# ON PARTIAL ORDERINGS HAVING PRECALIBRE-$\aleph_1$ AND FRAGMENTS OF MARTIN'S AXIOM

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ABSTRACT. We define a countable antichain condition (ccc) property for partial orderings, weaker than precalibre-$\aleph_1$, and show that Martin's axiom restricted to the class of partial orderings that have the property does not imply Martin's axiom for $\sigma$-linked partial orderings. This yields a new solution to an old question of the first author about the relative strength of Martin's axiom for $\sigma$-centered partial orderings together with the assertion that every Aronszajn tree is special. We also answer a question of J. Steprans and S. Watson (1988) by showing that, by a forcing that preserves cardinals, one can destroy the precalibre-$\aleph_1$ property of a partial ordering while preserving its ccc-ness.

A question asked in [1] is if $MA$($\sigma$-centered) plus "Every Aronszajn tree is special" implies $MA$($\sigma$-linked). The interest in this question originated in the result of Harrington-Shelah [5] showing that if $\aleph_1$ is accessible to reals, i.e., there exists a real number $x$ such that the cardinal $\aleph_1$ in the model $L[x]$ is equal to the real $\aleph_1$, then $MA$ implies that there exists a $\Delta^1_3(x)$ set of real numbers that does not have the Baire property. The hypothesis that $\aleph_1$ is accessible to reals is necessary, for if $\aleph_1$ is inaccessible to reals and $MA$ holds, then $\aleph_1$ is actually weakly-compact in $L$ ([5]), and K. Kunen showed that starting form a weakly compact cardinal one can get a model where $MA$ holds and every projective set of reals has the Baire property. In [1], using Todorčević's $\rho$-functions ([12]), it was shown that $MA$($\sigma$-centered) plus "Every Aronszajn tree is special" is sufficient to produce a $\Delta^1_3(x)$ of real numbers without the Baire property, assuming $\aleph_1 = \aleph_1^{L[x]}$. Thus, it was natural to ask how weak is $MA$($\sigma$-centered) plus "Every Aronszajn tree is special" as compared to the full $MA$, and in particular if it implies $MA$($\sigma$-linked). The answer is negative, as it has been observed by D. Chodounsky and J. Zapletal that a finite-support iteration of $\sigma$-centered posets combined with the forcing that specializes Aronszajn trees has the Y-c.c. property, and therefore does not add random reals (see [2]). In the first part of the paper we give a new and stronger negative answer to the question, namely we show that a fragment of $MA$ that includes $MA$($\sigma$-centered), and even

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$MA$(3-Knaster), and implies "Every Aronszajn tree is special", does not imply $MA(\sigma$-linked). A partial ordering with the precalibre-$\aleph_1$ property plays the key role in the construction of the model.

In the second part of the paper we answer a question of Steprans-Watson [9]. They ask if it is possible to destroy the precalibre-$\aleph_1$ property of a partial ordering, while preserving its ccc-ness, in a forcing extension of the set-theoretic universe $V$ that preserves cardinals. This is a natural question considering that, as shown in [9], on the one hand, assuming $MA$ plus the Covering Lemma, every precalibre-$\aleph_1$ partial ordering has precalibre-$\aleph_1$ in every forcing extension of $V$ that preserves cardinals; and on the other hand the ccc property of a partial ordering having precalibre-$\aleph_1$ can always be destroyed while preserving $\aleph_1$, and consistently even preserving all cardinals.

We answer the Steprans-Watson question positively, and in a very strong sense. Namely, we show that it is consistent, modulo ZFC, that the Continuum Hypothesis holds and there exist a forcing notion $T$ of cardinality $\aleph_1$ that preserves $\aleph_1$ (and therefore it preserves all cardinals, cofinalities, and the cardinal arithmetic), and two precalibre-$\aleph_1$ partial orderings, such that forcing with $T$ preserves their ccc-ness, but it also forces that their product is not ccc and therefore they don't have precalibre-$\aleph_1$.

## 1. Preliminaries

Recall that a partially ordered set (or poset) $\mathbb{P}$ is *ccc* if every antichain of $\mathbb{P}$ is countable; it is *productive-ccc* if the product of $\mathbb{P}$ with any ccc poset is also ccc; it is *Knaster* (or has *property-$\mathcal{K}$*) if every uncountable subset of $\mathbb{P}$ contains an uncountable subset consisting of pairwise compatible elements. More generally, for $k \geq 2$, $\mathbb{P}$ is *$k$-Knaster* if every uncountable subset of $\mathbb{P}$ contains an uncountable subset such that any $k$-many of its elements have a common lower bound. Thus, Knaster is the same as 2-Knaster. $\mathbb{P}$ has *precalibre-$\aleph_1$* if every uncountable subset of $\mathbb{P}$ has an uncountable subset such that any finite set of its elements has a common lower bound; it is *$\sigma$-linked* (or *$\sigma$-2-linked*) if it can be partitioned into countably-many pieces so that each piece is pairwise compatible. More generally, for $k \geq 2$, $\mathbb{P}$ is $\sigma$-$k$-linked if it can be partitioned into countably-many pieces so that any $k$-many elements in the same piece have a common lower bound. Finally, $\mathbb{P}$ is *$\sigma$-centered* if it can be partitioned into countably-many pieces so that any finite number of elements in the same piece have a common lower bound. We have the following implications, for every $k \geq 2$:

$$\sigma\text{-centered} \Rightarrow \sigma\text{-}k\text{-linked} \Rightarrow k\text{-Knaster} \Rightarrow \text{productive-ccc} \Rightarrow \text{ccc},$$

and

$$\sigma\text{-centered} \Rightarrow \text{precalibre-}\aleph_1 \Rightarrow k\text{-Knaster}.$$

These are the only implications that can be proved in ZFC.

For any property $\Gamma$ of posets that implies the ccc, and an infinite cardinal $\kappa$, *Martin's Axiom for $\Gamma$ and for families of $\kappa$-many dense open sets*, denoted by $MA_\kappa(\Gamma)$, asserts: for every $\mathbb{P}$ that satisfies the property $\Gamma$ and every family $\{D_\alpha : \alpha < \kappa\}$ of dense open subsets of $\mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ that is *generic* for the family, that is, $G \cap D_\alpha \neq \varnothing$ for every $\alpha < \kappa$.

When $\kappa = \aleph_1$ we omit the subscript and write $MA(\Gamma)$ for $MA_{\aleph_1}(\Gamma)$. Also, for an infinite cardinal $\theta$, the notation $MA_{<\theta}(\Gamma)$ means: $MA_\kappa(\Gamma)$ for all $\kappa < \theta$. The axiom $MA_{\aleph_0}(\Gamma)$ is provable in ZFC; and it is consistent, modulo ZFC, that the Continuum Hypothesis fails and $MA_{<2^{\aleph_0}}(ccc)$ holds (see [7], or [6]). *Martin's axiom*, denoted by $MA$, is $MA(ccc)$.

Thus, we have the following implications, for every $k \geq 2$:

$$MA_\kappa(ccc) \Rightarrow MA_\kappa(\text{productive-ccc}) \Rightarrow$$

$$\Rightarrow MA_\kappa(k\text{-Knaster}) \Rightarrow MA_\kappa(\sigma\text{-}k\text{-linked}) \Rightarrow MA_\kappa(\sigma\text{-centered}),$$

and

$$MA_\kappa(k\text{-Knaster}) \Rightarrow MA_\kappa(\text{precalibre-}\aleph_1) \Rightarrow MA_\kappa(\sigma\text{-centered}).$$

Again, the arrows cannot be reversed (see [13], [10] for even finer distinctions, and also [11] for Borel examples).

For all the facts mentioned in the rest of the paper without a proof, as well as for all undefined notions and notations, see [6].

## 2. The property $Pr_k$

Let us consider the following property of partial orderings, weaker than the $k$-Knaster property.

**Definition 1.** *For $k \geq 2$, let $Pr_k(\mathbb{Q})$ mean that $\mathbb{Q}$ is a forcing notion such that if $p_\varepsilon \in \mathbb{Q}$, for all $\varepsilon < \aleph_1$, then we can find $\bar{u}$ such that:*

(a) $\bar{u} = \langle u_\xi : \xi < \aleph_1 \rangle$.
(b) *$u_\xi$ is a finite subset of $\aleph_1$.*
(c) *$u_{\xi_0} \cap u_{\xi_1} = \varnothing$, whenever $\xi_0 \neq \xi_1$.*
(d) *If $\xi_0 < \ldots < \xi_{k-1}$, then we can find $\varepsilon_l \in u_{\xi_l}$, for $l < k$, such that $\{p_{\varepsilon_l} : l < k\}$ have a common lower bound.*

Notice that $Pr_k(\mathbb{Q})$ implies that $\mathbb{Q}$ is ccc, and that $Pr_{k+1}(\mathbb{Q})$ implies $Pr_k(\mathbb{Q})$. Also note that if $\mathbb{Q}$ is $k$-Knaster, then $Pr_k(\mathbb{Q})$. For given a subset $\{p_\varepsilon : \varepsilon < \aleph_1\}$ of $\mathbb{Q}$, there exists an uncountable $X \subseteq \aleph_1$ such that $\{p_{\varepsilon_l} : l < k\}$ has a common lower bound, for every $\varepsilon_0 < \ldots < \varepsilon_{k-1}$ in $X$, so we can take $u_\xi$ to be the singleton that contains the $\xi$-th element of $X$. Finally, observe that if $\mathbb{Q}$ has precalibre-$\aleph_1$, then $Pr_k(\mathbb{Q})$ holds for every $k \geq 2$.

Recall that if $T$ is an Aronszajn tree on $\omega_1$, then the forcing that specializes $T$ consists of finite functions $p$ from $\omega_1$ into $\omega$ such that if $\alpha \neq \beta$ are in the domain of $p$ and are comparable in the tree ordering, then $p(\alpha) \neq p(\beta)$. The ordering is the reversed inclusion. It is consistent, modulo ZFC, that the specializing forcing is not productive-ccc, an example being the case when $T$ is a Suslin tree. However, we have the following:

**Lemma 2.** *If $T$ is an Aronszajn tree and $\mathbb{Q} = \mathbb{Q}_T$ is the forcing that specializes $T$ with finite conditions, then $Pr_k(\mathbb{Q})$ holds, for every $k \geq 2$.*

*Proof.* Without loss of generality, $T = (\omega_1, <_T)$. Let $p_\alpha \in \mathbb{Q}$, for $\alpha < \aleph_1$. By a $\Delta$-system argument we may assume that $\{dom(p_\alpha) : \alpha < \aleph_1\}$ forms a $\Delta$-system, with root $r$. Moreover, we may assume that for some fixed $n$, $|dom(p_\alpha) \setminus r| = n$, for all $\alpha < \omega_1$. Let $\langle \alpha_1, \ldots, \alpha_n \rangle$ be an enumeration of $dom(p_\alpha) \setminus r$. We may also assume that if $\alpha < \beta$, then the highest level of $T$

that contains some $\alpha_i$ $(1 \leq i \leq n)$ is strictly lower than the lowest level of $T$ that contains some $\beta_j$ $(1 \leq j \leq n)$.

Fix a uniform ultrafilter $D$ over $\omega_1$. For each $\alpha < \omega_1$ and $1 \leq i, j \leq n$, let

$$D_{\alpha,i,j} := \{\beta > \alpha : \alpha_i <_T \beta_j\}$$

and let

$$D_{\alpha,i,0} := \{\beta > \alpha : \alpha_i \not<_T \beta_j, \text{ all } j\}.$$

For every $\alpha$ and every $i$, there exists $j_{\alpha,i} \leq n$ such that $D_{\alpha,i,j_{\alpha,i}} \in D$. Moreover, for every $1 \leq i \leq n$, there exists $E_i \in D$ such that $j_{\alpha,i}$ is fixed, say with value $j_i$, for all $\alpha \in E_i$. We claim that $j_i = 0$, for all $1 \leq i \leq n$. For suppose $i$ is so that $j_i \neq 0$. Pick $\alpha < \beta < \gamma$ in $E_i \cap D_{\alpha,i,j_i} \cap D_{\beta,i,j_i}$. Then $\alpha_i, \beta_i <_T \gamma_{j_i}$, hence $\alpha_i <_T \beta_i$. This yields an $\omega_1$-chain in $T$, which is impossible. Now let $E := \bigcap_{1 \leq i \leq n} E_i \in D$.

We claim that for every $m$ and every $\alpha$ we can find $u \in [\omega_1 \setminus \alpha]^m$ such that if $\beta < \gamma$ are in $u$, then $\beta_i \not<_T \gamma_j$, for every $1 \leq i, j \leq n$. Indeed, given $m$ and $\alpha$, choose any $\beta^0 \in E \setminus \alpha$. Now given $\beta^0, \ldots, \beta^l$, all in $E$, let $\beta^{l+1} \in E \cap \bigcap_{1 \leq i \leq n} \bigcap_{l' \leq l} D_{\beta^{l'},i,0}$. Then the set $u := \{\beta^0, \ldots, \beta^{m-1}\}$ is as required.

We can now choose $\langle u_\xi : \xi < \aleph_1 \rangle$ pairwise-disjoint, with $|u_\alpha| > k{\cdot}n$, so that if $\xi_1 < \xi_2$, then $sup(u_{\xi_1}) < min(u_{\xi_2})$, and each $u_\xi$ is as above, i.e., if $\beta < \gamma$ are in $u_\xi$, then $\beta_i \not<_T \gamma_j$, for every $1 \leq i, j \leq n$. We claim that $\langle u_\xi : \xi < \aleph_1 \rangle$ is as required. So, suppose $\xi_0 < \ldots \xi_{k-1}$. We choose $\alpha^\ell \in u_{\xi_\ell}$ by downward induction on $\ell \in \{0, \ldots, k-1\}$ so that $\{p_{\alpha^\ell} : \ell < k\}$ has a common lower bound. Let $\alpha^{k-1}$ be any element of $u_{\xi_{k-1}}$. Now suppose $\alpha^{\ell+1}, \ldots, \alpha^{k-1}$ have been already chosen and we shall choose $\alpha^\ell$. We may assume that for each $\beta \in u_{\xi_\ell}$, $p_\beta$ is incompatible with $p_{\alpha^{\ell'}}$, some $\ell' \in \{\ell+1, \ldots, k-1\}$, for otherwise we could take as our $\alpha^\ell$ any $\beta \in u_{\xi_\ell}$ with $p_\beta$ compatible with all $p_{\alpha^{\ell'}}$, $\ell' \in \{\ell+1, \ldots, k-1\}$. Thus, for each $\beta \in u_{\xi_\ell}$ there exist $\ell' \in \{\ell+1, \ldots, k-1\}$ and $1 \leq i, j \leq n$ such that $\beta_i <_T \alpha_j^{\ell'}$. So, since $|u_{\xi_\beta}| > k \cdot n$, there must exist $\beta, \beta' \in u_{\xi_\ell}$ and $\ell'$ such that $\beta_i, \beta_{i'} <_T \alpha_j^{\ell'}$, for some $1 \leq i, i', j \leq n$ with $\beta_i \neq \beta_{i'}$. But this implies that $\beta_i$ and $\beta_{i'}$ are $<_T$-comparable, contradicting our choice of $u_{\xi_\ell}$. $\square$

We show next that the property $Pr_k$ for forcing notions is preserved under iterations with finite support, of any length.

**Lemma 3.** *For any $k \geq 2$, the property $Pr_k$ is preserved under finite-support forcing iterations. That is, if*

$$\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta; \alpha \leq \lambda,\ \beta < \lambda \rangle$$

*is a finite-support iteration of forcing notions such that $Pr_k(\mathbb{P}_0)$ and $\Vdash_{\mathbb{P}_\beta}$ "$Pr_k(\underset{\sim}{\mathbb{Q}}_\beta)$", for every $\beta < \lambda$, then $Pr_k(\mathbb{P}_\lambda)$.*

*Proof.* By induction on $\alpha \leq \lambda$. For $\alpha = 0$ it is trivial. If $\alpha$ is a limit ordinal with $cf(\alpha) \neq \aleph_1$, and $p_\varepsilon \in \mathbb{P}_\alpha$, for all $\varepsilon < \aleph_1$, then either uncountably many $p_\varepsilon$ have the same support (in the case $cf(\alpha) = \omega$) or the support of all $p_\varepsilon$ is bounded by some $\alpha' < \alpha$. In either case $Pr_k(\mathbb{P}_\alpha)$ follows easily from the induction hypothesis.

If $cf(\alpha) = \aleph_1$, then we may use a $\Delta$-system argument, as in the usual proof of the preservation of the ccc.

So, suppose $\alpha = \beta + 1$. Let $p_\varepsilon \in \mathbb{P}_\alpha$, for all $\varepsilon < \aleph_1$. Without loss of generality, we may assume that $\beta \in dom(p_\varepsilon)$, for all $\varepsilon < \aleph_1$.

Since $\mathbb{P}_\beta$ is ccc, there is $q \in \mathbb{P}_\beta$ such that

$$q \Vdash_{\mathbb{P}_\beta} \text{“}|\{\varepsilon : p_\varepsilon \restriction \beta \in \underset{\sim}{G}_\beta\}| = \aleph_1\text{”}.$$

Let $G \subseteq \mathbb{P}_\beta$ be generic over $V$ and with $q \in G$. In $V[G]$ we have that $p_\varepsilon(\beta)[G] \in \underset{\sim}{\mathbb{Q}}_\beta[G]$, and $Pr_k(\underset{\sim}{\mathbb{Q}}_\beta[G])$ holds. So, there is $\langle u^0_\xi : \xi < \aleph_1\rangle$ as in Definition 1 for the sequence $\langle p_\varepsilon(\beta)[G] : p_\varepsilon \restriction \beta \in G\rangle$. So,

$$q \Vdash_{\mathbb{P}_\beta} \text{“}\langle \underset{\sim}{u}^0_\xi : \xi < \aleph_1\rangle \text{ is as in Definition 1 for } \langle p_\varepsilon(\beta) : p_\varepsilon \restriction \beta \in \underset{\sim}{G}_\beta\rangle\text{”}.$$

For each $\xi$, let $(q_\xi, u^1_\xi)$ be such that

$q_\xi \in \mathbb{P}_\beta$ and $q_\xi \le q$.
$q_\xi \Vdash_{\mathbb{P}_\beta}$ “$\underset{\sim}{u}^0_\xi = u^1_\xi$”, so $u^1_\xi$ is finite.
$q_\xi \le p_\varepsilon \restriction \beta$, for every $\varepsilon \in u^1_\xi$. (This can be ensured because if $\varepsilon \in u^1_\xi$, then $q_\xi \Vdash_{\mathbb{P}_\beta}$ “$p_\varepsilon \restriction \beta \in \underset{\sim}{G}_\beta$”, so we may as well take $q_\xi \le p_\varepsilon \restriction \beta$.)

Now apply the induction hypothesis for $\mathbb{P}_\beta$ to obtain $\langle u^2_\zeta : \zeta < \aleph_1\rangle$ as in the definition of $Pr_k$ for the sequence $\langle q_\xi : \xi < \aleph_1\rangle$. We may assume, by refining the sequence if necessary, that $max(u^2_\zeta) < min(u^2_{\zeta'})$ whenever $\zeta < \zeta'$.

Let $u^*_\zeta := \bigcup\{u^1_\xi : \xi \in u^2_\zeta\}$. We claim that $\bar{u}^* = \langle u^*_\zeta : \zeta < \aleph_1\rangle$ is as in the definition, for the sequence $\langle p_\varepsilon : \varepsilon < \aleph_1\rangle$. Clearly, the $u^*_\zeta$ are finite and pairwise-disjoint. Moreover, given $\zeta_0 < \ldots < \zeta_{k-1}$, we can find $\xi_0 \in u^2_{\zeta_0}, \ldots, \xi_{k-1} \in u^2_{\zeta_{k-1}}$ such that in $\mathbb{P}_\beta$ there is a common lower bound $q_*$ to $\{q_{\xi_0}, \ldots, q_{\xi_k}\}$. Since $q_* \le q_{\xi_0}, \ldots, q_{\xi_{k-1}} \le q$, there are some $q_{**} \le q_*$ and $\varepsilon_l \in u^1_{\xi_l}$, for each $l < k$, such that for some $\mathbb{P}_\beta$-name $\underset{\sim}{p}$,

$$q_{**} \Vdash \text{“}\underset{\sim}{p} \le_{\underset{\sim}{\mathbb{Q}}_\beta} p_{\varepsilon_0}(\beta), \ldots, p_{\varepsilon_{k-1}}(\beta)\text{”}.$$

Then the condition $q_{**} * \underset{\sim}{p}$ is a common lower bound for the conditions $p_{\varepsilon_0}, \ldots, p_{\varepsilon_{k-1}}$. $\square$

## 3. On fragments of $MA$

We shall now prove that $MA(Pr_{k+1})$ does not imply $MA(\sigma\text{-}k\text{-linked})$, which yields a negative answer to the first question stated in the Introduction. The following is the main lemma.

**Lemma 4.** *For $k \ge 2$, there is a forcing notion $\mathbb{P}_* = \mathbb{P}^k_*$ and $\mathbb{P}_*$-names $\underset{\sim}{\mathcal{A}}$ and $\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \mathbb{Q}^k_{\underset{\sim}{\mathcal{A}}}$ such that*

(1) $\mathbb{P}_*$ *has precalibre-$\aleph_1$ and is of cardinality $\aleph_1$.*
(2) $\Vdash_{\mathbb{P}_*}$ “$\underset{\sim}{\mathcal{A}} \subseteq [\aleph_1]^{k+1}$”
(3) $\Vdash_{\mathbb{P}_*}$ “$\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \{v \in [\aleph_1]^{<\aleph_0} : [v]^{k+1} \cap \underset{\sim}{\mathcal{A}} = \varnothing\}$, *ordered by $\supseteq$, is $\sigma$-$k$-linked.*”
(4) $\Vdash_{\mathbb{P}_*}$ “$\underset{\sim}{I}_\alpha := \{v \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} : v \not\subseteq \alpha\}$ *is dense, all $\alpha < \aleph_1$.*”

(5) $\Vdash_{\mathbb{P}_*}$ "*If* $v_\alpha \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$ *is such that* $v_\alpha \not\subseteq \alpha$, *for* $\alpha < \aleph_1$; *and* $u_\xi \in [\aleph_1]^{<\aleph_0}$, *for* $\xi < \aleph_1$, *are non-empty and pairwise disjoint, then there exist* $\xi_0 < \ldots < \xi_k$ *such that for every* $\langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u_{\xi_\ell}$ *the set* $\bigcup_{\ell \leq k} v_{\alpha_\ell}$ *does not belong to* $\mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$."

*Proof.* We define $\mathbb{P}_*$ by: $p \in \mathbb{P}_*$ if and only if $p$ has the form $(u, A, h) = (u_p, A_p, h_p)$, where

(*a*) $u \in [\aleph_1]^{<\aleph_0}$,
(*b*) $A \subseteq [u]^{k+1}$, and
(*c*) $h : \wp_p \to \omega$, where $\wp_p := \{v \subseteq u : [v]^{k+1} \cap A = \varnothing\}$, is such that if $w_0, \ldots, w_{k-1} \in \wp_p$ and $h$ is constant on $\{w_0, \ldots, w_{k-1}\}$, then $w_0 \cup \ldots \cup w_{k-1} \in \wp_p$.

The order is given by: $p \leq q$ if and only if $u_q \subseteq u_p$, $A_q = A_p \cap [u_q]^{k+1}$, and $h_q \subseteq h_p$ (hence $\wp_q = \wp_p \cap \mathcal{P}(u_q)$ and $h_p \restriction \wp_q = h_q$).

(1): Clearly, $\mathbb{P}_*$ has cardinality $\aleph_1$, so let us show that it has precalibre-$\aleph_1$. Given $\{q_\xi = (u_\xi, A_\xi, h_\xi) : \xi < \aleph_1\} \subseteq \mathbb{P}_*$, and writing $\wp_\xi$ instead of the more cumbersome $\wp_{q_\xi}$, we can find an uncountable $W \subseteq \aleph_1$ such that:

(i) The set $\{u_\xi : \xi \in W\}$ forms a $\Delta$-system with heart $u_*$.
(ii) The sets $[u_*]^{k+1} \cap A_\xi$, for $\xi \in W$, are all the same. Hence the sets $\wp_\xi \cap \mathcal{P}(u_*)$, for $\xi \in W$, are also all the same.
(iii) The functions $h_\xi \restriction (\wp_\xi \cap \mathcal{P}(u_*))$, for $\xi \in W$, are all the same.
(iv) The ranges of $h_\xi$, for $\xi \in W$, are all the same, say $R$. So, $R$ is finite.
(v) For each $i \in R$, the sets $\{w \cap u_* : h_\xi(w) = i\}$, for $\xi \in W$, are the same.

We will show that every finite subset of $\{q_\xi : \xi \in W\}$ has a common lower bound. Given $\xi_0, \ldots, \xi_m \in W$, let $q = (u_q, A_q, h_q)$ be such that

- $u_q = \bigcup_{\ell \leq m} u_{\xi_\ell}$
- $A_q = \bigcup_{\ell \leq m} A_{\xi_\ell}$. Note that this implies that the $\wp_{\xi_\ell}$ are contained in $\wp_q = \{v \subseteq u_q : [v]^{k+1} \cap A_q = \varnothing\}$. Indeed, if, say, $w \in \wp_{\xi_\ell}$, then $[w]^{k+1} \cap A_{\xi_\ell} = \varnothing$, and we claim that also $[w]^{k+1} \cap A_{\xi_j} = \varnothing$, for $j \leq m$. For if $v \in [w]^{k+1} \cap A_{\xi_j}$, with $j \neq \ell$, then $v \subseteq u_*$, and therefore $v \in [u_*]^{k+1} \cap A_{\xi_j} = [u_*]^{k+1} \cap A_{\xi_\ell}$. Hence, $v \in [w]^{k+1} \cap A_{\xi_\ell}$, which is impossible because $[w]^{k+1} \cap A_{\xi_\ell}$ is empty.
- $h_q : \wp_q \to \omega$ is such that $h_q(v) = h_{\xi_\ell}(v)$ for all $v \in \wp_{\xi_\ell}$, and the $h_q(v)$ are all distinct and greater than $sup\{h_q(v) : v \in \bigcup_{\ell \leq m} \wp_{\xi_\ell}\}$, for $v \notin \bigcup_{\ell \leq m} \wp_{\xi_\ell}$. Notice that $h_q$ is well-defined because the restrictions $h_{\xi_\ell} \restriction (\wp_{\xi_\ell} \cap \mathcal{P}(u_*))$, for $\ell \leq m$, are all the same.

We claim that $q \in \mathbb{P}_*$. For this, we only need to show that if $\{w_0, \ldots, w_{k-1}\} \subseteq \wp_q$ and $h_q$ is constant on $\{w_0, \ldots, w_{k-1}\}$, then $[\bigcup_{j<k} w_j]^{k+1} \cap A_q = \varnothing$. So fix a set $\{w_0, \ldots, w_{k-1}\} \subseteq \wp_q$ and suppose $h_q$ is constant on it, say with constant value $i$. By definition of $h_q$ we must have $\{w_0, \ldots, w_{k-1}\} \subseteq \bigcup_{\ell \leq m} \wp_{\xi_\ell}$. Now suppose, towards a contradiction, that $v \in [\bigcup_{j<k} w_j]^{k+1} \cap A_{\xi_\ell}$, some $\ell \leq m$. Let $s = \{w_j : j < k\} \cap \wp_{\xi_\ell}$, and let $t = \{w_j : j < k\} \setminus s$. Thus, $v \subseteq \bigcup s \cup (\bigcup t \cap u_*)$, for if $\alpha \in v \setminus \bigcup s$, then $\alpha \in \bigcup t$ and $\alpha \in \bigcup \wp_{\xi_{\ell'}}$, for some $\ell' \neq \ell$, hence $\alpha \in u_\xi \cap u_{\xi'} = u_*$.

By (v),

$$\{w \cap u_* : h_{\xi_\ell}(w) = i\} = \{w \cap u_* : h_{\xi_{\ell'}}(w) = i\}$$

for every $\ell' \leq m$. So, for every $w_j \in t$, there exists $w'_j \in \wp_{\xi_\ell}$ such that $w_j \cap u_* = w'_j \cap u_*$ and $h_{\xi_\ell}(w'_j) = i$. Let $t' = s \cup \{w'_j : w_j \in t\}$. Note that $t' \subseteq \wp_{\xi_\ell}$ and $t' \subseteq \{w : h_{\xi_\ell}(w) = i\}$. So,

$$v \subseteq \bigcup t' \subseteq \bigcup \{w : h_{\xi_\ell}(w) = i\}.$$

Thus, $v \in [\bigcup\{w : h_{\xi_\ell}(w) = i\}]^{k+1} \cap A_{\xi_\ell}$. But this is impossible because $\bigcup\{w : h_{\xi_\ell}(w) = i\} \in \wp_{\xi_\ell}$ (since $h_{\xi_\ell}$ satisfies property (c) above) and therefore

$$[\bigcup\{w : h_{\xi_\ell}(w) = i\}]^{k+1} \cap A_{\xi_\ell} = \varnothing.$$

Now one can easily check that $q \leq q_{\xi_0}, \ldots, q_{\xi_m}$. And this shows that the set $\{q_\xi : \xi \in W\}$ is finite-wise compatible.

(2): Let

$$\underset{\sim}{\mathcal{A}} = \{(\check{v}, p) : v \in A_p,\ p \in \mathbb{P}_*\}.$$

Thus, $\underset{\sim}{\mathcal{A}}$ is a name for the set $\bigcup\{A_p : p \in G\}$, where $G$ is the $\mathbb{P}_*$-generic filter. Clearly, (2) holds.

(3): Let

$$\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \{(\check{v}, p) : v \in \wp_p,\ p \in \mathbb{P}_*\}.$$

Thus, $\mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$ is a name for the set $\bigcup\{\wp_p : p \in G\}$, where $G$ is the $\mathbb{P}_*$-generic filter. Clearly, $\Vdash_{\mathbb{P}_*}$ "$\mathbb{Q}_{\underset{\sim}{\mathcal{A}}} = \{v \in [\aleph_1]^{<\aleph_0} : [v]^{k+1} \cap \underset{\sim}{\mathcal{A}} = \varnothing\}$". Moreover, if $G$ is $\mathbb{P}_*$-generic over $V$, then, by (c), the function $\bigcup\{h_p : p \in G\}$ witnesses that the interpretation $i_G(\mathbb{Q}_{\underset{\sim}{\mathcal{A}}})$, ordered by $\supseteq$, is $\sigma$-$k$-linked.

(4): Clear.

(5): Suppose that $p \in \mathbb{P}_*$ forces $\dot{v}_\alpha \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}}$ is such that $\dot{v}_\alpha \not\subseteq \alpha$, all $\alpha < \aleph_1$; and it also forces $\dot{u}_\xi \in [\aleph_1]^{<\aleph_0}$, all $\xi < \aleph_1$, are non-empty and pairwise disjoint.

For each $\xi < \aleph_1$, let $q_\xi = (u_\xi, A_\xi, h_\xi) \leq p$ and let $u^*_\xi \in [\aleph_1]^{<\aleph_0}$ and $\bar{v}^*_\xi = \langle v^*_{\xi,\alpha} : \alpha \in u^*_\xi\rangle$, with $v^*_{\xi,\alpha} \in [\aleph_1]^{<\aleph_0}$, be such that

$$q_\xi \Vdash_{\mathbb{P}_*} \text{“}\dot{u}_\xi = u^*_\xi \text{ and } \dot{v}_\alpha = v^*_{\xi,\alpha}, \text{ for } \alpha \in u^*_\xi.\text{”}$$

We may assume, by extending $q_\xi$ if necessary, that $u^*_\xi \cup \bigcup_{\alpha\in u^*_\xi} v^*_{\xi,\alpha} \subseteq u_\xi$.

As in (1), we can find an uncountable $W \subseteq \aleph_1$ such that (i)-(v) hold for the set of conditions $\{q_\xi : \xi \in W\}$. Hence $\{q_\xi : \xi \in W\}$ is pairwise compatible (in fact, finite-wise compatible), from which it follows that the set $\{u^*_\xi : \xi \in W\}$ is pairwise disjoint. Now choose $\xi_0 < \ldots < \xi_k$ from $W$ so that

- The heart $u_*$ of the $\Delta$-system $\{u_\xi : \xi \in W\}$ is an initial segment of $u_{\xi_\ell}$, all $\ell \leq k$,
- $\sup(u_{\xi_\ell}) < \inf(u_{\xi_{\ell+1}} \setminus u_*)$, for all $\ell < k$, and
- $u^*_{\xi_\ell} \subseteq u_{\xi_\ell} \setminus u_*$, for all $\ell \leq k$.

For each $\sigma = \langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u^*_{\xi_\ell}$, pick $w_\sigma \in [\bigcup_{\ell \leq k} v^*_{\xi_\ell, \alpha_\ell}]^{k+1}$ such that $|w_\sigma \cap (v^*_{\xi_\ell,\alpha_\ell} \setminus \alpha_\ell)| = 1$, for all $\ell \leq k$. This is possible because $v^*_{\xi_\ell,\alpha_\ell} \not\subseteq \alpha_\ell$.

**Claim 5.** *$w_\sigma \not\subseteq u_{\xi_\ell}$, hence $w_\sigma \notin A_{\xi_\ell}$, for all $\sigma \in \prod_{\ell \leq k} u^*_{\xi_\ell}$ and all $\ell \leq k$.*

*Proof of Claim.* Fix $\sigma = \langle \alpha_\ell : \ell \leq k \rangle$ and $\ell \leq k$, and suppose, for a contradiction, that $w_\sigma \subseteq u_{\xi_\ell}$. Then $w_\sigma \subseteq u_{\xi_\ell} \setminus u_*$. If $\ell < k$, then since $\sup(u_{\xi_\ell}) < \inf(u_{\xi_{\ell+1}} \setminus u_*) \leq \inf(u^*_{\xi_{\ell+1}}) \leq \alpha_{\ell+1}$, we would have $w_\sigma \setminus \alpha_{\ell+1} = \varnothing$, which contradicts our choice of $w_\sigma$. But if $\ell = k$, then since $\sup(v^*_{\xi_{\ell-1},\alpha_{\ell-1}}) \leq \sup(u_{\xi_{\ell-1}}) < \inf(u_{\xi_\ell} \setminus u_*)$, we would have $w_\sigma \cap v^*_{\xi_{\ell-1},\alpha_{\ell-1}} = \varnothing$, which contradicts again our choice of $w_\sigma$. □

Now define $q = (u_q, A_q, h_q)$ as follows:

- $u_q = \bigcup_{\ell \leq k} u_{\xi_\ell}$
- $A_q = (\bigcup_{\ell \leq k} A_{\xi_\ell}) \cup \{w_\sigma : \sigma \in \prod_{\ell \leq k} u^*_{\xi_\ell}\}$. Note that since $w_\sigma \not\subseteq u_{\xi_\ell}$ (Claim 5), we have that $w_\sigma \notin \wp_{\xi_\ell}$, for all $\sigma \in \prod_{\ell \leq k} u^*_{\xi_\ell}$ and $\ell \leq k$. Hence, $\wp_{\xi_\ell} \subseteq \wp_q$, all $\ell \leq k$.
- $h_q : \wp_q \to \omega$ is such that $h_q(v) = h_{\xi_\ell}(v)$ for $v \in \wp_{\xi_\ell}$, for all $\ell \leq k$, and the $h_q(v)$ are all distinct and greater than $sup\{h_q(v) : v \in \bigcup_{\ell \leq k} \wp_{\xi_\ell}\}$, for $v \notin \bigcup_{\ell \leq k} \wp_{\xi_\ell}$.

As in (1), we can now check that $q \in \mathbb{P}_*$. Moreover, by Claim 5, $A_{\xi_\ell} = A_q \cap [u_{\xi_\ell}]^{k+1}$. Hence, $q \leq q_{\xi_\ell}$, all $\ell \leq k$, and so

$$q \Vdash_{\mathbb{P}_*} \text{“}\dot{u}_{\xi_\ell} = u^*_{\xi_\ell} \text{ and } \dot{v}_\alpha = v^*_{\xi_\ell,\alpha}, \text{ for } \alpha \in u^*_{\xi_\ell}.\text{”}$$

And since $w_\sigma \in [\bigcup_{\ell \leq k} v^*_{\alpha_\ell}]^{k+1} \cap A_q$, for every $\sigma \in \prod_{\ell \leq k} u^*_{\xi_\ell}$, we have that

$$q \Vdash_{\mathbb{P}_*} \text{“}\bigcup_{\ell \leq k} \dot{v}_{\alpha_\ell} \notin \mathbb{Q}_{\underset{\sim}{\mathcal{A}}}, \text{ for all } \langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} \dot{u}_{\xi_\ell}.\text{”}$$

□

**Lemma 6.** *Let $k \geq 2$ and let $\mathbb{P}_*$ be as in Lemma 4. Suppose $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_*$-name for a forcing notion that satisfies $Pr_{k+1}$. Then,*

$$\Vdash_{\mathbb{P}_* * \underset{\sim}{\mathbb{Q}}} \text{“\textit{There is no directed} } G \subseteq \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} \text{ \textit{such that} } \underset{\sim}{I}_\alpha \cap G \neq \varnothing, \text{ \textit{all} } \alpha < \aleph_1.\text{”}$$

*where $\underset{\sim}{I}_\alpha$ is a name for the dense open set $\{v \in \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} : v \not\subseteq \alpha\}$.*

*Proof.* Suppose, for a contradiction, that $p * \dot{q} \in \mathbb{P}_* * \underset{\sim}{\mathbb{Q}}$ and

$$p * \dot{q} \Vdash_{\mathbb{P}_* * \underset{\sim}{\mathbb{Q}}} \text{“There exists } G \subseteq \mathbb{Q}_{\underset{\sim}{\mathcal{A}}} \text{ directed, with } \underset{\sim}{I}_\alpha \cap G \neq \varnothing, \text{ all } \alpha < \aleph_1.\text{”}$$

Suppose $G_0 \subseteq \mathbb{P}_*$ is a filter generic over $V$, with $p \in G_0$. So, in $V[G_0]$, letting $q = i_{G_0}(\dot{q})$ and $\mathbb{Q} = i_{G_0}(\underset{\sim}{\mathbb{Q}})$, we have that for some $\mathbb{Q}$-name $\underset{\sim}{G}$,

$$q \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{G} \subseteq \mathbb{Q}_{\mathcal{A}} \text{ is directed and } I_\alpha \cap \underset{\sim}{G} \neq \varnothing, \text{ all } \alpha < \aleph_1.\text{”}$$

For each $\alpha < \aleph_1$, let $q_\alpha \leq q$, and let $v_\alpha \in [\aleph_1]^{<\aleph_0}$ be such that

$$q_\alpha \Vdash_{\mathbb{Q}} \text{“}\check{v}_\alpha \in I_\alpha \cap \underset{\sim}{G}\text{”}.$$

Thus, $v_\alpha \not\subseteq \alpha$, for all $\alpha < \aleph_1$.

Since $\mathbb{Q}$ satisfies $Pr_{k+1}$, there exists $\bar{u} = \langle u_\xi : \xi < \aleph_1 \rangle$ such that

(a) $u_\xi$ is a finite subset of $\aleph_1$, all $\xi < \aleph_1$,

(b) $u_{\xi_0} \cap u_{\xi_1} = \varnothing$ whenever $\xi_0 \neq \xi_1$, and
(c) if $\xi_0 < \ldots < \xi_k$, then we can find $\alpha_\ell \in u_{\xi_\ell}$, for $\ell \leq k$, such that $\{q_{\alpha_\ell} : \ell \leq k\}$ have a common lower bound.

By Lemma 4, we can find $\xi_0 < \ldots < \xi_k$ such that for every $\langle \alpha_\ell : \ell \leq k \rangle \in \prod_{\ell \leq k} u_{\xi_\ell}$ the set $\bigcup_{\ell \leq k} v_{\alpha_\ell}$ does not belong to $\mathbb{Q}_{\mathcal{A}}$.

By $(c)$, let $\alpha_\ell \in u_{\xi_\ell}$, for $\ell \leq k$, be such that $\{q_{\alpha_\ell} : \ell \leq k\}$ have a common lower bound, call it $r$. Then $r$ forces that $\{\check{v}_{\alpha_\ell} : \ell \leq k\} \subseteq \underset{\sim}{G}$. And since $r$ forces that $\underset{\sim}{G}$ is directed, it also forces that $\bigcup_{\ell \leq k} v_{\alpha_\ell} \in \mathbb{Q}_{\mathcal{A}}$. A contradiction. $\square$

All elements are now in place to prove the main result of this section.

**Theorem 7.** *Let $k \geq 2$. Assume $\lambda = \lambda^{<\theta}$, where $\theta = cf(\theta) > \aleph_1$. Then there is a finite-support iteration*

$$\bar{\mathbb{P}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta; \alpha \leq \lambda, \beta < \lambda \rangle$$

*where*

(1) *$\mathbb{P}_0$ is the forcing $\mathbb{P}_*$ from Lemma 4.*
(2) *$\Vdash_{\mathbb{P}_\beta}$ "$Pr_{k+1}(\underset{\sim}{\mathbb{Q}}_\beta)$", for every $0 < \beta < \lambda$.*
(3) *In $V^{\mathbb{P}_\lambda}$ the axiom $MA_{<\theta}(Pr_{k+1})$ holds, hence in particular (Lemma 2) every Aronszajn tree on $\omega_1$ is special.*
(4) *$\underset{\sim}{\mathbb{Q}}_{\mathcal{A}}$ witnesses that $MA(\sigma$-$k$-linked) fails in $V^{\mathbb{P}_\lambda}$.*

*Proof.* To obtain (3), we proceed in the standard way as in all iterations forcing (some fragment of) $MA$, that is, we iterate all posets with the $Pr_{k+1}$ property and having cardinality $< \theta$, which are given by some fixed bookkeeping function (see [6] or [7] for details).

Since after forcing with $\mathbb{P}_0$ the rest of the iteration $\bar{\mathbb{P}}$ has the property $Pr_{k+1}$ (Lemma 3), (4) follows immediately from Lemma 6. $\square$

**Corollary 8.** *For every $k \geq 2$, ZFC plus $MA(Pr_{k+1})$ does not imply $MA(\sigma$-$k$-linked).*

Thus, since $MA(Pr_{k+1})$ implies both $MA(\sigma$-centerd) and "Every Aronszajn tree is special", the corollary answers in the negative and in a strong way the question from [1]: Does $MA(\sigma$-centered) plus "Every Aronszajn tree is special" imply $MA(\sigma$-linked)?

## 4. On destroying precalibre-$\aleph_1$ while preserving the ccc

We turn now to the second question stated in the Introduction (Steprans-Watson [9]): Is it consistent that there exists a precalibre-$\aleph_1$ poset which is ccc but does not have precalibre-$\aleph_1$ in some forcing extension that preserves cardinals?

Note that the forcing extension cannot be ccc, since ccc forcing preserves the precalibre-$\aleph_1$ property. Also, as shown in [9], assuming $MA$ plus the Covering Lemma, every forcing that preserves cardinals also preserves the precalibre-$\aleph_1$ property. Moreover, the examples provided in [9] of cardinal-preserving forcing notions that destroy the precalibre-$\aleph_1$ they do so by actually destroying the ccc property.

A positive answer to Question 1 is provided by the following theorem. But first, let us recall a strong form of Jensen's diamond principle, *diamond-star relativized to a stationary set* $S$, which is also due to Jensen. For $S$ a stationary subset of $\omega_1$, let

$\diamondsuit^*_S$: There exists a sequence $\langle \mathcal{S}_\alpha : \alpha \in S\rangle$, where $\mathcal{S}_\alpha$ is a countable set of subsets of $\alpha$, such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$, for every $\alpha \in C \cap S$.

The principle $\diamondsuit^*_S$ holds in the constructible universe $L$, for every stationary $S \subseteq \omega_1$ (see [3], 3.5, for a proof in the case $S = \omega_1$, which can be easily adapted to any stationary $S$). Also, $\diamondsuit^*_S$ can be forced by a $\sigma$-closed forcing notion (see [7], Chapter VII, Exercises H18 and H20, where it is shown how to force the even stronger form of diamond known as $\diamondsuit^+_S$).

**Theorem 9.** *It is consistent, modulo ZFC, that the CH holds and there exist*

1. *A forcing notion $T$ of cardinality $\aleph_1$ that preserves cardinals.*
2. *Two posets $\mathbb{P}_0$ and $\mathbb{P}_1$ of cardinality $\aleph_1$ that have precalibre-$\aleph_1$ and such that*

$$\Vdash_T \text{“}\mathbb{P}_0, \mathbb{P}_1 \textit{ are ccc, but } \mathbb{P}_0 \times \mathbb{P}_1 \textit{ is not ccc.”}$$

*Hence* $\Vdash_T$ “$\mathbb{P}_0$ *and* $\mathbb{P}_1$ *don't have precalibre-*$\aleph_1$”.

*Proof.* Let $\{S_1, S_2\}$ be a partition of $\Omega := \{\delta < \omega_1 : \delta \text{ a limit}\}$ into two stationary sets. By a preliminary forcing, we may assume that $\diamondsuit^*_{S_1}$ holds. So, there exists $\langle \mathcal{S}_\alpha : \alpha \in S_1\rangle$, where $\mathcal{S}_\alpha$ is a countable set of subsets of $\alpha$, such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$, for every $\alpha \in C \cap S_1$. In particular, the CH holds. Using $\diamondsuit^*_{S_1}$, we can build an $S_1$-*oracle*, i.e., an $\subset$-increasing sequence $\bar{M} = \langle M_\delta : \delta \in S_1\rangle$, with $M_\delta$ countable and transitive, $\delta \in M_\delta$, $M_\delta \models$ “$ZFC^- + \delta$ is countable”, and such that for every $A \subseteq \omega_1$ there is a club $C_A \subseteq \omega_1$ such that $A \cap \delta \in M_\delta$, for every $\delta \in C_A \cap S_1$. (For the latter, one simply needs to require that $\mathcal{S}_\delta \subseteq M_\delta$, for all $\delta \in S_1$.) Moreover, we can build $\bar{M}$ so that it has the following additional property:

(∗) For every regular uncountable cardinal $\chi$ and a well ordering $<^*_\chi$ of $H(\chi)$, the set of all (universes of) countable $N \preceq \langle H(\chi), \in, <^*_\chi\rangle$ such that the Mostowski collapse of $N$ belongs to $M_\delta$, where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$.

The property (∗) will be needed to prove that the tree partial ordering $T$ (defined below) has many branches, and also to prove that the product partial ordering $\mathbb{Q} \times T$ (defined below) is $S_1$-proper (Claim 10), and so it does not collapse $\aleph_1$.

To ensure (∗), take a big-enough regular cardinal $\lambda$ and define the sequence $\bar{M}$ so that, for every $\delta \in S_1$, $M_\delta$ is the Mostowski collapse of a countable elementary substructure $X$ of $H(\lambda)$ that contains $\bar{M} \restriction \delta$, all ordinals $\leq \delta$, and all elements of $\mathcal{S}_\delta$. To see that (∗) holds, fix a regular uncountable cardinal $\chi$, a well ordering $<^*_\chi$ of $H(\chi)$, and a club $E \subseteq [H(\chi)]^{\aleph_0}$. Let $\bar{N} = \langle N_\alpha : \alpha < \aleph_1\rangle$ be an $\subset$-increasing and $\in$-increasing continuous chain of elementary substructures of $\langle H(\chi), \in, <^*_\chi\rangle$ with the universe of $N_\alpha$ in $E$, for all $\alpha < \aleph_1$. We shall find $\delta \in S_1$ such that the transitive collapse of $N_\delta$ belongs to $M_\delta$, where $\delta = N_\delta \cap \omega_1$.

Fix a bijection $h : \aleph_1 \to \bigcup_{\alpha<\aleph_1} N_\alpha$, and let $\Gamma : \aleph_1 \times \aleph_1 \to \aleph_1$ be the standard pairing function (cf. [6], 3). Observe that the set

$$D := \{\delta < \aleph_1 : \delta \text{ is closed under } \Gamma \text{ and } h \text{ maps } \delta \text{ onto } N_\delta\}$$

is a club. Now let

$X_1 := \{\Gamma(i,j) : h(i) \in h(j)\}$
$X_2 := \{\Gamma(\alpha,i) : h(i) \in N_\alpha\}$
$X_3 := \{\Gamma(i,j) : h(i) <^*_\chi h(j)\}$
$X := \{3j + i : j \in X_i \text{ and } i \in \{1,2,3\}\}$.

The set $S_1' := \{\delta \in S_1 : X \cap \delta \in M_\delta\}$ is stationary. Thus, since the set $C := \{\delta < \aleph_1 : \delta = N_\delta \cap \omega_1\}$ is a club, we can pick $\delta \in C \cap D \cap S_1'$. Since $\delta \in D$, the structure

$$Y := \langle X_2 \cap \delta, \{\langle i,j\rangle : \Gamma(i,j) \in X_1 \cap \delta\}, \{\langle i,j\rangle : \Gamma(i,j) \in X_3 \cap \delta\}\rangle$$

is isomorphic to $N_\delta$, and therefore $Y$ and $N_\delta$ have the same transitive collapse. And $Y$ belongs to $M_\delta$, because $\delta \in S_1'$. Hence, since $M_\delta \models ZFC^-$, the transitive collapse of $Y$ belongs to $M_\delta$. Finally, since $\delta \in C$, $\delta = N_\delta \cap \omega_1$.

We shall define now the forcing $T$. Let us write $\aleph_1^{<\aleph_1}$ for the set of all countable sequences of countable ordinals. Let

$T := \{\eta \in \aleph_1^{<\aleph_1} : \mathit{Range}(\eta) \subset S_1$, $\eta$ is increasing and continuous, of successor length, and if $\varepsilon < lh(\eta)$, then $\eta \restriction \varepsilon \in M_{\eta(\varepsilon)}\}$.

Let $\leq_T$ be the partial order on $T$ given by end-extension. Thus, $(T, \leq_T)$ is a tree. Note that, since $\delta \in M_\delta$ for every $\delta \in S_1$, if $\eta \in T$, then $\eta \in M_{supRange(\eta)}$. Also notice that if $\eta \in T$, then $\eta^\frown\langle\delta\rangle \in T$, for every $\delta \in S_1$ greater than $supRange(\eta)$. In particular, every node of $T$ of finite length has $\aleph_1$-many extensions of any bigger finite length. Now suppose $\alpha < \omega_1$ is a limit, and suppose, inductively, that for every successor $\beta < \alpha$, every node of $T$ of length $\beta$ has $\aleph_1$-many extensions of every higher successor length below $\alpha$. We claim that every $\eta \in T$ of length less than $\alpha$ has $\aleph_1$-many extensions in $T$ of length $\alpha + 1$ (and in fact, the set of their suprema is stationary). For every $\delta < \omega_1$, let $T_\delta := \{\eta \in T : supRange(\eta) < \delta\}$. Notice that $T_\delta$ is countable: otherwise, uncountably-many $\eta \in T_\delta$ would have the same $supRange(\eta)$, and therefore they would all belong to the model $M_{supRange(\eta)}$, which is impossible because it is countable. Now fix a node $\eta \in T$ of length less than $\alpha$, and let $B := \{b_\gamma : \gamma < \omega_1\}$ be an enumeration of all the *branches* (i.e., linearly-ordered subsets of $T$ closed under predecessors) $b$ of $T$ that contain $\eta$ and have length $\alpha$ (i.e., $\bigcup\{dom(\eta') : \eta' \in b\} = \alpha$). For a club $C$ of $\delta$ the set $\{b_\gamma : \gamma < \delta\}$ belongs to $M_\delta$. We shall next build a sequence $B^* := \langle b^*_\xi : \xi < \omega_1\rangle$ of branches from $B$ so that the set $supB^* := \langle supRange(\bigcup b^*_\xi) : \xi < \omega_1\rangle$ is the increasing enumeration of a club. To this end, start by fixing an increasing sequence $\langle \alpha_n : n < \omega\rangle$ of successor ordinals converging to $\alpha$, with $\alpha_0$ greater than the length of $\eta$. Then let $b^*_0 := b_0$. Given $b^*_\xi$, let $\gamma$ be the least ordinal such that $\bigcup b_\gamma(\alpha_0) > supRange(\bigcup b^*_\xi)$, and let $b^*_{\xi+1} := b_\gamma$. Finally, given $b^*_\xi$ for all $\xi < \delta$, where $\delta < \omega_1$ is a limit ordinal, pick an increasing sequence $\langle \xi_n : n < \omega\rangle$ converging to $\delta$.

By construction, the sequence $\langle supRange(\bigcup b^*_{\xi_n}) : n < \omega\rangle$ is increasing. Now let $f : \alpha \to \aleph_1$ be such that $f \restriction [0, \alpha_0] = \bigcup b^*_{\xi_0} \restriction [0, \alpha_0]$, and $f \restriction (\alpha_n, \alpha_{n+1}] = \bigcup b^*_{\xi_{n+1}} \restriction (\alpha_n, \alpha_{n+1}]$, for all $n < \omega$. Then set $b^*_\delta := \{f \restriction \beta : \beta < \alpha$ is a successor$\}$. One can easily check that $b^*_\delta$ is a branch of $T$ of length $\alpha$ with $supRange(\bigcup b^*_\delta) = sup\{supRange(\bigcup b^*_\xi) : \xi < \zeta\}$. Finally, notice that if $\delta \in S_1 \cap C$ is greater than $\alpha$ and belongs to the club enumerated by $supB^*$, then since $M_\delta \models$ "$\delta$ is countable", we can pick the sequences $\langle \alpha_n : n < \omega\rangle$ and $\langle \xi_n : n < \omega\rangle$ in $M_\delta$. Then the sequence $\langle b^*_{\xi_n} : n < \omega\rangle$ belongs to $M_\delta$, and therefore $(\bigcup b^*_\delta)^\frown\langle\delta\rangle \in T$.

By $(*)$ the set of all countable $N \preceq \langle H(\aleph_2), \in, <^*_{\aleph_2}\rangle$ that contain $B^*$ and $\langle \alpha_n : n < \omega\rangle$, with $\alpha \subseteq N$, and such that the Mostowski collapse of $N$ belongs to $M_\delta$, where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$. So, since the set $Lim(supB^*)$ of limit points of $supB^*$ is a club, there is such an $N$ with $\delta := N \cap \omega_1 \in Lim(supB^*)$. If $\bar{N}$ is the transitive collapse of $N$, we have that $B^* \restriction \delta \in \bar{N} \in M_\delta$, and so in $M_\delta$ we can build, as above, the branch $b^*_\delta$. Therefore, since $\delta = supRange(\bigcup b^*_\delta)$, we have that $\bigcup b^*_\delta \cup \{\langle \alpha, \delta\rangle\} \in T$ and extends $\eta$. We have thus shown that $\eta$ has $\aleph_1$-many extensions in $T$ of length $\alpha + 1$. Even more, the set $\{supRange(\bigcup b) : b$ is a branch of length $\alpha + 1$ that extends $\eta\}$ is stationary.

Note however that since the complement of $S_1$ is stationary, $T$ has no branch of length $\omega_1$, because the range of such a branch would be a club contained in $S_1$. But since every $\eta \in T$ has extensions of length $\alpha + 1$, for every $\alpha$ greater than or equal to the length of $\eta$, forcing with $(T, \geq_T)$ yields a branch of $T$ of length $\omega_1$.

In order to obtain the forcing notions $\mathbb{P}_0$ and $\mathbb{P}_1$ claimed by the theorem, we need first to force with the forcing $\mathbb{Q}$, which we define as follows. For $u$ a subset of $T$, let $[u]^2_T$ be the set of all pairs $\{\eta, \nu\} \subseteq u$ such that $\eta \neq \nu$ and $\eta$ and $\nu$ are $<_T$-comparable. Let

$$\mathbb{Q} := \{p : [u]^2_T \to \{0, 1\} : u \text{ is a finite subset of } T\},$$

ordered by reversed inclusion.

It is easily seen that $\mathbb{Q}$ is ccc, and it has cardinality $\aleph_1$, so forcing with $\mathbb{Q}$ does not collapse cardinals, does not change cofinalities, and preserves cardinal arithmetic. (In fact, $\mathbb{Q}$ is equivalent, as a forcing notion, to the poset for adding $\aleph_1$ Cohen reals, which is $\sigma$-centered, but we shall not make use of this fact.)

Notice that if $G \subseteq \mathbb{Q}$ is a generic filter over $V$, then $\bigcup G : [T]^2_T \to \{0, 1\}$.

Recall that, for $S \subseteq \aleph_1$ stationary, a forcing notion $\mathbb{P}$ is called *$S$-proper* if for all (some) large-enough regular cardinals $\chi$ and all (stationary-many) countable $\langle N, \in\rangle \preceq \langle H(\chi), \in\rangle$ that contain $\mathbb{P}$ and such that $N \cap \aleph_1 \in S$, and all $p \in \mathbb{P} \cap N$, there is a condition $q \leq p$ that is $(N, \mathbb{P})$-generic. If $\mathbb{P}$ is $S$-proper, then it does not collapse $\aleph_1$. (See [8], or [4] for details.)

**Claim 10.** *The forcing $\mathbb{Q} \times T$ is $S_1$-proper, hence it does not collapse $\aleph_1$.*

*Proof of the claim.* Let $\chi$ be a large-enough regular cardinal, and let $<^*_\chi$ be a well-ordering of $H(\chi)$. Let $N \preceq \langle H(\chi), \in, <^*_\chi\rangle$ be countable and such that $\mathbb{Q} \times T$ belongs to $N$, $\delta := N \cap \aleph_1 \in S_1$, and the Mostowski collapse of $N$

belongs to $M_\delta$. Fix $(q_0, \eta_0) \in (\mathbb{Q} \times T) \cap N$. It will be sufficient to find a condition $\eta_* \in T$ such that $\eta_0 \leq_T \eta_*$ and $(q_0, \eta_*)$ is $(N, \mathbb{Q} \times T)$-generic.

Let

$$\mathbb{Q}_\delta := \{p \in \mathbb{Q} : \text{ if } \{\eta, \nu\} \in dom(p), \text{ then } \eta, \nu \in T_\delta\}.$$

Thus, $\mathbb{Q}_\delta$ is countable. Moreover, notice that $T_\delta = T \cap N$, and therefore $\mathbb{Q}_\delta = \mathbb{Q} \cap N$. Hence, $T_\delta$ and $\mathbb{Q}_\delta$ are the Mostowski collapses of $T$ and $\mathbb{Q}$, respectively, and so they belong to $M_\delta$.

In $M_\delta$, let $\langle (p_n, D_n) : n < \omega \rangle$ list all pairs $(p, D)$ such that $p \in \mathbb{Q}_\delta$, and $D$ is a dense open subset of $\mathbb{Q}_\delta \times T_\delta$ that belongs to the Mostowski collapse of $N$. That is, $D$ is the Mostowski collapse of a dense open subset of $\mathbb{Q} \times T$ that belongs to $N$. .

Also in $M_\delta$, fix an increasing sequence $\langle \delta_n : n < \omega \rangle$ converging to $\delta$, and let

$$D'_n := \{(p, \nu) \in D_n : lh(\nu) > \delta_n\}.$$

Clearly, $D'_n$ is dense open.

Note that, as the Mostowski collapse of $N$ belongs to $M_\delta$, we have that $<^*_\chi \restriction (\mathbb{Q}_\delta \times T_\delta) = (<^*_\chi \restriction (\mathbb{Q} \times T)) \cap N \in M_\delta$.

Now, still in $M_\delta$, and starting with $(q_0, \eta_0)$, we inductively choose a sequence $\langle (q_n, \eta_n) : n < \omega \rangle$, with $q_n \in \mathbb{Q}_\delta$ and $\eta_n \in T_\delta$, and such that if $n = m + 1$, then:

(a) $p_n \geq q_n$ and $\eta_m <_T \eta_n$.
(b) $(q_n, \eta_n) \in D'_n$.
(c) $(q_n, \eta_n)$ is the $<^*_\chi$-least such that (a) and (b) hold.

Then, $\eta_* := (\bigcup_n \eta_n) \cup \{\langle \delta, \delta \rangle\} \in T$, and $\eta^* \in M_\delta$, hence $(q_0, \eta_*) \in \mathbb{Q} \times T$. Clearly, $(q_0, \eta_*) \leq (q_0, \eta_0)$. So, we only need to check that $(q_0, \eta_*)$ is $(N, \mathbb{Q} \times T)$-generic.

Fix an open dense $E \subseteq \mathbb{Q} \times T$ that belongs to $N$. We need to see that $E \cap N$ is predense below $(q_0, \eta_*)$. So, fix $(r, \nu) \leq (q_0, \eta_*)$. Since $\mathbb{Q}$ is ccc, $q_0$ is $(N, \mathbb{Q})$-generic, so we can find $r' \in \{p : (p, \eta) \in E, \text{ some } \eta\} \cap N$ that is compatible with $r$. Let $n$ be such that $p_n = r'$ and $D_n$ is the Mostowski collapse of $E$. Then $(p_n, \eta_n)$ belongs to the transitive collapse of $E$, hence to $E \cap N$, and is compatible with $(r, \nu)$, as $(p_n, \eta_*) \leq (p_n, \eta_n)$. $\square$

We thus conclude that if $G \subseteq \mathbb{Q}$ is a filter generic over $V$, then in $V[G]$ the forcing $T$ does not collapse $\aleph_1$, and therefore, being of cardinality $\aleph_1$, it preserves cardinals, cofinalities, and the cardinal arithmetic.

We shall now define the $\mathbb{Q}$-names for the forcing notions $\underset{\sim}{\mathbb{P}}_\ell$, for $\ell \in \{0, 1\}$, as follows: in $V^{\mathbb{Q}}$, let $\underset{\sim}{b} = \bigcup \underset{\sim}{G}$, where $\underset{\sim}{G}$ is the standard $\mathbb{Q}$-name for the $\mathbb{Q}$-generic filter over $V$. Then let

$$\underset{\sim}{\mathbb{P}}_\ell := \{(w, c) : w \subseteq T \text{ is finite, } c \text{ is a function from } w \text{ into } \omega \text{ such that if } \{\eta, \nu\} \in [w]^2_T \text{ and } \underset{\sim}{b}(\{\eta, \nu\}) = \ell, \text{ then } c(\eta) \neq c(\nu)\}.$$

A condition $(w, c)$ is stronger than a condition $(v, d)$ if and only if $w \supseteq v$ and $c \supseteq d$.

We shall show that if $G$ is $\mathbb{Q}$-generic over $V$, then in the extension $V[G]$, the partial orderings $\mathbb{P}_\ell = \underset{\sim}{\mathbb{P}}_\ell[G]$, for $\ell \in \{0, 1\}$, and $T$ are as required.

**Claim 11.** *In $V[G]$, $\mathbb{P}_\ell$ has precalibre-$\aleph_1$.*

*Proof of the claim.* Assume $p_\alpha = (w_\alpha, c_\alpha) \in \mathbb{P}_\ell$, for $\alpha < \omega_1$. We shall find an uncountable $S \subseteq \aleph_1$ such that $\{p_\alpha : \alpha \in S\}$ is finite-wise compatible. For each $\delta \in S_2$, let

$$s_\delta := \{\eta \restriction (\gamma+1) : \eta \in w_\delta, \text{ and } \gamma \text{ is maximal such that } \gamma < lh(\eta) \wedge \eta(\gamma) < \delta\}.$$

As $\eta$ is an increasing and continuous sequence of ordinals from $S_1$, hence disjoint from $S_2$, the set $s_\delta$ is well-defined. Notice that $s_\delta$ is a finite subset of $T_\delta := \{\eta \in T : supRange(\eta) < \delta\}$, which is countable.

Let $s^1_\delta := w_\delta \cap T_\delta$. Note that $s^1_\delta \subseteq s_\delta$.

Let $f : S_2 \to \omega_1$ be given by $f(\delta) = max\{supRange(\eta) : \eta \in s_\delta\}$. Thus, $f$ is regressive, hence constant on a stationary $S_3 \subseteq S_2$. Let $\delta_0$ be the constant value of $f$ on $S_3$. Then, $s_\delta \subseteq T_{\delta_0}$, for every $\delta \in S_3$. So, since $T_{\delta_0}$ is countable, there exist $S_4 \subseteq S_3$ stationary and $s_*$ such that $s_\delta = s_*$, for every $\delta \in S_4$. Further, there is a stationary $S_5 \subseteq S_4$ and $s^1_*$ and $c_*$ such that for all $\delta \in S_5$,

$$s^1_\delta = s^1_*, \; c_\delta \restriction s^1_* = c_*, \text{ and } \forall \alpha < \delta (w_\alpha \subseteq T_\delta).$$

Hence, if $\delta_1 < \delta_2$ are from $S_5$, then not only $w_{\delta_1} \cap w_{\delta_2} = s^1_*$, but also if $\eta_1 \in w_{\delta_1} - s^1_*$ and $\eta_2 \in w_{\delta_2} - s^1_*$, then $\eta_1$ and $\eta_2$ are $<_T$-incomparable: for suppose otherwise, say $\eta_1 <_T \eta_2$. If $\gamma + 1 = lh(\eta_1)$, then $\eta_2 \restriction (\gamma+1) = \eta_1 <_T \eta_2$, and $\eta_2(\gamma) = \eta_1(\gamma) < \delta_2$, by choice of $S_5$. Hence, by the definition of $s_{\delta_2}$, $\eta_2 \restriction (\gamma+1) = \eta_1$ is an initial segment of some member of $s_{\delta_2} = s_*$, and so it belongs to $T_{\delta_1}$, hence $\eta_1 \in s^1_*$, contradicting the assumption that $\eta_1 \notin s^1_*$.

So, $\{p_\delta : \delta \in S_5\}$ is as required. $\square$

It only remains to show that forcing with $T$ over $V[G]$ preserves the ccc-ness of $\mathbb{P}_0$ and $\mathbb{P}_1$, but makes their product not ccc.

**Claim 12.** *If $G_T$ is $T$-generic over $V[G]$, then in the generic extension $V[G][G_T]$, the forcing $\mathbb{P}_\ell$ is ccc.*

*Proof of the claim.* First notice that, by the Product Lemma (see [6], 15.9), $G$ is $\mathbb{Q}$-generic over $V[G_T]$, and $V[G][G_T] = V[G_T][G]$. Now suppose $\underset{\sim}{A} = \{(\underset{\sim}{w}_\alpha, \underset{\sim}{c}_\alpha) : \alpha < \omega_1\} \in V[G_T]$ is a $\mathbb{Q}$-name for an uncountable subset of $\mathbb{P}_\ell$. For each $\alpha < \omega_1$, let $p_\alpha \in \mathbb{Q}$ and $(w_\alpha, c_\alpha)$ be such that $p_\alpha \Vdash$ "$(\underset{\sim}{w}_\alpha, \underset{\sim}{c}_\alpha) = (w_\alpha, c_\alpha)$". Let $u_\alpha$ be such that $dom(p_\alpha) = [u_\alpha]^2_T$. By extending $p_\alpha$, if necessary, we may assume that $w_\alpha \subseteq u_\alpha$, for all $\alpha < \omega_1$. We shall find $\alpha \neq \beta$ and a condition $p$ that extends both $p_\alpha$ and $p_\beta$ and forces that $(w_\alpha, c_\alpha)$ and $(w_\beta, c_\beta)$ are compatible. For this, first extend $(w_\alpha, c_\alpha)$ to $(u_\alpha, d_\alpha)$ by letting $d_\alpha$ give different values in $\omega \setminus Range(c_\alpha)$ to all $\eta \in u_\alpha \setminus w_\alpha$. We may assume that the set $\{u_\alpha : \alpha < \omega_1\}$ forms a $\Delta$-system with root $r$. Moreover, we may assume that $p_\alpha$ restricted to $[r]^2_T$ is the same for all $\alpha < \omega_1$, and also that $d_\alpha$ restricted to $r$ is the same for all $\alpha < \omega_1$. Now pick $\alpha \neq \beta$ and let $p : [u_\alpha \cup u_\beta]^2_T \to \{0,1\}$ be such that $p \restriction [u_\alpha]^2_T = p_\alpha$, $p \restriction [u_\beta]^2_T = p_\beta$, and $p(\{\eta, \nu\}) \neq \ell$, for all other pairs in $[u_\alpha \cup u_\beta]^2_T$. Then, $p$ extends both $p_\alpha$ and $p_\beta$, and forces that $(u_\alpha, d_\alpha)$ and $(u_\beta, d_\beta)$ are compatible, hence it forces that $(w_\alpha, c_\alpha)$ and $(w_\beta, c_\beta)$ are compatible. $\square$

But in $V[G][G_T]$, the product $\mathbb{P}_0 \times \mathbb{P}_1$ is not ccc. For let $\eta^* = \bigcup G_T$. For every $\alpha < \omega_1$, let $p^\ell_\alpha := (\{\eta^* \restriction (\alpha+1)\}, c^\ell_\alpha) \in \mathbb{P}_\ell$, where $c^\ell_\alpha(\eta^* \restriction (\alpha+1)) = 0$. Then the set $\{(p^0_\alpha, p^1_\alpha) : \alpha < \omega_1\}$ is an uncountable antichain.

$\square$

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