : i < i(*) \rangle$ we have: for some such interval $[a_i, a_{i+1})$ there are b_1, b_2 as above. For those for which we cannot define $(F(a_1), F(a_2))$ we can define it up to a good approximation. If there are enough, (this may include "pseudo cases" in respect to F) we can replace $\varphi_1(M)$ by $\varphi'_1(M) = \{a_i : i < i(*)\}$ and $\varphi'_2(-)$ defined by the function above.

So $|\varphi'(M)|$ is significantly smaller than $|\varphi_1(M)|$, essentially $\xi(\varphi'_1) = \xi(\varphi_1) - \xi(a_{i+1} - a_i) \sim \xi(\varphi_1) - \xi(\varphi_2) + \zeta, \zeta$ quite small. But we are over-compensating so we decrease $\varphi_2(x)$ to $\varphi'_2(x)$ which is quite closed to $\{F(a_i) : [a_i, a_{i+1})$ is of the pseudo second kind$\}$ and $\xi(\varphi'_2)$ is essentially $\xi(\varphi_2) - \xi(\varphi_2) + \zeta \sim \zeta$. So both lose similarly in the $\xi(-)$ measure but now, if we have arranged the numbers correctly $\xi(\varphi'_2) > 2\xi(\varphi'_2)$, a case we know to solve.

If there are not enough i's of the pseudo second kind, the function essentially inflates the image getting a finite cardinality arithmetic contradiction.

Theorem 3.10. *Assume $\diamondsuit_{\aleph_1}$. If M is a countable model of* PA*, <u>then</u> M has an elementary extension N of cardinality $\aleph_1$ such that $E^5_N = E^3_N$, i.e. is 3*-o.r.

Proof. Without loss of generality M has universe a countable ordinal. As we are assuming $\diamondsuit_{\aleph_1}$, we choose F_α a partial function from α to α for $\alpha < \aleph_1$, i.e. $\bar{F} = \langle F_\alpha, \alpha < \aleph_1\rangle$ such that for every partial function $F : \aleph_1 \to \aleph_1$, for stationarily many countable limit ordinals δ we have $F_\delta = F{\restriction}\delta$.

We now choose $\mathbf{a}_\alpha \in \mathrm{AP}_{\aleph_0}$ by induction on $\alpha < \aleph_1$ such that

(*a*) • $M_{\mathbf{a}_0} = M$
- $\Gamma_{\mathbf{a}_0} = \emptyset$

(*b*) $\langle \mathbf{a}_\beta : \beta \le \alpha\rangle$ is $\le_{\mathrm{AP}}$-increasing continuous

(*c*) if $\alpha = \delta + 1, \delta$ is a countable limit ordinal, $M_{\mathbf{a}_\delta}$ has universe δ and for some a_δ, b_δ the tuple $(\mathbf{a}_\delta, a_\delta, b_\delta, F_\delta)$ satisfies the assumptions of 3.7 on $(\mathbf{a}, a_*, b_*, F)$, they are necessarily unique (see 3.7(A)(c)), <u>then</u> $\mathbf{a}_{\delta+1}$ satisfies its conclusion (for some c_δ).

Why can we carry the induction?

For $\alpha = 0$ recall clause (a).

For $\alpha = 1$, as $\Gamma_{\mathbf{a}_0} = \emptyset$ let $M_{\mathbf{a}_1}$ be a countable model such that $M = M_{\mathbf{a}_0} \prec M_{\mathbf{a}_1}, M \ne M_{\mathbf{a}_1}$ and without loss of generality the universe of $M_{\mathbf{a}_1}$ is a countable ordinal.

Lastly, let $\Gamma_{\mathbf{a}_1} = \emptyset$.

For α a limit ordinal use 3.6(2), i.e. choose the union, this is obvious.

For $\alpha = \beta + 1$, if clause (c) apply use Claim 3.7.

For $\alpha = \beta + 1 > 1$ when clause (c) does not apply, this is easier than 3.7 (or choose (a_*, h_*, F) such that $(\mathbf{a}_\beta, a_*, b_*, F)$ are as in the assumption 3.7, this is possible because $M_{\mathbf{a}_\beta}$ is non-standard, see the case $\alpha = 1$, and note that $a, b \in M_{\mathbf{a}_\beta}\backslash\mathbb{N} \Rightarrow aE^5_{M_{\mathbf{a}}}b$ because $M_{\mathbf{a}}$ is countable; so we can use 3.7).

Having carried the induction let $N = \cup\{M_{\mathbf{a}_\alpha} : \alpha < \aleph_1\}$.

Clearly N is a model of T of cardinality $\aleph_1$. We know that $E^3_N \subseteq E^5_N$ by 2.4. Toward contradiction assume $a_* E^5_N b_*$ but $\neg(a_* E^3_N b_*)$ where $a_*, b_* \in N\backslash M$. Without loss of generality $b_* < a_*$ and let F be an order-isomorphism from $N_{<a_*}$ onto $N_{<b_*}$. So $S = \{\delta : F{\restriction}\delta = F_\delta\}$ is stationary and $E = \{\delta : a_*, b_* \in M_{\mathbf{a}_\delta}, M_{\mathbf{a}_\delta}$ has universe δ and F maps $M^{\mathbf{a}_\delta}_{<a_*}$ onto $M^{\mathbf{a}_\delta}_{<b_*}\}$ is a club of $\aleph_1$.

Choose $\delta \in S \cap E$ and use the choice of $\mathbf{a}_{\delta+1}$, i.e. clause (c) to get a contradiction. $\square_{3.10}$

Theorem 3.11. *Assume $\lambda = \lambda^{<\lambda}$ and $\diamondsuit_S$ where $S = S^{\lambda^+}_\lambda = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$.*

For any model M of PA *there is a λ-saturated model N of* $\mathrm{Th}(M)$ *of cardinality λ^+ such that $E^5_N \subseteq E^4_N$.*

Proof. Similar to 3.10 only the parallel of 3.7 is much easier. $\square_{3.11}$

Conjecture 3.12. 1) Assume λ is strong limit singular of cofinality $\aleph_0$ and $\diamondsuit_S$ where $S = S^{\lambda^+}_{\aleph_0} = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \aleph_0\}$ and $\square_\lambda$. If M is a model of PA, then $\mathrm{Th}(M)$ has a λ-universal model N of cardinality λ^+ which is 3-o.r.
2) Any model M of PA has a 3-o.r. elementary extension.

§ 4. Weaker version of PA

We may wonder what is the weakest version of PA needed in the various results so below we define some variants and then remark when they suffice. But say when we add the function 2^x, we prefer to add to the vocabulary a new function symbol and the relevant axioms (rather than an axiom stating that some definition of it has those properties). So we shall comment what version of PA is needed in the results of §1,§2.

Convention 4.1. A model is a model of PA_{-4} (see below) of vocabulary τ_{PA} if not said otherwise.

Definition 4.2. We define the first order theories PA_ℓ for $\ell \in \{-1, \dots, -4\}$ and let $\mathrm{PA}_\ell^{\mathrm{com}}$ be the set of completions of PA_ℓ.

Let PA_ℓ consist of the following first order sentences in the vocabulary $\{0, 1, <, +, \times\}$ of $\mathbb{N}$:

- (a) for $\ell \le 4$, the obvious axioms of addition and product and order, that is axioms describing the non-negative part of a discrete ordered ring,
- (b) if $\ell \le 3$ we also add division with remainder by any $n \in \mathbb{N}$,
- (c) if $\ell \le 2$ also add division with remainder,
- (d) if $\ell \le 1$, we add a unary function F_2 written 2^x with the obvious axioms for $x \mapsto 2^x$, including $(\forall x)(\exists y)(2^y \le x < 2^{y+1})$.

Claim 4.3. *For M, N are models of PA_{-4}; we still have 1.5(1),(1A),(1B),(3),(4) and 1.7 and 2.1(1),(2),(3),(5) and 1.12(1),(2),(3).*

Claim 4.4. *Claim 1.5(2) holds <u>when</u> M is a model of PA_{-3}.*

Proof. The only difference is why can we choose c_1, c_2 there?

Now if we assume $M \models \mathrm{PA}$ this is obvious, but we are assuming $M \models \mathrm{PA}_{-3}$, still we can divide $b - a$ by $n - 1$ and then get c_1 and $m < n - 1$ such that $b - a = (n - 1) \times c_2 + m$. Let $c_2 = a - c_1$ so $b = a + (n - 1) \times c_2 + m = c_1 + n \times c_2 + m$. We still have to justify using $a - c_2$, i.e. showing $c_2 \le a$, but otherwise $b - a = (n - 1) \times c_2 + m \ge (n - 1) \times a + m$, i.e. $b \ge n \times a + m$, contradiction. $\square_{4.4}$

Theorem 4.5. *If M_1, M_2 are models of PA_{-1} then 1.8 holds, i.e. if M_2 is 2-order-rigid and M_1, M_2 are order-isomorphic <u>then</u> M_1, M_2 are almost $\{<, +\}$-isomorphic.*

Proof. As in 1.8 with the following minor additions:

- in the proof of $(*)_3$ we use $M_2 \models \mathrm{PA}_{-4}$,
- in the proof of $(*)_5$ we use $M_\ell \models \mathrm{PA}_{-2}$,
- in the proof of $(*)_0$ we use $M_2 \models \mathrm{PA}_{-2}$,
- in the proof of $(*)_7$ we use $M_1 \models \mathrm{PA}_{-4}$,
- in the proof of $(*)_8, \circledast_2$ we use $M_\ell \models \mathrm{PA}_{-1}$.

$\square_{4.5}$

Claim 4.6. *1) If $M \models \mathrm{PA}_{-3}$ then 1.12(4) holds.*
2) If $M \models \mathrm{PA}_{-1}$ <u>then</u> 1.12(5) holds.

Proof. Straightforward. $\square_{4.6}$

We may wonder (see Definition 1.3(2)).

Question 4.7. Is there a 2-o.r. model of PA_{-1}?

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